1.11 Consider the following 9-point signals, $0 \le n \le 8$.

(a) [3, 2, 1, 0, 0, 0, 0, 2, 1]
(b) [3, 2, 1, 0, 0, 0, 0, −2, −1]
(c) [3, 2, 1, 0, 0, 0, 0, −2, −1]
(d) [0, 2, 1, 0, 0, 0, 0, −2, −1]
(e) [0, 2, 1, 0, 0, 0, 0, 2, 1]
(f) [3, 2, 1, 0, 0, 0, 0, 1, 2]
(g) [3, 2, 1, 0, 0, 0, 0, −1, −2]
(h) [0, 2, 1, 0, 0, 0, 0, −1, −2]
(i) [0, 2, 1, 0, 0, 0, 0, 1, 2]

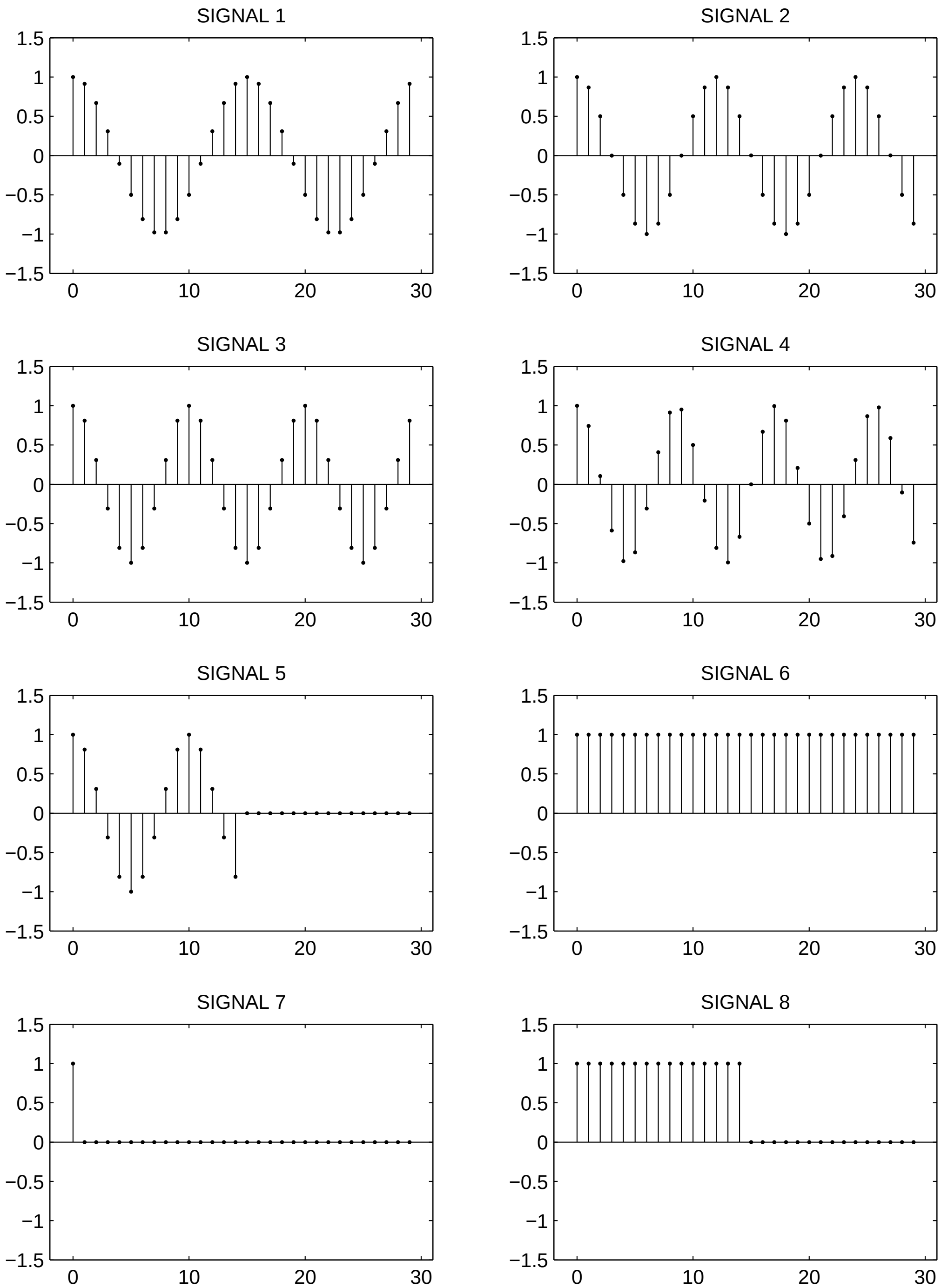
Which of these signals have a real-valued 9-point DFT? Which of these signals have an imaginary-valued 9-point DFT? Do not use MATLAB or any computer to solve this problem and do not explicitly compute the DFT; instead use the properties of the DFT.

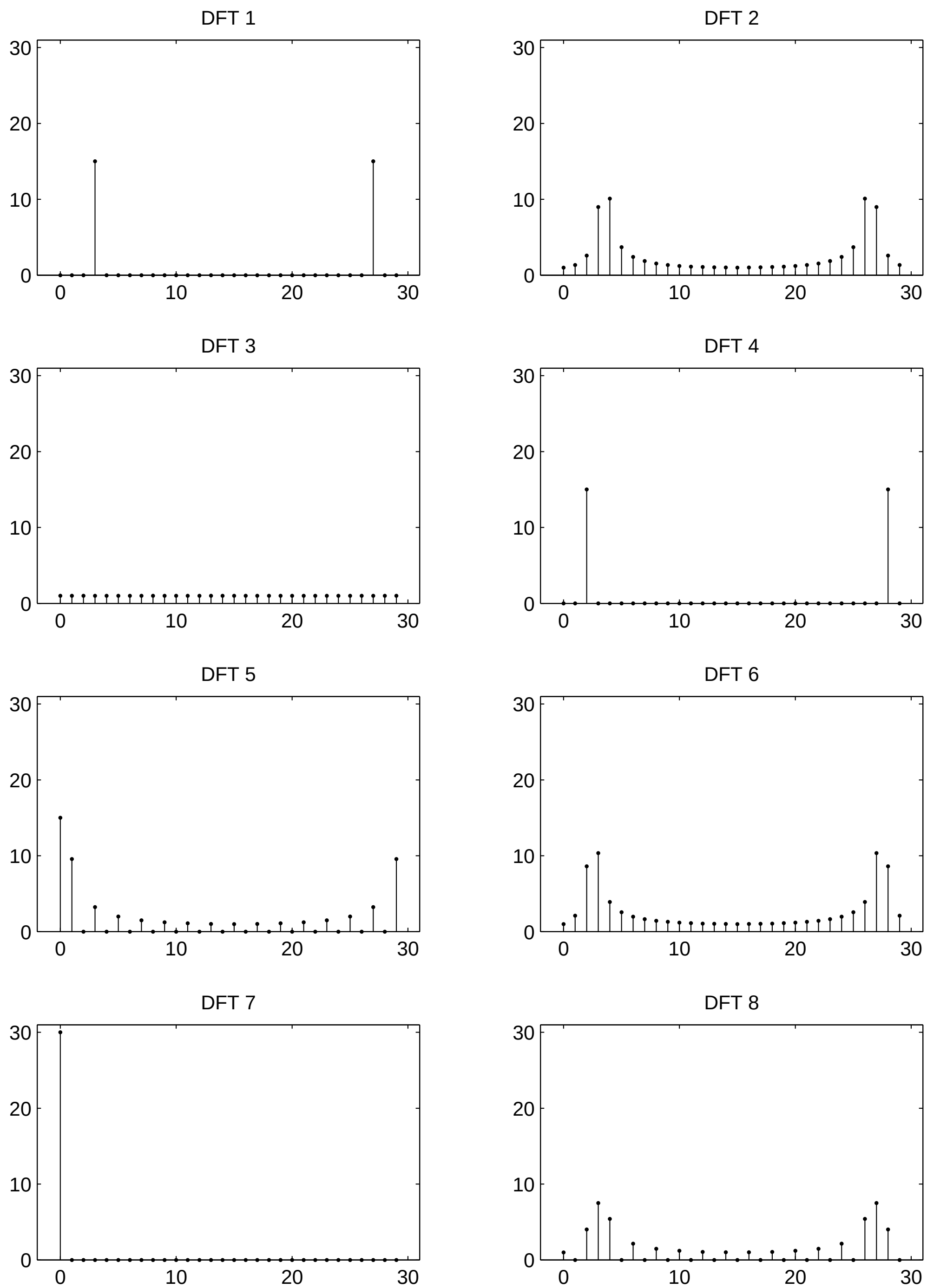
Solution:

Signals (f) and (i) both have purely real-valued DFT. Signal (h) has a purly imaginary-valued DFT.

1.12 **Matching.** Match each discrete-time signal with its DFT by filling out the following table. You should be able to do this problem with out using a computer.

Signal	DFT
1	
2	
3	
4	
5	
6	
7	
8	


SIGNAL 1
SIGNAL 2
SIGNAL 3
SIGNAL 4
SIGNAL 5
SIGNAL 6
SIGNAL 7
SIGNAL 8



Solution:

Signal 1 has exactly two cycles of a cosine, so you would expect $X(2)$ *and* $X(-2)$ *to be nonzero, and other DFT coefficients to be 0; that gives DFT 4. Note that* $X(-2)$ *is really* $X(N-2)$*.*

Signal 2 has two and a half cycles of a cosine, so you would expect the DFT to have a peak at index $k = 2.5$*, but that is not an integer — there is no DFT coefficient at that index. So the largest DFT coefficients would be at* $k = 2$ *and* $k = 3$ *and there would be 'leakage'. There would also be a peak* *around $k = N - 2.5$. This gives DFT 6.*

Similar reasons are used for signals 3 and 4.

The DFT of a constant is an impulse, so signal 6 corresponds to DFT 7. The DFT of an impulse is a constant, so signal 7 corresponds to DFT 3.

The DTFT of a rectangular pulse is a digital sinc function, so the DFT of a rectangular pulse is samples of the sinc function. So signal 8 corresponds to DFT 5.

That leaves signal 5 and DFT 8. Signal 5 can be written as a cosine times a rectangular pulse, so the DFT of signal 5 will be the convolution of a DFT of a cosine with the DFT of rectangular pulse — that is a sum of two shifted digital sinc functions.

Signal	DFT
1	4
2	6
3	1
4	2
5	8
6	7
7	3
8	5

1.25 The analog signal $x(t)$ is band-limited to 40 Hz. Suppose the signal is sampled at the rate of 100 samples per second and that at this rate 200 samples are collected. Then 200 zeros are appended to the 200 samples to form a 400-point vector. Then the 400-point DFT of this vector is computed to get $X(k)$ for $0 \le k \le 399$.

(a) Which DFT coefficients are free of aliasing?

(b) The DFT coefficient $X(50)$ represents the spectrum of the analog signal at what frequency f? (Give your answer in Hz).

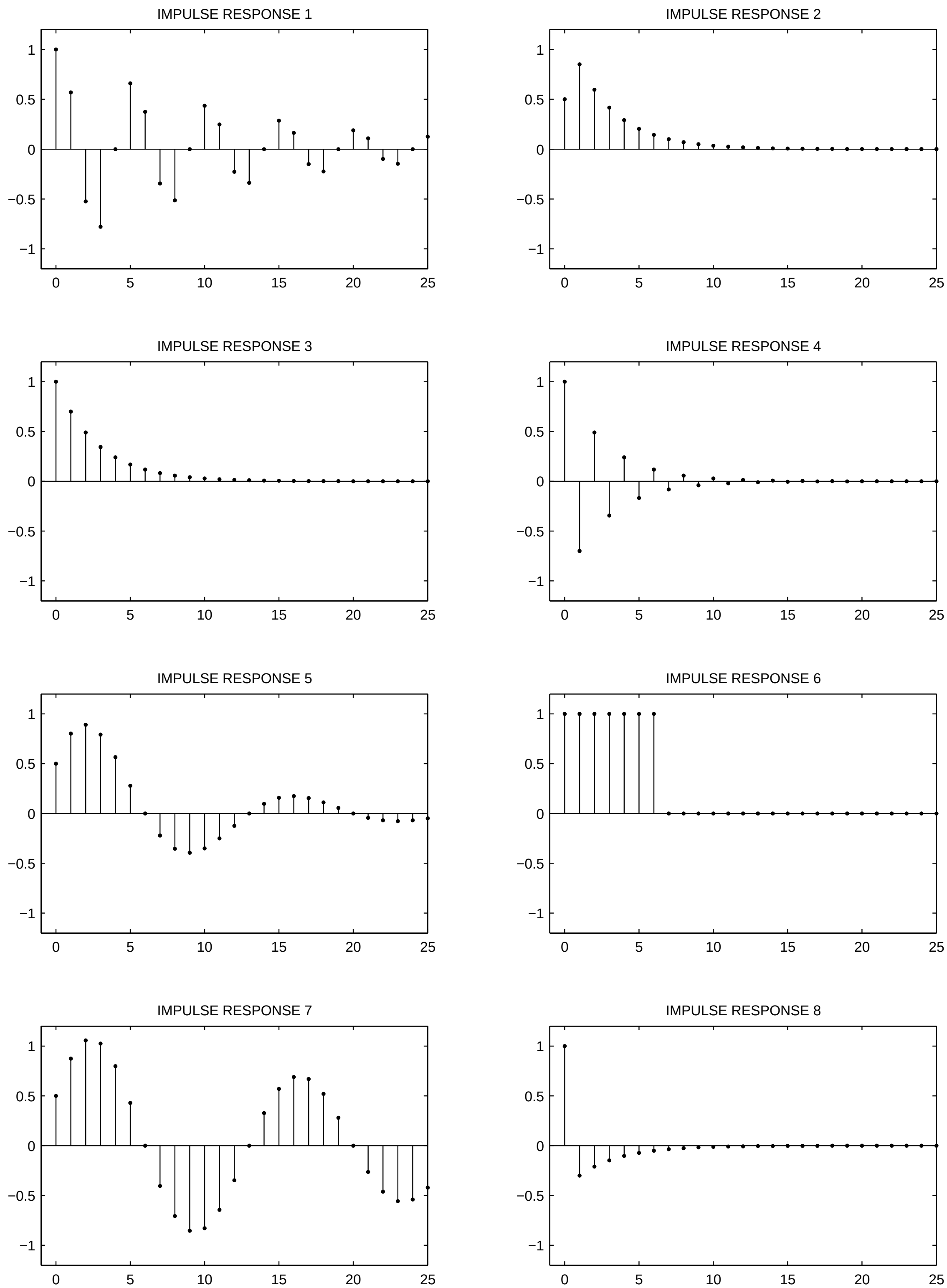
Solution:

(a) All of the DFT coefficients are free of aliasing. The sampling rate is more that twice the maximum signal frequency.

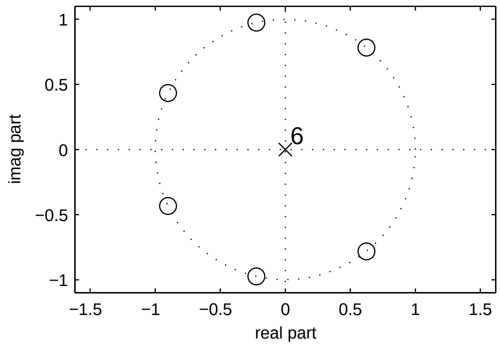
(b) The DFT bin width is $100/400$ *or 0.25 Hz. The 50th DFT coefficient corresponds to the frequency* 50 *times 0.25 Hz or* $\boxed{12.5 \text{ Hz}}$.

3.3 **Matching.** The diagrams on the following three pages show the impulse responses, pole-zero diagrams, and frequency responses magnitudes of 8 discrete-time causal LTI systems. But the diagrams are out of order. Match each diagram by filling out the following table.

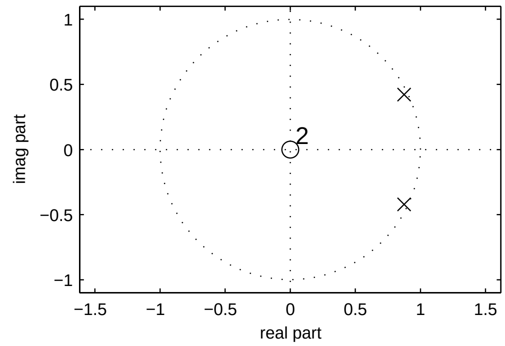
Impulse response	Pole-zero	Frequency response
1		
2		
3		
4		
5		
6		
7		
8		


IMPULSE RESPONSE 1
IMPULSE RESPONSE 2
IMPULSE RESPONSE 3
IMPULSE RESPONSE 4
IMPULSE RESPONSE 5
IMPULSE RESPONSE 6
IMPULSE RESPONSE 7
IMPULSE RESPONSE 8

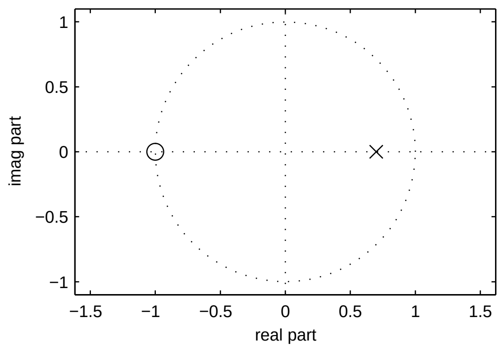
POLE−ZERO DIAGRAM 1



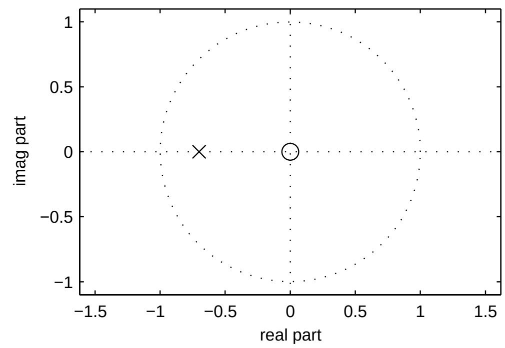
POLE−ZERO DIAGRAM 2



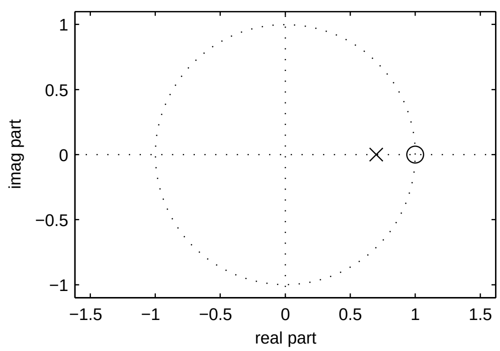
POLE−ZERO DIAGRAM 3



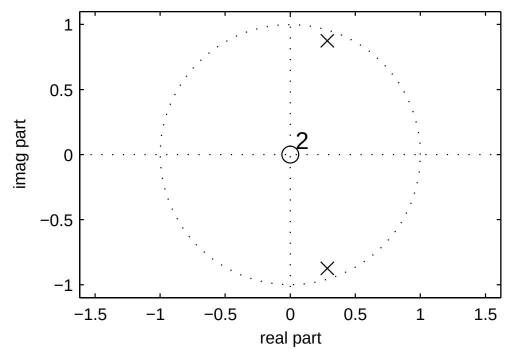
POLE−ZERO DIAGRAM 4



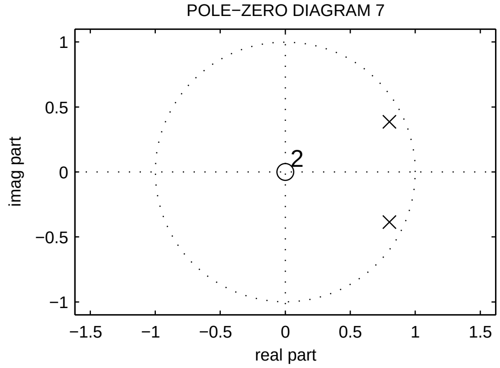
POLE−ZERO DIAGRAM 5



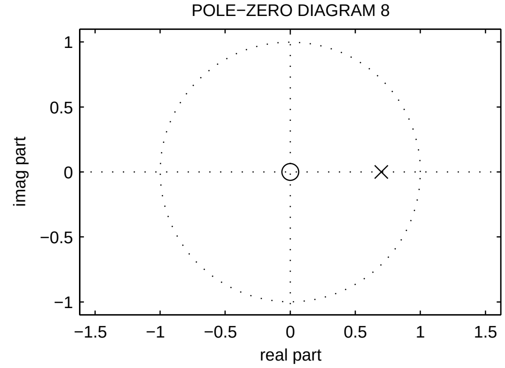
POLE−ZERO DIAGRAM 6

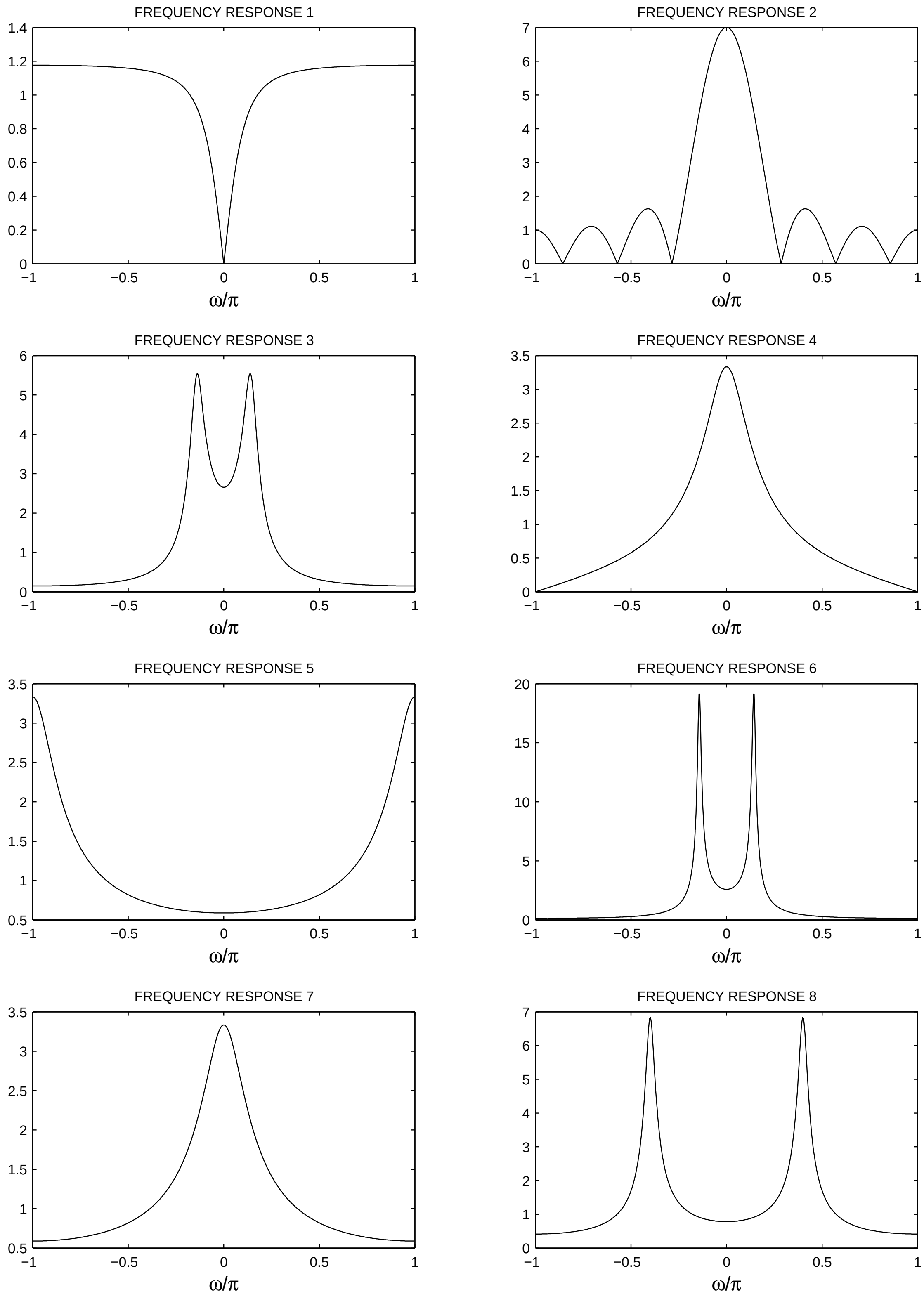


POLE−ZERO DIAGRAM 7



POLE−ZERO DIAGRAM 8




FREQUENCY RESPONSE 1
FREQUENCY RESPONSE 2
FREQUENCY RESPONSE 3
FREQUENCY RESPONSE 4
FREQUENCY RESPONSE 5
FREQUENCY RESPONSE 6
FREQUENCY RESPONSE 7
FREQUENCY RESPONSE 8
ω/π

Solution:

Impulse response	*Pole-zero*	*Frequency response*
1	*6*	*8*
2	*3*	*4*
3	*8*	*7*
4	*4*	*5*
5	*7*	*3*
6	*1*	*2*
7	*2*	*6*
8	*5*	*1*

3.6 An FIR digital filter has the transfer function

$$H(z) = (1 - z^{-1})^3 \, (1 + z^{-1})^3$$

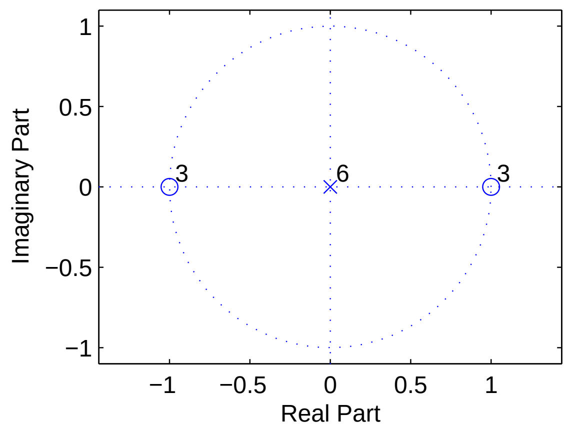
(a) Sketch the pole-zero diagram of this system.

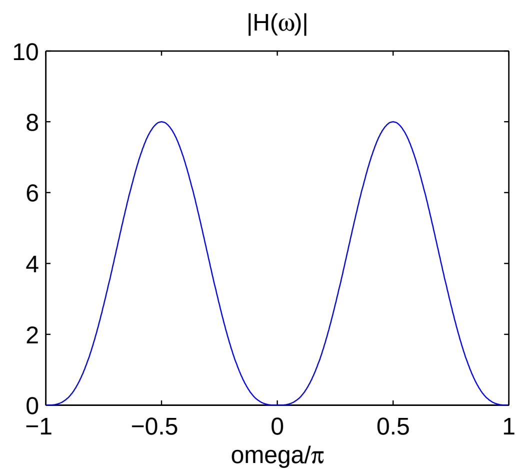
(b) Sketch $|H^f(\omega)|$.

(c) Would you classify this as a low-pass, high-pass, band-pass, or band-stop filter? Please briefly explain.

Solution:

Note that because the zero at $z = 1$ is of third order, not only is $H^f(\omega = 0)$ equal to one, but so is its first and second derivative, so the frequency response is flat at $\omega = 0$. The same is true for $\omega = \pi$.





4 Linear-Phase FIR Digital Filters

4.5 For the transfer function

$$H(z) = z^{-1} + z^{-6}$$

of an FIR linear-phase filter,

(a) sketch the impulse response

(b) what is the type of the filter (I, II, III, or IV)?

(c) sketch the frequency response magnitude $|H^f(\omega)|$.

(d) sketch the zero diagram

Solution:

This is a Type 2 FIR filter.

To find the zeros of $H(z)$,

$$z^{-1} + z^{-6} = 0 \tag{17}$$

$$z^5 + 1 = 1 \tag{18}$$

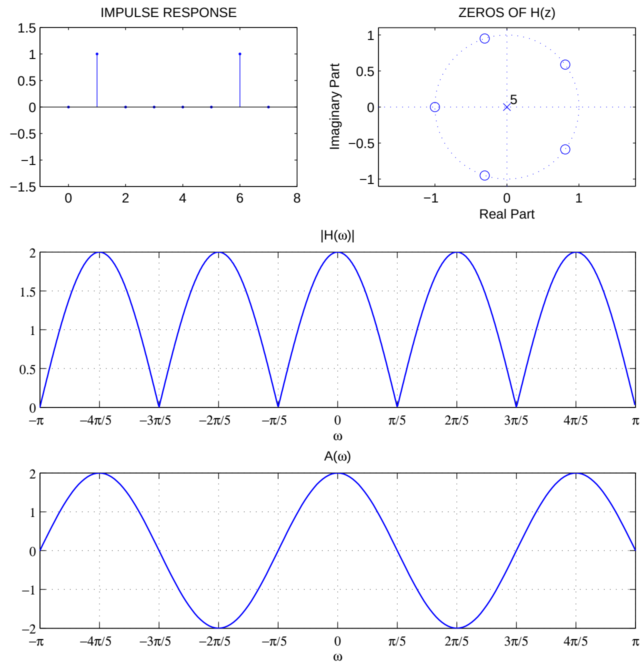
$$z^5 = -1 \tag{19}$$

$$z^5 = e^{j\,\pi} \tag{20}$$

$$z^5 = e^{j\,\pi + j\,2\,\pi\,k} \tag{21}$$

$$z = e^{j\,\pi/5 + j\,(2\,\pi/5)\,k} \tag{22}$$

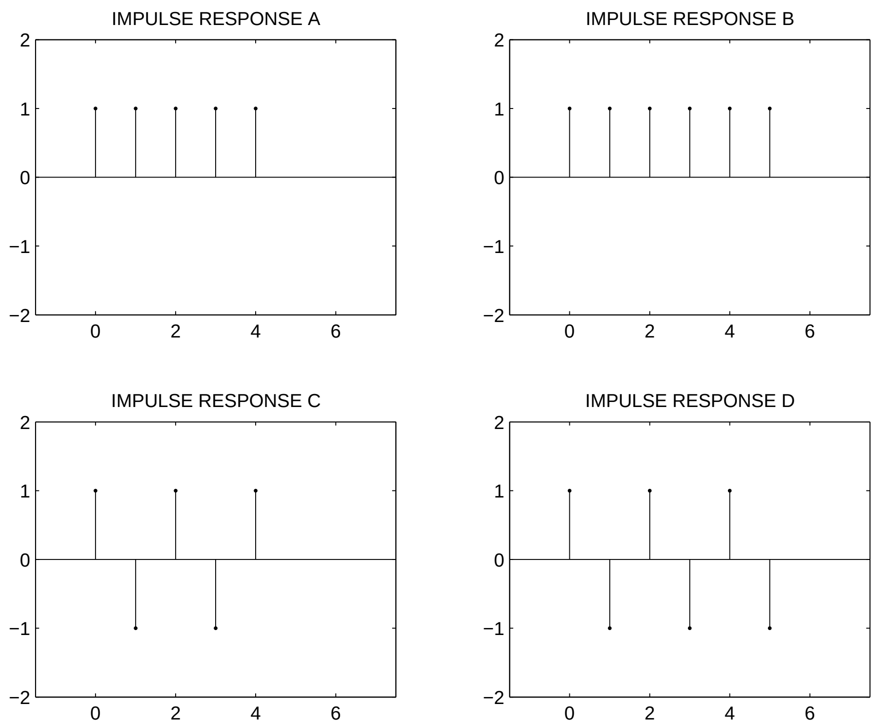
which for different integer values of k gives the values $z = e^{j\,\pi/5}$, $z = e^{j\,3\,\pi/5}$, $z = e^{j\,5\,\pi/5} = -1$, $z = e^{j\,7\,\pi/5}$, $z = e^{j\,9\,\pi/5}$, and which are shown in the zero diagram.

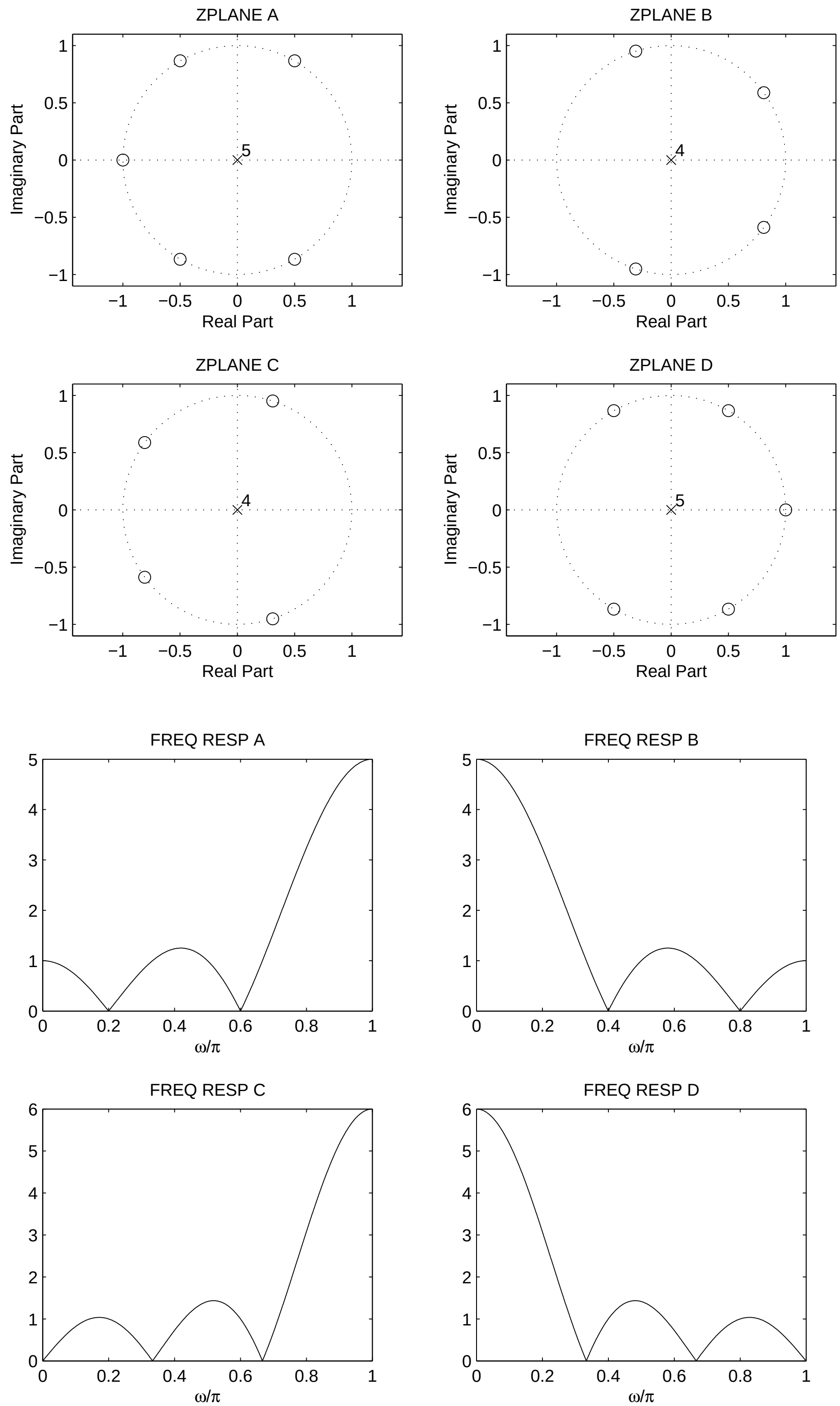


All the zeros lie on the unit circle, with equal spacing between them. From that, we can sketch the frequency response.

4.8 **Matching.** Match each impulse response with its frequency response and zero diagram by filling out the following table. You should do this problem with out using a computer.

Impulse Response	Zero Diagram	Frequency Response
A		
B		
C		
D		




ZPLANE A
ZPLANE B
ZPLANE C
ZPLANE D
Imaginary Part
Real Part
5
4
FREQ RESP A
FREQ RESP B
FREQ RESP C
FREQ RESP D
ω/π

Solution:

Impulse Response	*Zero Diagram*	*Frequency Response*
A	C	B
B	A	D
C	B	A
D	D	C

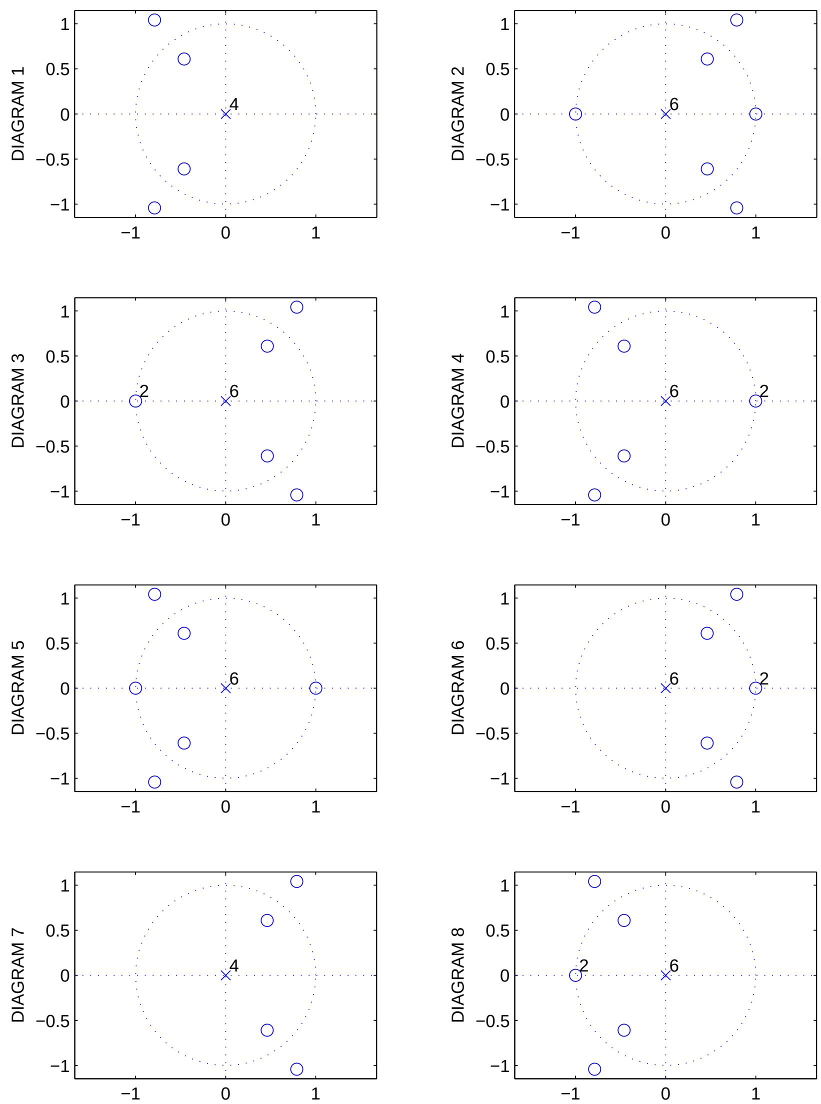
4.11 **FIR Filter Matching Problem.**

The following Matlab code fragment defines the impulse responses of four different FIR digital filters.

```
>> h1 = [2 7 12.5 12.5 7 2];
>> h2 = h1 .* ((-1).^(0:5));
>> h3 = conv(h2,[1 -1]);
>> h4 = h3 .* ((-1).^(0:6));
```

Without consulting MATLAB, match each of the two filters, `h3` and `h4`, with their pole-zero diagrams shown below.

Impulse Response	Pole-Zero Diagram
`h3`	
`h4`	



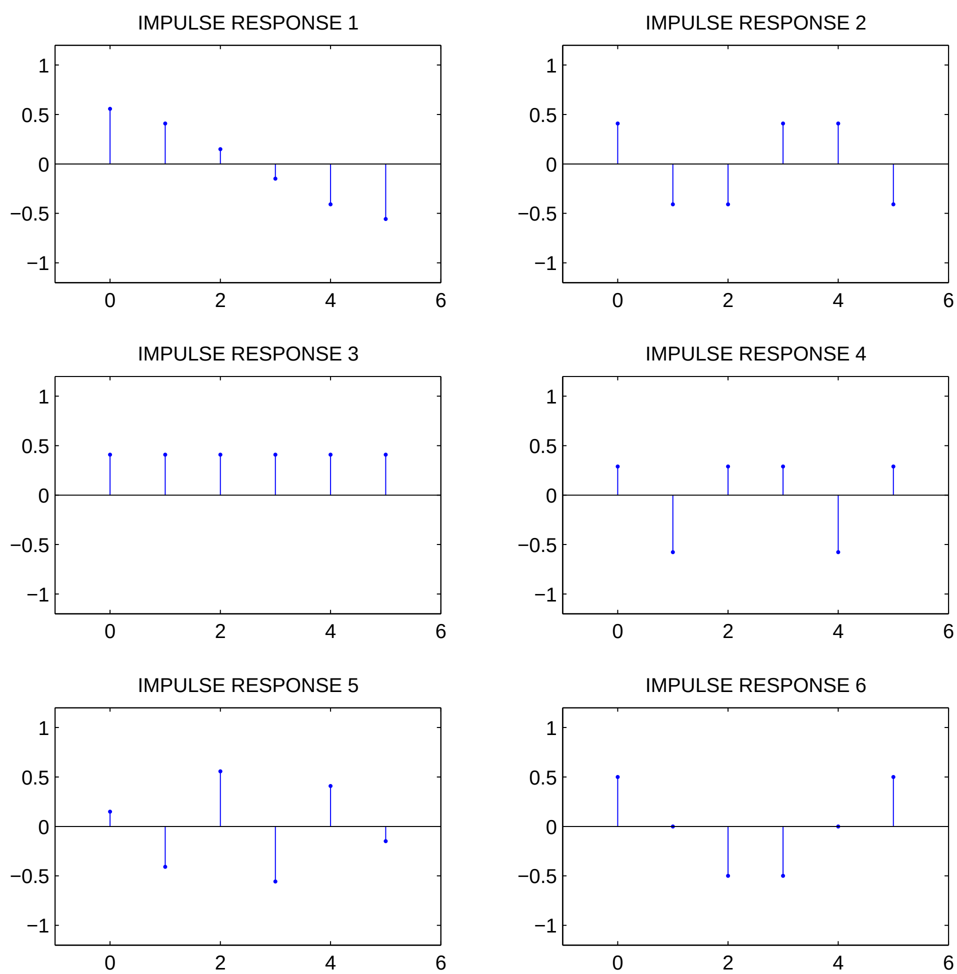
Solution:

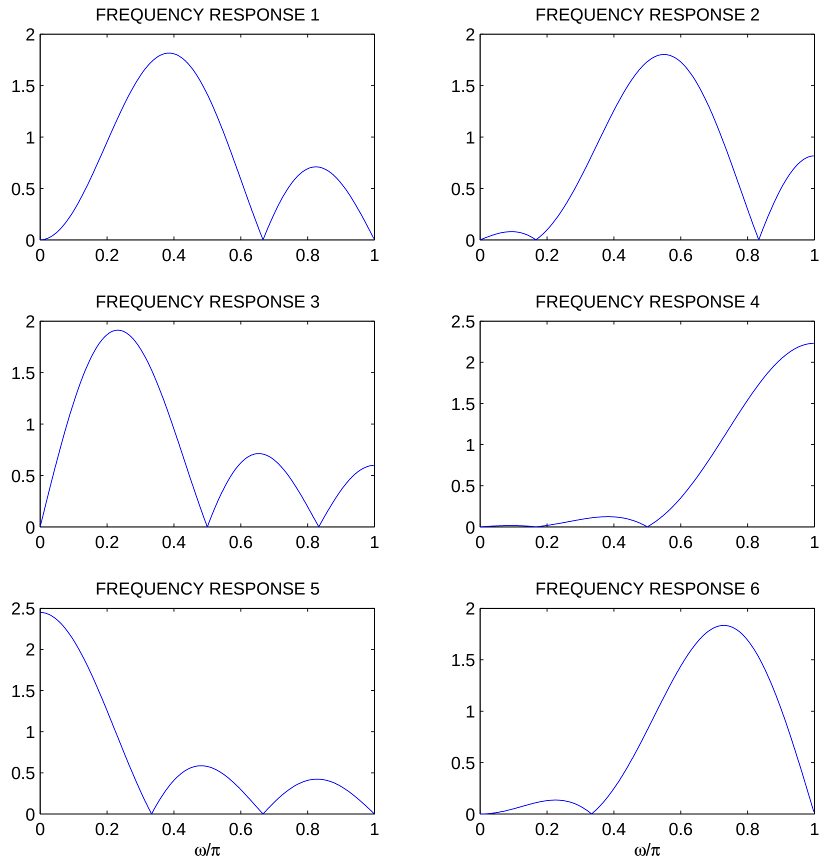
Note that h1 is a lowpass filter of Type 2. Also, from the code, we see that $h_2(n) = (-1)^n\, h_1(n)$*, so h2 is a highpass filter of Type 4. From the code we have that* $H_3(z) = H_2(z)\,(1 - z^{-1})$ *which puts a null in the frequency response at* $\omega = 0$*. Since h2 already has a zero at* $z = 1$*, h3 will have a double zero at* $z = 1$ *and will be a Type 1 filter. So h3 will correspond to pole-zero diagram 4 or 6. (This is not such a good exercise!) From the code we have* $h_4(n) = (-1)^n\, h_3(n)$ *or* $H_4(z) = H_3(-z)$ *which will negate all the zeros of* $H_3(z)$ *so h4 will correspond to pole-zero diagram 3 or 8.*

4.12 **Matching Problem.**

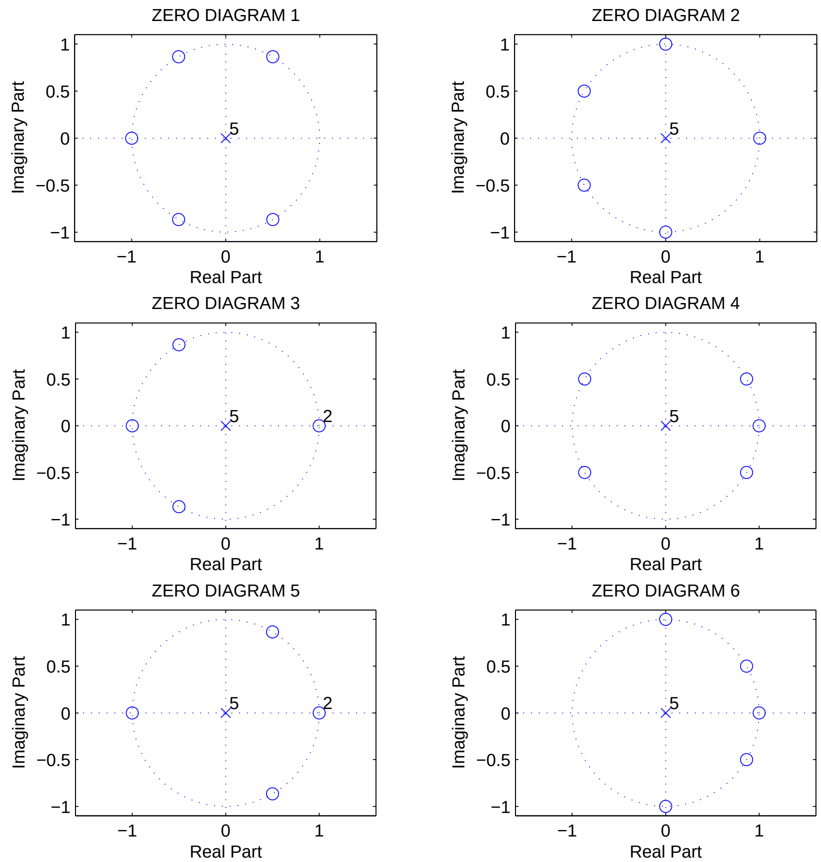
The following figures show 6 impulse responses, frequency responses, and zero diagrams. Match each frequency response and zero diagram to the corresponding impulse response.

Impulse Response	Frequency Response	Zero Diagram
1		
2		
3		
4		
5		
6		





FREQUENCY RESPONSE 1
FREQUENCY RESPONSE 2
FREQUENCY RESPONSE 3
FREQUENCY RESPONSE 4
FREQUENCY RESPONSE 5
FREQUENCY RESPONSE 6
ω/π
ω/π



Solution:

Although each impulse is one of the four types of FIR linear phase filter, in this problem, it is instructive to look at the number of sign changes in the impulse response. Roughly, the more sign changes an impulse response have, the higher its frequency content is. Impulse response 3 has no sign changes and it is a lowpass filter — that impulse response has the lowest frequency content. Then impulse response 1 has one sign change. Impulres response 6 has 2 sign changes etc. So you can match them to each of the frequency responses — each frequency response has its passband at a different frequency. From the frequency response, the zero diagram can be directly found.

Impulse Response	*Frequency Response*	*Zero Diagram*
1	3	2
2	2	4
3	5	1
4	6	5
5	4	6
6	1	3

4.15 A student is asked to design a Type I and a Type II low-pass FIR linear-phase filter using DFT-based interpolation. The student turns in the work shown on the next page, which has two problems.

(a) The student does not provide an explanation.

(b) One of the solutions is correct, the other solution is fake (the student just *made up* the vector `h`, it does not follow from the MATLAB commands).

Identify which solution is correct and provide an explanation for it. Why does the fake solution not work? You should be able to do this problem with out actually using MATLAB.

```
TYPE I FIR IMPULSE RESPONSE:

>> H = [1 1 1 1 1 0 0 0 0 0 0 1 1 1 1]';
>> v = ifft(H);
>> h = [v(9:15); v(1:8)]

h =

    0.0394
   -0.0667
         0
    0.0853
   -0.0667
   -0.0963
    0.3050
    0.6000
    0.3050
   -0.0963
   -0.0667
    0.0853
         0
   -0.0667
    0.0394
```

```
TYPE II FIR IMPULSE RESPONSE:

>> H = [1 1 1 1 1 0 0 0 0 0 1 1 1 1]';
>> v = ifft(H);
>> h = [v(9:14); v(1:8)]

h =

   -0.0318
   -0.0494
    0.0891
   -0.0255
   -0.1287
    0.2892
    0.6429
    0.6429
    0.2892
   -0.1287
   -0.0255
    0.0891
   -0.0494
   -0.0318
```

Solution:

The Type II filter solution is incorrect. The Type I filter solution is right.

For the Type I solution, note that the vector H is of length 15 and that it is real and circularly symmetric. That means that the inverse DFT, (v in the code) is also real and circularly symmetric, and also of length 15. The vector v will have the following symmetry pattern:

```
v = [a b c d e f g h h g f e d c b]
```

So the vector h produced by the 3rd line of MATLAB code in the Type I solution will have the symmetry pattern:

```
v = [h g f e d c b a b c d e f g h]
```

which is indeed a Type I impulse response.

On the other hand, for the Type II solution, note that the vector H is of length 14 and that it is also real and circularly symmetric. Therefore, the inverse DFT, (v in the MATLAB code) is also real and circularly symmetric, and also of length 14. The vector v will then have the following symmetry pattern:

```
v = [a b c d e f g h g f e d c b]
```

So the vector h produced by the 3rd line of MATLAB code in the Type II solution will have the symmetry pattern:

```
v = [g f e d c b a b c d e f g h]
```

which is not a symmetric impulse response. The actual result of the MATLAB commands for the Type II filter is:

```
h =

   -0.0318
   -0.0494
    0.0891
   -0.0255
   -0.1287
    0.2892
    0.6429
    0.2892
   -0.1287
   -0.0255
    0.0891
   -0.0494
   -0.0318
    0.0714
```

To design a Type II FIR filter using DFT-based interpolation, we can use a phase-shift of A prior to using the DFT. In that case, the ones at the end of the A vector should be -1.

4.16 *Optional*: Mitra 4.19. (But use the frequencies 0.2π, 0.4π, 0.9π)

Solution:

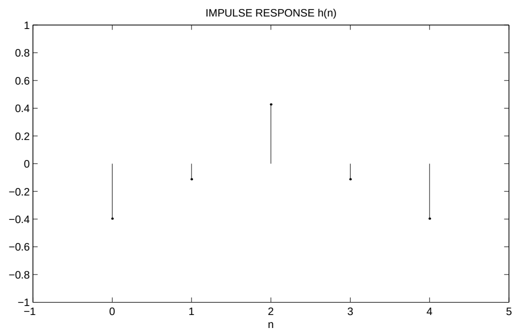
This is an interpolation problem:

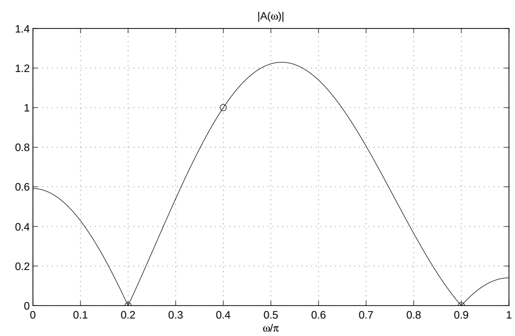
$$A(\omega) = 0 \text{ for } \omega = 0.2\pi,\ 0.9\pi, \quad \text{and} \quad A(0.4\pi) = 1.$$

We can use the general interpolation approach described in the notes. The impulse response we get as a solution is

$$h(n) = [-0.3968,\ -0.1127,\ 0.4276,\ -0.1127,\ -0.3968]$$

The frequency response we obtain is shown in the figure.






```
N = 5;
M = (N-1)/2;
wk = [0.2 0.4 0.9]'*pi;
Ak = [0 1 0]';
C = cos(wk*[0:M]);
a = C\Ak;
h = (1/2)*[a([M:-1:1]+1); 2*a([0]+1); a([1:M]+1)];

figure(1)
clf
subplot(2,1,1)
stem(0:N-1,h,'.')
xlabel('n')
title('IMPULSE RESPONSE h(n)')
axis([-1 5 -1 1])
[A,w] = firamp(h,1);
subplot(2,1,2)
```

5.3 (Porat 6.1) We saw that the height of the largest side-lobe of the rectangular window is about -13.5 dB relative to the main-lobe. What is the relative height of the smallest side-lobe? You may assume the window length N is odd. (Hint: At what frequency will it be located?).

Solution:

The smallest side lobe of the Dirichlet kernel is at $\omega = \pi$. *The formula is*

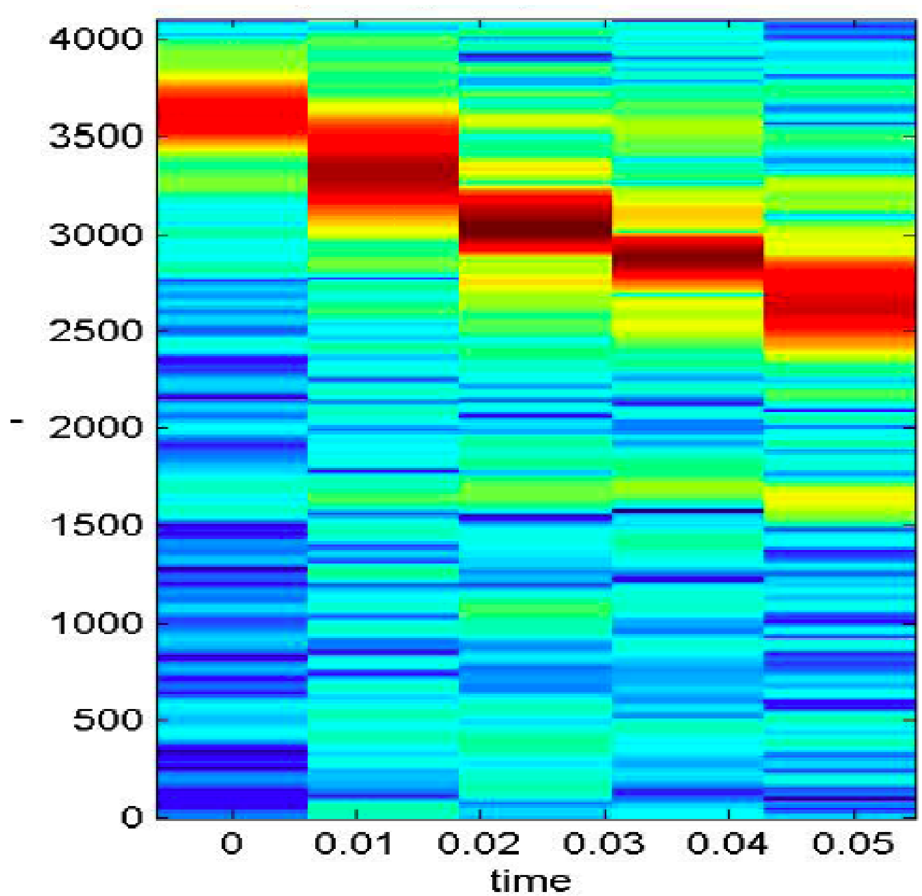
$$S^f(\omega) = \frac{\sin\left(\frac{N}{2}\,\omega\right)}{\sin\left(\frac{1}{2}\,\omega\right)}.$$

so

$$S^f(\pi) = \frac{\sin\left(\frac{N}{2}\,\pi\right)}{\sin\left(\frac{1}{2}\,\pi\right)} = 1$$

when N *is odd. Recall that* $S^f(0) = N$. *The relative height of the smallest side-lobe is therefore* $1/N$.

5.15 The following spectrogram is taken from the HW submitted by an EL 713 student.



Which of the following three parameters would you suggest the student modify to best improve the appearance of this spectrogram? What change would you make to that parameter? The three parameters are:

R = block length.

L = time lapse between blocks.

N = FFT length. (Each block is zero-padded to length N.)

Explain your answer.

Solution:

The student should reduce L the time-lapse between blocks. (Equivalently, the overlap fraction should be increased.) Making that change will reduce the step like appearance of the spectrogram.

5.17 The following figures show a signal and its spectrogram, computed with different sets of parameters,

$$R \in \{30, 60\}, \qquad N \in \{64, 256\}$$

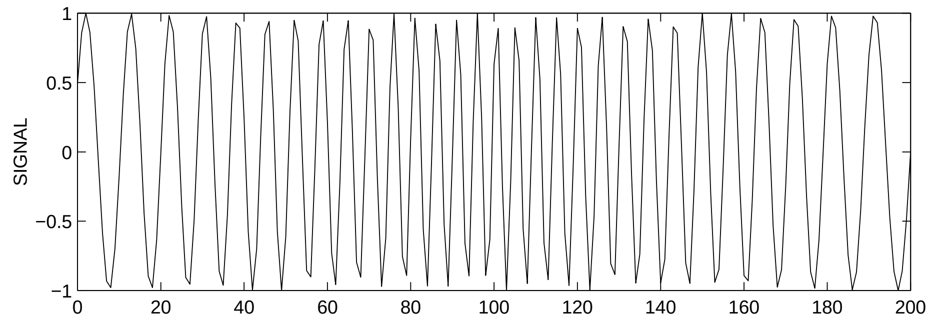
where

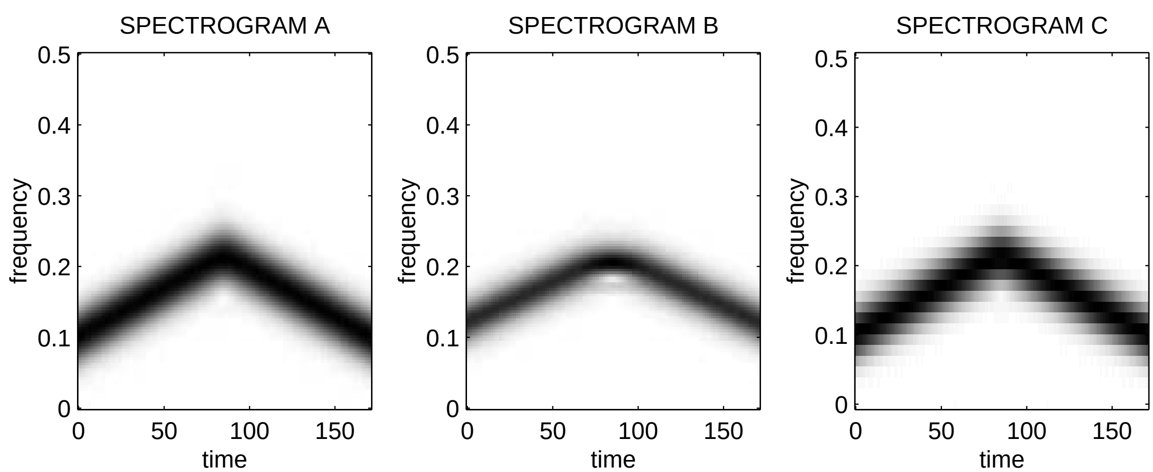
R = block length

N = FFT length (`nfft` in the Matlab `specgram` function). (Each block is zero-padded to length N.)

(a) For each of the spectrograms, indicate what you think R and N are, and explain your choices.

(b) Describe how the spectrogram would change if the time-skip (L in the notes), is increased.





Solution:

For spectrogram B, the vertical width of the bar with respect to the frequency axis is less than for spectrogram A, so spectorgram B uses a longer block length than spectrogram A. In addition, the point in the middle where the frequency starts to decrease is blurred, which also suggests that spectrogram B uses a longer block length.

The horizontal stripe effect in spectrogram C is due to discontinuities along the frequency axis — so for spectrogram C there is less zero padding that for A and B. Except for the stripe artifact, spectrogram C appears to be similar to spectram A, so it appears that they use the same block length.

Spectrogram	R	N
A	*30*	*256*
B	*60*	*256*
C	*30*	*64*