Edexcel
GCE A Level Maths

Further Maths 3

Vectors

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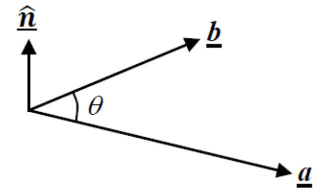
5 Vectors

Vector product

The vector, or cross, product of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ is

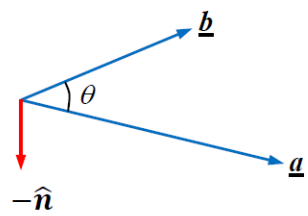
$$\underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}} = ab \sin\theta \, \underline{\hat{\boldsymbol{n}}}$$

where $\underline{\hat{\boldsymbol{n}}}$ is a *unit* (length 1) vector which is *perpendicular* to both $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$, and θ is the angle between $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$.



The direction of $\underline{\hat{\boldsymbol{n}}}$ is that in which a right hand corkscrew would move when turned through the angle θ from $\underline{\boldsymbol{a}}$ to $\underline{\boldsymbol{b}}$.

Notice that $\underline{\boldsymbol{b}} \times \underline{\boldsymbol{a}} = ab \sin\theta \, (-\underline{\hat{\boldsymbol{n}}})$, where $-\underline{\hat{\boldsymbol{n}}}$ is in the opposite direction to $\underline{\hat{\boldsymbol{n}}}$, since the corkscrew would move in the opposite direction when moving from $\underline{\boldsymbol{b}}$ to $\underline{\boldsymbol{a}}$.



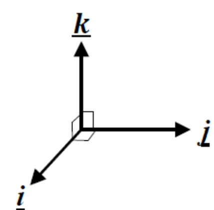
Thus $\underline{\boldsymbol{b}} \times \underline{\boldsymbol{a}} = -\underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}}$.

The vectors $\underline{\boldsymbol{i}}, \underline{\boldsymbol{j}}$ and $\underline{\boldsymbol{k}}$

For unit vectors, $\underline{\boldsymbol{i}}, \underline{\boldsymbol{j}}$ and $\underline{\boldsymbol{k}}$, in the directions of the axes

$\underline{\boldsymbol{i}} \times \underline{\boldsymbol{j}} = \underline{\boldsymbol{k}}, \quad \underline{\boldsymbol{j}} \times \underline{\boldsymbol{k}} = \underline{\boldsymbol{i}}, \quad \underline{\boldsymbol{k}} \times \underline{\boldsymbol{i}} = \underline{\boldsymbol{j}},$

$\underline{\boldsymbol{i}} \times \underline{\boldsymbol{k}} = -\underline{\boldsymbol{j}}, \quad \underline{\boldsymbol{j}} \times \underline{\boldsymbol{i}} = -\underline{\boldsymbol{k}}, \quad \underline{\boldsymbol{k}} \times \underline{\boldsymbol{j}} = -\underline{\boldsymbol{i}}.$



Properties

$\underline{\boldsymbol{a}} \times \underline{\boldsymbol{a}} = \underline{\mathbf{0}}$ — since $\theta = 0$

$\underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}} = \underline{\mathbf{0}} \quad \Rightarrow \quad \underline{\boldsymbol{a}}$ is parallel to $\underline{\boldsymbol{b}}$ — since $\sin\theta = 0 \;\Rightarrow\; \theta = 0$ or π

or $\underline{\boldsymbol{a}}$ or $\underline{\boldsymbol{b}} = \underline{\mathbf{0}}$

$\underline{\boldsymbol{a}} \times (\underline{\boldsymbol{b}} + \underline{\boldsymbol{c}}) = \underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}} + \underline{\boldsymbol{a}} \times \underline{\boldsymbol{c}}$ — remember the brilliant demo with the straws!

$\underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}}$ is perpendicular to both $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ — from the definition

Component form

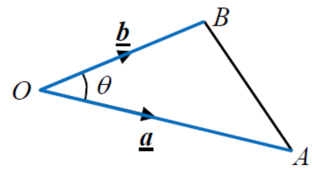
Using the above we can show that

$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Applications of the vector product

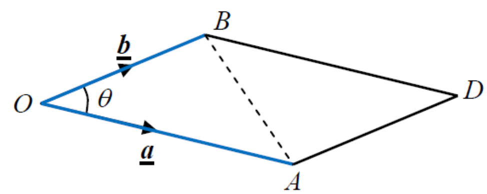
Area of triangle $OAB = \frac{1}{2} ab \sin\theta$

$\Rightarrow$ area of triangle $OAB = \frac{1}{2} \left| \underline{\mathbf{a}} \times \underline{\mathbf{b}} \right|$



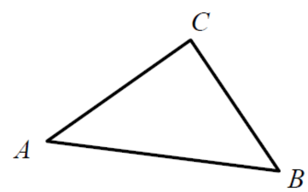
Area of parallelogram $OADB$ is twice the area of the triangle OAB

$\Rightarrow$ area of parallelogram $OADB = \left| \underline{\mathbf{a}} \times \underline{\mathbf{b}} \right|$



Example: A is $(-1, 2, 1)$, B is $(2, 3, 0)$ and C is $(3, 4, -2)$.

Find the area of the triangle ABC.



Solution: The area of the triangle $ABC = \left| \frac{1}{2} \overrightarrow{AB} \times \overrightarrow{AC} \right|$

$$\overrightarrow{AB} = \underline{\mathbf{b}} - \underline{\mathbf{a}} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \text{ and } \overrightarrow{AC} = \underline{\mathbf{c}} - \underline{\mathbf{a}} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$$

$$\Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 3 & 1 & -1 \\ 4 & 2 & -3 \end{vmatrix} = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}$$

$$\Rightarrow \text{area } ABC = \left| \frac{1}{2} \overrightarrow{AB} \times \overrightarrow{AC} \right| = \frac{1}{2}\sqrt{1^2 + 5^2 + 3^2} = \frac{1}{2}\sqrt{35}$$

Volume of a parallelepiped

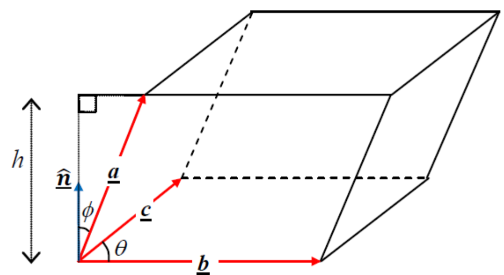
In the parallelepiped,

the base is parallel to $\underline{\boldsymbol{b}}$ and $\underline{\boldsymbol{c}}$

$\underline{\hat{\boldsymbol{n}}}$ is a unit vector perpendicular to the base

and the height $\underline{\boldsymbol{h}} = h\,\underline{\hat{\boldsymbol{n}}}$,

where $h = \pm a \cos\phi = \pm\,\underline{\boldsymbol{a}}\,.\,\underline{\hat{\boldsymbol{n}}}$
$\pm$ because ϕ might be obtuse



The area of base $= bc\sin\theta$

$\Rightarrow$ volume $\quad V = \pm h \times bc\sin\theta$

$\Rightarrow \qquad \pm V = a\cos\phi \times bc\sin\theta$

$$\underline{\boldsymbol{a}}\,.\,(\underline{\boldsymbol{b}}\times\underline{\boldsymbol{c}}) = \underline{\boldsymbol{a}}\,.\,(bc\sin\theta\ \underline{\hat{\boldsymbol{n}}}) = \underline{\boldsymbol{a}}\,.\,\underline{\hat{\boldsymbol{n}}}\,(bc\sin\theta\,)$$

$\Rightarrow\ \underline{\boldsymbol{a}}\,.\,(\underline{\boldsymbol{b}}\times\underline{\boldsymbol{c}}) = a\cos\phi \times bc\sin\theta = \pm V$

$\Rightarrow$ volume of parallelepiped $= \left|\underline{\boldsymbol{a}}\,.\,(\underline{\boldsymbol{b}}\times\underline{\boldsymbol{c}})\right|$

Triple scalar product

$$\left|\underline{\boldsymbol{a}}\,.\,(\underline{\boldsymbol{b}}\times\underline{\boldsymbol{c}})\right| = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\cdot\begin{pmatrix} b_2c_3 - b_3c_2 \\ -b_1c_3 + b_3c_1 \\ b_1c_2 - b_2c_1 \end{pmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(-b_1c_3 + b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

By expanding the determinants we can show that

$\underline{\boldsymbol{a}}\,.\,(\underline{\boldsymbol{b}}\times\underline{\boldsymbol{c}}) = (\underline{\boldsymbol{a}}\times\underline{\boldsymbol{b}})\,.\,\underline{\boldsymbol{c}}$ keep the order of $\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}, \underline{\boldsymbol{c}}$ but change the order of the $\times$ and **.**

For this reason the triple scalar product is written as $\{\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}, \underline{\boldsymbol{c}}\}$

$$\{\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}, \underline{\boldsymbol{c}}\} = \underline{\boldsymbol{a}}\,.\,(\underline{\boldsymbol{b}}\times\underline{\boldsymbol{c}}) = (\underline{\boldsymbol{a}}\times\underline{\boldsymbol{b}})\,.\,\underline{\boldsymbol{c}}$$

It can also be shown that a cyclic change of the order of $\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}, \underline{\boldsymbol{c}}$ does not change the value, but interchanging two of the vectors multiplies the value by -1.

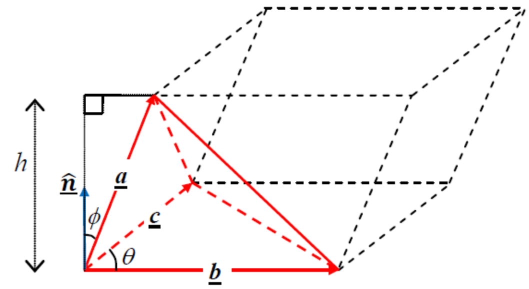
$$\Rightarrow \quad \{\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}, \underline{\boldsymbol{c}}\} = \{\underline{\boldsymbol{c}}, \underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}\} = \{\underline{\boldsymbol{b}}, \underline{\boldsymbol{c}}, \underline{\boldsymbol{a}}\} = -\{\underline{\boldsymbol{a}}, \underline{\boldsymbol{c}}, \underline{\boldsymbol{b}}\} = -\{\underline{\boldsymbol{c}}, \underline{\boldsymbol{b}}, \underline{\boldsymbol{a}}\} = -\{\underline{\boldsymbol{b}}, \underline{\boldsymbol{a}}, \underline{\boldsymbol{c}}\}$$

Volume of a tetrahedron

The volume of a tetrahedron is

$$\frac{1}{3}\text{ Area of base } \times h$$

The height of the tetrahedron is the same as the height of the parallelepiped, but its base has half the area



$\Rightarrow$ volume of tetrahedron $= \frac{1}{6}$ volume of parallelepiped

$\Rightarrow$ volume of tetrahedron $= \frac{1}{6}|\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}|$

Example: Find the volume of the tetrahedron $ABCD$,

given that A is (1, 0, 2), B is (–1, 2, 2), C is (1, 1, –3) and D is (4, 0, 3).

Solution: Volume $= \frac{1}{6}|\{\overrightarrow{AD}, \overrightarrow{AC}, \overrightarrow{AB}\}|$

$$\overrightarrow{AD} = \underline{\boldsymbol{d}} - \underline{\boldsymbol{a}} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}, \quad \overrightarrow{AB} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \{\overrightarrow{AD}, \overrightarrow{AC}, \overrightarrow{AB}\} = \begin{vmatrix} 3 & 0 & 1 \\ 0 & 1 & -5 \\ -2 & 2 & 0 \end{vmatrix} = 3 \times 10 + 2 = 32$$

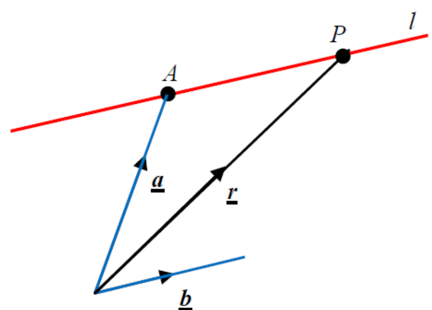
$\Rightarrow$ volume of tetrahedron is $\frac{1}{6} \times 32 = 5\frac{1}{3}$

Equations of straight lines

Vector equation of a line

$\underline{\boldsymbol{r}} = \underline{\boldsymbol{a}} + \lambda \underline{\boldsymbol{b}}$ is the equation of a line through the point A and parallel to the vector $\underline{\boldsymbol{b}}$,

or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l \\ m \\ n \end{pmatrix} + \lambda \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$.



Cartesian equation of a line in 3-D

Eliminating λ from the above equation we obtain

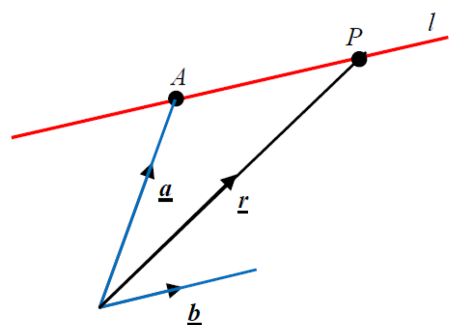
$$\frac{x-l}{\alpha} = \frac{y-m}{\beta} = \frac{z-n}{\gamma} \quad (= \lambda)$$

is the equation of a line through the point (l, m, n) and parallel to the vector $\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$.

This strange form of equation is really the intersection of the planes

$$\frac{x-l}{\alpha} = \frac{y-m}{\beta} \quad \text{and} \quad \frac{y-m}{\beta} = \frac{z-n}{\gamma} \qquad \left(\text{and } \frac{x-l}{\alpha} = \frac{z-n}{\gamma}\right).$$

Vector product equation of a line



$\overrightarrow{AP} = \underline{\boldsymbol{r}} - \underline{\boldsymbol{a}}$ and is parallel to the vector $\underline{\boldsymbol{b}}$

$\Rightarrow \quad \overrightarrow{AP} \times \underline{\boldsymbol{b}} = \underline{\boldsymbol{0}}$

$\Rightarrow \quad (\underline{\boldsymbol{r}} - \underline{\boldsymbol{a}}) \times \underline{\boldsymbol{b}} = \underline{\boldsymbol{0}}$ is the equation of a line through A and parallel to $\underline{\boldsymbol{b}}$.

or $\quad \underline{\boldsymbol{r}} \times \underline{\boldsymbol{b}} = \underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}} = \underline{\boldsymbol{c}}$ is the equation of a line parallel to $\underline{\boldsymbol{b}}$.

Notice that all three forms of equation refer to *a line through the point A and parallel to the vector $\underline{\boldsymbol{b}}$*.

Example: A straight line has Cartesian equation

$$x = \frac{2y+4}{5} = \frac{3-z}{2}.$$

Find its equation (i) in the form $\underline{\boldsymbol{r}} = \underline{\boldsymbol{a}} + \lambda\underline{\boldsymbol{b}}$, (ii) in the form $\underline{\boldsymbol{r}} \times \underline{\boldsymbol{b}} = \underline{\boldsymbol{c}}$.

Solution:

First re-write the equation in the *standard* manner

$$\Rightarrow \quad \frac{x-0}{1} = \frac{y--2}{2.5} = \frac{z-3}{-2}$$

$\Rightarrow \quad$ the line passes through A, $(0, -2, 3)$, and is parallel to $\underline{\boldsymbol{b}}$, $\begin{pmatrix}1\\2.5\\-2\end{pmatrix}$ or $\begin{pmatrix}2\\5\\-4\end{pmatrix}$

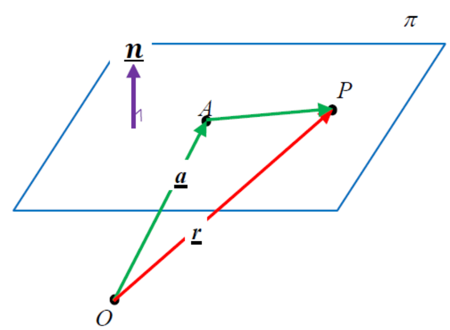
(i) $\quad \underline{r} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$

(ii) $\quad \left(\underline{r} - \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}\right) \times \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} = \underline{\mathbf{0}}$

$$\Rightarrow \quad \underline{r} \times \begin{pmatrix} 1 \\ 2.5 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 0 & -2 & 3 \\ 2 & 5 & -4 \end{vmatrix} = \begin{pmatrix} -7 \\ 6 \\ 4 \end{pmatrix}$$

$$\Rightarrow \quad \underline{r} \times \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} = \begin{pmatrix} -7 \\ 6 \\ 4 \end{pmatrix}.$$

Equation of a plane



Scalar product form

Let $\underline{n}$ be a vector perpendicular to the plane π.

Let A be a fixed point in the plane, and P be a general point, (x, y, z), in the plane.

Then $\overrightarrow{AP}$ is parallel to the plane, and therefore perpendicular to $\underline{n}$

$\Rightarrow \quad \overrightarrow{AP} \cdot \underline{n} = 0 \quad \Rightarrow \quad (\underline{r} - \underline{a}) \cdot \underline{n} = 0$

$\Rightarrow \quad \underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n} =$ a constant, d

$\Rightarrow \quad \underline{r} \cdot \underline{n} = d$ is the equation of a plane **perpendicular to the vector** $\underline{n}$.

Cartesian form

If $\underline{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then $\underline{r} \cdot \underline{n} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = ax + by + cz$

$\Rightarrow \quad ax + by + cz = d$ is the Cartesian equation of a plane **perpendicular to** $\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix}$.

Example: Find the scalar product form and the Cartesian equation of the plane through the points A, (3, 2, 5), B, (–1, 0, 3) and C, (2, 1, –2).

Solution: We first need a vector perpendicular to the plane.

A, (3, 2, 5), B, (–1, 0, 3) and C, (2, 1, –2) lie in the plane

$\Rightarrow \quad \overrightarrow{AB} = \begin{pmatrix} -4 \\ -2 \\ -2 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -1 \\ -1 \\ -7 \end{pmatrix}$ are parallel to the plane

$\Rightarrow \quad \overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to the plane

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{\boldsymbol{i}} & \underline{\boldsymbol{j}} & \underline{\boldsymbol{k}} \\ -4 & -2 & -2 \\ -1 & -1 & -7 \end{vmatrix} = \begin{pmatrix} 12 \\ -26 \\ 2 \end{pmatrix} = 2 \times \begin{pmatrix} 6 \\ -13 \\ 1 \end{pmatrix}$$ using smaller numbers

$\Rightarrow \quad 6x - 13y + z = d$

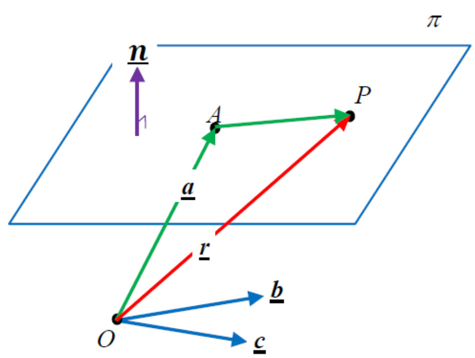
but A, (3, 2, 5) lies in the plane $\quad \Rightarrow \quad d = 6 \times 3 - 13 \times 2 + 5 = -3$

$\Rightarrow$ *Cartesian equation* is $6x - 13y + z = -3$

and *scalar product* equation is $\underline{\boldsymbol{r}} \cdot \begin{pmatrix} 6 \\ -13 \\ 1 \end{pmatrix} = -3$.

Vector equation of a plane

$\underline{\boldsymbol{r}} = \underline{\boldsymbol{a}} + \lambda\, \underline{\boldsymbol{b}} + \mu\, \underline{\boldsymbol{c}}$ is the equation of a plane, π, through A and parallel to the vectors $\underline{\boldsymbol{b}}$ and $\underline{\boldsymbol{c}}$.

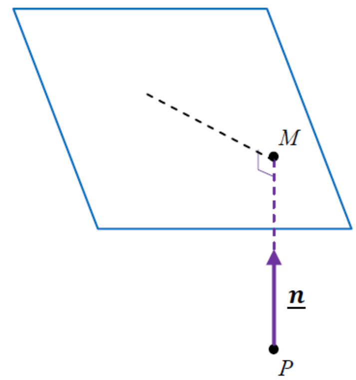


Example: Find the vector equation of the plane through the points A, (1, 4, –2), B, (1, 5, 3) and C, (4, 7, 2).

Solution: We want the plane through A, (1, 4, –2), parallel to $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$

$\Rightarrow$ vector equation is $\underline{\boldsymbol{r}} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$.

Distance from a point to a plane



Example: Find the distance from the point P $(-2, 3, 5)$ to the plane $4x - 3y + 12z = 21$.

Solution: Let M be the foot of the perpendicular from P to the plane. The distance of the origin from the plane is PM.

We must first find the intersection of the line PM with the plane.

PM is perpendicular to the plane

and so is parallel to $\underline{n} = \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix}$.

$\Rightarrow$ the line PM is $\underline{r} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} = \begin{pmatrix} -2+4\lambda \\ 3-3\lambda \\ 5+12\lambda \end{pmatrix}$,

and the point of intersection of PM with the plane is given by

$$4(-2+4\lambda) - 3(3-3\lambda) + 12(5+12\lambda) = 21$$

$\Rightarrow$ $-8 + 16\lambda - 9 + 9\lambda + 60 + 144\lambda = 21$

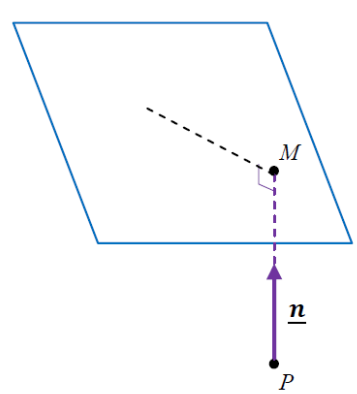
$\Rightarrow$ $\lambda = \frac{-22}{169}$

$\Rightarrow$ $\overrightarrow{PM} = \frac{-22}{169}\begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix}$

$\Rightarrow$ distance $= \left|\overrightarrow{PM}\right| = \frac{22}{169}\sqrt{4^2 + 3^2 + 12^2} = \frac{22}{13}$

The distance of the P from the plane is $\frac{22}{13}$.

Distance from any point to a plane



The above technique can be used to find the formula:-

distance, s, from the point P (α, β, γ) to the plane

$n_1x + n_2y + n_3z + d = 0$ is given by

$$s = \left|\frac{n_1\alpha + n_2\beta + n_3\gamma + d}{\sqrt{n_1{}^2 + n_2{}^2 + n_3{}^2}}\right|$$

This formula is in your formula booklets, but **not** in your text books.

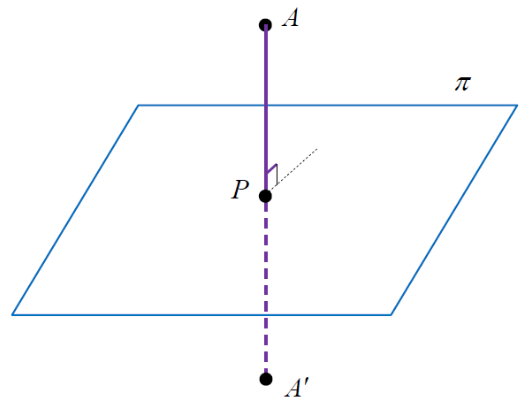
Reflection of a point in a plane

Example: Find the reflection of the point A (10, 1, 7) in the plane π, $\underline{r}.\begin{pmatrix}3\\-2\\1\end{pmatrix} = 7$.

Solution: Find the point of intersection, P, of the line through A and perpendicular to π with the plane π. Then find $\overrightarrow{AP}$, to give $\overrightarrow{OA'} = \overrightarrow{OA} + 2\overrightarrow{AP}$.

Line through A perpendicular to π is

$$\underline{r} = \begin{pmatrix}10\\1\\7\end{pmatrix} + \lambda\begin{pmatrix}3\\-2\\1\end{pmatrix}$$



This meets the plane π when

$3(10+3\lambda) - 2(1-2\lambda) + (7+\lambda) = 7$

$\Rightarrow$ $30 + 9\lambda - 2 + 4\lambda + 7 + \lambda = 7$

$\Rightarrow$ $\lambda = -2$

$\Rightarrow$ $\overrightarrow{OP} = \begin{pmatrix}10\\1\\7\end{pmatrix} + (-2)\begin{pmatrix}3\\-2\\1\end{pmatrix}$

$\Rightarrow$ $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (-2)\begin{pmatrix}3\\-2\\1\end{pmatrix} = \begin{pmatrix}-6\\4\\-2\end{pmatrix}$

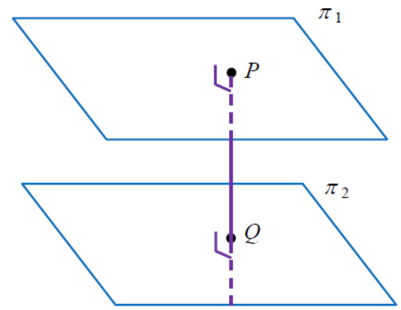
$\Rightarrow$ $\overrightarrow{OA'} = \overrightarrow{OA} + 2\overrightarrow{AP} = \begin{pmatrix}10\\1\\7\end{pmatrix} + 2\begin{pmatrix}-6\\4\\-2\end{pmatrix} = \begin{pmatrix}-2\\9\\3\end{pmatrix}$

$\Rightarrow$ the reflection of A is A', (–2, 9, 3)

Distance between parallel planes

Example: Find the distance between the parallel planes

π_1: $2x - 6y + 3z = 9$ and π_2: $2x - 6y + 3z = 5$



Solution: Take any point, P, on one of the planes, and then use the above formula for the shortest distance, PQ, between the planes.

By inspection the point P (0, 0, 3) lies on π_1

$\Rightarrow$ shortest distance s from P to the plane π_2 is $\left|\frac{n_1\alpha + n_2\beta + n_3\gamma + d}{\sqrt{{n_1}^2 + {n_2}^2 + {n_3}^2}}\right|$

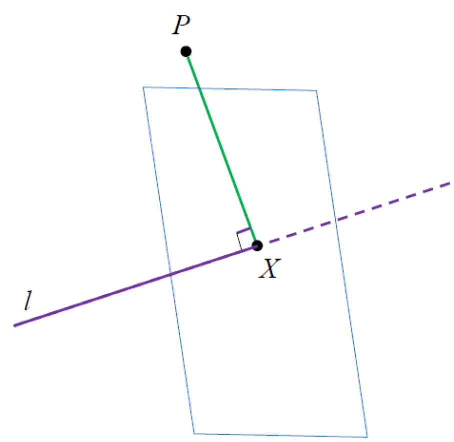
$$\Rightarrow \quad \text{shortest distance } s = \left|\frac{2\times 0 - 6\times 0 + 3\times 3 - 5}{\sqrt{2^2 + 6^2 + 3^2}}\right| = \frac{4}{7}$$

The distance between the planes is $\frac{4}{7}$.

Shortest distance from a point to a line

Example: Find the shortest distance from the point

P (3, –2, 4) to the line l, $\underline{r} = \begin{pmatrix}-2\\3\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\6\end{pmatrix}$



Solution: Any plane $2x - 3y + 6z = d$ must be perpendicular to the line l. If we make this plane pass through P and if it meets the line l in the point X, then PX must be perpendicular to the line l, and so PX is the shortest distance from P to the line l.

Plane passes through P (3, –2, 4)

$\Rightarrow$ $2x - 3y + 6z = 2 \times 3 - 3 \times (-2) + 6 \times 4 = 36$

$\Rightarrow$ $2x - 3y + 6z = 36$

l meets plane $\Rightarrow$ $2(-2 + 2\lambda) - 3(3 - 3\lambda) + 6(6\lambda) = 36$

$\Rightarrow$ $-4 + 4\lambda - 9 + 9\lambda + 36\lambda = 36$ $\Rightarrow$ $\lambda = 1$

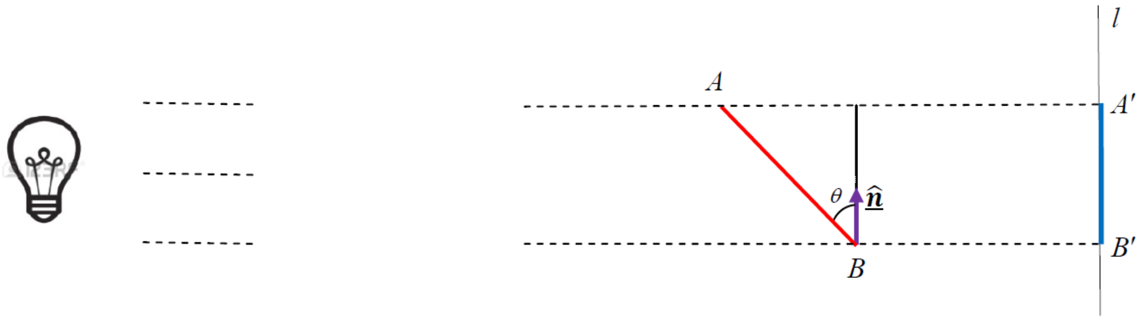
$\Rightarrow$ X is the point (–2, 0, 6)

$$\overrightarrow{PX} = \begin{pmatrix}0\\0\\6\end{pmatrix} - \begin{pmatrix}3\\-2\\4\end{pmatrix} = \begin{pmatrix}-3\\2\\2\end{pmatrix}$$

$\Rightarrow$ shortest distance is $PX = \sqrt{3^2 + 2^2 + 2^2} = \sqrt{17}$

Projections – an alternative approach

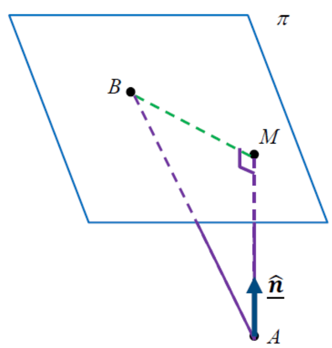
Imagine a light bulb causing a rod, AB, to make a shadow, $A'B'$, on the line l. If the light bulb is far enough away, we can think of all the light rays as parallel, and, if the rays are all perpendicular to the line l, the shadow is the *projection* of the rod onto l (strictly speaking an *orthogonal* projection).



The length of the shadow, $B'A'$, is $|BA \cos \theta| = |\overrightarrow{BA} \cdot \hat{\underline{n}}|$, where $\hat{\underline{n}}$ is a unit vector parallel to the line l.

Modulus signs are needed in case $\hat{\underline{n}}$ is in the opposite direction.

Shortest distance from a point from a plane.



To find AM, the shortest distance from A to the plane π,

For any point, B, on π AM is the projection of AB onto the line AM

$$\Rightarrow \quad AM = |\overrightarrow{AB} \cdot \hat{\underline{n}}|$$

Example: Find the shortest distance from the point A (–2, 3, 5) to the plane $4x - 3y + 12z = 21$.

Solution: By inspection B (0, –7, 0) lies on the plane

$$\Rightarrow \quad \overrightarrow{AB} = \begin{pmatrix} 0 \\ -7 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -10 \\ -5 \end{pmatrix}$$

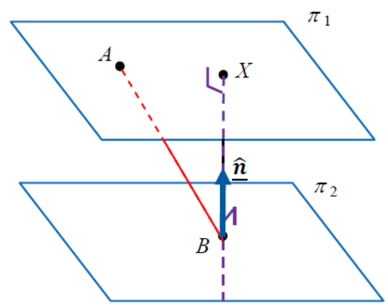
$$\underline{n} = \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} \quad \Rightarrow \quad n = \sqrt{4^2 + 3^2 + 12^2} = 13$$

$$\Rightarrow \quad \text{shortest distance} = |\overrightarrow{AB} \cdot \hat{\underline{n}}| = \left| \begin{pmatrix} 2 \\ -10 \\ -5 \end{pmatrix} \cdot \frac{1}{13} \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} \right| = \frac{22}{13}$$

Distance between parallel planes

Example: Find the distance between the parallel planes

π_1: $2x - 6y + 3z = 9$ and π_2: $2x - 6y + 3z = 5$



Solution: Take any point, B, on one of the planes, π_2, and then consider the line BX perpendicular to both planes; BX is then the shortest distance between the planes.

Then choose any point, A, on π_1, and BX is now the projection of AB onto BX

$\Rightarrow$ shortest distance $= BX = |\overrightarrow{AB} \,.\, \hat{\underline{\boldsymbol{n}}}|$

or shortest distance $= |(\underline{\boldsymbol{b}} - \underline{\boldsymbol{a}}).\hat{\underline{\boldsymbol{n}}}|$, for any two points A and B, one on each plane, where $\hat{\underline{\boldsymbol{n}}}$ is a unit vector perpendicular to both planes.

By inspection the point A (0, 0, 3) lies on π_1, and the point B (2·5, 0, 0) lies on π_2

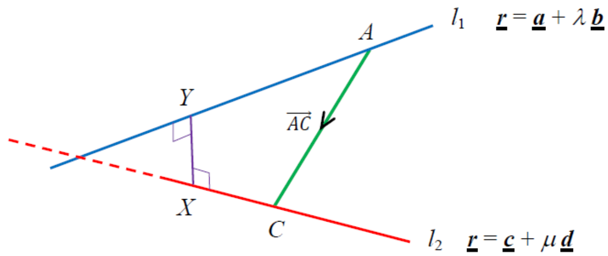
$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 2\cdot5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2\cdot5 \\ 0 \\ 3 \end{pmatrix}$$

$$\underline{n} = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} \quad \Rightarrow \quad n = \sqrt{2^2 + 6^2 + 3^2} = 7$$

$$\Rightarrow \quad \text{shortest distance} = \left| \begin{pmatrix} -2\cdot5 \\ 0 \\ 3 \end{pmatrix} . \tfrac{1}{7} \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} \right| = \tfrac{4}{7}$$

Shortest distance between two skew lines

It can be shown that there must be a line joining two skew lines which is perpendicular to both lines.



This line is XY and is the shortest distance between the lines.

The vector $\underline{n} = \underline{b} \times \underline{d}$ is perpendicular to both lines

$\Rightarrow$ the unit vector $\hat{\underline{n}} = \dfrac{\underline{b}\times\underline{d}}{|\underline{b}\times\underline{d}|}$

Now imagine two parallel planes π_1 and π_2, both perpendicular to $\hat{\underline{n}}$, one containing the line l_1 and the other containing the line l_2.

A and C are points on l_1 and l_2, and therefore on π_1 and π_2.

We now have two parallel planes with two points, A and C, one on each plane, and the planes are both perpendicular to $\hat{\underline{n}}$.

As in the example for the distance between parallel planes,

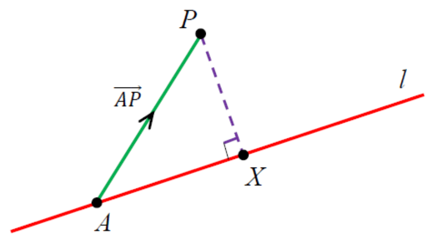
the shortest distance $d = |\overrightarrow{AC} \cdot \hat{\underline{n}}|$

$$\Rightarrow \quad d = \left|(\underline{c} - \underline{a}) \cdot \frac{\underline{b} \times \underline{d}}{|\underline{b} \times \underline{d}|}\right|$$

This result is not in your formula booklet, SO LEARN IT – please

Shortest distance from a point to a line

In trying to find the shortest distance from a point P to a line l, $\underline{r} = \underline{a} + \lambda\underline{b}$, we do not know $\hat{\underline{n}}$, the direction of the line through P perpendicular to l.



Some lateral thinking is needed.

We do know A, a point on the line, and $\hat{\underline{b}}$, the direction of the line l

$\Rightarrow$ $|\overrightarrow{AP}\,.\,\hat{\underline{b}}| = AX$, the projection of AP onto l

and we can now find $PX = \sqrt{AP^2 - AX^2}$, using Pythagoras

Example: Find the shortest distance from the point P (3, –2, 4)

to the line l, $\boldsymbol{r} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

Solution: If l is $\underline{r} = \underline{a} + \lambda\underline{b}$, then $\underline{a} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

$\Rightarrow$ $b = \sqrt{2^2 + 3^2 + 6^2} = 7$, $\Rightarrow$ $\hat{\underline{b}} = \frac{1}{7}\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

and $\overrightarrow{AP} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 4 \end{pmatrix}$

$\Rightarrow$ $AX = |\overrightarrow{AP}\,.\,\hat{\underline{b}}| = \left|\begin{pmatrix} 5 \\ -5 \\ 4 \end{pmatrix} . \frac{1}{7}\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}\right| = \frac{10+15+24}{7} = 7$

$\Rightarrow$ $PX = \sqrt{AP^2 - AX^2} = \sqrt{(5^2 + 5^2 + 4^2) - 7^2}$

$= \sqrt{17}$

Line of intersection of two planes

Example: Find an equation for the line of intersection of the planes

$$x + y + 2z = 4 \qquad \textbf{I}$$

and

$$2x - y + 3z = 4 \qquad \textbf{II}$$

Solution: Eliminate one variable –

I + **II** $\Rightarrow$ $3x + 5z = 8$

We are *not* expecting a unique solution, so put one variable, z say, equal to λ and find the other variables in terms of λ.

$$z = \lambda \;\Rightarrow\; x = \frac{8-5\lambda}{3}$$

$$\textbf{I} \Rightarrow \quad y = 4 - x - 2z = 4 - \frac{8-5\lambda}{3} - 2\lambda = \frac{4-\lambda}{3}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8/3 \\ 4/3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -5/3 \\ -1/3 \\ 1 \end{pmatrix}$$

or $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8/3 \\ 4/3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ -1 \\ 3 \end{pmatrix}$$ making the numbers nicer in the **direction vector only**

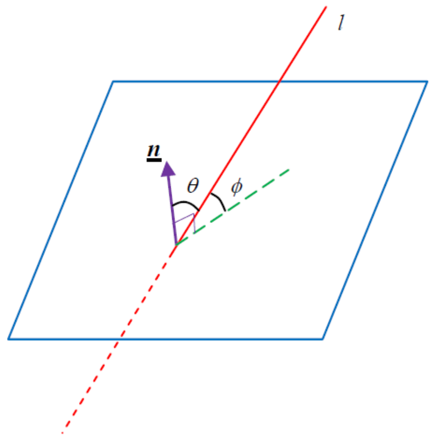
which is the equation of a line through $\left(\frac{8}{3}, \frac{4}{3}, 0\right)$ and parallel to $\begin{pmatrix} -5 \\ -1 \\ 3 \end{pmatrix}$.

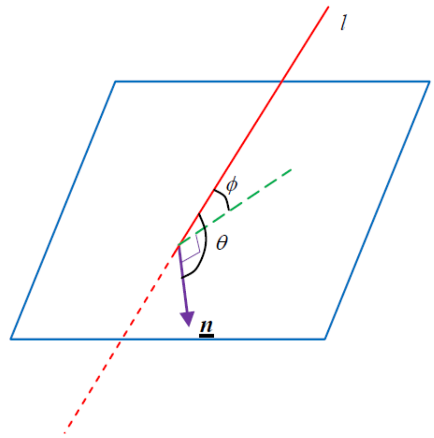
Angle between line and plane

Let the acute angle between the line and the plane be ϕ.

First find the angle between the line and the normal vector, θ.

There are two possibilities – as shown below:





(i) $\underline{n}$ and the angle ϕ are on the same side of the plane

$\Rightarrow \quad \phi = 90 - \theta$

(ii) $\underline{n}$ and the angle ϕ are on opposite sides of the plane

$\Rightarrow \quad \phi = \theta - 90$

Example: Find the angle between the line $\frac{x+1}{2} = \frac{y-2}{1} = \frac{z-3}{-2}$ and the plane $2x + 3y - 7z = 5$.

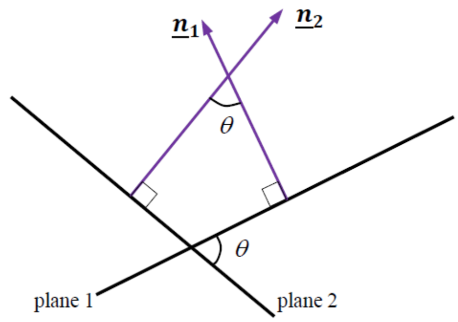
Solution: The line is parallel to $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, and the normal vector to the plane is $\begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix}$.

$$\underline{a} \cdot \underline{b} = ab \cos\theta \quad \Rightarrow \quad 21 = \sqrt{2^2 + 1^2 + 2^2}\,\sqrt{2^2 + 3^2 + 7^2}\ \cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{7}{\sqrt{62}} \quad \Rightarrow \quad \theta = 27.3^\circ$$

$\Rightarrow$ the angle between the line and the plane, $\phi = 90 - 27.3 = 62.7^\circ$

Angle between two planes



If we look 'end-on' at the two planes, we can see that the angle between the planes, θ, equals the angle between the normal vectors.

Example: Find the angle between the planes

$$2x + y + 3z = 5 \quad \text{and} \quad 2x + 3y + z = 7$$

Solution: The normal vectors are $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

$$\underline{a} \cdot \underline{b} = ab \cos\theta \quad \Rightarrow \quad 10 = \sqrt{2^2 + 1^2 + 3^2} \times \sqrt{2^2 + 1^2 + 3^2} \cos\theta$$

$$\Rightarrow \cos\theta = \frac{10}{14} \quad \Rightarrow \quad \theta = 44.4^\circ$$

1. The plane Π passes through the points

$$A(-1, -1, 1),\ B(4, 2, 1) \text{ and } C\ (2, 1, 0).$$

(*a*) Find a vector equation of the line perpendicular to Π which passes through the point D (1, 2, 3).

(3)

(*b*) Find the volume of the tetrahedron *ABCD*.

(3)

(*c*) Obtain the equation of Π in the form $\mathbf{r.n} = p$.

(3)

The perpendicular from D to the plane Π meets Π at the point E.

(*d*) Find the coordinates of E.

(4)

(*e*) Show that $DE = \dfrac{11\sqrt{35}}{35}$.

(2)

The point D' is the reflection of D in Π.

(*f*) Find the coordinates of D'.

(3)

[P6 June 2002 Qn 7]

2. Referred to a fixed origin O, the position vectors of three non-collinear points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. By considering $\overrightarrow{AB} \times \overrightarrow{AC}$, prove that the area of ΔABC can be expressed in the form $\frac{1}{2}\left|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}\right|$.

(5)

[P6 June 2003 Qn 1]

3. The plane Π_1 passes through the P, with position vector $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, and is perpendicular to the line L with equation

$$\mathbf{r} = 3\mathbf{i} - 2\mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}).$$

(*a*) Show that the Cartesian equation of Π_1 is $x - 5y - 3z = -6$.

(4)

The plane Π_2 contains the line L and passes through the point Q, with position vector $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

(*b*) Find the perpendicular distance of Q from Π_1.

(4)

(*c*) Find the equation of Π_2 in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$.

(4)

[P6 June 2003 Qn 7]

4. The points A, B and C lie on the plane Π and, relative to a fixed origin O, they have position vectors

$$\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}, \quad \mathbf{b} = -\mathbf{i} + 2\mathbf{j}, \quad \mathbf{c} = 5\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$$

respectively.

(*a*) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(4)

(*b*) Find an equation of Π in the form $\mathbf{r.n} = p$.

(2)

The point D has position vector $5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

(*c*) Calculate the volume of the tetrahedron $ABCD$.

(4)

[P6 June 2004 Qn 3]

5. (*a*) (i) Explain why, for any two vectors **a** and **b**, $\mathbf{a.b} \times \mathbf{a} = 0$.

(2)

(ii) Given vectors **a**, **b** and **c** such that $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, where $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{c}$, show that

$$\mathbf{b} - \mathbf{c} = \lambda\mathbf{a}, \quad \text{where } \lambda \text{ is a scalar.}$$

(2)

(*b*) **A**, **B** and **C** are 2×2 matrices.

(i) Given that $\mathbf{AB} = \mathbf{AC}$, and that **A** is not singular, prove that $\mathbf{B} = \mathbf{C}$.

(2)

(ii) Given that $\mathbf{AB} = \mathbf{AC}$, where $\mathbf{A} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$, find a matrix **C** whose elements are all non-zero.

(3)

[FP3/P6 June 2005 Qn 2]

6. The line l_1 has equation

$$\mathbf{r} = \mathbf{i} + 6\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{k})$$

and the line l_2 has equation

$$\mathbf{r} = 3\mathbf{i} + p\mathbf{j} + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k}), \text{ where } p \text{ is a constant.}$$

The plane Π_1 contains l_1 and l_2.

(*a*) Find a vector which is normal to Π_1.

(2)

(*b*) Show that an equation for Π_1 is $6x + y - 4z = 16$.

(2)

(*c*) Find the value of p.

(1)

The plane Π_2 has equation $\mathbf{r.}(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 2$.

(*d*) Find an equation for the line of intersection of Π_1 and Π_2, giving your answer in the form

$$(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}.$$

(5)

[FP3/P6 June 2005 Qn 3]

7. The plane Π passes through the points

$$P(-1, 3, -2),\ \ Q(4, -1, -1) \text{ and } R(3, 0, c), \text{ where } c \text{ is a constant.}$$

(*a*) Find, in terms of *c*, $\overrightarrow{RP} \times \overrightarrow{RQ}$.

(3)

Given that $\overrightarrow{RP} \times \overrightarrow{RQ} = 3\mathbf{i} + d\mathbf{j} + \mathbf{k}$, where *d* is a constant,

(*b*) find the value of *c* and show that $d = 4$,

(2)

(*c*) find an equation of Π in the form $\mathbf{r.n} = p$, where *p* is a constant.

(3)

The point *S* has position vector $\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$. The point S' is the image of *S* under reflection in Π.

(*d*) Find the position vector of S'.

(5)

[FP3/P6 January 2006 Qn 7]

8. The points *A*, *B* and *C* lie on the plane Π_1 and, relative to a fixed origin *O*, they have position vectors

$$\mathbf{a} = \mathbf{i} + 3\mathbf{j} - \mathbf{k},\quad \mathbf{b} = 3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} \quad \text{and} \quad \mathbf{c} = 5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$$

respectively.

(*a*) Find $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$.

(4)

(*b*) Find an equation for Π_1, giving your answer in the form $\mathbf{r.n} = p$.

(2)

The plane Π_2 has cartesian equation $x + z = 3$ and Π_1 and Π_2 intersect in the line *l*.

(*c*) Find an equation for *l*, giving your answer in the form $(\mathbf{r} - \mathbf{p}) \times \mathbf{q} = \mathbf{0}$.

(4)

The point *P* is the point on *l* that is the nearest to the origin *O*.

(*d*) Find the coordinates of *P*.

(4)

[FP3 June 2006 Qn 7]

9. The points A, B and C have position vectors, relative to a fixed origin O,

$$\mathbf{a} = 2\mathbf{i} - \mathbf{j},$$

$$\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k},$$

$$\mathbf{c} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k},$$

respectively. The plane Π passes through A, B and C.

(*a*) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(4)

(*b*) Show that a cartesian equation of Π is $3x - y + 2z = 7$.

(2)

The line l has equation $(\mathbf{r} - 5\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \mathbf{0}$. The line l and the plane Π intersect at the point T.

(*c*) Find the coordinates of T.

(5)

(*d*) Show that A, B and T lie on the same straight line.

(3)

[FP3 June 2007 Qn 7]

10.

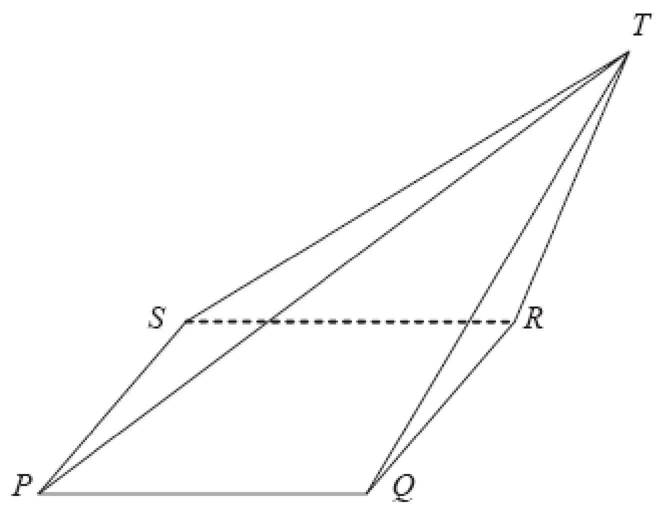


Figure 1

Figure 1 shows a pyramid $PQRST$ with base $PQRS$.

The coordinates of P, Q and R are P (1, 0, –1), Q (2, –1, 1) and R (3, –3, 2).

Find

(a) $\overrightarrow{PQ} \times \overrightarrow{PR}$

(3)

(b) a vector equation for the plane containing the face $PQRS$, giving your answer in the form $\mathbf{r} \cdot \mathbf{n} = d$.

(2)

The plane Π contains the face PST. The vector equation of Π is $\mathbf{r} \cdot (\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}) = 6$.

(c) Find cartesian equations of the line through P and S.

(5)

(d) Hence show that PS is parallel to QR.

(2)

Given that $PQRS$ is a parallelogram and that T has coordinates (5, 2, –1),

(e) find the volume of the pyramid $PQRST$.

(3)

[FP3 June 2008 Qn 7]

11.

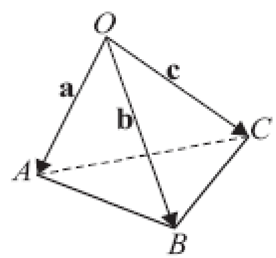


Figure 1

The points A, B and C have position vectors **a**, **b** and **c** respectively, relative to a fixed origin O, as shown in Figure 1.

It is given that

$$\mathbf{a} = \mathbf{i} + \mathbf{j}, \quad \mathbf{b} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{c} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}\,.$$

Calculate

(a) $\mathbf{b} \times \mathbf{c}$,

(3)

(b) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$,

(2)

(c) the area of triangle OBC,

(2)

(d) the volume of the tetrahedron $OABC$.

(1)

[FP3 June 2009 Qn 2]

12. The lines l_1 and l_2 have equations

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} \alpha \\ -4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}.$$

If the lines l_1 and l_2 intersect, find

(*a*) the value of α,

(4)

(*b*) an equation for the plane containing the lines l_1 and l_2, giving your answer in the form $ax + by + cz + d = 0$, where a, b, c and d are constants.

(4)

For other values of α, the lines l_1 and l_2 do not intersect and are skew lines.

Given that $\alpha = 2$,

(*c*) find the shortest distance between the lines l_1 and l_2 .

(3)

[FP3 June 2009 Qn 7]

13. The plane Π has vector equation

$$\mathbf{r} = 3\mathbf{i} + \mathbf{k} + \lambda(-4\mathbf{i} + \mathbf{j}) + \mu(6\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

(*a*) Find an equation of Π in the form $\mathbf{r.n} = p$, where $\mathbf{n}$ is a vector perpendicular to Π and p is a constant.

(5)

The point P has coordinates (6, 13, 5). The line l passes through P and is perpendicular to Π. The line l intersects Π at the point N.

(*b*) Show that the coordinates of N are (3, 1, –1).

(4)

The point R lies on Π and has coordinates (1, 0, 2).

(*c*) Find the perpendicular distance from N to the line PR. Give your answer to 3 significant figures.

(5)

[FP3 June 2010 Qn 7]

14. The plane P has equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

(a) Find a vector perpendicular to the plane P.

(2)

The line l passes through the point A (1, 3, 3) and meets P at (3, 1, 2).

The acute angle between the plane P and the line l is α.

(b) Find α to the nearest degree.

(4)

(c) Find the perpendicular distance from A to the plane P.

(4)

[FP3 June 2011 Qn 6]

15. The position vectors of the points A, B and C relative to an origin O are $\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, $7\mathbf{i} - 3\mathbf{k}$ and $4\mathbf{i} + 4\mathbf{j}$ respectively.

Find

(a) $\overrightarrow{AC} \times \overrightarrow{BC}$,

(4)

(b) the area of triangle ABC,

(2)

(c) an equation of the plane ABC in the form $\mathbf{r}.\ \mathbf{n} = p$.

(2)

[FP3 June 2012 Qn 3]

16. The plane Π_1 has vector equation

$$\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 5$$

(a) Find the perpendicular distance from the point (6, 2, 12) to the plane Π_1.

(3)

The plane Π_2 has vector equation

$$\mathbf{r} = \lambda(2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} - 2\mathbf{k}), \text{ where } \lambda \text{ and } \mu \text{ are scalar parameters.}$$

(b) Find the acute angle between Π_1 and Π_2 giving your answer to the nearest degree.

(5)

(c) Find an equation of the line of intersection of the two planes in the form $\mathbf{r} \times \mathbf{a} = \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

(6)

[FP3 June 2013 Qn 8]

17. Two skew lines l_1 and l_2 have equations

$$l_1\text{: } \mathbf{r} = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + \lambda(4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$$

$$l_2\text{: } \mathbf{r} = (3\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}) + \mu(-4\mathbf{i} + 6\mathbf{j} + \mathbf{k})$$

respectively, where λ and μ are real parameters.

(*a*) Find a vector in the direction of the common perpendicular to l_1 and l_2.

(2)

(*b*) Find the shortest distance between these two lines.

(5)

[FP3 June 2013_R Qn 2]

18. The plane Π_1 has vector equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix},$$

where s and t are real parameters.

The plane Π_1 is transformed to the plane Π_2 by the transformation represented by the matrix $\mathbf{T}$, where

$$\mathbf{T} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

Find an equation of the plane in the form $\mathbf{r}.\ \mathbf{n} = p$.

(9)

[FP3 June 2013_R Qn 4]

19. The plane Π_1 has vector equation

$$\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 5$$

(*a*) Find the perpendicular distance from the point (6, 2, 12) to the plane Π_1.

(3)

The plane Π_2 has vector equation

$\mathbf{r} = \lambda(2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$, where λ and μ are scalar parameters.

(*b*) Find the acute angle between Π_1 and Π_2 giving your answer to the nearest degree.

(5)

(*c*) Find an equation of the line of intersection of the two planes in the form $\mathbf{r} \times \mathbf{a} = \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

(6)

FP3 June 2013 Q8

20. The plane Π_1 has vector equation $\mathbf{r}.\begin{pmatrix}2\\1\\3\end{pmatrix}=5$.

The plane Π_2 has vector equation $\mathbf{r}.\begin{pmatrix}-1\\2\\4\end{pmatrix}=7$.

(*a*) Find a vector equation for the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and λ is a scalar parameter.

(6)

The plane Π_3 has cartesian equation

$$x - y + 2z = 31$$

(*b*) Using your answer to part (*a*), or otherwise, find the coordinates of the point of intersection of the planes Π_1, Π_2 and Π_3.

(3)

FP3 June 2014_R Q8

21. The line l passes through the point $P(2, 1, 3)$ and is perpendicular to the plane Π whose vector equation is

$$\mathbf{r}.(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = 3$$

Find

(*a*) a vector equation of the line l,

(2)

(*b*) the position vector of the point where l meets Π.

(4)

(*c*) Hence find the perpendicular distance of P from Π.

(2)

FP3 June 2014 Q1

22. The position vectors of the points A, B and C from a fixed origin O are

$$\mathbf{a} = \mathbf{i} - \mathbf{j}, \quad \mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \mathbf{c} = 2\mathbf{j} + \mathbf{k}$$

respectively.

(*a*) Using vector products, find the area of the triangle ABC.

(4)

(*b*) Show that $\frac{1}{6}\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = 0$.

(3)

(*c*) Hence or otherwise, state what can be deduced about the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(1)

FP3 June 2014 Q8

23. The points A, B and C have position vectors $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ respectively.

(*a*) Find a vector equation of the straight line AB.

(2)

(*b*) Find a cartesian form of the equation of the straight line AB.

(2)

The plane Π contains the points A, B and C.

(*c*) Find a vector equation of Π in the form $\mathbf{r}.\mathbf{n} = p$.

(4)

(*d*) Find the perpendicular distance from the origin to Π.

(2)

FP3 June 2015 Q5

24. The plane Π_1 has equation

$$x - 5y - 2z = 3.$$

The plane Π_2 has equation

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k}),$$

where λ and μ are scalar parameters.

(*a*) Show that Π_1 is perpendicular to Π_2.

(4)

(*b*) Find a cartesian equation for Π_2.

(2)

(*c*) Find an equation for the line of intersection of Π_1 and Π_2 giving your answer in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors to be found.

(6)

FP3 June 2016 Q8

25. The plane $\tilde{O}_1$ has equation $x - 2y - 3z = 5$ and the plane $\tilde{O}_2$ has equation $6x + y - 4z = 7$

(*a*) Find, to the nearest degree, the acute angle between $\tilde{O}_1$ and $\tilde{O}_2$

(3)

The point P has coordinates $(2, 3, -1)$. The line l is perpendicular to $\tilde{O}_1$ and passes through the point P. The line l intersects $\tilde{O}_2$ at the point Q.

(*b*) Find the coordinates of Q.

(4)

The plane $\tilde{O}_3$ passes through the point Q and is perpendicular to $\tilde{O}_1$ and $\tilde{O}_2$

(*c*) Find an equation of the plane $\tilde{O}_3$ in the form $\mathbf{r}.\mathbf{n} = p$

(4)

FP3 June 2017 Q5

26. The line l has equation

$$\mathbf{r} = (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + \lambda(3\mathbf{i} + 2\mathbf{j} + \mathbf{k}), \text{ where } \lambda \text{ is a scalar parameter,}$$

and the plane Π has equation

$$\mathbf{r}.(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 19$$

(*a*) Find the coordinates of the point of intersection of l and Π.

(4)

The perpendicular to Π from the point A (2, 1, –2) meets Π at the point B.

(*b*) Verify that the coordinates of B are (4, 3, –6).

(3)

The point A (2, 1, –2) is reflected in the plane Π to give the image point A'.

(*c*) Find the coordinates of the point A'.

(2)

(*d*) Find an equation for the line obtained by reflecting the line l in the plane Π, giving your answer in the form

$$\mathbf{r} \times \mathbf{a} = \mathbf{b},$$

where **a** and **b** are vectors to be found.

(4)

F3 IAL June 2014 Q8

27. The plane Π_1 contains the point (3, 3, –2) and the line $\dfrac{x-1}{2} = \dfrac{y-2}{-1} = \dfrac{z+1}{4}$

(*a*) Show that a cartesian equation of the plane Π_1 is

$$3x - 10y - 4z = -13$$

(5)

The plane Π_2 is parallel to the plane Π_1

The point (α, 1, 1), where α is a constant, lies in Π_2

Given that the shortest distance between the planes Π_1 and Π_2 is $\dfrac{1}{\sqrt{5}}$

(*b*) find the possible values of α.

(6)

F3 IAL June 2015 Q7

28. The coordinates of the points A, B and C relative to a fixed origin O are (1, 2, 3), (–1, 3, 4) and (2, 1, 6) respectively. The plane Π contains the points A, B and C.

(*a*) Find a cartesian equation of the plane Π.

(5)

The point D has coordinates (k, 4, 14) where k is a positive constant.
Given that the volume of the tetrahedron $ABCD$ is 6 cubic units,

(*b*) find the value of k.

(4)

F3 IAL June 2016 Q6

29. With respect to a fixed origin O, the points A(–1, 5, 1), B(1, 0, 3), C(2, –1, 2) and D(3, 6, –1) are the vertices of a tetrahedron.

(*a*) Find the volume of the tetrahedron $ABCD$.

(4)

The plane P contains the points A, B and C.

(*b*) Find a cartesian equation of P.

(4)

The point T lies on the plane P.
The line DT is perpendicular to P.

(*c*) Find the exact coordinates of the point T.

(4)

F3 IAL June 2017 Q9