Educational Aspects of Uncertainty Calculation with Software Tools

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Abstract

Despite its importance, uncertainty of measurements and parameters is frequently neglected by practitioners in the design of systems even in safety critical applications. Thus, problems arising from uncertainty may only be identified late in the design process or even remain. This can lead to additional costs and increased risks. Although there exists numerous tools to support uncertainty calculation, reasons for limited usage in early design phases may be low awareness of the existence of the tools and insufficient training in the practical application.

In order to enhance the widespread use of such tool support we suggest a teaching concept for uncertainty calculation in measurement science education that is directly based on the utilization of software tools. Although the developed material is currently based on the GUM (Guide to the expression of uncertainty in measurement) method we believe that it is also useful with other methods. Additionally, the concept goes beyond the scope of measurement uncertainty quantification demonstrating that it is also useful for system analysis and optimization.

Keywords: GUM, measurement uncertainty, optimization

1. Introduction and Motivation

The fact that measurement results are more than just numeric values is well known and accepted when it comes to physicals units. Thus, it is com-

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mon practice to report the unit together with the numeric result of a measurement. However, it is not as common to emphasize that measurement results are usually composed by realizations of random variables. The ideal way to represent random variables is to provide the probability distribution . However, this may be difficult, impractical or even impossible in many situations. As an alternative, the uncertainty attributed to the measurement result may be reported in terms of certain parameters of the probability density function. In the simplest case we could just report a single additional parameter to indicate if the distribution is narrow or wide; i.e. a non-negative parameter characterizing the dispersion of the quantity values being attributed to a measurand, based on the information used [1]. In other words, it provides information about the remaining uncertainty about the measurand [2]. The idea to develop a new guide for the treatment of uncertainty in measurement was to overcome some of the limitations that are associated with the previously used term error [3] and led to the the change in the treatment of measurement uncertainty from an Error Approach (sometimes called Traditional Approach or True Value Approach) to an Uncertainty Approach [1]. With respect to metrology, the uncertainty reflects the fact that measurements can only provide incomplete knowledge and that a measurement is only useful when the lack of knowledge is somehow quantified. This is particularly true with respect to safety and reliability. Consider, for example, a monitoring system that should validate that a certain parameter lies within a certain interval. If the measurement uncertainty of the monitoring system becomes larger than the interval to be monitored, then the monitoring system can *never* be used to validate that the parameter is actually *within* the interval; it can only be used to validate that the parameter (with high probability) resides *outside* of the interval. This may not be apparent for a user or even for a developer of such a system, in particular considering that the engineer may not be an expert in stochastics and uncertainty quantification. Therefore, it seems to be reasonable to provide a method that is commonly accepted by practitioners and experts, can easily be applied for a wide range of problems and still provides good results (even if they may not be optimal in a theoretical sense).

In 1977, as it was recognized the existence of a lack of international consensus on the expression of uncertainty in measurement, the world's highest authority in metrology, the Comité International des Poids et Mesures (CIPM), requested the Bureau International des Poids et Mesures (BIPM) to address the problem in conjunction with the national standards laboratories and to make a recommendation. The effort finally led to the development of the Guide to the Expression of Uncertainty in Measurement (GUM) [3]. According to the GUM, the ideal method should be universal (applicable to all kinds of measurements and to all types of input data used in measurements), internally consistent (directly derivable from the components that contribute to it), and transferable (possibility to directly use the uncertainty evaluated for one result as a component in evaluating the uncertainty of another measurement in which the first result is used).

With respect to one of the initial requirements for such a recommendation - i.e. the approach has to be universal - the GUM [3] treats all uncertainty contributions identically, more or less as if the distributions were Gaussian and the relations were linear. The Central Limit Theorem is significant in this context because it shows the very important role played by the variances of the input quantities' probability distributions, compared with that played by the higher moments of the distributions, in determining the form of the resulting convolved distribution of Y. Further, it implies that the convolved distribution converges towards the normal distribution even for comparatively small numbers of contributing parameters. For instance, the convolution of as few as three rectangular distributions of equal width is approximately normal [3].

However, the GUM working group was aware that there are limitations of the GUM [3] method and in supplements [3, 4, 5] suggested to use Monte Carlo sampling in certain cases. A recent survey [6] on current research activities in the field of measurement uncertainty reports that most recent work addresses the GUM. Consequently, the present paper focuses on this approach, which has a wide acceptance within the field of metrology. Simplicity of tools that implement the method is also crucial for the acceptance, as stated e.g. in [7]. Similarly, the authors of [8] emphasize the beneficial role that tools may play to eventually make uncertainty propagation an inherent component of computational procedures instead of an optional addendum. With the same motivation, we aim to bring students in touch with such tools early in their curriculum.

Our approach uses a tool that integrates well into a mathematical programming environment with which our students are familiar. We currently use Matlab, but the approach may also be used with other environments, e.g. [9] for students well trained in Java. The basic educational concept was presented in [10]. In the present paper, we discuss additional aspects such as the numeric representation of uncertainty and the utilization of the concept beyond the scope of classical measurement uncertainty quantification.

Our educational concept is directly applicable to two toolboxes for Matlab [11]; i.e. Metas.UncLib MatLab toolbox [12] and a toolbox developed by our group. Both toolboxes include an implementation of the GUM tree method [13]/automatic differentiation [14]. The toolboxes are similar in basic usage and basic functionality. Differences mainly relate to reporting of uncertainty and analysis of uncertainty contributions (in part as a response on student feedbacks). Furthermore, to keep things transparent for the students they can have a look into the MatLab source code rather than obtaining a "black box".

It should be noted that there have been many discussions about the GUM and several alternative approaches exist as discussed e.g. in [15], [16] and recently in [17]. Additionally, a revision of the GUM [18, 19] is in preparation. However, the teaching concept that we present in this paper can be used with different approaches as long as it is possible to implement them in an automatic tool. In this context it will be important to outline the methods and explain their advantages and disadvantages to the students. However, the main objective is to sensitize students for the concept of uncertainty such that it becomes a part of practices of daily life. This can be achieved with different approaches and we currently use the classical GUM approach.

2. Software Tool Concept

2.1. Assigning and Reporting Uncertainty

In principle, the GUM [3] has two different types of uncertainty evaluation. The Type A evaluation uses statistical methods, i.e. the uncertainties are obtained from experiments by drawing samples from the distribution and calculate the standard uncertainty based on the empirical data. In the Type B evaluation, the uncertainty of input quantities is known a-priori. In order to obtain the combined standard uncertainty attributed to the final measurement result it is necessary to determine the individual contributions of the input quantities.

In this paper, we focus on the determination of the combined standard uncertainty based on the standard uncertainty of the input quantities. Our examples are based on Type B uncertainties, which represents a common case where the prior knowledge is provided by the manufactures of the devices, e.g. instrumental measurement uncertainty for voltmeters or sensors in the respective datasheets. However, the standard uncertainty of the input quantities could also be determined using Type A evaluation. Transducers may even provide uncertainty information in a Transducer Electronic Datasheet [20]. In addition, other impact factors such as material properties and mechanical geometries may come with known uncertainties. The aim is to combine all the individual contributions in a simple way.

Dealing with uncertainty in mathematic programming environments such as MatLab[11] or R [21] requires several steps:

- Attributing standard uncertainties to the input quantities, i.e. indicating them as realizations of random variables. Assigning uncertainty to a variable, i.e. create a random variable.
- Keeping track of uncertainties during calculations.
- Reporting the results including the uncertainties.

As the second and third step are the job of the software tool, the main novelty for students is that they additionally need the assignment step in order to obtain results including the corresponding combined standard uncertainty.

The requirements on the developed software tool, supporting uncertainty calculations, are explained and summarized in the following. The tool shall be usable in the typical developer or engineering software environment. Only if the tool is smoothly integrated into the existing tool landscape wide acceptance can be achieved. In our approach, we integrated the functions into MatLab. MatLab has been chosen due to the high usage in education and research as well as in in industry. Assigning uncertainty requires connecting the uncertainty of a parameter to the estimated value of the parameter itself, or to consider the uncertainty during the evaluation of the mathematical model representing the measurement. A smart approach allows for tying the uncertainty directly to the variable. Thus, our toolbox extends the datatypes of MatLab by a new class called unc. The estimated value and uncertainty are assigned to a variable of this class unc. For instance, a value of 3.67 and a corresponding standard uncertainty of 0.35 are assigned to a variable x according to Figure 1.

With respect to reporting uncertainty, [3] suggests four different ways to report (combined) standard uncertainty. All of them include a textual explanation of how the reported result should be implemented. We believe that it is not very practical to provide a full textual explanation for each result,

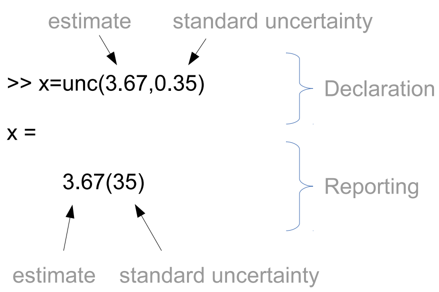

Figure 1: Assigning and reporting of a variable x with the value and the standard uncertainty.

in particular when larger numbers or even matrices are to be reported in a mathematical programming environment. Therefore, a textual explanation is only displayed once, when a first uncertain variable is reported. All following reports of standard uncertainty use the numerical representation alone.

In order to keep the representation short and compact, we decided to use a representation where the (combined) standard uncertainty is reported in parentheses after the numerical value of the measurement result, where the number in parentheses is the numerical value of (the combined standard uncertainty) u referred to the corresponding last digits of the quoted result. Using this representation, the number of digits for expressing the uncertainty also defines the number of reported digits for the numerical value of the measurement result. The number of digits can be configured in our toolbox. In previous courses we used three digits. However, [3] suggests to use only two digits. Therefore, in all examples in the present paper we follow the suggestion of [3]. Examples for assigning and reporting of uncertainty are given in Table 1. Please note that in contrast to the recommended way according to [3] we do not report a unit (and thus it is not necessary to assign it).

2.2. Calculation of the Standard Uncertainty

With the use of a tool during the introduction of uncertainty calculation we aim to put the focus on the meaning and implications of uncertainty rather than on how standard uncertainty is calculated. This may be compared, e.g. with the introduction to trigonometric functions. In order to understand the meaning of sine, cosine and tangent it is initially not important to know how these are calculated numerically. Similarly, our teaching concept initially

Value	Standard Uncertainty	Displayed Result
-144.587	15.1	-145(15)
0.001351	0.0000001543	0.00135100(15)
0.555550123	0.061002	0.556(61)

Table 1: Examples for reporting uncertainty in MatLab. Please note that two significant digits are used to report the standard uncertainty, following the recommendation given in [3].

does not focus on how the calculation is actually performed. In this paper, we use the linearization approach as suggested in the GUM [3]. In the background, for each variable that is assigned or created by mathematical operations a list of input quantities is maintained and the derivatives (often called sensitivity coefficients) of the variable with respect to these input quantities calculated analytically and stored with a reference to that input quantity. This also allows for a later analysis of the contributions of the individual input quantities. The verification of the software tool has been based on a regression suite including several hand written test cases and the corresponding expected results. Therefore, the tool should currently only be used for educational purposes, for uncritical cases or as an auxiliary means, e.g. during system optimization. Nevertheless, in upcoming versions the verification can be extended by the method proposed e.g. in [22]. Besides this simple approach described above, it is also possible to use more accurate yet more complex methods, such as higher order derivatives or even Monte-Carlo sampling [12]. We believe that a fast operation of the tools is mandatory for a wide acceptance (we are even considering the use in real time applications). This favors the linearization approach, yet an automatic justification of the applicability would be desirable. Such an automatic justification would rise a warning when the linearization is a poor approximation of the function (e.g. for x^2 at $x = 0$).

3. Teaching Concept

The course in question, Measurement Science, Sensors and Actuators, comprising a lecture and an exercise, is part of the bachelor curriculum Information Technology at Alpen-Adria-University Klagenfurt and usually attended by students in their third year. Consequently, we expect that the students are familiar with basic concepts of electrical engineering, have some experience with measurement devices and also have background from a mathematically oriented course on stochastic. Therefore, they are familiar with the concept of random variables, Ohms law and electrical networks. Thus, we decided to actually start the course with the concept of measurement uncertainty before we introduce the SI system. Consequently, the discussion on "good" definitions for base units can already be based on the uncertainty concept. Furthermore, traceability is also directly linked to this discussion. This should provide a holistic view of how measurement science work and that knowing the uncertainty is as important as knowing the estimate of some parameter in question.

Our proposed introduction to the concept of uncertainty is illustrated in Table 2. Starting with a discussion of the difference between measurement results and indications and the interpretation of indications as a realizations of random variables that provide some information about the parameter in question, we introduce the GUM [3] including terms such as standard uncertainty, combined uncertainty and determination by means of Taylor series expansion.

In order to get familiar with the difference between ordinary numbers and measurement results with attributed standard uncertainties we use and explain introductory examples as shown in Figure 2.

After the introductory examples, our first practical example is the determination of a resistance value and corresponding combined standard uncertainty from the measurement of voltage and current with respective rectangular distribution using Ohm's law

$$R = \frac{U}{I} \tag{1}$$

as the measurement model and the corresponding equation for uncorrelated input quantities and linearization

$$u_c^2(R) = \left(\frac{\partial R}{\partial I}\right)^2 u^2(I) + \left(\frac{\partial R}{\partial U}\right)^2 u^2(U) \tag{2}$$

with the partial derivatives to be computed in the measured values of the input quantities. As a second step, systematic errors due to the inner resistance of the measurement instruments are included in the measurement model.

```
>> a=unc(100,1);
>> b=unc(100,1);
>> a-b

ans =

    0.0(1.4)
```

```
>> a=unc(100,1);
>> b=a;
>> a-b

ans =

    0(0)
```

```
>> a=unc(100,1);
>> b=unc(100,1);
>> a/b

ans =

    1.000(14)
```

```
>> a=unc(100,1);
>> b=a;
>> a/b

ans =

    1(0)
```

Figure 2: Introductory examples. The construction of a variable using unc generates independent variables, whereas an assignment means that two variables are fully correlated. Consequently, the difference between two random variables with equal value and uncertainty is zero with non-zero uncertainty when the variables are independent. However, when two variables are actually identical (i.e. $b = a$), then they are fully correlated, i.e. all random influences affect both, a and b, equivalently. Therefore, the uncertainty of their difference vanishes. The same applies to ratios. For partially correlated variables, i.e. where some but not all random influences affect both, a and b, equivalently, the resulting uncertainty is somewhere in between of those of the previous cases, depending on the degree of correlation. By this, the benefits of the differential measurement method as well as the ratiometric method are easily illustrated.

After this simple example, we apply the method to the Wheatstone Bridge circuit as another common method to evaluate resistance. There we also introduce software tools for the evaluation of uncertainty. This is described in the next section.

3.1. The Wheatstone Bridge with Uncertainties

Bridge circuits for the determination of unknown impedances and as realization of the compensation method are important building blocks in measurement science and thus usually treated in introductory courses. We use the Wheatstone Bridge to emphasize how tools for uncertainty calculation may change the way how the material is presented to students. Figure 3 shows the circuit of a Wheatstone Bridge.

First, we start from the classical result for the equation to determinate the value of the unknown resistor R_1, i.e.

$$R_1 = R_2 \frac{R_3}{R_4} \tag{3}$$

and directly apply the GUM method to this, leading to

$$u_C{}^2(R_1) = \frac{\partial R_1}{\partial R_2}^2 u^2(R_2) + \frac{\partial R_1}{\partial R_3}^2 u^2(R_3) + \frac{\partial R_1}{\partial R_4}^2 u^2(R_4) \tag{4}$$

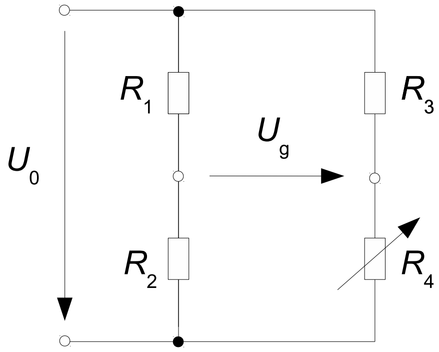

Figure 3: Example for a Wheatstone Bridge.

and let the students do an interpretation of the results. The aim is to point out that apparently some important influences are missing, e.g., it seems that the choice of U_0 and the accuracy of the instrument that measures U_g would not be important with respect to the uncertainty. This is obviously not correct. This leads to a more detailed analysis, showing that it may not be sufficient to only consider a measurement equation to fully determine the standard uncertainty.

The students are instructed to derive a more complete measurement model such as

$$R_1 = \frac{U_0 R_2 (R_3 + R_4)}{U_0 R_4 + U_g (R_3 + R_4)} - R_2 \tag{5}$$

Here we emphasize that a value that is measured to be zero may still be significantly different from zero and must thus not be omitted for uncertainty considerations. In the next step the measurement equation is derived with respect to U_0, U_g, R_2, R_3, R_4, leading to the sensitivity coefficients $C_{R2} = \frac{\partial R_1}{\partial R_2}$, $C_{R3} = \frac{\partial R_1}{\partial R_3}$ and so on. The resulting uncertainty of the resistor R_1 can be calculated as follows:

$$u_C{}^2(R_1) = C_{U0}^2 u^2(U_0) + C_{Ug}^2 u^2(U_g) + C_{R2}^2 u^2(R_2) + C_{R3}^2 u^2(R_3) + C_{R4}^2 u^2(R_4) \tag{6}$$

with the uncertainty $u(X)$ of the respective input quantities and the sensitivity coefficients calculated previously.

Following this analysis we use the toolboxes for MatLab and the datatype "uncertain". With this, we perform the same calculation as above, but numerically and step by step as shown in Figure 4 emphasizing that it does not require any additional effort to obtain the combined standard uncertainty

```
>> R2 = unc(1e3,5,'R2');
>> R3 = unc(1e3,5,'R3');
>> R4 = unc(1.001e3,1,'R4');
>> U0 = unc(5,0.1,'U0');
>> Ug = unc(0,0.01,'Ug');
>> I2=U0/(R3+R4);
>> I1=(I2*R4+Ug)/R2;
>> R1 = U0/I1-R2

R1 =

    999(11)
```

Figure 4: Calculation of the measurement result according to Figure 3 including the standard uncertainty of the unknown resistor R_1 using Metas.UncLib toolbox for MatLab [12].

but providing the standard uncertainty for the input quantities. It is also shown that the toolbox could also be applied to equation (3) but leading to the same incorrect result as above (Fig. 5).

```
>> R2*R3/R4

ans =

    999.0(7.1)
```

Figure 5: Incorrect determination of the standard uncertainty of R_1 due to direct application of the GUM to the classical solution (3) of the Wheatstone Bridge (Figure 3).

The reason - failing to correctly consider the uncertainty of the voltage measurement (the imbalance of the bridge) - is otherwise often not obvious for students. Consequently, several rules for use of uncertain measurement results but also for the derivation of measurement equations (including their simplification) can be derived. This allows obtaining the well known result according to equation (3) but clearly highlighting that other parameters that do not occur in the equation may significantly contribute to the uncertainty.

However, the benefit is not just the automatic calculation of the standard uncertainty. Additionally, with respect to analysis and optimization the toolboxes also provide means to determine the contribution of the uncertain input variables to the combined uncertainty of the result. This is shown in Figure 6. In the present example, the main contribution to the uncertainty comes from the uncertainty of the voltage measurement, which is in practice not truly zero.

```
U0 = 5 V

>> disp_contribution(R1);

Uncertainty Contribution:
(Square root of the contribution of the variable X to the squared standard uncertainty of Y)
Variable Name | Contribution
-----------------------------
Ug .......... |   7.99
R2 .......... |      5
R3 .......... |      5
R4 .......... |  0.998
U0 .......... | 5.68e-15

U0 = 0.5 V
>> disp_contribution(R1);

Uncertainty Contribution:
(Square root of the contribution of the variable X to the squared standard uncertainty of Y)
Variable Name | Contribution
-----------------------------
Ug .......... |   79.9
R3 .......... |      5
R2 .......... |      5
R4 .......... |  0.998
U0 .......... | 4.55e-14
```

Figure 6: Analysis of the contributions of the various sources of uncertainty to the standard uncertainty of R_1 using MatLab and an uncertainty toolbox. The reported ~~2~~contribution values represent the contribution $u_i(R_1)$ of each input quantity i. The toolbox also provides a function to report the sensitivity coefficients. This example aims to demonstrate interdependencies between parameters. Even though the contribution of the uncertainty of the voltage U_0 to the uncertainty of R_1 is very low, the choice of U_0 has a strong impact on the contribution of U_g.

In this example, the contribution of U_0 to the uncertainty is close to zero. By reducing U_0 to one tenth of its original value, we show that although its contribution to the standard uncertainty is still negligible we see an increase of the combined standard uncertainty as the contributions of other input quantities increase. Here we aim to emphasize such interdependencies and how they are easily studied with the tools.

3.2. Hall Sensor with Uncertainties

In order to demonstrate the influence of correlated uncertainties and the benefit of calibration we use a simple Hall sensor example. Starting from the equation for the Hall voltage U_{gem},

$$U_{gem} = V\, S\, I\, B \tag{7}$$

with amplifier gain V, Hall sensor sensitivity S, bias current I and magnetic flux density B [23], we can determine the flux density B from the Hall voltage under consideration of the random sensor offset U_{oH} and random amplifier offset U_{oIV} using

$$B = \frac{\frac{U_{gem}}{V} - U_{oIV} - U_{oH}}{I\, S} \tag{8}$$

For the example data, the flux density signal is much smaller than the corresponding standard uncertainty as shown in Figure 8. Looking at the uncertainty contributions provided in Figure 7 it can be seen that the major contribution results from the offsets. However, as all output voltages are subject to the same offsets, an offset calibration can be used, e.g. by exposing the sensor to a known flux density for the first sample. The effect is also shown in Figure 8. Now, the uncertainty contributions, provided in Figure 7 for the calibrated case, show that the major contributions from the offsets are eliminated and the uncertainty of the calibration flux density becomes the main contribution. The uncertainty of the voltage (ADC) appears twice as two voltage values (current sample and calibration sample) are now used to calculate one flux density result.

3.3. Transient Characteristics of a Digital to Analog Converter

Besides the determination of measurement uncertainty, the concept can generally be applied where uncertainties are relevant for a result and even for the evolution of uncertainty over time. We demonstrate this in the example of a simple model of the analog output stage of a digital to analog converter (DAC) as shown in Figure 9. The input x represents the static output signal of the DAC, y represents the actual output signal that is subject to a filter characteristic described by a low-pass comprising a resistor R and a capacitor C. A discrete time model (Euler method) is given by

$$y[n] = y[n-1] + a\,(x[n] - y[n-1]) \tag{9}$$

with n is the discrete step and $a = 1/RCdt$ with time step dt.

As all real components also the filter's resistor and the capacitor are subject to uncertainties. In order to study their impact it is only necessary to

```
% assign parameters and corresponding standard uncertainties
I=unc(1e-3,1e-5,'Bias Current');
U_oIV=unc(0,1e-3,'Amplifier Offset');
U_oH=unc(0,1e-3,'Hall Sensor Offset');
V=unc(100,1,'IV Amplification');
S=unc(2e-3,1e-4,'Hall Sensor Sensitvity');
B_K=unc(0,3,'Calibration Flux Density');


% calculate the Flux Density from the measurements
% just as if there were no uncertainty...
B=(Ugem/V-U_oIV-U_oH)/I/S;
% now apply calibration
BK=B(:)-B(1)+B_K;

>> disp_contribution(B(10))

Uncertainty Contribution:
(Square root of the contribution of the variable X
to the squared standard uncertainty of Y)
Variable Name            | Contribution
---------------------------------------
Hall Sensor Offset .... | 500
Amplifier Offset ...... | 500
Hall Sensor Sensitvity  | 7.47
IV Amplification ...... | 1.49
Bias Current .......... | 1.49
ADC ................... | 0.5


>> disp_contribution(BK(10))

Uncertainty Contribution:
(Square root of the contribution of the variable X
to the squared standard uncertainty of Y)
Variable Name              | Contribution
-----------------------------------------
Calibration Flux Density  | 3
Hall Sensor Sensitvity .. | 2.47
ADC ..................... | 0.5
ADC ..................... | 0.5
IV Amplification ........ | 0.494
Bias Current ............ | 0.494
```

Figure 7: Code for the determination of the flux density from Hall voltages with and without calibration

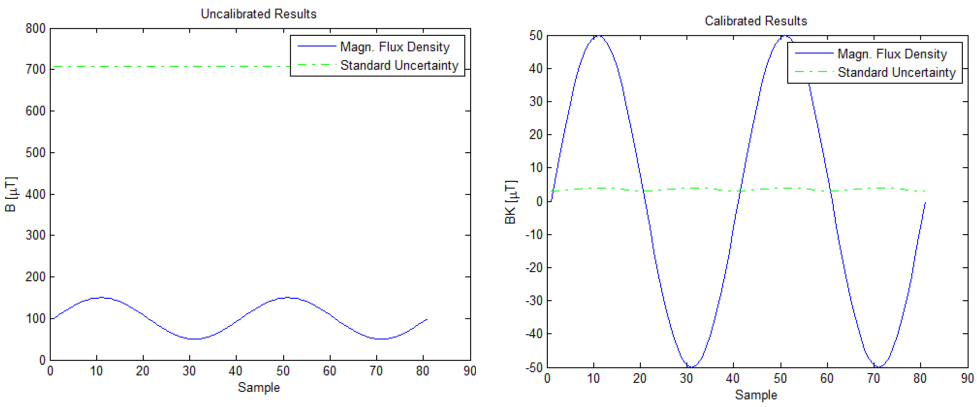

Figure 8: Measurement results for the flux density obtained with a Hall sensor with and without calibration

assign the uncertainty to these parameters and the toolbox does the calculation. Without a high effort, students can thus focus on the analysis of the results. The commands to study this model are given in Figure 10 and results are shown in Figure 11 ("Time Invariant"), where the output value y as well as the uncertainty of the output $u(y)$ are plotted. Obviously, the uncertainty of the output result is not constant. The uncertainty at the beginning of the signal is initially low, fast increases and finally decreases until converging to zero. Thereby, the maximum of the uncertainty is in the dynamic region of the output signal. This example aims to illustrate that transient behavior of a DAC is subject to large variations even when the static uncertainty is very low.

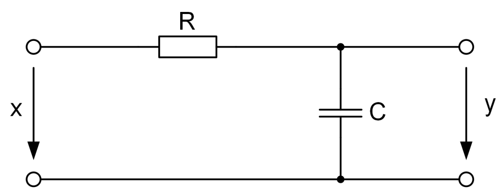

Figure 9: Lumped element model of the output of a DAC circuit consisting of a resistor R and a capacitor C.

As a continuation of the example and in order to demonstrate once again the difference between correlated and uncorrelated variables, we modify the previous example such that the variables R, C and x are drawn at each time

```
%% Low Pass filter
dt=0.1e-6;           % sampling period
C = unc(10e-9,1e-9);% Capacitance
R = unc(820,40);     % Resistance
a = 1/(R*C)*dt;      % Coefficient

% Simulation
y=unc(zeros(1,1000),zeros(1,1000));
x=unc(ones(1,1000),1e-3*ones(1,1000));
x(:)=unc(1,1e-3);
y(1)=unc(0,1e-3);
for n=2:1:1000
    y(n)=y(n-1)+a*(x(n)-y(n-1));
end
```

Figure 10: MatLab code to simulate the model according to Figure 9 (Euler method) with consideration of the uncertainties.

step. This means that we obtain a time variant system. The results emphasize the counter-intuitive result that the introduction of more uncertain variables (i.e. one at each time step) actually reduces the uncertainty of the result. The results are also shown in Figure 11. Additionally, the analysis shows that the contributions of all filter coefficients (”old” and ”new”) are equal whereas the contribution of ”old” input values is much lower than for ”new” input values.

4. Discussion of Advantages and Disadvantages and Student Feedback

We consider toolboxes for the calculation of the combined standard uncertainty just as an additional feature of an electronic calculator. In order to illustrate this, consider e.g. the calculation of a non-linear function such as a trigonometric function. A user of the calculator does not need to precisely know how the calulator determines the numeric value. It is more important that the user can interpret and understand the results (e.g. the cosine of an angle). Similarily, at this point we aim to put the focus on the interpretation of the result.

As with the electronic calculator in general, a frequently observed disadvantage is a loss of the ’feeling’ whether a result is in the correct order of magnitude, which otherwise could help to identify mistakes.

Our aim with respect to uncertainty is to put the focus on the fact that measurement results are actually composed of realizations of random vari-

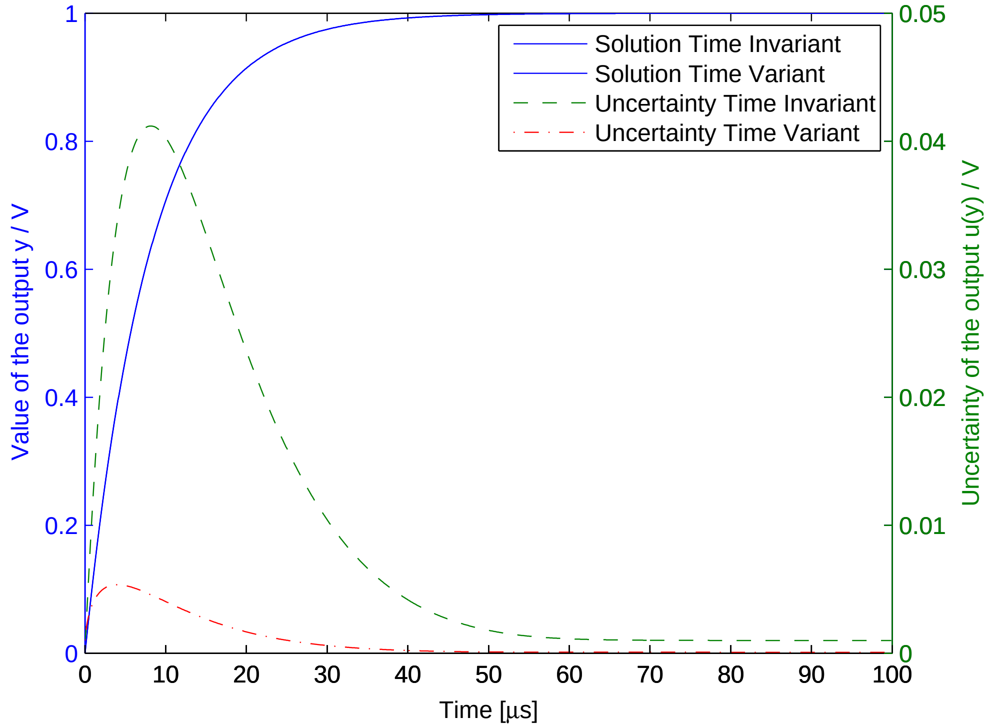

Figure 11: Simulation results according to the model shown in Figure 9 for time variant and time invariant parameters. Please note that the linearization approach leads to identical solutions for both cases, yet the standard uncertainties are different. During the transient period, the uncertainty is significantly higher than for stabilized states for both cases. In the time variant case, model parameters and the input signal can change over time due to independent realizations of the uncertainty. Even though this means that we have 3000 instead of three uncertain input quantities, the resulting uncertainty of y is lower. This - for students usually counter-intuitive effect - is due to the averaging of uncorrelated input quantities.

ables and that the GUM aims to describe the probability density function using the standard uncertainty Consequently, the effort to perform the calculations is left to tools. As a drawback of this approach, students may not be aware of the limitations of these tools as they are used to fully rely on the results from electronic calculators. Therefore, it is important to continuously emphasize the limitations of the tools.

The positive students' feedback reasons from the initial introduction of the uncertainties, which are directly coupled to the variables. This allows copying the uncertainties given in the specification directly to the variable itself. Since the students are familiar with MatLab, the barrier of using the tool is low. They can work in the known environment without any additional training. Nevertheless, the most important positive point is that the students' attention can be unglued from the already known mathematical calculations and guided towards the measurement model construction and the analysis of the contribution of the various sources of uncertainty. Especially, the simple analysis of the contributions of the various input quantities allows for immediate highlighting of the main contributors to the uncertainty. The students get a feeling on what shall be tried to improve first. Based on students feedback using [12] we implemented a different uncertainty reporting approach and a different approach for analysis in our toolbox as described above, aiming to get a more intuitive tool.

5. Conclusion

We propose an approach for considering uncertainty in measurement science education from the very beginning using special tools. The approach does not offer different or even better results; it is currently simply based on the GUM method. However, the use of tool support allows for emphasizing the meaning and implications of uncertainty. Furthermore, we show that the use of the tools is not restricted to determination of measurement uncertainty. We believe that this approach may help to increase the awareness with respect to uncertainty, has a high practical applicability in particular with respect to safety engineering and may thus lead to consideration of uncertainty in early design phases.

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Section	Content	Educational Objective
1	Introduction to Uncertainty	Understand measurements as realizations of random variables
2	Introduction to GUM [3], Error Propagation by means of Taylor Series	Know GUM and its application to measurement equations
3	Example Resistor Measurement (Voltage/Current)	Practical experience with the GUM
4	Consideration of systematic errors	Understand importance of measurement model
5	Example Wheatstone Bridge - Simple	Identify pitfalls in approach according to section 2
6	Example Wheatstone Bridge - More Accurate	Practice, learn that a correct consideration of uncertainties may be time consuming if done "by hand"
7	Example Wheatstone Bridge - Tool Based	Understand the concepts of "uncertain" as a datatype
8	Example Wheatstone Bridge - Analysis of Sources of Uncertainty	Understand that tools can ease the analysis of measurement chains
9	Example Wheatstone Bridge - Influence of Parameters	Understand that parameters that do not directly contribute to the uncertainty may do so through other parameters.
11	Example Hall Sensor	Understand correlated and uncorrelated uncertainties and calibration approach.
10	Discussion	Understand that uncertainty must always be considered in measurement. Awareness that tools are available that simplify most of the calculations. Awareness that the tools and GUM have limitations

Table 2: Steps of the proposed introduction to the concept of uncertainty.