EL3210 Multivariable Feedback Control

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https://people.kth.se/~jacobsen/multi_17.shtml

Lecture 1: Introduction, classical SISO feedback control

The Practical

- 8 lectures
  - slides on homepage
  - reading assignments on homepage
- Course literature
  - *Skogestad and Postlethwaite, Multivariable Feedback Control, 2nd ed.*
  - *Supporting text: Zhou, Doyle and Glover, Robust and Optimal Control*
- 8 homeworks, compulsory
  - download from homepage after each lecture, hand in within one week
  - require Matlab with Robust Control toolbox
- 1-day take home open book exam, within 6 weeks after last lecture

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Course Content

Feedback control of MIMO LTI systems under model uncertainty

- *frequency domain* analysis and design;
  - extension of classical SISO methods to MIMO systems
  - optimal control problems formulated in input-output space
- *input-output controllability*; what can be achieved with feedback in a given system?



- *robustness:* stability and performance under model uncertainty

Course goals

After completed course you should be able to

- quantify the performance that can be achieved with feedback for a given system
- analyze feedback systems with respect to stability and performance in the presence of structured and unstructured model uncertainty
- design/synthesize controllers for robust performance

Lecture Plan

L1: Introduction, classical SISO feedback control (Ch.1-2)

L2: Performance limitations in SISO feedback (Ch. 5)

L3: Introduction to MIMO systems, excerpts from Linear Systems Theory (Ch. 3-4)

L4: Performance limitations in MIMO feedback (Ch. 6)

L5: Uncertainty and robust stability (Ch. 7-8)

L6: Robust performance (Ch. 7-8)

L7: Controller synthesis and design (Ch. 9-10)

L8: Alternative formulations (LMIs, IQCs, ...), summary (Ch. 10, 12)

Todays Lecture

- The Control Problem
- A Historical Perspective
- Brief introduction to norms
- Brief recap of classical control

The Control Problem

Control problems usually formulated in terms of signal tracking

$$y = Gu + G_d d$$

y - output / controlled variable

u - input / manipulated variable

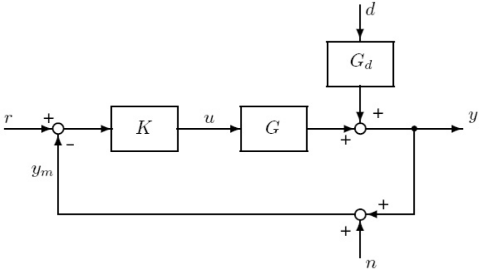
d - disturbance

r - reference, setpoint

- *Regulator problem:* attenuate effect of d on y
- *Servo problem:* make y follow r

Control objective: make $e = r - y$ "small" using feedback $u = \mathcal{C}(y, r)$

Why Feedback?



Why not $u = G^{-1}r - G^{-1}G_d d$?

1. model uncertainty - uncertain knowledge of system behavior
2. unmeasured disturbances
3. instability

Cost of feedback:

- potentially induce instability
- feed measurement noise into process

Fact 1: Feedback has its limitations

- feedback is a simple and potentially very powerful tool for tailoring the dynamic behavior of a system, but with hard limitations to what can be achieved
- control performance depends on controller *and* system

Ziegler and Nichols (1943): *In the application of automatic controllers, it is important to realize that controller and process form a unit; credit or discredit for results obtained are attributable to one as much as the other. . . . The finest controller made, when applied to a miserably designed process, may not deliver the desired performance. True, on badly designed processes, advanced controllers are able to eke out better results than older models, but on these processes, there is a definite end point which can be approached by instrumentation and it falls short of perfection.*

Approaches to Control Design

- "Traditional":
  1. specify desired performance
  2. design controller that meets specifications
  3. if 2 fails, try more advanced controller and repeat from 2
- This course:
  1. specify desired performance
  2. determine achievable performance
  3. if conflict between 1 and 2, change specifications or modify system
  4. design controller using your favorite method

Fact 2: Models are always uncertain

Models (G, G_d) always inaccurate, e.g., true system

$$G_p = G + E$$

with $E =$ “uncertainty”, or “perturbation” (unknown)

Definitions for closed loop:

- **Nominal stability (NS):** stable with no model uncertainty
- **Nominal performance (NP):** satisifies performance requirements with no model uncertainty
- **Robust stability (RS):** stable for “all” possible perturbations E
- **Robust performance (RP):** satisifies performance requirements for “all” possible perturbations E

System Representations

- State-space representation

$$\dot{x} = Ax(t) + Bu(t), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^p$$
$$y(t) = Cx(t) + Du(t), \quad y \in \mathbb{R}^l$$

- Transfer-function

$$Y(s) = G(s)U(s) \; ; \quad G(s) = C(sI - A)^{-1}B + D$$

- Frequency response

$$Y(j\omega) = G(j\omega)U(j\omega)$$

Sometimes we write

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \; ; \quad G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

A Brief History of Control

- Classical, 30's-50's: frequency domain methods
  Bode, Nyquist, Nichols, . . .
  - + yields insight (loop shaping)
  - + address model uncertainty (gain and phase margins)
  - ÷ only applicable to SISO systems
- "Modern", 60's-70's: state-space optimal control
  Bellman, Pontryagin, Kalman, . . .
  - + control cast as time-domain optimization problem
  - + applicable to MIMO systems (LQG)
  - ÷ can not accomodate for unmodeled dynamics
  - ÷ LQG has no guaranteed stability margins
  - ÷ no clear link to classical methods

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Famous paper (and abstract...) (IEEE Trans AC, 1978)

Guaranteed Margins for LQG Regulators

JOHN C. DOYLE

Abstract—There are none.

Introduction

Considerable attention has been given lately to the issue of robustness of linear–quadratic (LQ) regulators. The recent work by Safonov and Athans [1] has extended to the multivariable case the now well-known guarantee of 60° phase and 6 dB gain margin for such controllers. However, for even the single-input, single-output case there has remained the question of whether there exist any guaranteed margins for the full LQG (Kalman filter in the loop) regulator. By counterexample, this note answers that question; there are none.

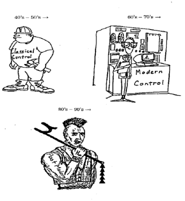
A standard two-state single-input single-output LQG control problem is posed for which the resulting closed-loop regulator has arbitrarily small gain margin.

A Brief History of Control

- "Postmodern", 80's-90's: robust control
  Zames, Francis, Doyle, . . .
  + frequency domain methods for MIMO systems
  + explicitly adress model uncertainty
  + control cast as optimization problem ($\mathcal{H}_2, \mathcal{H}_\infty$)
  + links classical and modern approaches; *"formulate and analyze in input-output domain, compute in state-space"*

  ÷ high order controllers, computational issues, ...

It was the introduction of norms in control, in particular the $\mathcal{H}_\infty$-norm, that paved the way for analyzing fundamental limitations and robustness in MIMO systems

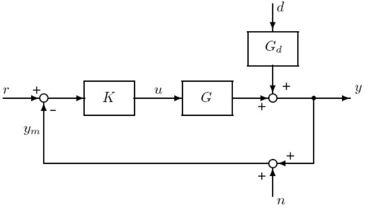


A Brief History of Control

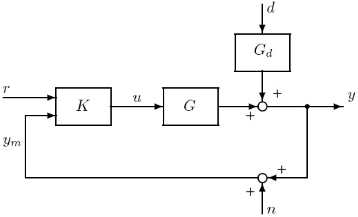
- Post 90's
  - analysis/synthesis using convex optimization, e.g., LMIs
  - combining $\mathcal{H}_2$ for performance with $\mathcal{H}_\infty$ for robustness
  - beyond LTI systems, e.g., *Integral Quadratic Constraints* (IQCs)
  - ...

Control Structures

- 1-Degree of freedom

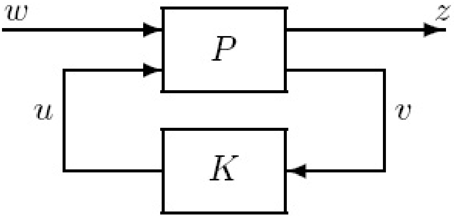


- 2-Degrees of freedom



Control Structures

- General control structure



P - generalized system, *K* - controller

w - exogeneous inputs (*d*, *r*, *n*)	*z* - exogeneous outputs (*e*, *u*)
u - manipulated inputs	*v* - measurements, setpoints

Objective: minimize *gain* from *w* to *z*. With appropriate weights/scaling, make *gain* smaller than 1

Brief on Norms (more in Lec 3)

A real valued function $\|\cdot\|$ on a linear space H, over the field of real or complex numbers, is called a *norm* on H if it satisfies

(i) $\|x\| \geq 0$

(ii) $\|x\| = 0$ if and only if $x = 0$

(iii) $\|ax\| = |a|\|x\|$ for any scalar a

(iv) $\|x + y\| \leq \|x\| + \|y\|$

for any $x, y \in H$

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Vector and Matrix Norms

- For $x \in \mathbb{C}^n$ the p-norm is

$$\|x\|_p = \left(\Sigma_{i=1}^n |x_i|^p\right)^{1/p}$$

We will mainly consider $p = 2$, the Euclidian norm

$$\|x\|_2 = |x| = \sqrt{x^H x}$$

- For $A \in \mathbb{C}^{m \times n}$ the (induced) p-norm is

$$\|A\|_p = \sup_{x \in \mathbb{C}^n, x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

We will mainly consider $p = 2$

$$\|A\|_2 = \bar{\sigma}(A) = \max_i \sqrt{\lambda_i(A^H A)}$$

Operator Norms

- For a vector valued signal $x(t)$ the L_p-norm is

$$\|x(t)\|_p = \left(\int_{-\infty}^{\infty} \Sigma_i |x_i(\tau)|^p d\tau\right)^{1/p}$$

We will mainly consider $p = 2$, i.e., the L_2-norm

$$\|x(t)\|_2 = \sqrt{\int_{-\infty}^{\infty} x(\tau)^T x(\tau) d\tau} = \sqrt{\int_{-\infty}^{\infty} |x|^2 d\tau}$$

In frequency domain (by Parsevals thm)

$$\|x\|_2 = \frac{1}{2\pi}\sqrt{\int_{-\infty}^{\infty} x(i\omega)^H x(i\omega) d\omega}$$

Operator Norms

For a transfer-function $G(s)$

- the $\mathcal{H}_2$-norm is

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathtt{tr}(G(j\omega)^H G(j\omega))d\omega}$$

- and the $\mathcal{H}_\infty$-norm is

$$\|G(s)\|_\infty = \sup_\omega \bar{\sigma}(G(j\omega))$$

$\mathcal{H}$ denotes *Hardy space*, $\mathcal{H}_\infty$ ($\mathcal{H}_2$) is the set of stable and (strictly) proper transfer-functions

The $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms

- The $\mathcal{H}_\infty$-norm is an induced norm from L_2 to L_2, i.e., the L_2-gain

$$y = Gu \; ; \quad \|G(s)\|_\infty = \sup_{u(t)\neq 0} \frac{\|y\|_2}{\|u\|_2}$$

- The $\mathcal{H}_2$-norm is not an induced norm. But, e.g., equals amplification from a white noise input to the 2-norm of the output

We will consider both norms for design later, but the fact that the $\mathcal{H}_\infty$-norm is an induced norm makes it useful for analyzing performance limitations (Lec 2) as well as robustness (Lec 5) .

Scaling - simplifies analysis and design

Unscaled model:

$$\hat{y} = \hat{G}\hat{u} + \hat{G}_d\hat{d} \; ; \quad \hat{e} = \hat{r} - \hat{y}$$

Scale all variables so that expected/allowed magnitude is less than 1:

$$u = \frac{\hat{u}}{\hat{u}_{max}} \; ; \quad d = \frac{\hat{d}}{\hat{d}_{max}} \; ; \quad y = \frac{\hat{y}}{\hat{e}_{max}} \; ; \quad e = \frac{\hat{e}}{\hat{e}_{max}} \; ; \quad r = \frac{\hat{r}}{\hat{e}_{max}}$$

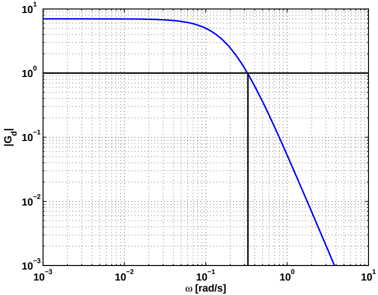
Introduce $D_d = \hat{d}_{max} \; ; \quad D_u = \hat{u}_{max} \; ; \quad D_e = \hat{e}_{max}$

$$y = \underbrace{D_e^{-1}\hat{G}D_u}_{G}\, u + \underbrace{D_e^{-1}\hat{G}_d D_d}_{G_d}\, d$$

- In the scaled model, all signals should have magnitude less than 1, i.e., expected $|d| < 1$ and acceptable $|e| < 1$

Example: scaled frequency response

Bode plot for $|G_d(j\omega)|$:

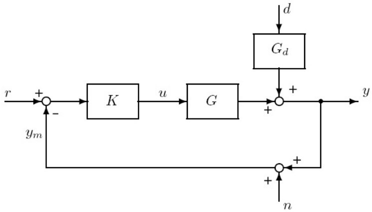


- Need disturbance attenuation for frequencies where $|G_d(j\omega)| > 1$, i.e., for $\omega < 0.33\ rad/s$
- Or, equivalently, we require

$$\|SG_d\|_\infty < 1$$

Next: classical control revisited

Closed-Loop Transfer Functions - 1-DOF structure



Closed-loop transfer-functions

$$y = \underbrace{(1+GK)^{-1}GK}_{T}\, r + \underbrace{(1+GK)^{-1}}_{S}\, G_d d - \underbrace{(1+GK)^{-1}GK}_{T}\, n$$

control error

$$e = r - y = -Sr + SG_d d - Tn$$

input

$$u = KSr - KSG_d d - KSn$$

The Sensitivity Functions

Introduce the loop gain $L = GK$

$$S = (1 + L)^{-1} \; ; \quad T = (1 + L)^{-1} L$$

$$\Downarrow$$

$$S + T = 1$$

S - the sensitivity function

T - the complimentary sensitivity function

The name "Sensitivity"

- Bode: relative sensitivity of T to model perturbations (uncertainty)

$$S = (dT/T)/(dG/G)$$

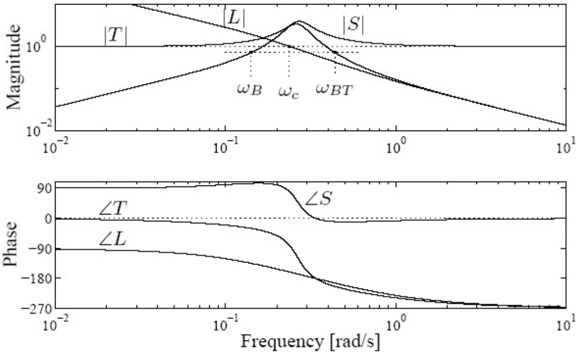
- But, also effect of feedback on sensitivity to disturbances

$$y = SG_d d$$

$|S(j\omega)| < 1$: feedback reduces disturbance sensitivity

$|S(j\omega)| > 1$: feedback increases disturbance sensitivity

Frequency Plots



Definitions:

- crossover frequency ω_c: $|L(j\omega_c)| = 1$
- bandwidth ω_B: $|S(j\omega_B)| = 1/\sqrt{2}$
- bandwidth for T, ω_{BT}: $|T(j\omega_{BT})| = 1/\sqrt{2}$

Bandwidth and Crossover Frequency

- Effective feedback for frequencies where $|S(j\omega)| < 1$, i.e., up to bandwidth ω_B
$$|S(j\omega)| < 1,\ \omega \in [0, \omega_B]$$
- Bandwidth and crossover frequencies:
$$\omega_B < \omega_c < \omega_{BT}$$
Proof: see (2.53) in book
- Typically assume
$$\omega_c \approx \omega_B$$

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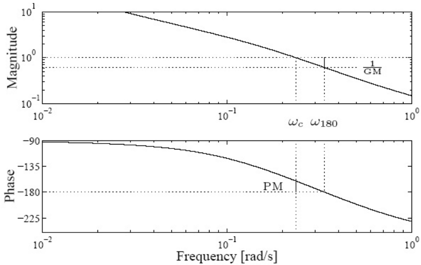
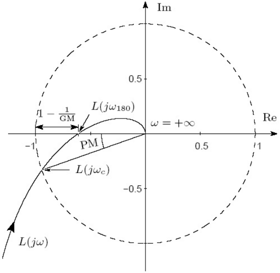
Stability margins

Gain margin GM and phase margin PM - robustness measures

Bode diagram



Nyquist diagram



Sensitivity peaks

Stability margins and performance are related

- $|S|^{-1} = |1 + L(j\omega)|$ is distance from $L(j\omega)$ to critical point -1 in Nyquist diagram
- define $M_S = \max_\omega |S(j\omega)|$; $\quad M_T = \max_\omega |T(j\omega)|$, then

$$M_S \geq \frac{1}{PM} \qquad M_S \geq \frac{GM}{GM - 1}$$

$$M_T \geq \frac{1}{PM} \qquad M_T \geq \frac{1}{GM - 1}$$

  - obtained by considering the loop gain at L at ω_c and ω_{180}, respectively

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Controller Design

Three main approaches:

1. **Shaping transfer-functions**
   a. **Loop shaping (classic):** use controller K to shape loop gain $L(j\omega)$
   b. **Shaping the closed loop:** shape S, T etc, using optimization based methods

2. **Signal based approaches:** minimize signals, i.e., control error e and input u, given characteristics of inputs d, r, n.

3. **Numerical optimization:** optimize "real" control objectives, e.g., rise time and overshoot for step responses.

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Classic Loop Shaping - shaping $|L|$

recall

$$e = -\underbrace{\frac{1}{1+L}}_{S} r + \underbrace{\frac{1}{I+L}}_{S} d - \underbrace{\frac{L}{I+L}}_{T} n$$

Fundamental trade-offs:

- setpoint following: $|L|$ large
- disturbance attenuation: $|L|$ large
- noise propagation: $|L|$ small

Also:

$$u = KSr - KSG_d d - KSn$$

- input usage: $|K|$ small

Resolving the trade-off

Typically: make $|L|$ large in frequency range where disturbances and setpoints important, and $|L|$ small for higher frequencies,

$$|L| >> 1, \omega \in [0, \omega_B]$$
$$|L| << 1, \omega > \omega_B$$

i.e., want $|L|$ to drop off steeply around $\omega_B \approx \omega_c$

But, slope of $|L|$ and phase $\arg L$ coupled

Bode Relation

$$\arg L(j\omega_0) \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\ ln\,|L|}{d\ ln\,\omega} ln \left| \frac{\omega + \omega_0}{\omega - \omega_0} \right| \frac{d\omega}{\omega}$$

- equality for minimum phase systems
- with slope $N = d\ ln\,|L|/d\ ln\,\omega$,

$$\arg L(j\omega_0) \approx \frac{\pi}{2} N(j\omega_0)$$

Thus, slope around crossover ω_c should be at most -2, less to yield some phase margin.

A Procedure for Loop Shaping

1. First try

$$L(s) = \frac{\omega_c}{s} \quad \Rightarrow \quad K = \frac{\omega_c}{s} G^{-1}(s)$$

   yields

$$y = \frac{\omega_c}{s + \omega_c} r$$

   But, bad disturbance rejection if G_d slow

2. For disturbances

$$e = SG_d d$$

   Require

$$|SG_d| < 1 \quad \forall \omega$$

   corresponds to

$$|1 + L| > |G_d| \quad \forall \omega$$

A Procedure for Loop Shaping

cont. If $|G_d| > 1$ we get approximately

$$|L| > |G_d|$$

Simple choice

$$L = G_d \quad \Rightarrow \quad K = G^{-1} G_d$$

and with integral action

$$K = \frac{s + \omega_I}{s} G^{-1} G_d$$

3. High frequency correction

$$K = \frac{s + \omega_I}{s} G^{-1} G_d \frac{\tau s + 1}{\frac{\tau}{\gamma} s + 1}$$

to improve stability, i.e., modify slope of $|L|$ around ω_c

4. To improve setpoint tracking, add prefilter $K_r(s)$ on setpoint $\Rightarrow$ 2-DOF control structure

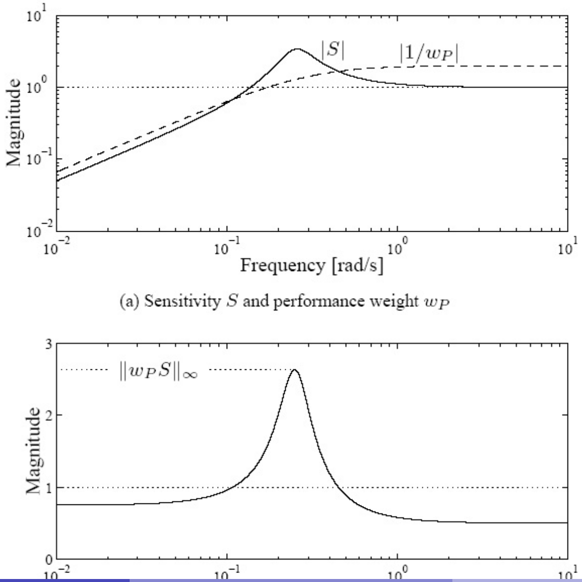
Shaping the Closed Loop

- Shaping $L = GK$ is just a means of achieving a desired closed-loop
- Alternative: find controller that minimizes a weighted sensitivity, e.g.,

$$\min_K \left(\max_\omega |w_P S| \right) = \min_K \|w_P S\|_\infty$$

  - $\|wpS\|_\infty < 1 \quad \Rightarrow \quad |S| < 1/|w_p| \quad \forall\omega$

Weighted Sensitivity

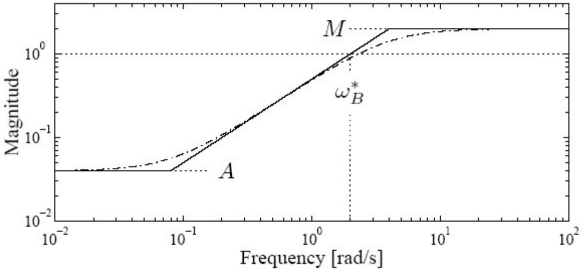


Performance weight

Typical choice for weigth

$$w_p = \frac{s/M + \omega_B}{s + \omega_B A}$$

Magnitude of $1/|w_p|$:



Control objective satisfied if $\|w_p S\|_\infty < 1$

Next Time

- Fundamental performance limitations and tradeoffs in SISO feedback
- *Controllability analysis:* what is achievable performance, e.g., ω_B, M, for a given system?

Homework:

- Exercise 1 (download from the course homepage). Hand in next Friday.
- Read Chapter 5 (and 1-2) in Skogestad and Postlethwaite