EDWARD M. PURCELL AND
DAVID J. MORIN

Electricity and Magnetism

THIRD EDITION

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Electricity and Magnetism

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EDWARD M. PURCELL (1912–1997) was the recipient of many awards for his scientific, educational, and civic work. In 1952 he shared the Nobel Prize for Physics for the discovery of nuclear magnetic resonance in liquids and solids, an elegant and precise method of determining the chemical structure of materials that serves as the basis for numerous applications, including magnetic resonance imaging (MRI). During his career he served as science adviser to Presidents Dwight D. Eisenhower, John F. Kennedy, and Lyndon B. Johnson.

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THIRD EDITION

ELECTRICITY AND MAGNETISM

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CONTENTS

Preface to the third edition of Volume 2 xiii

Preface to the second edition of Volume 2 xvii

Preface to the first edition of Volume 2 xxi

CHAPTER 1
ELECTROSTATICS: CHARGES AND FIELDS 1

1.1 Electric charge 1
1.2 Conservation of charge 4
1.3 Quantization of charge 5
1.4 Coulomb's law 7
1.5 Energy of a system of charges 11
1.6 Electrical energy in a crystal lattice 14
1.7 The electric field 16
1.8 Charge distributions 20
1.9 Flux 22
1.10 Gauss's law 23
1.11 Field of a spherical charge distribution 26
1.12 Field of a line charge 28
1.13 Field of an infinite flat sheet of charge 29
1.14 The force on a layer of charge 30
1.15 Energy associated with the electric field 33
1.16 Applications 35

Chapter summary 38
Problems 39
Exercises 47

CHAPTER 2
THE ELECTRIC POTENTIAL 58

2.1 Line integral of the electric field 59
2.2 Potential difference and the potential function 61
2.3 Gradient of a scalar function 63
2.4 Derivation of the field from the potential 65
2.5 Potential of a charge distribution 65
2.6 Uniformly charged disk 68
2.7 Dipoles 73
2.8 Divergence of a vector function 78
2.9 Gauss's theorem and the differential form of Gauss's law 79
2.10 The divergence in Cartesian coordinates 81
2.11 The Laplacian 85
2.12 Laplace's equation 86
2.13 Distinguishing the physics from the mathematics 88
2.14 The curl of a vector function 90
2.15 Stokes' theorem 92
2.16 The curl in Cartesian coordinates 93
2.17 The physical meaning of the curl 95
2.18 Applications 100
Chapter summary 103
Problems 105
Exercises 112

CHAPTER 3
ELECTRIC FIELDS AROUND CONDUCTORS 124

3.1 Conductors and insulators 125
3.2 Conductors in the electrostatic field 126
3.3 The general electrostatic problem and the uniqueness theorem 132
3.4 Image charges 136
3.5 Capacitance and capacitors 141
3.6 Potentials and charges on several conductors 147
3.7 Energy stored in a capacitor 149
3.8 Other views of the boundary-value problem 151
3.9 Applications 153
Chapter summary 155

Problems 155
Exercises 163

CHAPTER 4
ELECTRIC CURRENTS 177

4.1 Electric current and current density 177
4.2 Steady currents and charge conservation 180
4.3 Electrical conductivity and Ohm's law 181
4.4 The physics of electrical conduction 189
4.5 Conduction in metals 198
4.6 Semiconductors 200
4.7 Circuits and circuit elements 204
4.8 Energy dissipation in current flow 207
4.9 Electromotive force and the voltaic cell 209
4.10 Networks with voltage sources 212
4.11 Variable currents in capacitors and resistors 215
4.12 Applications 217
Chapter summary 221
Problems 222
Exercises 226

CHAPTER 5
THE FIELDS OF MOVING CHARGES 235

5.1 From Oersted to Einstein 236
5.2 Magnetic forces 237
5.3 Measurement of charge in motion 239
5.4 Invariance of charge 241
5.5 Electric field measured in different frames of reference 243
5.6 Field of a point charge moving with constant velocity 247
5.7 Field of a charge that starts or stops 251
5.8 Force on a moving charge 255
5.9 Interaction between a moving charge and other moving charges 259
Chapter summary 267
Problems 268
Exercises 270

CHAPTER 6
THE MAGNETIC FIELD 277

6.1 Definition of the magnetic field 278
6.2 Some properties of the magnetic field 286

6.3	Vector potential	293
6.4	Field of any current-carrying wire	296
6.5	Fields of rings and coils	299
6.6	Change in **B** at a current sheet	303
6.7	How the fields transform	306
6.8	Rowland's experiment	314
6.9	Electrical conduction in a magnetic field: the Hall effect	314
6.10	Applications	317
	Chapter summary	322
	Problems	323
	Exercises	331

CHAPTER 7
ELECTROMAGNETIC INDUCTION 342

7.1	Faraday's discovery	343
7.2	Conducting rod moving through a uniform magnetic field	345
7.3	Loop moving through a nonuniform magnetic field	346
7.4	Stationary loop with the field source moving	352
7.5	Universal law of induction	355
7.6	Mutual inductance	359
7.7	A reciprocity theorem	362
7.8	Self-inductance	364
7.9	Circuit containing self-inductance	366
7.10	Energy stored in the magnetic field	368
7.11	Applications	369
	Chapter summary	373
	Problems	374
	Exercises	380

CHAPTER 8
ALTERNATING-CURRENT CIRCUITS 388

8.1	A resonant circuit	388
8.2	Alternating current	394
8.3	Complex exponential solutions	402
8.4	Alternating-current networks	405
8.5	Admittance and impedance	408
8.6	Power and energy in alternating-current circuits	415
8.7	Applications	418
	Chapter summary	420
	Problems	421
	Exercises	424

CHAPTER 9
MAXWELL'S EQUATIONS AND ELECTROMAGNETIC WAVES 430

9.1 "Something is missing" 430
9.2 The displacement current 433
9.3 Maxwell's equations 436
9.4 An electromagnetic wave 438
9.5 Other waveforms; superposition of waves 441
9.6 Energy transport by electromagnetic waves 446
9.7 How a wave looks in a different frame 452
9.8 Applications 454
Chapter summary 455
Problems 457
Exercises 461

CHAPTER 10
ELECTRIC FIELDS IN MATTER 466

10.1 Dielectrics 467
10.2 The moments of a charge distribution 471
10.3 The potential and field of a dipole 474
10.4 The torque and the force on a dipole in an external field 477
10.5 Atomic and molecular dipoles; induced dipole moments 479
10.6 Permanent dipole moments 482
10.7 The electric field caused by polarized matter 483
10.8 Another look at the capacitor 489
10.9 The field of a polarized sphere 492
10.10 A dielectric sphere in a uniform field 495
10.11 The field of a charge in a dielectric medium, and Gauss's law 497
10.12 A microscopic view of the dielectric 500
10.13 Polarization in changing fields 504
10.14 The bound-charge current 505
10.15 An electromagnetic wave in a dielectric 507
10.16 Applications 509
Chapter summary 511
Problems 513
Exercises 516

CHAPTER 11
MAGNETIC FIELDS IN MATTER 523

11.1 How various substances respond to a magnetic field 524

11.2 The absence of magnetic “charge” 529
11.3 The field of a current loop 531
11.4 The force on a dipole in an external field 535
11.5 Electric currents in atoms 540
11.6 Electron spin and magnetic moment 546
11.7 Magnetic susceptibility 549
11.8 The magnetic field caused by magnetized matter 551
11.9 The field of a permanent magnet 557
11.10 Free currents, and the field **H** 559
11.11 Ferromagnetism 565
11.12 Applications 570
Chapter summary 573
Problems 575
Exercises 577

CHAPTER 12
SOLUTIONS TO THE PROBLEMS 586

12.1 Chapter 1 586
12.2 Chapter 2 611
12.3 Chapter 3 636
12.4 Chapter 4 660
12.5 Chapter 5 678
12.6 Chapter 6 684
12.7 Chapter 7 707
12.8 Chapter 8 722
12.9 Chapter 9 734
12.10 Chapter 10 744
12.11 Chapter 11 755

Appendix A:
Differences between SI and Gaussian units 762

Appendix B:
SI units of common quantities 769

Appendix C:
Unit conversions 774

Appendix D:
SI and Gaussian formulas 778

Appendix E:
Exact relations among SI and Gaussian units 789

Appendix F:
Curvilinear coordinates 791

Appendix G:
A short review of special relativity 804

Appendix H:
Radiation by an accelerated charge 812

Appendix I:
Superconductivity 817

Appendix J:
Magnetic resonance 821

Appendix K:
Helpful formulas/facts 825

References 831

Index 833

Preface to the third edition of Volume 2

For 50 years, physics students have enjoyed learning about electricity and magnetism through the first two editions of this book. The purpose of the present edition is to bring certain things up to date and to add new material, in the hopes that the trend will continue. The main changes from the second edition are (1) the conversion from Gaussian units to SI units, and (2) the addition of many solved problems and examples.

The first of these changes is due to the fact that the vast majority of courses on electricity and magnetism are now taught in SI units. The second edition fell out of print at one point, and it was hard to watch such a wonderful book fade away because it wasn't compatible with the way the subject is presently taught. Of course, there are differing opinions as to which system of units is "better" for an introductory course. But this issue is moot, given the reality of these courses.

For students interested in working with Gaussian units, or for instructors who want their students to gain exposure to both systems, I have created a number of appendices that should be helpful. Appendix A discusses the differences between the SI and Gaussian systems. Appendix C derives the conversion factors between the corresponding units in the two systems. Appendix D explains how to convert formulas from SI to Gaussian; it then lists, side by side, the SI and Gaussian expressions for every important result in the book. A little time spent looking at this appendix will make it clear how to convert formulas from one system to the other.

The second main change in the book is the addition of many solved problems, and also many new examples in the text. Each chapter ends with "problems" and "exercises." The solutions to the "problems" are located in Chapter 12. The only official difference between the problems

and exercises is that the problems have solutions included, whereas the exercises do not. (A separate solutions manual for the exercises is available to instructors.) In practice, however, one difference is that some of the more theorem-ish results are presented in the problems, so that students can use these results in other problems/exercises.

Some advice on using the solutions to the problems: problems (and exercises) are given a (very subjective) difficulty rating from 1 star to 4 stars. If you are having trouble solving a problem, it is critical that you don't look at the solution too soon. Brood over it for a while. If you do finally look at the solution, don't just read it through. Instead, cover it up with a piece of paper and read one line at a time until you reach a hint to get you started. Then set the book aside and work things out for real. That's the only way it will sink in. It's quite astonishing how unhelpful it is simply to read a solution. You'd *think* it would do some good, but in fact it is completely ineffective in raising your understanding to the next level. Of course, a careful reading of the text, including perhaps a few problem solutions, is necessary to get the basics down. But if Level 1 is understanding the basic concepts, and Level 2 is being able to *apply* those concepts, then you can read and read until the cows come home, and you'll never get past Level 1.

The overall structure of the text is essentially the same as in the second edition, although a few new sections have been added. Section 2.7 introduces dipoles. The more formal treatment of dipoles, along with their applications, remains in place in Chapter 10. But because the fundamentals of dipoles can be understood using only the concepts developed in Chapters 1 and 2, it seems appropriate to cover this subject earlier in the book. Section 8.3 introduces the important technique of solving differential equations by forming complex solutions and then taking the real part. Section 9.6.2 deals with the Poynting vector, which opens up the door to some very cool problems.

Each chapter concludes with a list of "everyday" applications of electricity and magnetism. The discussions are brief. The main purpose of these sections is to present a list of fun topics that deserve further investigation. You can carry onward with some combination of books/internet/people/pondering. There is effectively an infinite amount of information out there (see the references at the beginning of Section 1.16 for some starting points), so my goal in these sections is simply to provide a springboard for further study.

The intertwined nature of electricity, magnetism, and relativity is discussed in detail in Chapter 5. Many students find this material highly illuminating, although some find it a bit difficult. (However, these two groups are by no means mutually exclusive!) For instructors who wish to take a less theoretical route, it is possible to skip directly from Chapter 4 to Chapter 6, with only a brief mention of the main result from Chapter 5, namely the magnetic field due to a straight current-carrying wire.

The use of non-Cartesian coordinates (cylindrical, spherical) is more prominent in the present edition. For setups possessing certain symmetries, a wisely chosen system of coordinates can greatly simplify the calculations. Appendix F gives a review of the various vector operators in the different systems.

Compared with the second edition, the level of difficulty of the present edition is slightly higher, due to a number of hefty problems that have been added. If you are looking for an extra challenge, these problems should keep you on your toes. However, if these are ignored (which they certainly can be, in any standard course using this book), then the level of difficulty is roughly the same.

I am grateful to all the students who used a draft version of this book and provided feedback. Their input has been invaluable. I would also like to thank Jacob Barandes for many illuminating discussions of the more subtle topics in the book. Paul Horowitz helped get the project off the ground and has been an endless supplier of cool facts. It was a pleasure brainstorming with Andrew Milewski, who offered many ideas for clever new problems. Howard Georgi and Wolfgang Rueckner provided much-appreciated sounding boards and sanity checks. Takuya Kitagawa carefully read through a draft version and offered many helpful suggestions. Other friends and colleagues whose input I am grateful for are: Allen Crockett, David Derbes, John Doyle, Gary Feldman, Melissa Franklin, Jerome Fung, Jene Golovchenko, Doug Goodale, Robert Hart, Tom Hayes, Peter Hedman, Jennifer Hoffman, Charlie Holbrow, Gareth Kafka, Alan Levine, Aneesh Manohar, Kirk McDonald, Masahiro Morii, Lev Okun, Joon Pahk, Dave Patterson, Mara Prentiss, Dennis Purcell, Frank Purcell, Daniel Rosenberg, Emily Russell, Roy Shwitters, Nils Sorensen, Josh Winn, and Amir Yacoby.

I would also like to thank the editorial and production group at Cambridge University Press for their professional work in transforming the second edition of this book into the present one. It has been a pleasure working with Lindsay Barnes, Simon Capelin, Irene Pizzie, Charlotte Thomas, and Ali Woollatt.

Despite careful editing, there is zero probability that this book is error free. A great deal of new material has been added, and errors have undoubtedly crept in. If anything looks amiss, please check the webpage www.cambridge.org/Purcell-Morin for a list of typos, updates, etc. And please let me know if you discover something that isn't already posted. Suggestions are always welcome.

David Morin

Preface to the second edition of Volume 2

This revision of "Electricity and Magnetism," Volume 2 of the Berkeley Physics Course, has been made with three broad aims in mind. First, I have tried to make the text clearer at many points. In years of use teachers and students have found innumerable places where a simplification or reorganization of an explanation could make it easier to follow. Doubtless some opportunities for such improvements have still been missed; not too many, I hope.

A second aim was to make the book practically independent of its companion volumes in the Berkeley Physics Course. As originally conceived it was bracketed between Volume I, which provided the needed special relativity, and Volume 3, "Waves and Oscillations," to which was allocated the topic of electromagnetic waves. As it has turned out, Volume 2 has been rather widely used alone. In recognition of that I have made certain changes and additions. A concise review of the relations of special relativity is included as Appendix A. Some previous introduction to relativity is still assumed. The review provides a handy reference and summary for the ideas and formulas we need to understand the fields of moving charges and their transformation from one frame to another. The development of Maxwell's equations for the vacuum has been transferred from the heavily loaded Chapter 7 (on induction) to a new Chapter 9, where it leads naturally into an elementary treatment of plane electromagnetic waves, both running and standing. The propagation of a wave in a dielectric medium can then be treated in Chapter 10 on Electric Fields in Matter.

A third need, to modernize the treatment of certain topics, was most urgent in the chapter on electrical conduction. A substantially rewritten

Chapter 4 now includes a section on the physics of homogeneous semiconductors, including doped semiconductors. Devices are not included, not even a rectifying junction, but what is said about bands, and donors and acceptors, could serve as starting point for development of such topics by the instructor. Thanks to solid-state electronics the physics of the voltaic cell has become even more relevant to daily life as the number of batteries in use approaches in order of magnitude the world's population. In the first edition of this book I unwisely chose as the example of an electrolytic cell the one cell—the Weston standard cell—which advances in physics were soon to render utterly obsolete. That section has been replaced by an analysis, with new diagrams, of the lead-acid storage battery—ancient, ubiquitous, and far from obsolete.

One would hardly have expected that, in the revision of an elementary text in classical electromagnetism, attention would have to be paid to new developments in particle physics. But that is the case for two questions that were discussed in the first edition, the significance of charge quantization, and the apparent absence of magnetic monopoles. Observation of proton decay would profoundly affect our view of the first question. Assiduous searches for that, and also for magnetic monopoles, have at this writing yielded no confirmed events, but the possibility of such fundamental discoveries remains open.

Three special topics, optional extensions of the text, are introduced in short appendixes: Appendix B: Radiation by an Accelerated Charge; Appendix C: Superconductivity; and Appendix D: Magnetic Resonance.

Our primary system of units remains the Gaussian CGS system. The SI units, ampere, coulomb, volt, ohm, and tesla are also introduced in the text and used in many of the problems. Major formulas are repeated in their SI formulation with explicit directions about units and conversion factors. The charts inside the back cover summarize the basic relations in both systems of units. A special chart in Chapter 11 reviews, in both systems, the relations involving magnetic polarization. The student is not expected, or encouraged, to memorize conversion factors, though some may become more or less familiar through use, but to look them up whenever needed. There is no objection to a "mixed" unit like the ohm-cm, still often used for resistivity, providing its meaning is perfectly clear.

The definition of the meter in terms of an assigned value for the speed of light, which has just become official, simplifies the exact relations among the units, as briefly explained in Appendix E.

There are some 300 problems, more than half of them new.

It is not possible to thank individually all the teachers and students who have made good suggestions for changes and corrections. I fear that some will be disappointed to find that their suggestions have not been followed quite as they intended. That the net result is a substantial improvement I hope most readers familiar with the first edition will agree.

Mistakes both old and new will surely be found. Communications pointing them out will be gratefully received.

It is a pleasure to thank Olive S. Rand for her patient and skillful assistance in the production of the manuscript.

Edward M. Purcell

Preface to the first edition of Volume 2

The subject of this volume of the Berkeley Physics Course is electricity and magnetism. The sequence of topics, in rough outline, is not unusual: electrostatics; steady currents; magnetic field; electromagnetic induction; electric and magnetic polarization in matter. However, our approach is different from the traditional one. The difference is most conspicuous in Chaps. 5 and 6 where, building on the work of Vol. I, we treat the electric and magnetic fields of moving charges as manifestations of relativity and the invariance of electric charge. This approach focuses attention on some fundamental questions, such as: charge conservation, charge invariance, the meaning of field. The only formal apparatus of special relativity that is really necessary is the Lorentz transformation of coordinates and the velocity-addition formula. It is essential, though, that the student bring to this part of the course some of the ideas and attitudes Vol. I sought to develop—among them a readiness to look at things from different frames of reference, an appreciation of invariance, and a respect for symmetry arguments. We make much use also, in Vol. II, of arguments based on superposition.

Our approach to electric and magnetic phenomena in matter is primarily "microscopic," with emphasis on the nature of atomic and molecular dipoles, both electric and magnetic. Electric conduction, also, is described microscopically in the terms of a Drude-Lorentz model. Naturally some questions have to be left open until the student takes up quantum physics in Vol. IV. But we freely talk in a matter-of-fact way about molecules and atoms as electrical structures with size, shape, and stiffness, about electron orbits, and spin. We try to treat carefully a question that is sometimes avoided and sometimes beclouded in introductory texts, the meaning of the macroscopic fields **E** and **B** inside a material.

In Vol. II, the student's mathematical equipment is extended by adding some tools of the vector calculus—gradient, divergence, curl, and the Laplacian. These concepts are developed as needed in the early chapters.

In its preliminary versions, Vol. II has been used in several classes at the University of California. It has benefited from criticism by many people connected with the Berkeley Course, especially from contributions by E. D. Commins and F. S. Crawford, Jr., who taught the first classes to use the text. They and their students discovered numerous places where clarification, or something more drastic, was needed; many of the revisions were based on their suggestions. Students' criticisms of the last preliminary version were collected by Robert Goren, who also helped to organize the problems. Valuable criticism has come also from J. D. Gavenda, who used the preliminary version at the University of Texas, and from E. F. Taylor, of Wesleyan University. Ideas were contributed by Allan Kaufman at an early stage of the writing. A. Felzer worked through most of the first draft as our first "test student."

The development of this approach to electricity and magnetism was encouraged, not only by our original Course Committee, but by colleagues active in a rather parallel development of new course material at the Massachusetts Institute of Technology. Among the latter, J. R. Tessman, of the MIT Science Teaching Center and Tufts University, was especially helpful and influential in the early formulation of the strategy. He has used the preliminary version in class, at MIT, and his critical reading of the entire text has resulted in many further changes and corrections.

Publication of the preliminary version, with its successive revisions, was supervised by Mrs. Mary R. Maloney. Mrs. Lila Lowell typed most of the manuscript. The illustrations were put into final form by Felix Cooper.

The author of this volume remains deeply grateful to his friends in Berkeley, and most of all to Charles Kittel, for the stimulation and constant encouragement that have made the long task enjoyable.

Edward M. Purcell

1

Electrostatics: charges and fields

Overview The existence of this book is owed (both figuratively and literally) to the fact that the building blocks of matter possess a quality called *charge.* Two important aspects of charge are *conservation* and *quantization.* The electric force between two charges is given by *Coulomb's law.* Like the gravitational force, the electric force falls off like $1/r^2$. It is *conservative,* so we can talk about the potential energy of a system of charges (the work done in assembling them). A very useful concept is the *electric field,* which is defined as the force per unit charge. Every point in space has a unique electric field associated with it. We can define the *flux* of the electric field through a given surface. This leads us to *Gauss's law,* which is an alternative way of stating Coulomb's law. In cases involving sufficient symmetry, it is much quicker to calculate the electric field via Gauss's law than via Coulomb's law and direct integration. Finally, we discuss the *energy density* in the electric field, which provides another way of calculating the potential energy of a system.

1.1 Electric charge

Electricity appeared to its early investigators as an extraordinary phenomenon. To draw from bodies the "subtle fire," as it was sometimes called, to bring an object into a highly electrified state, to produce a steady flow of current, called for skillful contrivance. Except for the spectacle of lightning, the ordinary manifestations of nature, from the freezing of water to the growth of a tree, seemed to have no relation to the curious behavior of electrified objects. We know now that electrical

forces largely determine the physical and chemical properties of matter over the whole range from atom to living cell. For this understanding we have to thank the scientists of the nineteenth century, Ampère, Faraday, Maxwell, and many others, who discovered the nature of electromagnetism, as well as the physicists and chemists of the twentieth century who unraveled the atomic structure of matter.

Classical electromagnetism deals with electric charges and currents and their interactions as if all the quantities involved could be measured independently, with unlimited precision. Here *classical* means simply "nonquantum." The quantum law with its constant h is ignored in the classical theory of electromagnetism, just as it is in ordinary mechanics. Indeed, the classical theory was brought very nearly to its present state of completion before Planck's discovery of quantum effects in 1900. It has survived remarkably well. Neither the revolution of quantum physics nor the development of special relativity dimmed the luster of the electromagnetic field equations Maxwell wrote down 150 years ago.

Of course the theory was solidly based on experiment, and because of that was fairly secure within its original range of application – to coils, capacitors, oscillating currents, and eventually radio waves and light waves. But even so great a success does not guarantee validity in another domain, for instance, the inside of a molecule.

Two facts help to explain the continuing importance in modern physics of the classical description of electromagnetism. First, special relativity required no revision of classical electromagnetism. Historically speaking, special relativity *grew out* of classical electromagnetic theory and experiments inspired by it. Maxwell's field equations, developed long before the work of Lorentz and Einstein, proved to be entirely compatible with relativity. Second, quantum modifications of the electromagnetic forces have turned out to be unimportant down to distances less than 10^{-12} meters, 100 times smaller than the atom. We can describe the repulsion and attraction of particles in the atom using the same laws that apply to the leaves of an electroscope, although we need quantum mechanics to predict how the particles will behave under those forces. For still smaller distances, a fusion of electromagnetic theory and quantum theory, called *quantum electrodynamics,* has been remarkably successful. Its predictions are confirmed by experiment down to the smallest distances yet explored.

It is assumed that the reader has some acquaintance with the elementary facts of electricity. We are not going to review all the experiments by which the existence of electric charge was demonstrated, nor shall we review all the evidence for the electrical constitution of matter. On the other hand, we do want to look carefully at the experimental foundations of the basic laws on which all else depends. In this chapter we shall study the physics of stationary electric charges – *electrostatics.*

Certainly one fundamental property of electric charge is its existence in the two varieties that were long ago named *positive* and *negative.*

The observed fact is that all charged particles can be divided into two classes such that all members of one class repel each other, while attracting members of the other class. If two small electrically charged bodies *A* and *B*, some distance apart, attract one another, and if *A* attracts some third electrified body *C*, then we always find that *B* repels *C*. Contrast this with gravitation: there is only one kind of gravitational mass, and every mass attracts every other mass.

One may regard the two kinds of charge, positive and negative, as opposite manifestations of one quality, much as *right* and *left* are the two kinds of handedness. Indeed, in the physics of elementary particles, questions involving the sign of the charge are sometimes linked to a question of handedness, and to another basic symmetry, the relation of a sequence of events, *a*, then *b*, then *c*, to the temporally reversed sequence *c*, then *b*, then *a*. It is only the duality of electric charge that concerns us here. For every kind of particle in nature, as far as we know, there can exist an *antiparticle,* a sort of electrical "mirror image." The antiparticle carries charge of the opposite sign. If any other intrinsic quality of the particle has an opposite, the antiparticle has that too, whereas in a property that admits no opposite, such as mass, the antiparticle and particle are exactly alike.

The electron's charge is negative; its antiparticle, called a *positron,* has a positive charge, but its mass is precisely the same as that of the electron. The proton's antiparticle is called simply an *antiproton*; its electric charge is negative. An electron and a proton combine to make an ordinary hydrogen atom. A positron and an antiproton could combine in the same way to make an atom of antihydrogen. Given the building blocks, positrons, antiprotons, and antineutrons,[1] there could be built up the whole range of antimatter, from antihydrogen to antigalaxies. There is a practical difficulty, of course. Should a positron meet an electron or an antiproton meet a proton, that pair of particles will quickly vanish in a burst of radiation. It is therefore not surprising that even positrons and antiprotons, not to speak of antiatoms, are exceedingly rare and short-lived in our world. Perhaps the universe contains, somewhere, a vast concentration of antimatter. If so, its whereabouts is a cosmological mystery.

The universe around us consists overwhelmingly of matter, not antimatter. That is to say, the abundant carriers of negative charge are electrons, and the abundant carriers of positive charge are protons. The proton is nearly 2000 times heavier than the electron, and very different, too, in some other respects. Thus matter at the atomic level incorporates negative and positive electricity in quite different ways. The positive charge is all in the atomic nucleus, bound within a massive structure no more than 10^{-14} m in size, while the negative charge is spread, in

[1] Although the electric charge of each is zero, the neutron and its antiparticle are not interchangeable. In certain properties that do not concern us here, they are opposite.

effect, through a region about 10^4 times larger in dimensions. It is hard to imagine what atoms and molecules – and all of chemistry – would be like, if not for this fundamental electrical asymmetry of matter.

What we call negative charge, by the way, could just as well have been called positive. The name was a historical accident. There is nothing essentially negative about the charge of an electron. It is not like a negative integer. A negative integer, once multiplication has been defined, differs essentially from a positive integer in that its square is an integer of opposite sign. But the product of two charges is not a charge; there is no comparison.

Two other properties of electric charge are essential in the electrical structure of matter: charge is *conserved,* and charge is *quantized.* These properties involve *quantity* of charge and thus imply a measurement of charge. Presently we shall state precisely how charge can be measured in terms of the force between charges a certain distance apart, and so on. But let us take this for granted for the time being, so that we may talk freely about these fundamental facts.

1.2 Conservation of charge

The total charge in an isolated system never changes. By *isolated* we mean that no matter is allowed to cross the boundary of the system. We could let light pass into or out of the system, since the "particles" of light, called *photons,* carry no charge at all. Within the system charged particles may vanish or reappear, but they always do so in pairs of equal and opposite charge. For instance, a thin-walled box in a vacuum exposed to gamma rays might become the scene of a "pair-creation" event in which a high-energy photon ends its existence with the creation of an electron and a positron (Fig. 1.1). Two electrically charged particles have been newly created, but the net change in total charge, in and on the box, is zero. An event that *would* violate the law we have just stated would be the creation of a positively charged particle *without* the simultaneous creation of a negatively charged particle. Such an occurrence has never been observed.

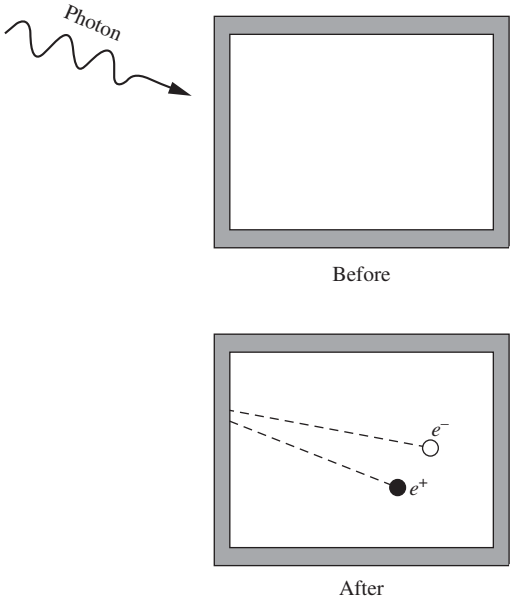


Figure 1.1.
Charged particles are created in pairs with equal and opposite charge.

Of course, if the electric charges of an electron and a positron were not precisely equal in magnitude, pair creation would still violate the strict law of charge conservation. That equality is a manifestation of the particle–antiparticle duality already mentioned, a universal symmetry of nature.

One thing will become clear in the course of our study of electromagnetism: nonconservation of charge would be quite incompatible with the structure of our present electromagnetic theory. We may therefore state, either as a postulate of the theory or as an empirical law supported without exception by all observations so far, the charge conservation law:

The total electric charge in an isolated system, that is, the algebraic sum of the positive and negative charge present at any time, never changes.

Sooner or later we must ask whether this law meets the test of relativistic invariance. We shall postpone until Chapter 5 a thorough discussion of this important question. But the answer is that it does, and not merely in the sense that the statement above holds in any given inertial frame, but in the stronger sense that observers in different frames, measuring the charge, obtain the same number. In other words, the total electric charge of an isolated system is a relativistically invariant number.

1.3 Quantization of charge

The electric charges we find in nature come in units of one magnitude only, equal to the amount of charge carried by a single electron. We denote the magnitude of that charge by e. (When we are paying attention to sign, we write $-e$ for the charge on the electron itself.) We have already noted that the positron carries precisely that amount of charge, as it must if charge is to be conserved when an electron and a positron annihilate, leaving nothing but light. What seems more remarkable is the apparently exact equality of the charges carried by all other charged particles – the equality, for instance, of the positive charge on the proton and the negative charge on the electron.

That particular equality is easy to test experimentally. We can see whether the net electric charge carried by a hydrogen molecule, which consists of two protons and two electrons, is zero. In an experiment carried out by J. G. King,[2] hydrogen gas was compressed into a tank that was electrically insulated from its surroundings. The tank contained about $5 \cdot 10^{24}$ molecules (approximately 17 grams) of hydrogen. The gas was then allowed to escape by means that prevented the escape of any ion – a molecule with an electron missing or an extra electron attached. If the charge on the proton differed from that on the electron by, say, one part in a billion, then each hydrogen molecule would carry a charge of $2 \cdot 10^{-9}e$, and the departure of the whole mass of hydrogen would alter the charge of the tank by $10^{16}e$, a gigantic effect. In fact, the experiment could have revealed a residual molecular charge as small as $2 \cdot 10^{-20}e$, and none was observed. This proved that the proton and the electron do not differ in magnitude of charge by more than 1 part in 10^{20}.

Perhaps the equality is really *exact* for some reason we don't yet understand. It may be connected with the possibility, suggested by certain

[2] See King (1960). References to previous tests of charge equality will be found in this article and in the chapter by V. W. Hughes in Hughes (1964).

theories, that a proton can, *very* rarely, decay into a positron and some uncharged particles. If that were to occur, even the slightest discrepancy between proton charge and positron charge would violate charge conservation. Several experiments designed to detect the decay of a proton have not yet, as of this writing, registered with certainty a single decay. If and when such an event is observed, it will show that exact equality of the magnitude of the charge of the proton and the charge of the electron (the positron's antiparticle) can be regarded as a corollary of the more general law of charge conservation.

That notwithstanding, we now know that the *internal* structure of all the strongly interacting particles called *hadrons* – a class that includes the proton and the neutron – involves basic units called *quarks,* whose electric charges come in multiples of $e/3$. The proton, for example, is made with three quarks, two with charge $2e/3$ and one with charge $-e/3$. The neutron contains one quark with charge $2e/3$ and two quarks with charge $-e/3$.

Several experimenters have searched for single quarks, either free or attached to ordinary matter. The fractional charge of such a quark, since it cannot be neutralized by any number of electrons or protons, should betray the quark's presence. So far no fractionally charged particle has been conclusively identified. The present theory of the strong interactions, called *quantum chromodynamics,* explains why the liberation of a quark from a hadron is most likely impossible.

The fact of charge quantization lies outside the scope of classical electromagnetism, of course. We shall usually ignore it and act as if our point charges q could have any strength whatsoever. This will not get us into trouble. Still, it is worth remembering that classical theory cannot be expected to explain the structure of the elementary particles. (It is not certain that present quantum theory can either!) What holds the electron together is as mysterious as what fixes the precise value of its charge. Something more than electrical forces must be involved, for the electrostatic forces between different parts of the electron would be repulsive.

In our study of electricity and magnetism we shall treat the charged particles simply as carriers of charge, with dimensions so small that their extension and structure is, for most purposes, quite insignificant. In the case of the proton, for example, we know from high-energy scattering experiments that the electric charge does not extend appreciably beyond a radius of 10^{-15} m. We recall that Rutherford's analysis of the scattering of alpha particles showed that even heavy nuclei have their electric charge distributed over a region smaller than 10^{-13} m. For the physicist of the nineteenth century a "point charge" remained an abstract notion. Today we are on familiar terms with the atomic particles. The graininess of electricity is so conspicuous in our modern description of nature that we find a point charge less of an artificial idealization than a smoothly varying distribution of charge density. When we postulate such smooth charge distributions, we may think of them as averages over very

large numbers of elementary charges, in the same way that we can define the macroscopic density of a liquid, its lumpiness on a molecular scale notwithstanding.

1.4 Coulomb's law

As you probably already know, the interaction between electric charges at rest is described by Coulomb's law: two stationary electric charges repel or attract one another with a force proportional to the product of the magnitude of the charges and inversely proportional to the square of the distance between them.

We can state this compactly in vector form:

$$\mathbf{F}_2 = k\frac{q_1 q_2 \hat{\mathbf{r}}_{21}}{r_{21}^2}. \tag{1.1}$$

Here q_1 and q_2 are numbers (scalars) giving the magnitude and sign of the respective charges, $\hat{\mathbf{r}}_{21}$ is the unit vector in the direction[3] from charge 1 to charge 2, and $\mathbf{F}_2$ is the force acting on charge 2. Thus Eq. (1.1) expresses, among other things, the fact that like charges repel and unlike charges attract. Also, the force obeys Newton's third law; that is, $\mathbf{F}_2 = -\mathbf{F}_1$.

The unit vector $\hat{\mathbf{r}}_{21}$ shows that the force is parallel to the line joining the charges. It could not be otherwise unless space itself has some built-in directional property, for with two point charges alone in empty and isotropic space, no other direction could be singled out.

If the point charge itself had some internal structure, with an axis defining a direction, then it would have to be described by more than the mere scalar quantity q. It is true that some elementary particles, including the electron, do have another property, called *spin.* This gives rise to a magnetic force between two electrons in addition to their electrostatic repulsion. This magnetic force does not, in general, act in the direction of the line joining the two particles. It decreases with the inverse fourth power of the distance, and at atomic distances of 10^{-10} m the Coulomb force is already about 10^4 times stronger than the magnetic interaction of the spins. Another magnetic force appears if our charges are moving – hence the restriction to stationary charges in our statement of Coulomb's law. We shall return to these magnetic phenomena in later chapters.

Of course we must assume, in writing Eq. (1.1), that both charges are well localized, each occupying a region small compared with r_{21}. Otherwise we could not even define the distance r_{21} precisely.

The value of the constant k in Eq. (1.1) depends on the units in which r, $\mathbf{F}$, and q are to be expressed. In this book we will use the International System of Units, or "SI" units for short. This system is based on the

[3] The convention we adopt here may not seem the natural choice, but it is more consistent with the usage in some other parts of physics and we shall try to follow it throughout this book.

meter, kilogram, and second as units of length, mass, and time. The SI unit of charge is the *coulomb* (C). Some other SI electrical units that we will eventually become familiar with are the volt, ohm, ampere, and tesla. The official definition of the coulomb involves the magnetic force, which we will discuss in Chapter 6. For present purposes, we can define the coulomb as follows. Two like charges, each of 1 coulomb, repel one another with a force of $8.988 \cdot 10^9$ newtons when they are 1 meter apart. In other words, the k in Eq. (1.1) is given by

$$k = 8.988 \cdot 10^9 \ \frac{\mathrm{N\,m^2}}{\mathrm{C^2}}. \tag{1.2}$$

In Chapter 6 we will learn where this seemingly arbitrary value of k comes from. In general, approximating k by $9 \cdot 10^9\ \mathrm{N\,m^2/C^2}$ is quite sufficient. The magnitude of e, the fundamental quantum of electric charge, happens to be about $1.602 \cdot 10^{-19}$ C. So if you wish, you may think of a coulomb as defined to be the magnitude of the charge contained in $6.242 \cdot 10^{18}$ electrons.

Instead of k, it is customary (for historical reasons) to introduce a constant ϵ_0 which is defined by

$$k \equiv \frac{1}{4\pi\epsilon_0} \implies \boxed{\epsilon_0 \equiv \frac{1}{4\pi k} = 8.854 \cdot 10^{-12}\ \frac{\mathrm{C^2}}{\mathrm{N\,m^2}}} \quad \left(\text{or } \frac{\mathrm{C^2\,s^2}}{\mathrm{kg\,m^3}}\right). \tag{1.3}$$

In terms of ϵ_0, Coulomb's law in Eq. (1.1) takes the form

$$\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2 \hat{\mathbf{r}}_{21}}{r_{21}^2}} \tag{1.4}$$

The constant ϵ_0 will appear in many expressions that we will meet in the course of our study. The 4π is included in the definition of ϵ_0 so that certain formulas (such as Gauss's law in Sections 1.10 and 2.9) take on simple forms. Additional details and technicalities concerning ϵ_0 can be found in Appendix E.

Another system of units that comes up occasionally is the *Gaussian* system, which is one of several types of cgs systems, short for centimeter–gram–second. (In contrast, the SI system is an mks system, short for meter–kilogram–second.) The Gaussian unit of charge is the "electrostatic unit," or esu. The esu is defined so that the constant k in Eq. (1.1) *exactly* equals 1 (and this is simply the number 1, with no units) when r_{21} is measured in cm, $\mathbf{F}$ in dynes, and the q values in esu. Figure 1.2 gives some examples using the SI and Gaussian systems of units. Further discussion of the SI and Gaussian systems can be found in Appendix A.

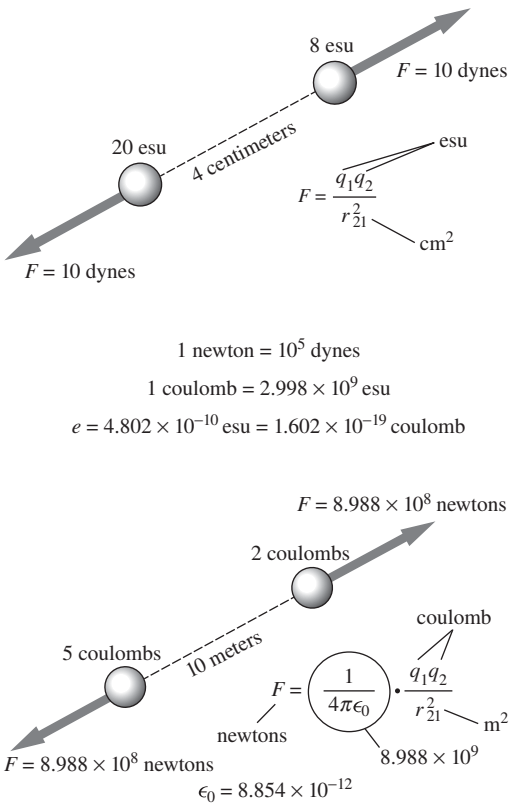


Figure 1.2.
Coulomb's law expressed in Gaussian electrostatic units (top) and in SI units (bottom). The constant ϵ_0 and the factor relating coulombs to esu are connected, as we shall learn later, with the speed of light. We have rounded off the constants in the figure to four-digit accuracy. The precise values are given in Appendix E.

Example (Relation between 1 coulomb and 1 esu) Show that 1 coulomb equals $2.998 \cdot 10^9$ esu (which generally can be approximated by $3 \cdot 10^9$ esu).

Solution From Eqs. (1.1) and (1.2), two charges of 1 coulomb separated by a distance of 1 m exert a (large!) force of $8.988 \cdot 10^9$ N $\approx 9 \cdot 10^9$ N on each other. We can convert this to the Gaussian unit of force via

$$1\,\text{N} = 1\,\frac{\text{kg m}}{\text{s}^2} = \frac{(1000\,\text{g})(100\,\text{cm})}{\text{s}^2} = 10^5\,\frac{\text{g cm}}{\text{s}^2} = 10^5\,\text{dynes}. \tag{1.5}$$

The two 1 C charges therefore exert a force of $9 \cdot 10^{14}$ dynes on each other. How would someone working in Gaussian units describe this situation? In Gaussian units, Coulomb's law gives the force simply as q^2/r^2. The separation is 100 cm, so if 1 coulomb equals N esu (with N to be determined), the $9 \cdot 10^{14}$ dyne force between the charges can be expressed as

$$9 \cdot 10^{14}\,\text{dyne} = \frac{(N\,\text{esu})^2}{(100\,\text{cm})^2} \implies N^2 = 9 \cdot 10^{18} \implies N = 3 \cdot 10^9. \tag{1.6}$$

Hence,[4]

$$1\,\text{C} = 3 \cdot 10^9\,\text{esu}. \tag{1.7}$$

The magnitude of the electron charge is then given approximately by $e = 1.6 \cdot 10^{-19}$ C $\approx 4.8 \cdot 10^{-10}$ esu.

If we had used the more exact value of k in Eq. (1.2), the "3" in our result would have been replaced by $\sqrt{8.988} = 2.998$. This looks suspiciously similar to the "2.998" in the speed of light, $c = 2.998 \cdot 10^8$ m/s. This is no coincidence. We will see in Section 6.1 that Eq. (1.7) can actually be written as $1\,\text{C} = (10\{c\})$ esu, where we have put the c in brackets to signify that it is just the number $2.998 \cdot 10^8$ without the units of m/s.

On an everyday scale, a coulomb is an extremely large amount of charge, as evidenced by the fact that if you have two such charges separated by 1 m (never mind how you would keep each charge from flying apart due to the self repulsion!), the above force of $9 \cdot 10^9$ N between them is about one million tons. The esu is a much more reasonable unit to use for everyday charges. For example, the static charge on a balloon that sticks to your hair is on the order of 10 or 100 esu.

The only way we have of detecting and measuring electric charges is by observing the interaction of charged bodies. One might wonder, then, how much of the apparent content of Coulomb's law is really only definition. As it stands, the significant physical content is the statement of inverse-square dependence and the implication that electric charge

[4] We technically shouldn't be using an "=" sign here, because it suggests that the units of a coulomb are the same as those of an esu. This is not the case; they are units in different systems and cannot be expressed in terms of each other. The proper way to express Eq. (1.7) is to say, "1 C is equivalent to $3 \cdot 10^9$ esu." But we'll usually just use the "=" sign, and you'll know what we mean. See Appendix A for further discussion of this.

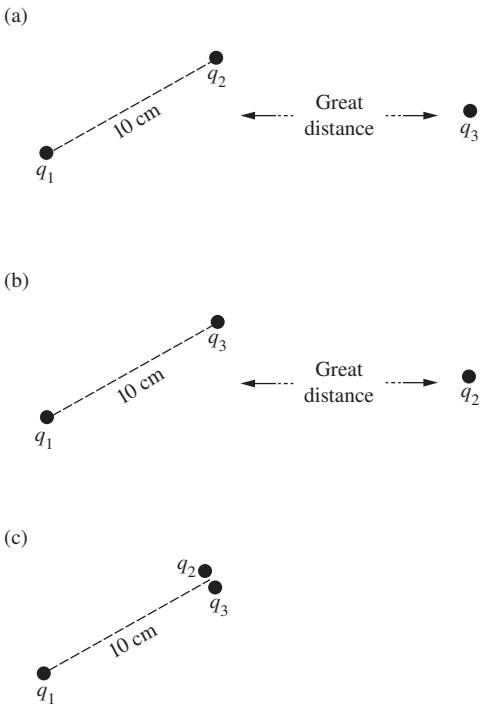


Figure 1.3.
The force on q_1 in (c) is the sum of the forces on q_1 in (a) and (b).

is *additive* in its effect. To bring out the latter point, we have to consider *more* than two charges. After all, if we had only two charges in the world to experiment with, q_1 and q_2, we could never measure them separately. We could verify only that F is proportional to $1/r_{21}^2$. Suppose we have *three* bodies carrying charges q_1, q_2, and q_3. We can measure the force on q_1 when q_2 is 10 cm away from q_1, with q_3 very far away, as in Fig. 1.3(a). Then we can take q_2 away, bring q_3 into q_2's former position, and again measure the force on q_1. Finally, we can bring q_2 and q_3 very close together and locate the combination 10 cm from q_1. We find by measurement that the force on q_1 is equal to the sum of the forces previously measured. This is a significant result that could *not* have been predicted by logical arguments from symmetry like the one we used above to show that the force between two point charges *had* to be along the line joining them. *The force with which two charges interact is not changed by the presence of a third charge.*

No matter how many charges we have in our system, Coulomb's law in Eq. (1.4) can be used to calculate the interaction of every pair. This is the basis of the principle of *superposition,* which we shall invoke again and again in our study of electromagnetism. Superposition means combining two sets of sources into one system by adding the second system "on top of" the first without altering the configuration of either one. Our principle ensures that the force on a charge placed at any point in the combined system will be the vector sum of the forces that each set of sources, acting alone, causes to act on a charge at that point. This principle must not be taken lightly for granted. There may well be a domain of phenomena, involving very small distances or very intense forces, where superposition *no longer holds.* Indeed, we know of quantum phenomena in the electromagnetic field that do represent a failure of superposition, seen from the viewpoint of the classical theory.

Thus the physics of electrical interactions comes into full view only when we have *more* than two charges. We can go beyond the explicit statement of Eq. (1.1) and assert that, with the three charges in Fig. 1.3 occupying any positions whatsoever, the force on any one of them, such as q_3, is correctly given by the following equation:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_3 q_1 \hat{\mathbf{r}}_{31}}{r_{31}^2} + \frac{1}{4\pi\epsilon_0}\frac{q_3 q_2 \hat{\mathbf{r}}_{32}}{r_{32}^2}. \tag{1.8}$$

The experimental verification of the inverse-square law of electrical attraction and repulsion has a curious history. Coulomb himself announced the law in 1786 after measuring with a torsion balance the force between small charged spheres. But 20 years earlier Joseph Priestly, carrying out an experiment suggested to him by Benjamin Franklin, had noticed the absence of electrical influence within a hollow charged container and made an inspired conjecture: "May we not infer from this experiment that the attraction of electricity is subject to the same laws with that of gravitation and is therefore according to the square of the

distances; since it is easily demonstrated that were the earth in the form of a shell, a body in the inside of it would not be attracted to one side more than the other." (Priestly, 1767).

The same idea was the basis of an elegant experiment in 1772 by Henry Cavendish. Cavendish charged a spherical conducting shell that contained within it, and temporarily connected to it, a smaller sphere. The outer shell was then separated into two halves and carefully removed, the inner sphere having been first disconnected. This sphere was tested for charge, the absence of which would confirm the inverse-square law. (See Problem 2.8 for the theory behind this.) Assuming that a deviation from the inverse-square law could be expressed as a difference in the exponent, $2 + \delta$, say, instead of 2, Cavendish concluded that δ must be less than 0.03. This experiment of Cavendish remained largely unknown until Maxwell discovered and published Cavendish's notes a century later (1876). At that time also, Maxwell repeated the experiment with improved apparatus, pushing the limit down to $\delta < 10^{-6}$. The present limit on δ is a fantastically small number – about one part in 10^{16}; see Crandall (1983) and Williams *et al.* (1971).

Two hundred years after Cavendish's experiment, however, the question of interest changed somewhat. Never mind how perfectly Coulomb's law works for charged objects in the laboratory – is there a range of distances where it completely breaks down? There are two domains in either of which a breakdown is conceivable. The first is the domain of very small distances, distances less than 10^{-16} m, where electromagnetic theory as we know it may not work at all. As for very large distances, from the geographical, say, to the astronomical, a test of Coulomb's law by the method of Cavendish is obviously not feasible. Nevertheless we do observe certain large-scale electromagnetic phenomena that prove that the laws of classical electromagnetism work over very long distances. One of the most stringent tests is provided by planetary magnetic fields, in particular the magnetic field of the giant planet Jupiter, which was surveyed in the mission of Pioneer 10. The spatial variation of this field was carefully analyzed[5] and found to be entirely consistent with classical theory out to a distance of at least 10^5 km from the planet. This is tantamount to a test, albeit indirect, of Coulomb's law over that distance.

To summarize, we have every reason for confidence in Coulomb's law over the stupendous range of 24 decades in distance, from 10^{-16} to 10^8 m, if not farther, and we take it as the foundation of our description of electromagnetism.

1.5 Energy of a system of charges

In principle, Coulomb's law is all there is to electrostatics. Given the charges and their locations, we can find all the electrical forces. Or, given

[5] See Davis *et al.* (1975). For a review of the history of the exploration of the outer limit of classical electromagnetism, see Goldhaber and Nieto (1971).

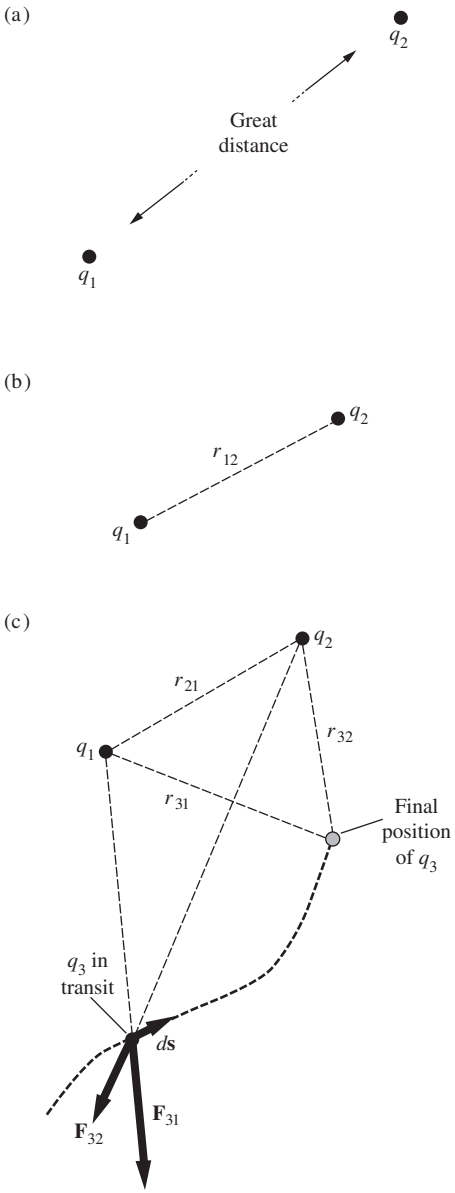


Figure 1.4.
Three charges are brought near one another. First q_2 is brought in; then, with q_1 and q_2 fixed, q_3 is brought in.

that the charges are free to move under the influence of other kinds of forces as well, we can find the equilibrium arrangement in which the charge distribution will remain stationary. In the same sense, Newton's laws of motion are all there is to mechanics. But in both mechanics and electromagnetism we gain power and insight by introducing other concepts, most notably that of energy.

Energy is a useful concept here because electrical forces are *conservative.* When you push charges around in electric fields, no energy is irrecoverably lost. Everything is perfectly reversible. Consider first the work that must be done *on* the system to bring some charged bodies into a particular arrangement. Let us start with two charged bodies or particles very far apart from one another, as indicated in Fig. 1.4(a), carrying charges q_1 and q_2. Whatever energy may have been needed to create these two concentrations of charge originally we shall leave entirely out of account. How much work does it take to bring the particles slowly together until the distance between them is r_{12}?

It makes no difference whether we bring q_1 toward q_2 or the other way around. In either case the work done is the integral of the product: force times displacement, where these are signed quantities. The force that has to be applied to move one charge toward the other is equal and opposite to the Coulomb force. Therefore,

$$W = \int (\text{applied force}) \cdot (\text{displacement})$$
$$= \int_{r=\infty}^{r_{12}} \left(-\frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \right) dr = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}}. \tag{1.9}$$

Note that because r is changing from ∞ to r_{12}, the differential dr is negative. We know that the overall sign of the result is correct, because the work done on the system must be positive for charges of like sign; they have to be pushed together (consistent with the minus sign in the applied force). Both the displacement and the applied force are negative in this case, resulting in positive work being done on the system. With q_1 and q_2 in coulombs, and r_{12} in meters, Eq. (1.9) gives the work in joules.

This work is the same whatever the path of approach. Let's review the argument as it applies to the two charges q_1 and q_2 in Fig. 1.5. There we have kept q_1 fixed, and we show q_2 moved to the same final position along two different paths. Every spherical shell, such as the one indicated between r and $r + dr$, must be crossed by both paths. The increment of work involved, $-\mathbf{F} \cdot d\mathbf{s}$ in this bit of path (where $\mathbf{F}$ is the Coulomb force), is the same for the two paths.[6] The reason is that $\mathbf{F}$ has the same magnitude at both places and is directed radially from q_1, while

[6] Here we use for the first time the scalar product, or "dot product," of two vectors. A reminder: the scalar product of two vectors $\mathbf{A}$ and $\mathbf{B}$, written $\mathbf{A} \cdot \mathbf{B}$, is the number $AB\cos\theta$, where A and B are the magnitudes of the vectors $\mathbf{A}$ and $\mathbf{B}$, and θ is the angle between them. Expressed in terms of Cartesian components of the two vectors, $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$.

$ds = dr/\cos\theta$; hence $\mathbf{F}\cdot d\mathbf{s} = F\,dr$. Each increment of work along one path is matched by a corresponding increment on the other, so the sums must be equal. Our conclusion holds even for paths that loop in and out, like the dotted path in Fig. 1.5. (Why?)

Returning now to the two charges as we left them in Fig. 1.4(b), let us bring in from some remote place a third charge q_3 and move it to a point P_3 whose distance from charge 1 is r_{31}, and from charge 2, r_{32}. The work required to effect this will be

$$W_3 = -\int_{\infty}^{P_3} \mathbf{F}_3 \cdot d\mathbf{s}. \tag{1.10}$$

Thanks to the additivity of electrical interactions, which we have already emphasized,

$$\begin{aligned} -\int \mathbf{F}_3 \cdot d\mathbf{s} &= -\int (\mathbf{F}_{31} + \mathbf{F}_{32}) \cdot d\mathbf{s} \\ &= -\int \mathbf{F}_{31} \cdot d\mathbf{s} - \int \mathbf{F}_{32} \cdot d\mathbf{s}. \end{aligned} \tag{1.11}$$

That is, the work required to bring q_3 to P_3 is the sum of the work needed when q_1 is present alone and that needed when q_2 is present alone:

$$W_3 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{r_{31}} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{32}}. \tag{1.12}$$

The total work done in assembling this arrangement of three charges, which we shall call U, is therefore

$$U = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right). \tag{1.13}$$

We note that q_1, q_2, and q_3 appear symmetrically in the expression above, in spite of the fact that q_3 was brought in last. We would have reached the same result if q_3 had been brought in first. (Try it.) Thus U is independent of the *order* in which the charges were assembled. Since it is independent also of the route by which each charge was brought in, U must be a unique property of the final arrangement of charges. We may call it the *electrical potential energy* of this particular system. There is a certain arbitrariness, as always, in the definition of a potential energy. In this case we have chosen the zero of potential energy to correspond to the situation with the three charges already in existence but infinitely far apart from one another. The potential energy *belongs to the configuration as a whole.* There is no meaningful way of assigning a certain fraction of it to one of the charges.

It is obvious how this very simple result can be generalized to apply to any number of charges. If we have N different charges, in any arrangement in space, the potential energy of the system is calculated by summing over all pairs, just as in Eq. (1.13). The zero of potential energy, as in that case, corresponds to all charges far apart.

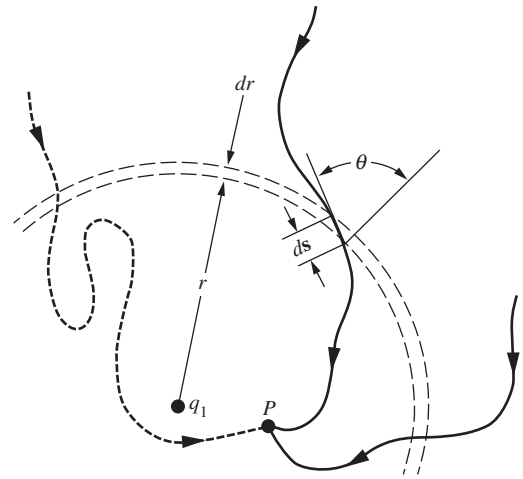


Figure 1.5.
Because the force is central, the sections of different paths between $r + dr$ and r require the same amount of work.

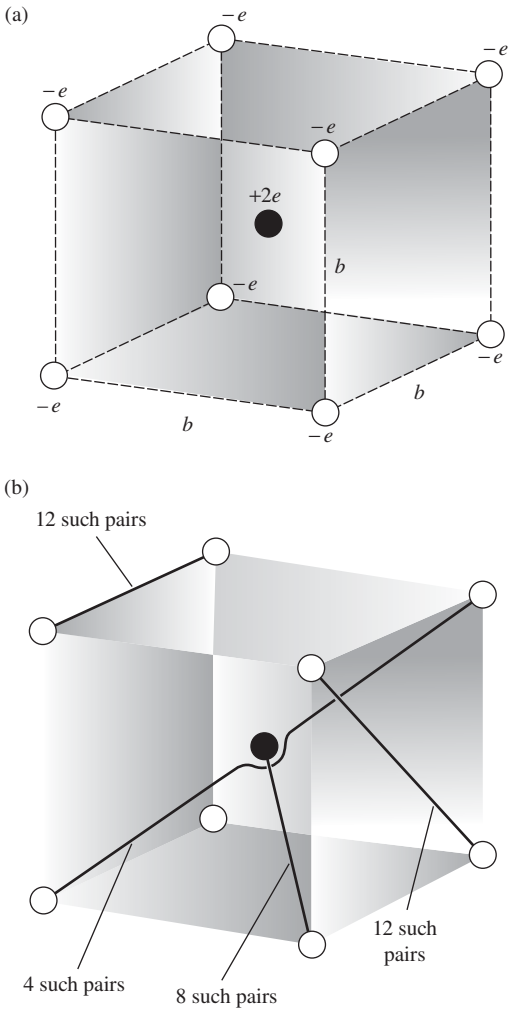


Figure 1.6.
(a) The potential energy of this arrangement of nine point charges is given by Eq. (1.14).
(b) Four types of pairs are involved in the sum.

Example (Charges in a cube) What is the potential energy of an arrangement of eight negative charges on the corners of a cube of side b, with a positive charge in the center of the cube, as in Fig. 1.6(a)? Suppose each negative charge is an electron with charge $-e$, while the central particle carries a double positive charge, $2e$.

Solution Figure 1.6(b) shows that there are four different types of pairs. One type involves the center charge, while the other three involve the various edges and diagonals of the cube. Summing over all pairs yields

$$U = \frac{1}{4\pi\epsilon_0}\left(8\cdot\frac{(-2e^2)}{(\sqrt{3}/2)b} + 12\cdot\frac{e^2}{b} + 12\cdot\frac{e^2}{\sqrt{2}\,b} + 4\cdot\frac{e^2}{\sqrt{3}\,b}\right) \approx \frac{1}{4\pi\epsilon_0}\frac{4.32e^2}{b}. \tag{1.14}$$

The energy is positive, indicating that work had to be done on the system to assemble it. That work could, of course, be recovered if we let the charges move apart, exerting forces on some external body or bodies. Or if the electrons were simply to fly apart from this configuration, the *total kinetic energy* of all the particles would become equal to U. This would be true whether they came apart simultaneously and symmetrically, or were released one at a time in any order. Here we see the power of this simple notion of the total potential energy of the system. Think what the problem would be like if we had to compute the resultant vector force on every particle at every stage of assembly of the configuration! In this example, to be sure, the geometrical symmetry would simplify that task; even so, it would be more complicated than the simple calculation above.

One way of writing the instruction for the sum over pairs is this:

$$U = \frac{1}{2}\sum_{j=1}^{N}\sum_{k\neq j}\frac{1}{4\pi\epsilon_0}\frac{q_j q_k}{r_{jk}}. \tag{1.15}$$

The double-sum notation, $\sum_{j=1}^{N}\sum_{k\neq j}$, says: take $j=1$ and sum over $k=2,3,4,\ldots,N$; then take $j=2$ and sum over $k=1,3,4,\ldots,N$; and so on, through $j=N$. Clearly this includes every pair *twice,* and to correct for that we put in front the factor $1/2$.

1.6 Electrical energy in a crystal lattice

These ideas have an important application in the physics of crystals. We know that an ionic crystal like sodium chloride can be described, to a very good approximation, as an arrangement of positive ions (Na^+) and negative ions (Cl^-) alternating in a regular three-dimensional array or lattice. In sodium chloride the arrangement is that shown in Fig. 1.7(a). Of course the ions are not point charges, but they are nearly spherical distributions of charge and therefore (as we shall prove in Section 1.11) the electrical forces they exert on one another are the same as if each ion

were replaced by an equivalent point charge at its center. We show this electrically equivalent system in Fig. 1.7(b). The electrostatic potential energy of the lattice of charges plays an important role in the explanation of the stability and cohesion of the ionic crystal. Let us see if we can estimate its magnitude.

We seem to be faced at once with a sum that is enormous, if not doubly infinite; any macroscopic crystal contains 10^{20} atoms at least. Will the sum converge? Now what we hope to find is the potential energy per unit volume or mass of crystal. We confidently expect this to be independent of the size of the crystal, based on the general argument that one end of a macroscopic crystal can have little influence on the other. Two grams of sodium chloride ought to have twice the potential energy of one gram, and the shape should not be important so long as the surface atoms are a small fraction of the total number of atoms. We would be *wrong* in this expectation if the crystal were made out of ions of one sign only. Then, 1 g of crystal would carry an enormous electric charge, and putting two such crystals together to make a 2 g crystal would take a fantastic amount of energy. (You might estimate how much!) The situation is saved by the fact that the crystal structure is an alternation of equal and opposite charges, so that any macroscopic bit of crystal is very nearly neutral.

To evaluate the potential energy we first observe that every positive ion is in a position equivalent to that of every other positive ion. Furthermore, although it is perhaps not immediately obvious from Fig. 1.7, the arrangement of positive ions around a negative ion is exactly the same as the arrangement of negative ions around a positive ion, and so on. Hence we may take one ion as a center, it matters not which kind, sum over *its* interactions with all the others, and simply multiply by the total number of ions of both kinds. This reduces the double sum in Eq. (1.15) to a single sum and a factor N; we must still apply the factor $1/2$ to compensate for including each pair twice. That is, the energy of a sodium chloride lattice composed of a total of N ions is

$$U = \frac{1}{2} N \sum_{k=2}^{N} \frac{1}{4\pi\epsilon_0} \frac{q_1 q_k}{r_{1k}}. \tag{1.16}$$

Taking the positive ion at the center as in Fig. 1.7(b), our sum runs over all its neighbors near and far. The leading terms start out as follows:

$$U = \frac{1}{2} N \frac{1}{4\pi\epsilon_0} \left(-\frac{6e^2}{a} + \frac{12e^2}{\sqrt{2}\,a} - \frac{8e^2}{\sqrt{3}\,a} + \cdots \right). \tag{1.17}$$

The first term comes from the 6 nearest chlorine ions, at distance a, the second from the 12 sodium ions on the cube edges, and so on. It is clear, incidentally, that this series does not converge *absolutely*; if we were so

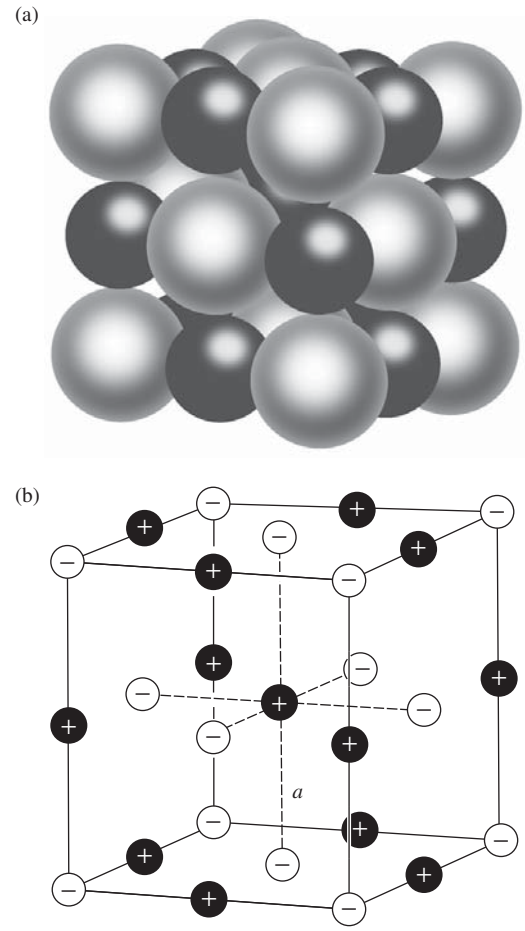


Figure 1.7.
A portion of a sodium chloride crystal, with the ions Na^+ and Cl^- shown in about the right relative proportions (a), and replaced by equivalent point charges (b).

foolish as to try to sum all the positive terms first, that sum would diverge. To evaluate such a sum, we should arrange it so that as we proceed outward, including ever more distant ions, we include them in groups that represent nearly neutral shells of material. Then if the sum is broken off, the more remote ions that have been neglected will be such an even mixture of positive and negative charges that we can be confident their contribution would have been small. This is a crude way to describe what is actually a somewhat more delicate computational problem. The numerical evaluation of such a series is easily accomplished with a computer. The answer in this example happens to be

$$U = \frac{-0.8738Ne^2}{4\pi\epsilon_0 a}. \tag{1.18}$$

Here N, the number of ions, is twice the number of NaCl molecules.

The negative sign shows that work would have to be *done* to take the crystal apart into ions. In other words, the electrical energy helps to explain the cohesion of the crystal. If this were the whole story, however, the crystal would collapse, for the potential energy of the charge distribution is obviously *lowered* by shrinking all the distances. We meet here again the familiar dilemma of classical – that is, nonquantum – physics. No system of stationary particles can be in stable equilibrium, according to classical laws, under the action of electrical forces alone; we will give a proof of this fact in Section 2.12. Does this make our analysis useless? Not at all. Remarkably, and happily, in the quantum physics of crystals the electrical potential energy can still be given meaning, and can be computed very much in the way we have learned here.

1.7 The electric field

Suppose we have some arrangement of charges, $q_1, q_2, \ldots, q_N$, fixed in space, and we are interested not in the forces they exert on one another, but only in their effect on some other charge q_0 that might be brought into their vicinity. We know how to calculate the resultant force on this charge, given its position which we may specify by the coordinates x, y, z. The force on the charge q_0 is

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{N}\frac{q_0 q_j \hat{\mathbf{r}}_{0j}}{r_{0j}^2}, \tag{1.19}$$

where $\mathbf{r}_{0j}$ is the vector from the jth charge in the system to the point (x, y, z). The force is proportional to q_0, so if we divide out q_0 we obtain a vector quantity that depends only on the structure of our original system of charges, $q_1, \ldots, q_N$, and on the position of the point (x, y, z). We call this vector function of x, y, z the *electric field* arising from the $q_1, \ldots, q_N$

and use the symbol **E** for it. The charges $q_1, \ldots, q_N$ we call *sources* of the field. We may take as the *definition* of the electric field **E** of a charge distribution, at the point (x, y, z),

$$\mathbf{E}(x, y, z) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{N} \frac{q_j \hat{\mathbf{r}}_{0j}}{r_{0j}^2}. \tag{1.20}$$

The force on some other charge q at (x, y, z) is then

$$\boxed{\mathbf{F} = q\mathbf{E}} \tag{1.21}$$

Figure 1.8 illustrates the vector addition of the field of a point charge of 2 C to the field of a point charge of -1 C, at a particular point in space. In the SI system of units, electric field strength is expressed in newtons per unit charge, that is, newtons/coulomb. In Gaussian units, with the esu as the unit of charge and the dyne as the unit of force, the electric field strength is expressed in dynes/esu.

After the introduction of the electric potential in Chapter 2, we shall have another, and completely equivalent, way of expressing the unit of electric field strength; namely, volts/meter in SI units and statvolts/centimeter in Gaussian units.

So far we have nothing really new. The electric field is merely another way of describing the system of charges; it does so by giving the force per unit charge, in magnitude and direction, that an exploring charge q_0 would experience at any point. We have to be a little careful with that interpretation. Unless the source charges are really immovable, the introduction of some finite charge q_0 may cause the source charges to shift their positions, so that the field itself, as defined by Eq. (1.20), is different. That is why we assumed fixed charges to begin our discussion. People sometimes define the field by requiring q_0 to be an "infinitesimal" test charge, letting **E** be the limit of $\mathbf{F}/q_0$ as $q_0 \to 0$. Any flavor of rigor this may impart is illusory. Remember that in the real world we have never observed a charge smaller than e! Actually, if we take Eq. (1.20) as our *definition* of **E**, without reference to a test charge, no problem arises and the sources need not be fixed. If the introduction of a new charge causes a shift in the source charges, then it has indeed brought about a change in the electric field, and if we want to predict the force on the new charge, we must use the new electric field in computing it.

Perhaps you still want to ask, what *is* an electric field? Is it something real, or is it merely a name for a factor in an equation that has to be multiplied by something else to give the numerical value of the force we measure in an experiment? Two observations may be useful here. First, since it works, it doesn't make any difference. That is not a frivolous answer, but a serious one. Second, the fact that the electric field vector

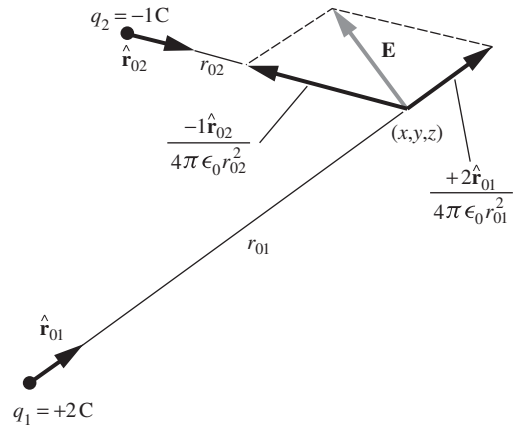


Figure 1.8.
The field at a point is the vector sum of the fields of each of the charges in the system.

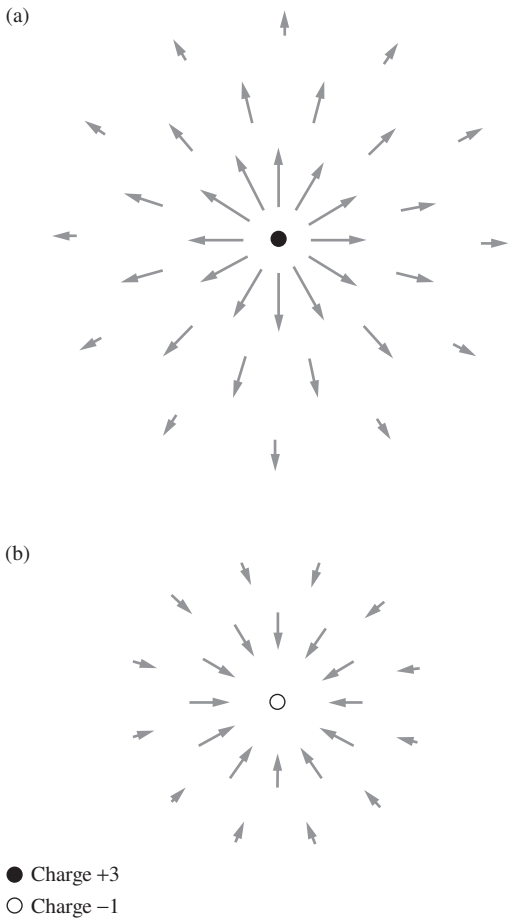


Figure 1.9.
(a) Field of a charge $q_1 = 3$. (b) Field of a charge $q_2 = -1$. Both representations are necessarily crude and only roughly quantitative.

at a point in space is all we need know to predict the force that will act on *any* charge at that point is by no means trivial. It might have been otherwise! If no experiments had ever been done, we could imagine that, in two different situations in which unit charges experience equal force, test charges of strength 2 units might experience unequal forces, depending on the nature of the other charges in the system. If that were true, the field description wouldn't work. The electric field attaches to every point in a system a *local property,* in this sense: if we know **E** in some small neighborhood, we know, *without further inquiry,* what will happen to any charges in that neighborhood. We do not need to ask what produced the field.

To visualize an electric field, you need to associate a vector, that is, a magnitude and direction, with every point in space. We shall use various schemes in this book, none of them wholly satisfactory, to depict vector fields.

It is hard to draw in two dimensions a picture of a vector function in three-dimensional space. We can indicate the magnitude and direction of **E** at various points by drawing little arrows near those points, making the arrows longer where E is larger.[7] Using this scheme, we show in Fig. 1.9(a) the field of an isolated point charge of 3 units and in Fig. 1.9(b) the field of a point charge of -1 unit. These pictures admittedly add nothing whatsoever to our understanding of the field of an isolated charge; anyone can imagine a simple radial inverse-square field without the help of a picture. We show them in order to combine (side by side) the two fields in Fig. 1.10, which indicates in the same manner the field of two such charges separated by a distance a. All that Fig. 1.10 can show is the field in a plane containing the charges. To get a full three-dimensional representation, one must imagine the figure rotated around the symmetry axis. In Fig. 1.10 there is one point in space where **E** is zero. As an exercise, you can quickly figure out where this point lies. Notice also that toward the edge of the picture the field points more or less radially outward all around. One can see that at a very large distance from the charges the field will look very much like the field from a positive point charge. This is to be expected because the separation of the charges cannot make very much difference for points far away, and a point charge of 2 units is just what we would have left if we superimposed our two sources at one spot.

Another way to depict a vector field is to draw *field lines.* These are simply curves whose tangent, at any point, lies in the direction of the field at that point. Such curves will be smooth and continuous except at singularities such as point charges, or points like the one in the example of Fig. 1.10 where the field is zero. A field line plot does not directly give

[7] Such a representation is rather clumsy at best. It is hard to indicate the point in space to which a particular vector applies, and the range of magnitudes of E is usually so large that it is impracticable to make the lengths of the arrows proportional to E.

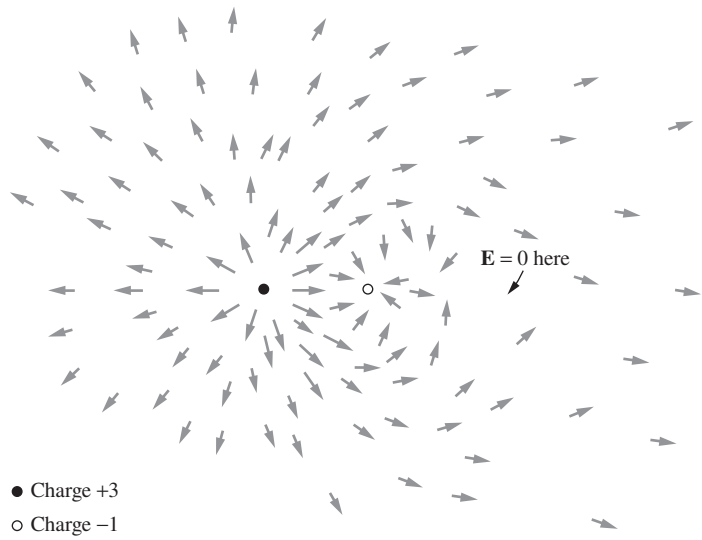


Figure 1.10.
The field in the vicinity of two charges, $q_1 = +3$, $q_2 = -1$, is the superposition of the fields in Figs. 1.9(a) and (b).

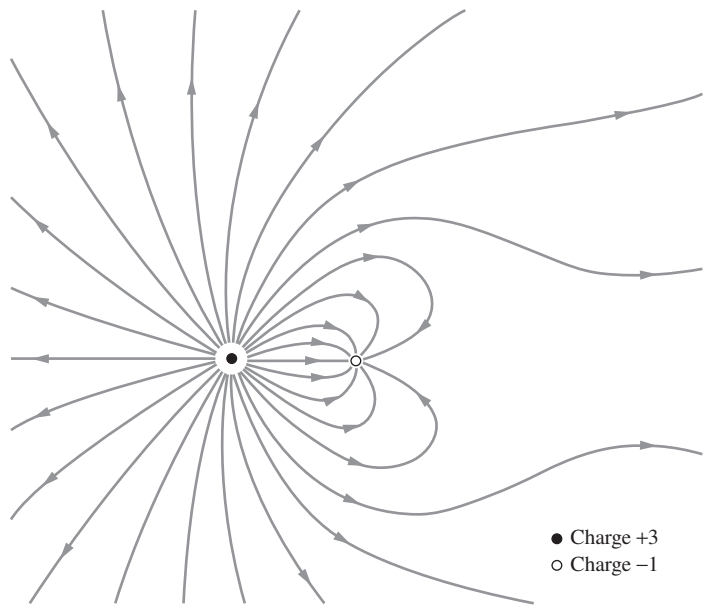


Figure 1.11.
Some field lines in the electric field around two charges, $q_1 = +3$, $q_2 = -1$.

the magnitude of the field, although we shall see that, in a general way, the field lines converge as we approach a region of strong field and spread apart as we approach a region of weak field. In Fig. 1.11 are drawn some field lines for the same arrangement of charges as in Fig. 1.10, a positive charge of 3 units and a negative charge of 1 unit. Again, we are restricted

by the nature of paper and ink to a two-dimensional section through a three-dimensional bundle of curves.

1.8 Charge distributions

This is as good a place as any to generalize from *point charges* to *continuous charge distributions.* A volume distribution of charge is described by a scalar charge-density function ρ, which is a function of position, with the dimensions *charge/volume.* That is, ρ times a volume element gives the amount of charge contained in that volume element. The same symbol is often used for mass per unit volume, but in this book we shall always give charge per unit volume first call on the symbol ρ. If we write ρ as a function of the coordinates x, y, z, then $\rho(x, y, z)\, dx\, dy\, dz$ is the charge contained in the little box, of volume $dx\, dy\, dz$, located at the point (x, y, z).

On an atomic scale, of course, the charge density varies enormously from point to point; even so, it proves to be a useful concept in that domain. However, we shall use it mainly when we are dealing with large-scale systems, so large that a volume element $dv = dx\, dy\, dz$ can be quite small relative to the size of our system, although still large enough to contain many atoms or elementary charges. As we have remarked before, we face a similar problem in defining the ordinary mass density of a substance.

If the source of the electric field is to be a continuous charge distribution rather than point charges, we merely replace the sum in Eq. (1.20) with the appropriate integral. The integral gives the electric field at (x, y, z), which is produced by charges at other points (x', y', z'):

$$\mathbf{E}(x, y, z) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(x', y', z')\hat{\mathbf{r}}\, dx'\, dy'\, dz'}{r^2}. \tag{1.22}$$

This is a volume integral. Holding (x, y, z) fixed, we let the variables of integration, x', y', and z', range over all space containing charge, thus summing up the contributions of all the bits of charge. The unit vector $\hat{\mathbf{r}}$ points from (x', y', z') to (x, y, z) – unless we want to put a minus sign before the integral, in which case we may reverse the direction of $\hat{\mathbf{r}}$. It is always hard to keep signs straight. Let's remember that the electric field points *away* from a positive source (Fig. 1.12).

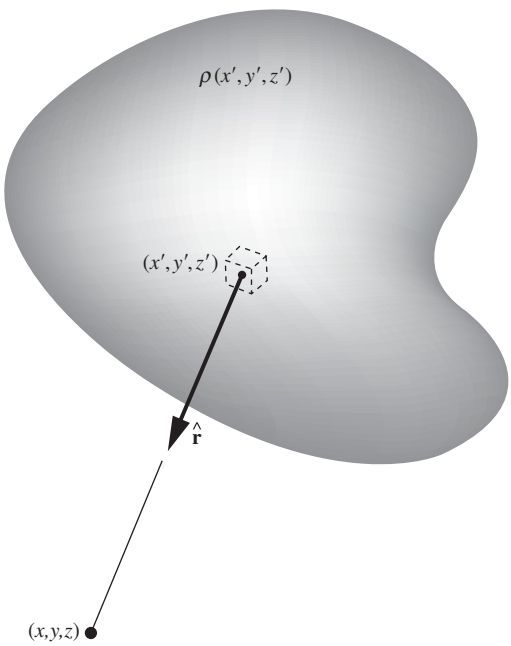


Figure 1.12.
Each element of the charge distribution $\rho(x', y', z')$ makes a contribution to the electric field $\mathbf{E}$ at the point (x, y, z). The total field at this point is the sum of all such contributions; see Eq. (1.22).

Example (Field due to a hemisphere) A solid hemisphere has radius R and uniform charge density ρ. Find the electric field at the center.

Solution Our strategy will be to slice the hemisphere into rings around the symmetry axis. We will find the electric field due to each ring, and then integrate over the rings to obtain the field due to the entire hemisphere. We will work with

polar coordinates (or, equivalently, spherical coordinates), which are much more suitable than Cartesian coordinates in this setup.

The cross section of a ring is (essentially) a little rectangle with side lengths dr and $r\,d\theta$, as shown in Fig. 1.13. The cross-sectional area is thus $r\,dr\,d\theta$. The radius of the ring is $r\sin\theta$, so the volume is $(r\,dr\,d\theta)(2\pi r\sin\theta)$. The charge in the ring is therefore $\rho(2\pi r^2\sin\theta\,dr\,d\theta)$. Equivalently, we can obtain this result by using the standard spherical-coordinate volume element, $r^2\sin\theta\,dr\,d\theta\,d\phi$, and then integrating over ϕ to obtain the factor of 2π.

Consider a tiny piece of the ring, with charge dq. This piece creates an electric field at the center of the hemisphere that points diagonally upward (if ρ is positive) with magnitude $dq/4\pi\epsilon_0 r^2$. However, only the vertical component survives, because the horizontal component cancels with the horizontal component from the diametrically opposite charge dq on the ring. The vertical component involves a factor of $\cos\theta$. When we integrate over the whole ring, the dq simply integrates to the total charge we found above. The (vertical) electric field due to a given ring is therefore

$$dE_y = \frac{\rho(2\pi r^2\sin\theta\,dr\,d\theta)}{4\pi\epsilon_0 r^2}\cos\theta = \frac{\rho\sin\theta\cos\theta\,dr\,d\theta}{2\epsilon_0}. \tag{1.23}$$

Integrating over r and θ to obtain the field due to the entire hemisphere gives

$$\begin{aligned} E_y &= \int_0^R \int_0^{\pi/2} \frac{\rho\sin\theta\cos\theta\,dr\,d\theta}{2\epsilon_0} = \frac{\rho}{2\epsilon_0}\left(\int_0^R dr\right)\left(\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\right) \\ &= \frac{\rho}{2\epsilon_0}\cdot R\cdot\frac{\sin^2\theta}{2}\bigg|_0^{\pi/2} = \frac{\rho R}{4\epsilon_0}. \end{aligned} \tag{1.24}$$

Note that the radius r canceled in Eq. (1.23). For given values of θ, $d\theta$, and dr, the volume of a ring grows like r^2, and this exactly cancels the r^2 in the denominator in Coulomb's law.

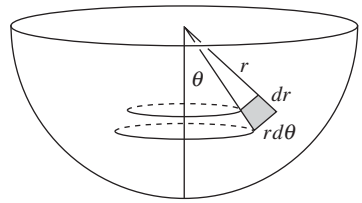


Figure 1.13.
Cross section of a thin ring. The hemisphere may be considered to be built up from rings.

REMARK As explained above, the electric field due to the hemisphere is vertical. This fact also follows from considerations of symmetry. We will make many symmetry arguments throughout this book, so let us be explicit here about how the reasoning proceeds. Assume (in search of a contradiction) that the electric field due to the hemisphere is *not* vertical. It must then point off at some angle, as shown in Fig. 1.14(a). Let's say that the **E** vector lies above a given dashed line painted on the hemisphere. If we rotate the system by, say, 180° around the symmetry axis, the field now points in the direction shown in Fig. 1.14(b), because it must still pass over the dashed line. But we have *exactly the same hemisphere* after the rotation, so the field must still point upward to the right. We conclude that the field due to the hemisphere points both upward to the left and upward to the right. This is a contradiction. The only way to avoid this contradiction is for the field to point along the symmetry axis (possibly in the negative direction), because in that case it doesn't change under the rotation.

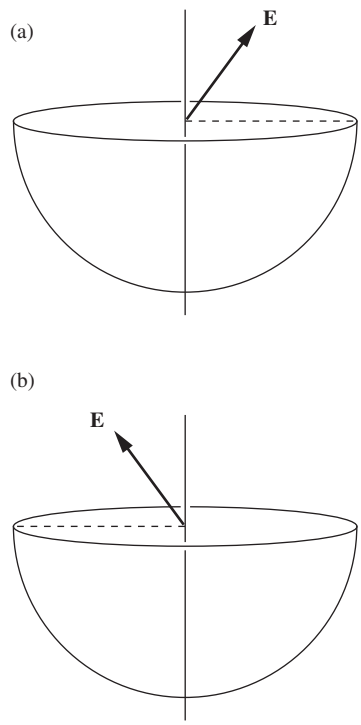


Figure 1.14.
The symmetry argument that explains why **E** must be vertical.

In the neighborhood of a true point charge the electric field grows infinite like $1/r^2$ as we approach the point. It makes no sense to talk about the field *at* the point charge. As our ultimate physical sources of field are

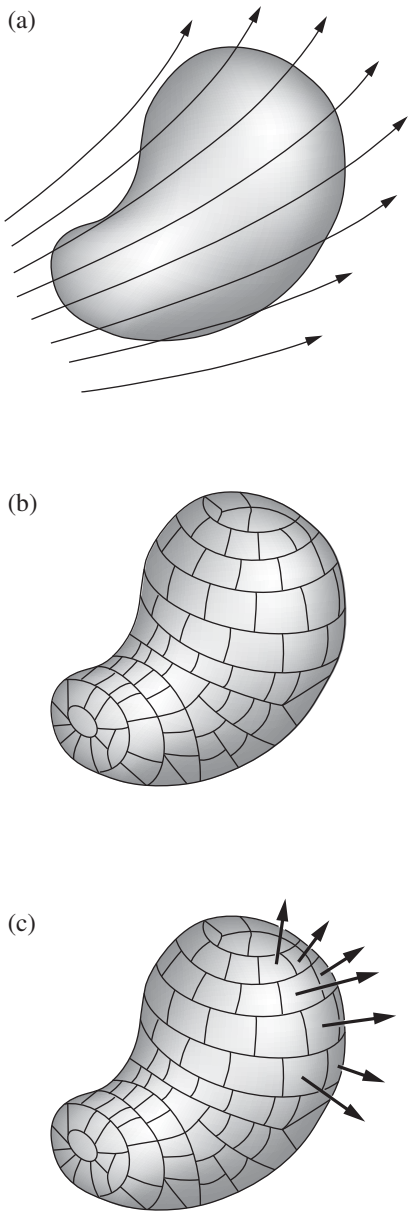


Figure 1.15.
(a) A closed surface in a vector field is divided (b) into small elements of area. (c) Each element of area is represented by an outward vector.

not, we believe, infinite concentrations of charge in zero volume, but instead finite structures, we simply ignore the mathematical singularities implied by our point-charge language and rule out of bounds the interior of our elementary sources. A continuous charge distribution $\rho(x', y', z')$ that is nowhere infinite gives no trouble at all. Equation (1.22) can be used to find the field at any point within the distribution. The integrand doesn't blow up at $r = 0$ because the volume element in the numerator equals $r^2 \sin\phi \, d\phi \, d\theta \, dr$ in spherical coordinates, and the r^2 here cancels the r^2 in the denominator in Eq. (1.22). That is to say, so long as ρ remains finite, the field will remain finite everywhere, even in the interior or on the boundary of a charge distribution.

1.9 Flux

The relation between the electric field and its sources can be expressed in a remarkably simple way, one that we shall find very useful. For this we need to define a quantity called *flux*.

Consider some electric field in space and in this space some arbitrary closed surface, like a balloon of any shape. Figure 1.15 shows such a surface, the field being suggested by a few field lines. Now divide the whole surface into little patches that are so small that over any one patch the surface is practically flat and the vector field does not change appreciably from one part of a patch to another. In other words, don't let the balloon be too crinkly, and don't let its surface pass right through a singularity[8] of the field such as a point charge. The area of a patch has a certain magnitude in square meters, and a patch defines a unique direction – the outward-pointing normal to its surface. (Since the surface is closed, you can tell its inside from its outside; there is no ambiguity.) Let this magnitude and direction be represented by a vector. Then for every patch into which the surface has been divided, such as patch number j, we have a vector $\mathbf{a}_j$ giving its area and orientation. The steps we have just taken are pictured in Figs. 1.15(b) and (c). Note that the vector $\mathbf{a}_j$ does not depend at all on the shape of the patch; it doesn't matter how we have divided up the surface, as long as the patches are small enough.

Let $\mathbf{E}_j$ be the electric field vector at the location of patch number j. The scalar product $\mathbf{E}_j \cdot \mathbf{a}_j$ is a number. We call this number the *flux* through that bit of surface. To understand the origin of the name, imagine a vector function that represents the velocity of motion in a fluid – say in a river, where the velocity varies from one place to another but is constant in time at any one position. Denote this vector field by $\mathbf{v}$, measured in

[8] By a singularity of the field we would ordinarily mean not only a point source where the field approaches infinity, but also any place where the field changes magnitude or direction discontinuously, such as an infinitesimally thin layer of concentrated charge. Actually this latter, milder, kind of singularity would cause no difficulty here unless our balloon's surface were to coincide with the surface of discontinuity over some finite area.

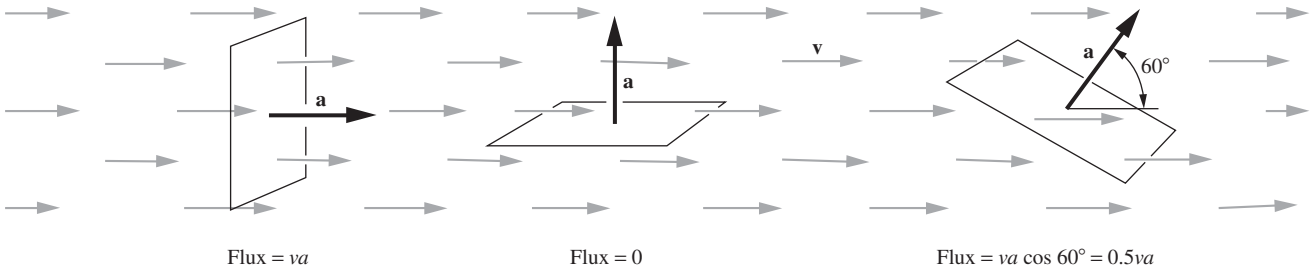


Figure 1.16.
The flux through the frame of area **a** is $\mathbf{v} \cdot \mathbf{a}$, where **v** is the velocity of the fluid. The flux is the volume of fluid passing through the frame, per unit time.

meters/second. Then, if **a** is the oriented area in square meters of a frame lowered into the water, $\mathbf{v} \cdot \mathbf{a}$ is the *rate of flow* of water through the frame in cubic meters per second (Fig. 1.16). The $\cos\theta$ factor in the standard expression for the dot product correctly picks out the component of **v** along the direction of **a**, or equivalently the component of **a** along the direction of **v**. We must emphasize that our definition of flux is applicable to any vector function, whatever physical variable it may represent.

Now let us add up the flux through all the patches to get the flux through the entire surface, a scalar quantity which we shall denote by Φ:

$$\Phi = \sum_{\text{all } j} \mathbf{E}_j \cdot \mathbf{a}_j. \tag{1.25}$$

Letting the patches become smaller and more numerous without limit, we pass from the sum in Eq. (1.25) to a surface integral:

$$\Phi = \int_{\substack{\text{entire}\\ \text{surface}}} \mathbf{E} \cdot d\mathbf{a}. \tag{1.26}$$

A surface integral of any vector function **F**, over a surface S, means just this: divide S into small patches, each represented by a vector outward, of magnitude equal to the patch area; at every patch, take the scalar product of the patch area vector and the local **F**; sum all these products, and the limit of this sum, as the patches shrink, is the surface integral. Do not be alarmed by the prospect of having to perform such a calculation for an awkwardly shaped surface like the one in Fig. 1.15. The surprising property we are about to demonstrate makes that unnecessary!

1.10 Gauss's law

Take the simplest case imaginable; suppose the field is that of a single isolated positive point charge q, and the surface is a sphere of radius r centered on the point charge (Fig. 1.17). What is the flux Φ through this surface? The answer is easy because the magnitude of **E** at every point on the surface is $q/4\pi\epsilon_0 r^2$ and its direction is the same as that of the outward normal at that point. So we have

$$\Phi = E \cdot (\text{total area}) = \frac{q}{4\pi\epsilon_0 r^2} \cdot 4\pi r^2 = \frac{q}{\epsilon_0}. \tag{1.27}$$

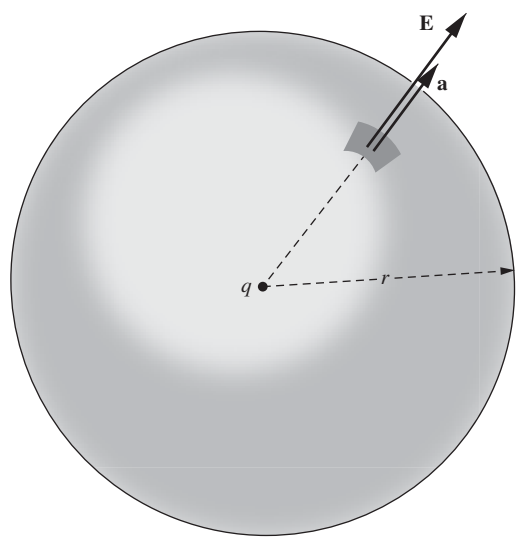


Figure 1.17.
In the field **E** of a point charge q, what is the outward flux over a sphere surrounding q?

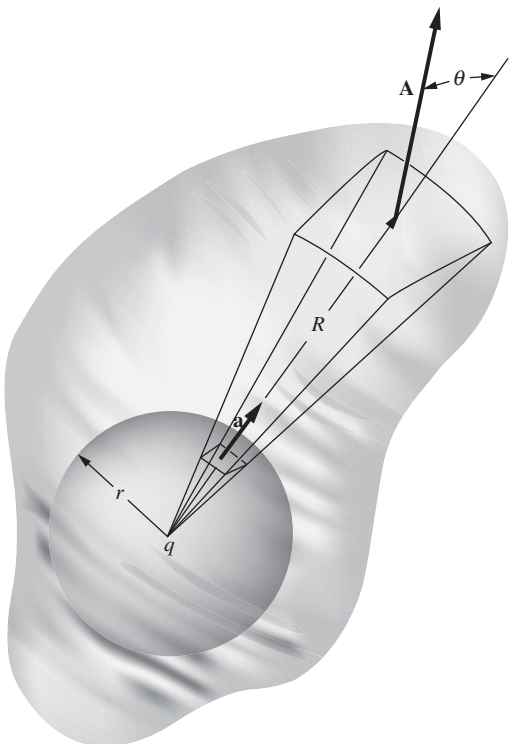


Figure 1.18.
Showing that the flux through any closed surface around q is the same as the flux through the sphere.

The flux is independent of the size of the sphere. Here for the first time we see the benefit of including the factor of $1/4\pi$ in Coulomb's law in Eq. (1.4). Without this factor, we would have an uncanceled factor of 4π in Eq. (1.27) and therefore also, eventually, in one of Maxwell's equations. Indeed, in Gaussian units Eq. (1.27) takes the form of $\Phi = 4\pi q$.

Now imagine a second surface, or balloon, enclosing the first, but *not* spherical, as in Fig. 1.18. We claim that *the total flux through this surface is the same as that through the sphere.* To see this, look at a cone, radiating from q, that cuts a small patch **a** out of the sphere and continues on to the outer surface, where it cuts out a patch **A** at a distance R from the point charge. The area of the patch **A** is larger than that of the patch **a** by two factors: first, by the ratio of the distance squared $(R/r)^2$; and second, owing to its inclination, by the factor $1/\cos\theta$. The angle θ is the angle between the outward normal and the radial direction (see Fig. 1.18). The electric field in that neighborhood is reduced from its magnitude on the sphere by the factor $(r/R)^2$ and is still radially directed. Letting $\mathbf{E}_{(R)}$ be the field at the outer patch and $\mathbf{E}_{(r)}$ be the field at the sphere, we have

$$\begin{aligned} &\text{flux through outer patch} = \mathbf{E}_{(R)} \cdot \mathbf{A} = E_{(R)} A \cos\theta, \\ &\text{flux through inner patch} = \mathbf{E}_{(r)} \cdot \mathbf{a} = E_{(r)} a. \end{aligned} \tag{1.28}$$

Using the above facts concerning the magnitude of $\mathbf{E}_{(R)}$ and the area of **A**, the flux through the outer patch can be written as

$$E_{(R)} A \cos\theta = \left[E_{(r)} \left(\frac{r}{R} \right)^2 \right] \left[a \left(\frac{R}{r} \right)^2 \frac{1}{\cos\theta} \right] \cos\theta = E_{(r)} a, \tag{1.29}$$

which equals the flux through the inner patch.

Now every patch on the outer surface can in this way be put into correspondence with part of the spherical surface, so the total flux must be the same through the two surfaces. That is, the flux through the new surface must be just q/ϵ_0. But this was a surface of *arbitrary* shape and size.[9] We conclude: the flux of the electric field through *any* surface enclosing a point charge q is q/ϵ_0. As a corollary we can say that the total flux through a closed surface is *zero* if the charge lies *outside* the surface. We leave the proof of this to the reader, along with Fig. 1.19 as a hint of one possible line of argument.

There is a way of looking at all this that makes the result seem obvious. Imagine at q a source that emits particles – such as bullets or photons – in all directions at a steady rate. Clearly the flux of particles through a window of unit area will fall off with the inverse square of the window's distance from q. Hence we can draw an analogy between the electric field strength E and the intensity of particle flow in bullets per unit area per

[9] To be sure, we had the second surface enclosing the sphere, but it didn't have to, really. Besides, the sphere can be taken as small as we please.

unit time. It is pretty obvious that the flux of bullets through any surface completely surrounding q is independent of the size and shape of that surface, for it is just the total number emitted per unit time. Correspondingly, the flux of E through the closed surface must be independent of size and shape. The common feature responsible for this is the inverse-square behavior of the intensity.

The situation is now ripe for superposition! Any electric field is the sum of the fields of its individual sources. This property was expressed in our statement, Eq. (1.19), of Coulomb's law. Clearly flux is an additive quantity in the same sense, for if we have a number of sources, $q_1, q_2, \ldots, q_N$, the fields of which, if each were present alone, would be $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_N$, then the flux Φ through some surface S in the actual field can be written

$$\Phi = \int_S \mathbf{E} \cdot d\mathbf{a} = \int_S (\mathbf{E}_1 + \mathbf{E}_2 + \cdots + \mathbf{E}_N) \cdot d\mathbf{a}. \qquad (1.30)$$

We have just learned that $\int_S \mathbf{E}_i \cdot d\mathbf{a}$ equals q_i/ϵ_0 if the charge q_i is inside S and equals zero otherwise. So every charge q inside the surface contributes exactly q/ϵ_0 to the surface integral of Eq. (1.30) and all charges outside contribute nothing. We have arrived at Gauss's law.

The flux of the electric field $\mathbf{E}$ through any closed surface, that is, the integral $\int \mathbf{E} \cdot d\mathbf{a}$ over the surface, equals $1/\epsilon_0$ times the total charge enclosed by the surface:

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \sum_i q_i = \frac{1}{\epsilon_0} \int \rho \, dv \qquad \text{(Gauss's law)} \qquad (1.31)$$

We call the statement in the box a *law* because it is equivalent to Coulomb's law and it could serve equally well as the basic law of electrostatic interactions, after charge and field have been defined. Gauss's law and Coulomb's law are not two independent physical laws, but the same law expressed in different ways.[10] In Gaussian units, the $1/\epsilon_0$ in Gauss's law is replaced with 4π.

Looking back over our proof, we see that it hinged on the inverse-square nature of the interaction and of course on the additivity of interactions, or superposition. Thus the theorem is applicable to any inverse-square field in physics, for instance to the gravitational field.

[10] There is one difference, inconsequential here, but relevant to our later study of the fields of moving charges. Gauss's law is obeyed by a wider class of fields than those represented by the electrostatic field. In particular, a field that is inverse-square in r but not spherically symmetrical can satisfy Gauss's law. In other words, Gauss's law alone does not imply the symmetry of the field of a point source which is implicit in Coulomb's law.

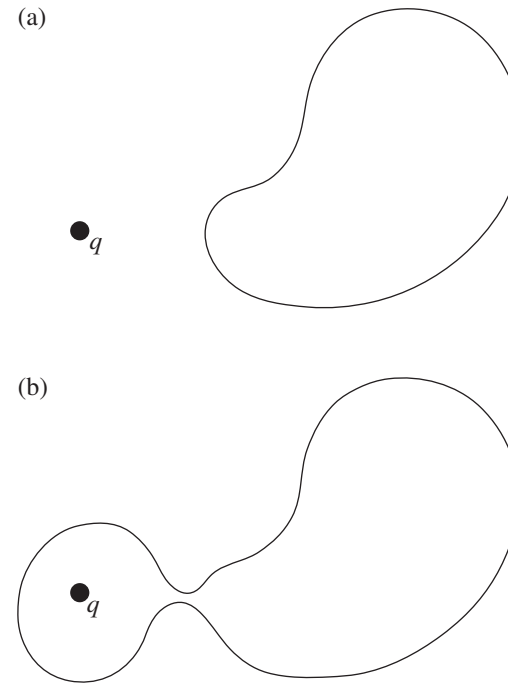


Figure 1.19.
To show that the flux through the closed surface in (a) is zero, you can make use of (b).

It is easy to see that Gauss's law would *not* hold if the law of force were, say, inverse-cube. For in that case the flux of electric field from a point charge q through a sphere of radius R centered on the charge would be

$$\Phi = \int \mathbf{E} \cdot d\mathbf{a} = \frac{q}{4\pi\epsilon_0 R^3} \cdot 4\pi R^2 = \frac{q}{\epsilon_0 R}. \tag{1.32}$$

By making the sphere large enough we could make the flux through it as small as we pleased, while the total charge inside remained constant.

This remarkable theorem extends our knowledge in two ways. First, it reveals a connection between the field and its sources that is the converse of Coulomb's law. Coulomb's law tells us how to derive the electric field if the charges are given; with Gauss's law we can determine how much charge is in any region if the field is known. Second, the mathematical relation here demonstrated is a powerful analytic tool; it can make complicated problems easy, as we shall see in the following examples. In Sections 1.11–1.13 we use Gauss's law to calculate the electric field due to various nicely shaped objects. In all of these examples the symmetry of the object will play a critical role.

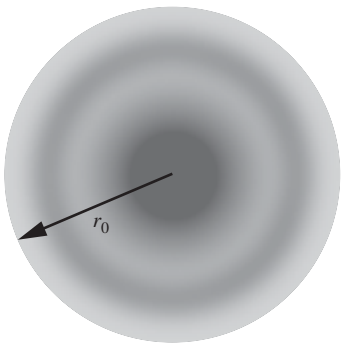


Figure 1.20.
A charge distribution with spherical symmetry.

1.11 Field of a spherical charge distribution

We can use Gauss's law to find the electric field of a spherically symmetrical distribution of charge, that is, a distribution in which the charge density ρ depends only on the radius from a central point. Figure 1.20 depicts a cross section through some such distribution. Here the charge density is high at the center, and is zero beyond r_0. What is the electric field at some point such as P_1 outside the distribution, or P_2 inside it (Fig. 1.21)? If we could proceed only from Coulomb's law, we should have to carry out an integration that would sum the electric field vectors at P_1 arising from each elementary volume in the charge distribution. Let's try a different approach that exploits both the symmetry of the system and Gauss's law.

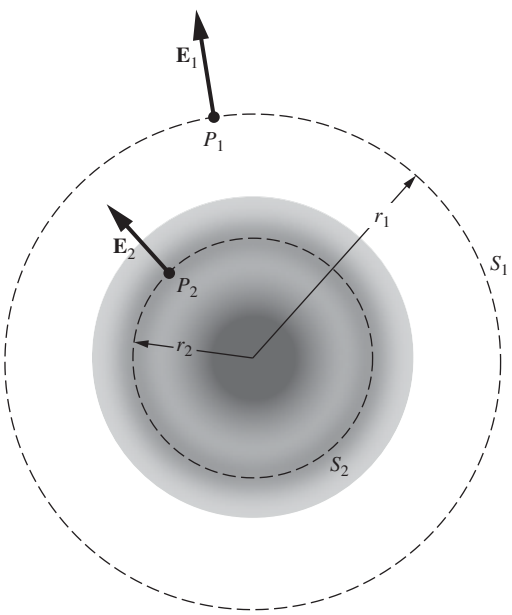


Figure 1.21.
The electric field of a spherical charge distribution.

Because of the spherical symmetry, the electric field at any point must be radially directed – no other direction is unique. Likewise, the field magnitude E must be the same at all points on a spherical surface S_1 of radius r_1, for all such points are equivalent. Call this field magnitude E_1. The flux through this surface S_1 is therefore simply $4\pi r_1^2 E_1$, and by Gauss's law this must be equal to $1/\epsilon_0$ times the charge enclosed by the surface. That is, $4\pi r_1^2 E_1 = (1/\epsilon_0) \cdot$ (charge inside S_1) or

$$E_1 = \frac{\text{charge inside } S_1}{4\pi\epsilon_0 r_1^2}. \tag{1.33}$$

Comparing this with the field of a point charge, we see that *the field at all points on S_1 is the same as if all the charge within S_1 were concentrated at the center.* The same statement applies to a sphere drawn

inside the charge distribution. The field at any point on S_2 is the same as if all charge within S_2 were at the center, and all charge *outside* S_2 absent. Evidently the field inside a "hollow" spherical charge distribution is zero (Fig. 1.22). Problem 1.17 gives an alternative derivation of this fact.

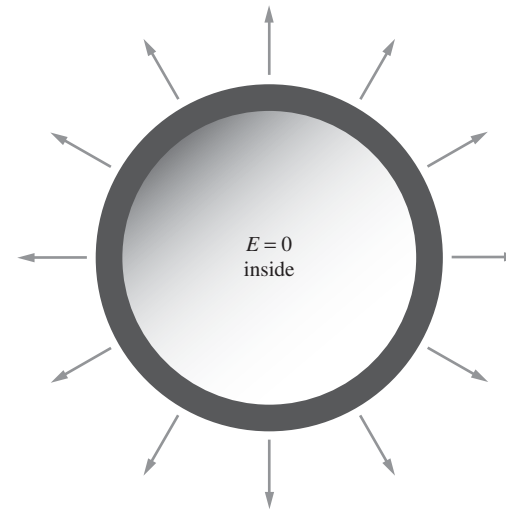


Figure 1.22.
The field is zero inside a spherical shell of charge.

Example (Field inside and outside a uniform sphere) A spherical charge distribution has a density ρ that is constant from $r=0$ out to $r=R$ and is zero beyond. What is the electric field for all values of r, both less than and greater than R?

Solution For $r \geq R$, the field is the same as if all of the charge were concentrated at the center of the sphere. Since the volume of the sphere is $4\pi R^3/3$, the field is therefore radial and has magnitude

$$E(r) = \frac{(4\pi R^3/3)\rho}{4\pi\epsilon_0 r^2} = \frac{\rho R^3}{3\epsilon_0 r^2} \qquad (r \geq R). \tag{1.34}$$

For $r \leq R$, the charge outside radius r effectively contributes nothing to the field, while the charge inside radius r acts as if it were concentrated at the center. The volume inside radius r is $4\pi r^3/3$, so the field inside the given sphere is radial and has magnitude

$$E(r) = \frac{(4\pi r^3/3)\rho}{4\pi\epsilon_0 r^2} = \frac{\rho r}{3\epsilon_0} \qquad (r \leq R). \tag{1.35}$$

In terms of the total charge $Q = (4\pi R^3/3)\rho$, this can be written as $Qr/4\pi\epsilon_0 R^3$. The field increases linearly with r inside the sphere; the r^3 growth of the effective charge outweighs the $1/r^2$ effect from the increasing distance. And the field decreases like $1/r^2$ outside the sphere. A plot of $E(r)$ is shown in Fig. 1.23. Note that $E(r)$ is continuous at $r = R$, where it takes on the value $\rho R/3\epsilon_0$. As we will see in Section 1.13, field discontinuities are created by surface charge densities, and there are no surface charges in this system. The field goes to zero at the center, so it is continuous there also. How should the density vary with r so that the magnitude $E(r)$ is uniform inside the sphere? That is the subject of Exercise 1.68.

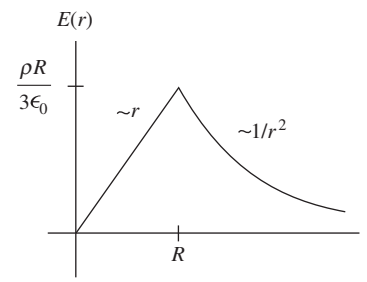


Figure 1.23.
The electric field due to a uniform sphere of charge.

The same argument applied to the gravitational field would tell us that the earth, assuming it is spherically symmetrical in its mass distribution, attracts outside bodies as if its mass were concentrated at the center. That is a rather familiar statement. Anyone who is inclined to think the principle expresses an obvious property of the center of mass must be reminded that the theorem is not even true, in general, for other shapes. A perfect cube of uniform density does *not* attract external bodies as if its mass were concentrated at its geometrical center.

Newton didn't consider the theorem obvious. He needed it as the keystone of his demonstration that the moon in its orbit around the earth and a falling body on the earth are responding to similar forces. The delay of nearly 20 years in the publication of Newton's theory of gravitation

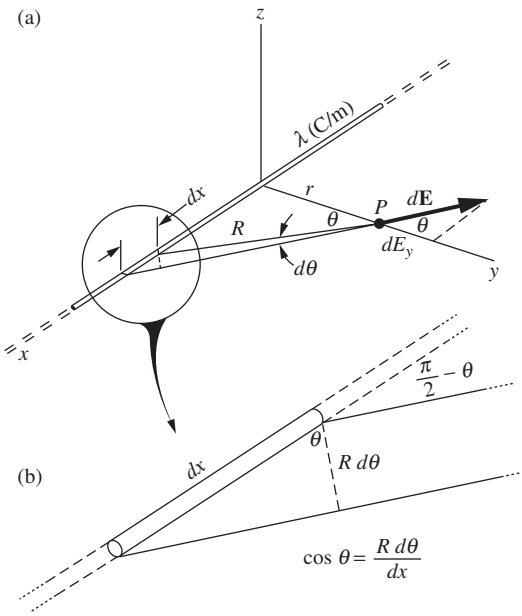


Figure 1.24.
(a) The field at P is the vector sum of contributions from each element of the line charge. (b) Detail of (a).

was apparently due, in part at least, to the trouble he had in proving this theorem to his satisfaction. The proof he eventually devised and published in the *Principia* in 1686 (Book I, Section XII, Theorem XXXI) is a marvel of ingenuity in which, roughly speaking, a tricky volume integration is effected without the aid of the integral calculus as we know it. The proof is a good bit longer than our whole preceding discussion of Gauss's law, and more intricately reasoned. You see, with all his mathematical resourcefulness and originality, Newton lacked Gauss's law – a relation that, once it has been shown to us, seems so obvious as to be almost trivial.

1.12 Field of a line charge

A long, straight, charged wire, if we neglect its thickness, can be characterized by the amount of charge it carries per unit length. Let λ, measured in coulombs/meter, denote this *linear charge density.* What is the electric field of such a line charge, assumed infinitely long and with constant linear charge density λ? We'll do the problem in two ways, first by an integration starting from Coulomb's law, and then by using Gauss's law.

To evaluate the field at the point P, shown in Fig. 1.24, we must add up the contributions from all segments of the line charge, one of which is indicated as a segment of length dx. The charge dq on this element is given by $dq = \lambda\, dx$. Having oriented our x axis along the line charge, we may as well let the y axis pass through P, which is a distance r from the nearest point on the line. It is a good idea to take advantage of symmetry at the outset. Obviously the electric field at P must point in the y direction, so that E_x and E_z are both zero. The contribution of the charge dq to the y component of the electric field at P is

$$dE_y = \frac{dq}{4\pi\epsilon_0 R^2}\cos\theta = \frac{\lambda\, dx}{4\pi\epsilon_0 R^2}\cos\theta, \tag{1.36}$$

where θ is the angle the electric field of dq makes with the y direction. The total y component is then

$$E_y = \int dE_y = \int_{-\infty}^{\infty} \frac{\lambda\cos\theta}{4\pi\epsilon_0 R^2}\, dx. \tag{1.37}$$

It is convenient to use θ as the variable of integration. Since Figs. 1.24(a) and (b) tell us that $R = r/\cos\theta$ and $dx = R\, d\theta/\cos\theta$, we have $dx = r\, d\theta/\cos^2\theta$. (This expression for dx comes up often. It also follows from $x = r\tan\theta \Longrightarrow dx = r\, d(\tan\theta) = r\, d\theta/\cos^2\theta$.) Eliminating dx and R from the integral in Eq. (1.37), in favor of θ, we obtain

$$E_y = \int_{-\pi/2}^{\pi/2} \frac{\lambda\cos\theta\, d\theta}{4\pi\epsilon_0 r} = \frac{\lambda}{4\pi\epsilon_0 r}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta = \frac{\lambda}{2\pi\epsilon_0 r}. \tag{1.38}$$

We see that the field of an infinitely long, uniformly dense line charge is proportional to the reciprocal of the distance from the line. Its direction

is of course radially outward if the line carries a positive charge, inward if negative.

Gauss's law leads directly to the same result. Surround a segment of the line charge with a closed circular cylinder of length L and radius r, as in Fig. 1.25, and consider the flux through this surface. As we have already noted, symmetry guarantees that the field is radial, so the flux through the ends of the "tin can" is zero. The flux through the cylindrical surface is simply the area, $2\pi rL$, times E_r, the field at the surface. On the other hand, the charge enclosed by the surface is just λL, so Gauss's law gives us $(2\pi rL)E_r = \lambda L/\epsilon_0$ or

$$E_r = \frac{\lambda}{2\pi\epsilon_0 r}, \tag{1.39}$$

in agreement with Eq. (1.38).

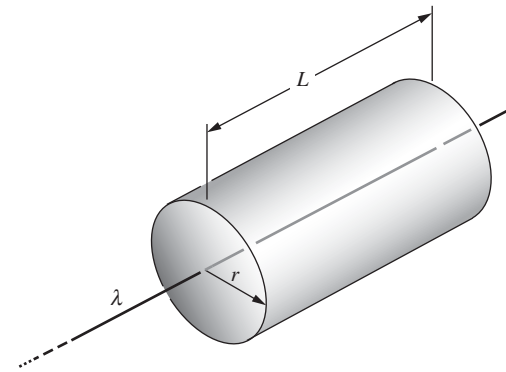


Figure 1.25.
Using Gauss's law to find the field of a line charge.

1.13 Field of an infinite flat sheet of charge

Electric charge distributed smoothly in a thin sheet is called a *surface charge distribution.* Consider a flat sheet, infinite in extent, with the constant surface charge density σ. The electric field on either side of the sheet, whatever its magnitude may turn out to be, must surely point perpendicular to the plane of the sheet; there is no other unique direction in the system. Also, because of symmetry, the field must have the same magnitude and the opposite direction at two points P and P' equidistant from the sheet on opposite sides. With these facts established, Gauss's law gives us at once the field intensity, as follows: draw a cylinder, as in Fig. 1.26 (actually, any shape with uniform cross section will work fine), with P on one side and P' on the other, of cross-sectional area A. The outward flux is found only at the ends, so that if E_P denotes the magnitude of the field at P, and $E_{P'}$ the magnitude at P', the outward flux is $AE_P + AE_{P'} = 2AE_P$. The charge enclosed is σA, so Gauss's law gives $2AE_P = \sigma A/\epsilon_0$, or

$$E_P = \frac{\sigma}{2\epsilon_0}. \tag{1.40}$$

We see that the field strength is independent of r, the distance from the sheet. Equation (1.40) could have been derived more laboriously by calculating the vector sum of the contributions to the field at P from all the little elements of charge in the sheet.

In the more general case where there are other charges in the vicinity, the field need not be perpendicular to the sheet, or symmetric on either side of it. Consider a very squat Gaussian surface, with P and P' infinitesimally close to the sheet, instead of the elongated surface in Fig. 1.26. We can then ignore the negligible flux through the cylindrical "side" of the pillbox, so the above reasoning gives $E_{\perp,P} + E_{\perp,P'} = \sigma/\epsilon_0$, where the "$\perp$" denotes the component perpendicular to the sheet. If you want

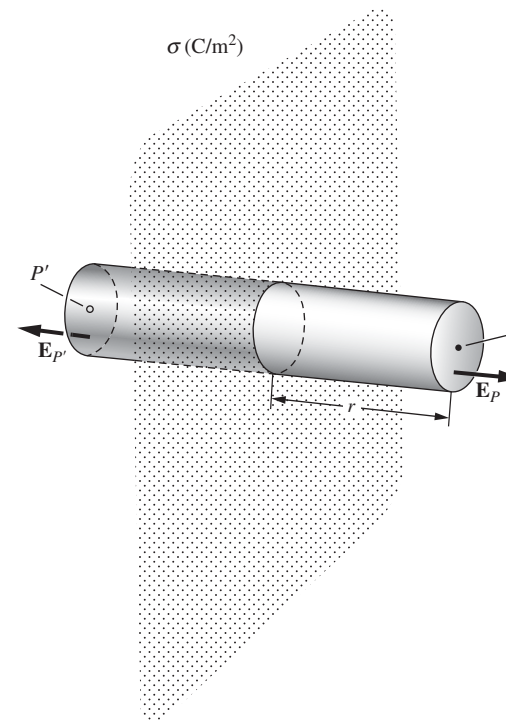


Figure 1.26.
Using Gauss's law to find the field of an infinite flat sheet of charge.

to write this in terms of vectors, it becomes $\mathbf{E}_{\perp,P} - \mathbf{E}_{\perp,P'} = (\sigma/\epsilon_0)\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the sheet, in the direction of P. In other words, the discontinuity in $\mathbf{E}_\perp$ across the sheet is given by

$$\Delta\mathbf{E}_\perp = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}. \tag{1.41}$$

Only the normal component is discontinuous; the parallel component is continuous across the sheet. So we can just as well replace the $\Delta\mathbf{E}_\perp$ in Eq. (1.41) with $\Delta\mathbf{E}$. This result is also valid for any finite-sized sheet, because from up close the sheet looks essentially like an infinite plane, at least as far as the normal component is concerned.

The field of an infinitely long line charge, we found, varies inversely as the distance from the line, while the field of an infinite sheet has the same strength at all distances. These are simple consequences of the fact that the field of a point charge varies as the inverse square of the distance. If that doesn't yet seem compellingly obvious, look at it this way: roughly speaking, the part of the line charge that is mainly responsible for the field at P in Fig. 1.24 is the near part – the charge within a distance of order of magnitude r. If we lump all this together and forget the rest, we have a concentrated charge of magnitude $q \approx \lambda r$, which ought to produce a field proportional to q/r^2, or λ/r. In the case of the sheet, the amount of charge that is "effective," in this sense, increases proportionally to r^2 as we go out from the sheet, which just offsets the $1/r^2$ decrease in the field from any given element of charge.

1.14 The force on a layer of charge

The sphere in Fig. 1.27 has a charge distributed over its surface with the uniform density σ, in $\mathrm{C/m^2}$. Inside the sphere, as we have already learned, the electric field of such a charge distribution is zero. Outside the sphere the field is $Q/4\pi\epsilon_0 r^2$, where Q is the total charge on the sphere, equal to $4\pi r_0^2\sigma$. So just outside the surface of the sphere the field strength is

$$E_{\text{just outside}} = \frac{\sigma}{\epsilon_0}. \tag{1.42}$$

Compare this with Eq. (1.40) and Fig. 1.26. In both cases Gauss's law is obeyed: the *change* in the normal component of $\mathbf{E}$, from one side of the layer to the other, is equal to σ/ϵ_0, in accordance with Eq. (1.41).

What is the electrical force experienced by the charges that make up this distribution? The question may seem puzzling at first because the field $\mathbf{E}$ arises from these very charges. What we must think about is the force on some small element of charge dq, such as a small patch of area dA with charge $dq = \sigma\, dA$. Consider, separately, the force on dq due to all the other charges in the distribution, and the force on the patch due to the charges within the patch itself. This latter force is surely zero. Coulomb repulsion between charges within the patch is just another example of

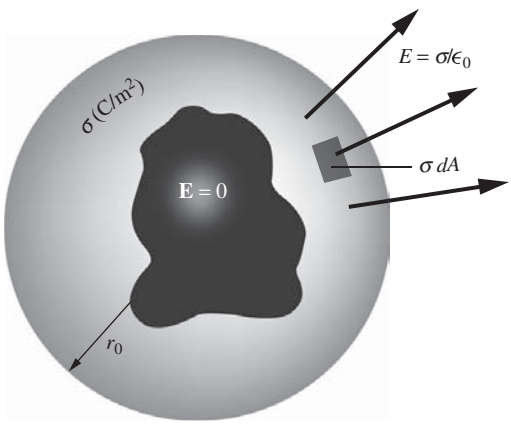


Figure 1.27.
A spherical surface with uniform charge density σ.

Newton's third law; the patch as a whole cannot push on itself. That simplifies our problem, for it allows us to use the entire electric field **E**, *including* the field due to all charges in the patch, in calculating the force $d\mathbf{F}$ on the patch of charge dq:

$$d\mathbf{F} = \mathbf{E}\, dq = \mathbf{E}\sigma\, dA. \tag{1.43}$$

But what E shall we use, the field $E = \sigma/\epsilon_0$ outside the sphere or the field $E = 0$ inside? The correct answer, as we shall prove in a moment, is the *average* of the two fields that is,

$$dF = \frac{1}{2}\big(\sigma/\epsilon_0 + 0\big)\sigma\, dA = \frac{\sigma^2\, dA}{2\epsilon_0}. \tag{1.44}$$

To justify this we shall consider a more general case, and one that will introduce a more realistic picture of a layer of surface charge. Real charge layers do not have zero thickness. Figure 1.28 shows some ways in which charge might be distributed through the thickness of a layer. In each example, the value of σ, the total charge per unit area of layer, is the same. These might be cross sections through a small portion of the spherical surface in Fig. 1.27 on a scale such that the curvature is not noticeable. To make it more general, however, we can let the field on the left be E_1 (rather than 0, as it was inside the sphere), with E_2 the field on the right. The condition imposed by Gauss's law, for given σ, is, in each case,

$$E_2 - E_1 = \frac{\sigma}{\epsilon_0}. \tag{1.45}$$

Now let us look carefully within the layer where the field is changing continuously from E_1 to E_2 and there is a volume charge density $\rho(x)$ extending from $x = 0$ to $x = x_0$, the thickness of the layer (Fig. 1.29). Consider a much thinner slab, of thickness $dx \ll x_0$, which contains per unit area an amount of charge $\rho\, dx$. If the area of this thin slab is A, the force on it is

$$dF = E\rho\, dx \cdot A. \tag{1.46}$$

Thus the total force per unit area of our original charge layer is

$$\frac{F}{A} = \int \frac{dF}{A} = \int_0^{x_0} E\rho\, dx. \tag{1.47}$$

But Gauss's law tells us via Eq. (1.45) that dE, the change in E through the thin slab, is just $\rho\, dx/\epsilon_0$. Hence $\rho\, dx$ in Eq. (1.47) can be replaced by $\epsilon_0\, dE$, and the integral becomes

$$\frac{F}{A} = \int_{E_1}^{E_2} \epsilon_0 E\, dE = \frac{\epsilon_0}{2}\big(E_2^2 - E_1^2\big). \tag{1.48}$$

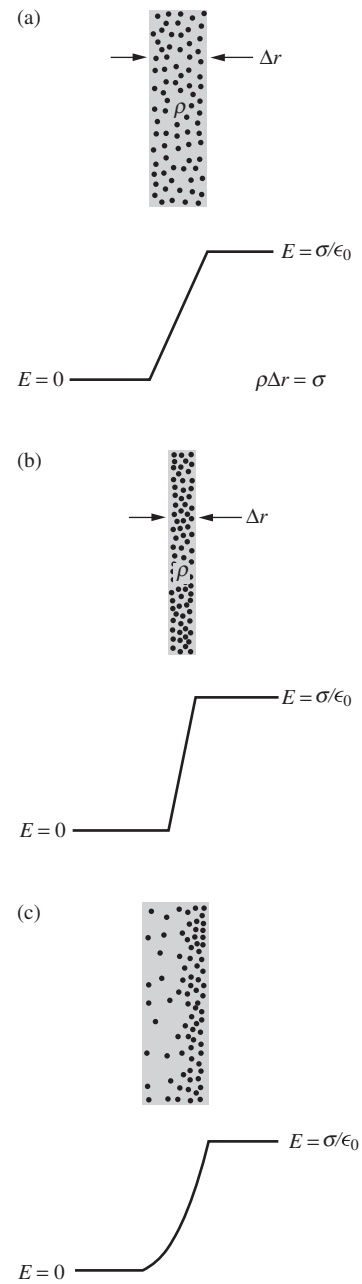


Figure 1.28.
The net change in field at a charge layer depends only on the total charge per unit area.

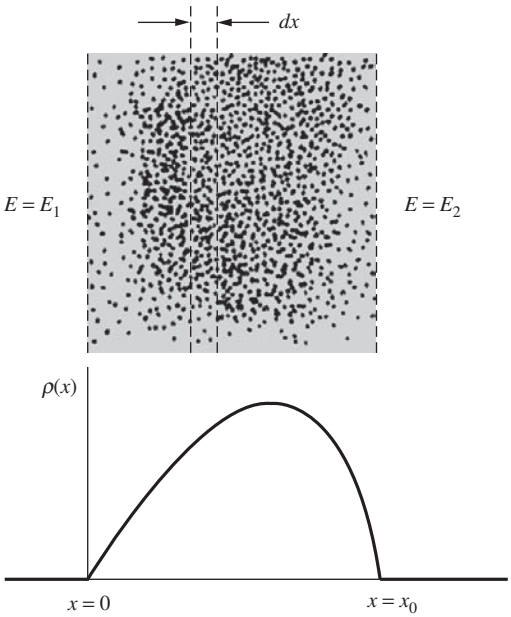


Figure 1.29.
Within the charge layer of density $\rho(x)$, $E(x+dx) - E(x) = \rho\, dx/\epsilon_0$.

Since $E_2 - E_1 = \sigma/\epsilon_0$, the force per unit area in Eq. (1.48), after being factored, can be expressed as

$$\boxed{\frac{F}{A} = \frac{1}{2}(E_1 + E_2)\sigma} \tag{1.49}$$

We have shown, as promised, that for given σ the force per unit area on a charge layer is determined by the mean of the external field on one side and that on the other.[11] This is independent of the thickness of the layer, as long as it is small compared with the total area, and of the variation $\rho(x)$ in charge density within the layer. See Problem 1.30 for an alternative derivation of Eq. (1.49).

The direction of the electrical force on an element of the charge on the sphere is, of course, outward whether the surface charge is positive or negative. If the charges do not fly off the sphere, that outward force must be balanced by some inward force, not included in our equations, that can hold the charge carriers in place. To call such a force "nonelectrical" would be misleading, for electrical attractions and repulsions are the dominant forces in the structure of atoms and in the cohesion of matter generally. The difference is that these forces are effective only at short distances, from atom to atom, or from electron to electron. Physics on that scale is a story of individual particles. Think of a charged rubber balloon, say 0.1 m in radius, with 10^{-8} C of negative charge spread as uniformly as possible on its outer surface. It forms a surface charge of density $\sigma = (10^{-8}\,\mathrm{C})/4\pi(0.1\,\mathrm{m})^2 = 8\cdot 10^{-8}\,\mathrm{C/m^2}$. The resulting outward force, per area of surface charge, is given by Eq. (1.44) as

$$\frac{dF}{dA} = \frac{\sigma^2}{2\epsilon_0} = \frac{(8\cdot 10^{-8}\,\mathrm{C/m^2})^2}{2\left(8.85\cdot 10^{-12}\,\mathrm{C^2/(N\,m^2)}\right)} = 3.6\cdot 10^{-4}\,\mathrm{N/m^2}. \tag{1.50}$$

In fact, our charge consists of about $6\cdot 10^{10}$ electrons attached to the rubber film, which corresponds to about 50 million extra electrons per square centimeter. So the "graininess" in the charge distribution is hardly apparent. However, if we could look at one of these extra electrons, we would find it roughly 10^{-4} cm – an enormous distance on an atomic scale – from its nearest neighbor. This electron would be stuck, electrically stuck, to a local molecule of rubber. The rubber molecule would be attached to adjacent rubber molecules, and so on. If you pull on the electron, the force is transmitted in this way to the whole piece of rubber. Unless, of course, you pull hard enough to tear the electron loose from the molecule to which it is attached. That would take an electric field many thousands of times stronger than the field in our example.

[11] Note that this is *not* necessarily the same as the average field within the layer, a quantity of no special interest or significance.

1.15 Energy associated with the electric field

Suppose our spherical shell of charge is compressed slightly, from an initial radius of r_0 to a smaller radius, as in Fig. 1.30. This requires that work be done against the repulsive force, which we found above to be $\sigma^2/2\epsilon_0$ newtons for each square meter of surface. The displacement being dr, the total work done is $(4\pi r_0^2)(\sigma^2/2\epsilon_0)\,dr$, or $(2\pi r_0^2\sigma^2/\epsilon_0)\,dr$. This represents an *increase* in the energy required to assemble the system of charges, the energy U we talked about in Section 1.5:

$$dU = \frac{2\pi r_0^2\sigma^2}{\epsilon_0}\,dr. \tag{1.51}$$

Notice how the electric field E has been changed. Within the shell of thickness dr, the field was zero and is now σ/ϵ_0. Beyond r_0 the field is unchanged. In effect we have created a field of strength $E = \sigma/\epsilon_0$ filling a region of volume $4\pi r_0^2\,dr$. We have done so by investing an amount of energy given by Eq. (1.51) which, if we substitute $\epsilon_0 E$ for σ, can be written like this:

$$dU = \frac{\epsilon_0 E^2}{2}\,4\pi r_0^2\,dr. \tag{1.52}$$

This is an instance of a general theorem which we shall not prove now (but see Problem 1.33): *the potential energy U of a system of charges, which is the total work required to assemble the system, can be calculated from the electric field itself simply by assigning an amount of energy* $(\epsilon_0 E^2/2)\,dv$ *to every volume element dv and integrating over all space where there is electric field*:

$$\boxed{U = \frac{\epsilon_0}{2}\int_{\substack{\text{entire}\\ \text{field}}} E^2\,dv} \tag{1.53}$$

E^2 is a scalar quantity, of course: $E^2 \equiv \mathbf{E}\cdot\mathbf{E}$.

One may think of this energy as "stored" in the field. The system being conservative, that amount of energy can of course be recovered by allowing the charges to go apart; so it is nice to think of the energy as "being somewhere" meanwhile. Our accounting comes out right if we think of it as stored in space with a density of $\epsilon_0 E^2/2$, in joules/m^3. There is no harm in this, but in fact we have no way of identifying, quite independently of anything else, the energy stored in a particular cubic meter of space. Only the total energy is physically measurable, that is, the work required to bring the charge into some configuration, starting from some other configuration. Just as the concept of electric field serves in place of Coulomb's law to explain the behavior of electric charges, so when we use Eq. (1.53) rather than Eq. (1.15) to express the total potential energy of an electrostatic system, we are merely using a different kind of bookkeeping. Sometimes a change in viewpoint, even if it is at

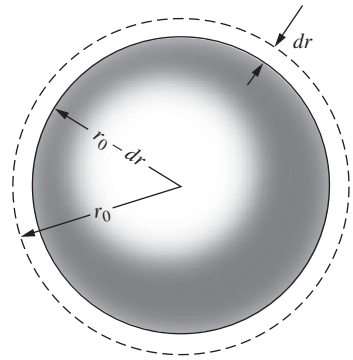


Figure 1.30.
Shrinking a spherical shell or charged balloon.

first only a change in bookkeeping, can stimulate new ideas and deeper understanding. The notion of the electric field as an independent entity will take form when we study the dynamical behavior of charged matter and electromagnetic radiation.

Example (Potential energy of a uniform sphere) What is the energy stored in a sphere of radius R with charge Q uniformly distributed throughout the interior?

Solution The electric field is nonzero both inside and outside the sphere, so Eq. (1.53) involves two different integrals. Outside the sphere, the field at radius r is simply $Q/4\pi\epsilon_0 r^2$, so the energy stored in the external field is

$$U_{\text{ext}} = \frac{\epsilon_0}{2}\int_R^\infty \left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 4\pi r^2\,dr = \frac{Q^2}{8\pi\epsilon_0}\int_R^\infty \frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon_0 R}. \tag{1.54}$$

The example in Section 1.11 gives the field at radius r inside the sphere as $E_r = \rho r/3\epsilon_0$. But the density equals $\rho = Q/(4\pi R^3/3)$, so the field is $E_r = (3Q/4\pi R^3)r/3\epsilon_0 = Qr/4\pi\epsilon_0 R^3$. The energy stored in the internal field is therefore

$$U_{\text{int}} = \frac{\epsilon_0}{2}\int_0^R \left(\frac{Qr}{4\pi\epsilon_0 R^3}\right)^2 4\pi r^2\,dr = \frac{Q^2}{8\pi\epsilon_0 R^6}\int_0^R r^4\,dr = \frac{Q^2}{8\pi\epsilon_0 R}\cdot\frac{1}{5}. \tag{1.55}$$

This is one-fifth of the energy stored in the external field. The total energy is the sum of U_{ext} and U_{int}, which we can write as $(3/5)Q^2/4\pi\epsilon_0 R$. We see that it takes three-fifths as much energy to assemble the sphere as it does to bring in two point charges Q to a separation of R. Exercise 1.61 presents an alternative method of calculating the potential energy of a uniformly charged sphere, by imagining building it up layer by layer.

We run into trouble if we try to apply Eq. (1.53) to a system that contains a point charge, that is, a finite charge q of zero size. Locate q at the origin of the coordinates. Close to the origin, E^2 will approach $q^2/(4\pi\epsilon_0)^2 r^4$. With $dv = 4\pi r^2\,dr$, the integrand $E^2\,dv$ will behave like dr/r^2, and our integral will blow up at the limit $r = 0$. That simply tells us that it would take infinite energy to pack finite charge into zero volume – which is true but not helpful. In the real world we deal with particles like electrons and protons. They are so small that for most purposes we can ignore their dimensions and think of them as point charges when we consider their electrical interaction with one another. How much energy it took to make such a particle is a question that goes beyond the range of classical electromagnetism. We have to regard the particles as supplied to us ready-made. The energy we are concerned with is the work done in moving them around.

The distinction is usually clear. Consider two charged particles, a proton and a negative pion, for instance. Let $\mathbf{E}_{\text{p}}$ be the electric field of the proton, $\mathbf{E}_\pi$ that of the pion. The total field is $\mathbf{E} = \mathbf{E}_{\text{p}} + \mathbf{E}_\pi$, and $\mathbf{E}\cdot\mathbf{E}$

equals $E_p^2 + E_\pi^2 + 2\mathbf{E}_p \cdot \mathbf{E}_\pi$. According to Eq. (1.53) the total energy in the electric field of this two-particle system is

$$U = \frac{\epsilon_0}{2} \int E^2 \, dv$$

$$= \frac{\epsilon_0}{2} \int E_p^2 \, dv + \frac{\epsilon_0}{2} \int E_\pi^2 \, dv + \epsilon_0 \int \mathbf{E}_p \cdot \mathbf{E}_\pi \, dv. \qquad (1.56)$$

The value of the first integral is a property of any isolated proton. It is a constant of nature which is not changed by moving the proton around. The same goes for the second integral, involving the pion's electric field alone. It is the third integral that directly concerns us, for it expresses the energy required to assemble the system *given* a proton and a pion as constituents.

The distinction could break down if the two particles interact so strongly that the electrical structure of one is distorted by the presence of the other. Knowing that both particles are in a sense composite (the proton consisting of three quarks, the pion of two), we might expect that to happen during a close approach. In fact, nothing much happens down to a distance of 10^{-15} m. At shorter distances, for strongly interacting particles like the proton and the pion, nonelectrical forces dominate the scene anyway.

That explains why we do not need to include "self-energy" terms like the first two integrals in Eq. (1.56) in our energy accounts for a system of elementary charged particles. Indeed, we want to omit them. We are doing just that, in effect, when we replace the actual distribution of discrete elementary charges (the electrons on the rubber balloon) by a perfectly continuous charge distribution.

1.16 Applications

Each chapter of this book concludes with a list of "everyday" applications of the topics covered in the chapter. The discussions are brief. It would take many pages to explain each item in detail; real-life physics tends to involve countless variations, complications, and subtleties. The main purpose here is just to say a few words to convince you that the applications are interesting and worthy of further study. You can carry onward with some combination of books/internet/people/pondering. There is effectively an infinite amount of information out there, so you should take advantage of it! Two books packed full of real-life applications are:

- *The Flying Circus of Physics* (Walker, 2007);
- *How Things Work* (Bloomfield, 2010).

And some very informative websites are:

- *The Flying Circus of Physics* website: www.flyingcircusofphysics.com;
- *How Stuff Works*: www.howstuffworks.com;

- *Explain That Stuff*: www.explainthatstuff.com;
- and *Wikipedia*, of course: www.wikipedia.org.

These websites can point you to more technical sources if you want to pursue things at a more advanced level.

With the exception of the gravitational force keeping us on the earth, and ignoring magnets for the time being, essentially all "everyday" forces are *electrostatic* in origin (with some quantum mechanics mixed in, to make things stable; see Earnshaw's theorem in Section 2.12). Friction, tension, normal force, etc., all boil down to the electric forces between the electrons in the various atoms and molecules. You can open a door by pushing on it because the forces between neighboring molecules in the door, and also in your hand, are sufficiently strong. We can ignore the gravitational force between everyday-sized objects because the gravitational force is so much weaker than the electric force (see Problem 1.1). Only if one of the objects is the earth does the gravitational force matter. And even in that case, it is quite remarkable that the electric forces between the molecules in, say, a wooden board that you might be standing on can completely balance the gravitational force on you due to the entire earth. However, this wouldn't be the case if you attempt to stand on a lake (unless it's frozen!).

If you want to give an object a net charge, a possible way is via the *triboelectric effect*. If certain materials are rubbed against each other, they can become charged. For example, rubbing wool and Teflon together causes the wool to become positively charged and the Teflon negatively charged. The mechanism is simple: the Teflon simply grabs electrons from the wool. The determination of which material ends up with extra electrons depends on the electronic structure of the molecules in the materials. It turns out that actual rubbing isn't necessary. Simply touching and separating the materials can produce an imbalance of charge. Triboelectric effects are mitigated by humid air, because the water molecules in the air are inclined to give or receive electrons, depending on which of these actions neutralizes the object. This is due to the fact that water molecules are *polar*, that is, they are electrically lopsided. (Polar molecules will be discussed in Chapter 10.)

The *electrical breakdown* of air occurs when the electric field reaches a strength of about $3 \cdot 10^6$ V/m. In fields this strong, electrons are ripped from molecules in the air. They are then accelerated by the field and collide with other molecules, knocking electrons out of these molecules, and so on, in a cascading process. The result is a *spark*, because eventually the electrons will combine in a more friendly manner with molecules and drop down to a lower energy level, emitting the light that you see. If you shuffle your feet on a carpet and then bring your finger close to a grounded object, you will see a spark.

The electric field near the surface of the earth is about 100 V/m, pointing downward. You can show that this implies a charge of $-5 \cdot 10^5$ C

on the earth. The atmosphere contains roughly the opposite charge, so that the earth-plus-atmosphere system is essentially neutral, as it must be. (Why?) If there were no regenerative process, charge would leak between the ground and the atmosphere, and they would neutralize each other in about an hour. But there *is* a regenerative process: *lightning*. This is a spectacular example of electrical breakdown. There are millions of lightning strikes per day over the surface of the earth, the vast majority of which transfer negative charge to the earth. A lightning strike is the result of the strong electric field that is produced by the buildup (or rather, the separation) of charge in a cloud. This separation arises from the charge carried on moving raindrops, although the exact process is rather complicated (see the interesting discussion in Chapter 9 of Feynman *et al.* (1977)). "Lightning" can also arise from the charge carried on dust particles in coal mines, flour mills, grain storage facilities, etc. The result can be a deadly explosion.

A more gentle form of electrical breakdown is *corona discharge*. Near the tip of a charged pointy object, such as a needle, the field is large but then falls off rapidly. (You can model the needle roughly as having a tiny charged sphere on its end.) Electrons are ripped off the needle (or off the air molecules) very close to the needle, but the field farther away isn't large enough to sustain the breakdown. So there is a slow leakage instead of an abrupt spark. This leakage can sometimes be seen as a faint glow. Examples are *St. Elmo's fire* at the tips of ship masts, and a glow at the tips of airplane wings.

Electrostatic paint sprayers can produce very even coats of paint. As the paint leaves the sprayer, an electrode gives it a charge. This causes the droplets in the paint mist to repel each other, helping to create a uniform mist with no clumping. If the object being painted is grounded (or given the opposite charge), the paint will be attracted to it, leading to less wasted paint, less mess, and less inhalation of paint mist. When painting a metal pipe, for example, the mist will wrap around and partially coat the back side, instead of just sailing off into the air.

Photocopiers work by giving the toner powder a charge, and giving certain locations on a drum or belt the opposite charge. These locations on the drum can be made to correspond to the locations of ink on the original paper. This is accomplished by coating the drum with a *photoconductive* material, that is, one that becomes conductive when exposed to light. The entire surface of the drum is given an initial charge and then exposed to light at locations corresponding to the white areas on the original page (accomplished by reflecting light off the page). The charge can be made to flow off these newly conductive locations on the drum, leaving charge only at the locations corresponding to the ink. When the oppositely charged toner is brought nearby, it is attracted to these locations on the drum. The toner is then transferred to a piece of paper, producing the desired copy.

Electronic paper, used in many eBook readers, works by using electric fields to rotate or translate small black and white objects.

One technique uses tiny spheres (about 10^{-4} m in diameter) that are black on one side and white on the other, with the sides being oppositely charged. Another technique uses similarly tiny spheres that are filled with many even tinier charged white particles along with a dark dye. In both cases, a narrow gap between sheets of electrodes (with one sheet being the transparent sheet that you look through) is filled with the spheres. By depositing a specific pattern of charge on the sheets, the color of the objects facing your eye can be controlled. In the first system, the black and white spheres rotate accordingly. In the second system, the tiny white particles pile up on one side of the sphere. In contrast with a standard LCD computer screen, electronic paper acts like normal paper, in that it doesn't produce its own light; an outside light source is needed to view the page. An important advantage of electronic paper is that it uses a very small amount of power. A battery is needed only when the page is refreshed, whereas an LCD screen requires continual refreshing.

CHAPTER SUMMARY

- Electric charge, which can be positive or negative, is both *conserved* and *quantized.* The force between two charges is given by *Coulomb's law:*

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2 \hat{\mathbf{r}}_{21}}{r_{21}^2}. \tag{1.57}$$

  Integrating this force, we find that the *potential energy* of a system of charges (the work necessary to bring them in from infinity) equals

$$U = \frac{1}{2}\sum_{j=1}^{N}\sum_{k\neq j}\frac{1}{4\pi\epsilon_0}\frac{q_j q_k}{r_{jk}}. \tag{1.58}$$

- The *electric field* due to a charge distribution is (depending on whether the distribution is continuous or discrete)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x', y', z')\hat{\mathbf{r}}\, dx'\, dy'\, dz'}{r^2} \quad \text{or} \quad \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{N}\frac{q_j\hat{\mathbf{r}}_j}{r_j^2}. \tag{1.59}$$

  The force on a test charge q due to the field is $\mathbf{F} = q\mathbf{E}$.
- The *flux* of an electric field through a surface S is

$$\Phi = \int_S \mathbf{E}\cdot d\mathbf{a}. \tag{1.60}$$

  Gauss's law states that the flux of the electric field $\mathbf{E}$ through any closed surface equals $1/\epsilon_0$ times the total charge enclosed by the

surface. That is (depending on whether the distribution is continuous or discrete),

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int \rho \, dv = \frac{1}{\epsilon_0} \sum_i q_i. \tag{1.61}$$

Gauss's law gives the fields for a sphere, line, and sheet of charge as

$$E_{\text{sphere}} = \frac{Q}{4\pi\epsilon_0 r^2}, \qquad E_{\text{line}} = \frac{\lambda}{2\pi\epsilon_0 r}, \qquad E_{\text{sheet}} = \frac{\sigma}{2\epsilon_0}. \tag{1.62}$$

More generally, the discontinuity in the normal component of $\mathbf{E}$ across a sheet is $\Delta E_\perp = \sigma/\epsilon_0$. Gauss's law is always valid, although it is useful for calculating the electric field only in cases where there is sufficient symmetry.

- The force per unit area on a layer of charge equals the density times the average of the fields on either side:

$$\frac{F}{A} = \frac{1}{2}(E_1 + E_2)\sigma. \tag{1.63}$$

- The *energy density* of an electric field is $\epsilon_0 E^2/2$, so the total energy in a system equals

$$U = \frac{\epsilon_0}{2} \int E^2 \, dv. \tag{1.64}$$

Problems

1.1 *Gravity vs. electricity* $*$

(a) In the domain of elementary particles, a natural unit of mass is the mass of a *nucleon,* that is, a proton or a neutron, the basic massive building blocks of ordinary matter. Given the nucleon mass as $1.67 \cdot 10^{-27}$ kg and the gravitational constant G as $6.67 \cdot 10^{-11}$ m^3/(kg s^2), compare the gravitational attraction of two protons with their electrostatic repulsion. This shows why we call gravitation a very *weak* force.

(b) The distance between the two protons in the helium nucleus could be at one instant as much as 10^{-15} m. How large is the force of electrical repulsion between two protons at that distance? Express it in newtons, and in pounds. Even stronger is the *nuclear* force that acts between any pair of hadrons (including neutrons and protons) when they are that close together.

1.2 *Zero force from a triangle* $**$

Two positive ions and one negative ion are fixed at the vertices of an equilateral triangle. Where can a fourth ion be placed, along the symmetry axis of the setup, so that the force on it will be zero? Is there more than one such place? You will need to solve something numerically.

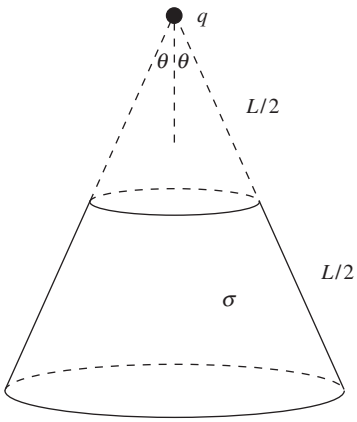


Figure 1.31.

1.3 *Force from a cone* ∗∗

(a) A charge q is located at the tip of a hollow cone (such as an ice cream cone without the ice cream) with surface charge density σ. The slant height of the cone is L, and the half-angle at the vertex is θ. What can you say about the force on the charge q due to the cone?

(b) If the top half of the cone is removed and thrown away (see Fig. 1.31), what is the force on the charge q due to the remaining part of the cone? For what angle θ is this force maximum?

1.4 *Work for a rectangle* ∗∗

Two protons and two electrons are located at the corners of a rectangle with side lengths a and b. There are two essentially different arrangements. Consider the work required to assemble the system, starting with the particles very far apart. Is it possible for the work to be positive for either of the arrangements? If so, how must a and b be related? You will need to solve something numerically.

1.5 *Stable or unstable?* ∗∗

In the setup in Exercise 1.37, is the charge $-Q$ at the center of the square in stable or unstable equilibrium? You can answer this by working with either forces or energies. The latter has the advantage of not involving components, although things can still get quite messy. However, the math is simple if you use a computer. Imagine moving the $-Q$ charge infinitesimally to the point (x, y), and use, for example, the `Series` operation in *Mathematica* to calculate the new energy of the charge, to lowest nontrivial order in x and y. If the energy decreases for at least one direction of displacement, then the equilibrium is unstable. (The equilibrium is certainly stable with respect to displacements perpendicular to the plane of the square, because the attractive force from the other charges is directed back toward the plane. The question is, what happens in the plane of the square?)

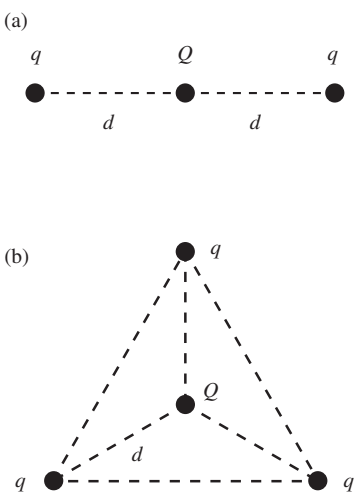


Figure 1.32.

1.6 *Zero potential energy for equilibrium* ∗∗

(a) Two charges q are each located a distance d from a charge Q, as shown in Fig. 1.32(a). What should the charge Q be so that the system is in equilibrium; that is, so that the force on each charge is zero? (The equilibrium is an unstable one, which can be seen by looking at longitudinal displacements of the (negative) charge Q. This is consistent with a general result that we will derive Section 2.12.)

(b) Same question, but now with the setup in Fig. 1.32(b). The three charges q are located at the vertices of an equilateral triangle.

(c) Show that the total potential energy in each of the above systems is zero.

(d) In view of the previous result, we might make the following conjecture: "The total potential energy of any system of charges in equilibrium is zero." Prove that this conjecture is indeed true. *Hint:* The goal is to show that zero work is required to move the charges out to infinity. Since the electrostatic force is conservative, you need only show that the work is zero for one particular set of paths of the charges. And there is indeed a particular set of paths that makes the result clear.

1.7 *Potential energy in a two-dimensional crystal* $**$
Use a computer to calculate numerically the potential energy, per ion, for an infinite two-dimensional square ionic crystal with separation a; that is, a plane of equally spaced charges of magnitude e and alternating sign (as with a checkerboard).

1.8 *Oscillating in a ring* $***$
A ring with radius R has uniform positive charge density λ. A particle with positive charge q and mass m is initially located at the center of the ring and is then given a tiny kick. If it is constrained to move in the plane of the ring, show that it undergoes simple harmonic motion (for small oscillations), and find the frequency. *Hint:* Find the potential energy of the particle when it is at a (small) radius, r, by integrating over the ring, and then take the negative derivative to find the force. You will need to use the law of cosines and also the Taylor series $1/\sqrt{1+\epsilon} \approx 1 - \epsilon/2 + 3\epsilon^2/8$.

1.9 *Field from two charges* $**$
A charge $2q$ is at the origin, and a charge $-q$ is at $x = a$ on the x axis.

(a) Find the point on the x axis where the electric field is zero.
(b) Consider the vertical line passing through the charge $-q$, that is, the line given by $x = a$. Locate, at least approximately, a point on this line where the electric field is parallel to the x axis.

1.10 *45-degree field line* $**$
A half-infinite line has linear charge density λ. Find the electric field at a point that is "even" with the end, a distance ℓ from it, as shown in Fig. 1.33. You should find that the field always points up at a 45° angle, independent of ℓ.

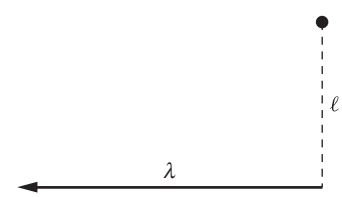


Figure 1.33.

1.11 *Field at the end of a cylinder* $**$
(a) Consider a half-infinite hollow cylindrical shell (that is, one that extends to infinity in one direction) with radius R and uniform surface charge density σ. What is the electric field at the midpoint of the end face?
(b) Use your result to determine the field at the midpoint of a half-infinite *solid* cylinder with radius R and uniform volume

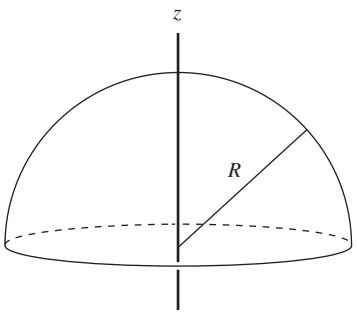


Figure 1.34.

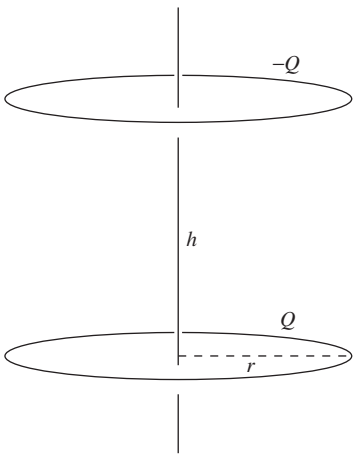


Figure 1.35.

charge density ρ, which can be considered to be built up from many cylindrical shells.

1.12 *Field from a hemispherical shell* ∗∗∗

A hemispherical shell has radius R and uniform surface charge density σ (see Fig. 1.34). Find the electric field at a point on the symmetry axis, at position z relative to the center, for any z value from $-\infty$ to ∞.

1.13 *A very uniform field* ∗∗∗

(a) Two rings with radius r have charge Q and $-Q$ uniformly distributed around them. The rings are parallel and located a distance h apart, as shown in Fig. 1.35. Let z be the vertical coordinate, with $z = 0$ taken to be at the center of the lower ring. As a function of z, what is the electric field at points on the axis of the rings?

(b) You should find that the electric field is an even function with respect to the $z = h/2$ point midway between the rings. This implies that, at this point, the field has a local extremum as a function of z. The field is therefore fairly uniform there; there are no variations to first order in the distance along the axis from the midpoint. What should r be in terms of h so that the field is *very* uniform?

By "very" uniform we mean that additionally there aren't any variations to second order in z. That is, the second derivative vanishes. This then implies that the leading-order change is *fourth* order in z (because there are no variations at any odd order, since the field is an even function around the midpoint). Feel free to calculate the derivatives with a computer.

1.14 *Hole in a plane* ∗∗

(a) A hole of radius R is cut out from a very large flat sheet with uniform charge density σ. Let L be the line perpendicular to the sheet, passing through the center of the hole. What is the electric field at a point on L, a distance z from the center of the hole? *Hint:* Consider the plane to consist of many concentric rings.

(b) If a charge $-q$ with mass m is released from rest on L, very close to the center of the hole, show that it undergoes oscillatory motion, and find the frequency ω of these oscillations. What is ω if $m = 1$ g, $-q = -10^{-8}$ C, $\sigma = 10^{-6}\,\mathrm{C/m^2}$, and $R = 0.1$ m?

(c) If a charge $-q$ with mass m is released from rest on L, a distance z from the sheet, what is its speed when it passes through the center of the hole? What does your answer reduce to for large z (or, equivalently, small R)?

1.15 *Flux through a circle* $**$

A point charge q is located at the origin. Consider the electric field flux through a circle a distance ℓ from q, subtending an angle 2θ, as shown in Fig. 1.36. Since there are no charges except at the origin, any surface that is bounded by the circle and that stays to the right of the origin must contain the same flux. (Why?) Calculate this flux by taking the surface to be:

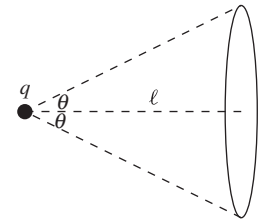


Figure 1.36.

(a) the flat disk bounded by the circle;

(b) the spherical cap (with the sphere centered at the origin) bounded by the circle.

1.16 *Gauss's law and two point charges* $**$

(a) Two point charges q are located at positions $x = \pm\ell$. At points close to the origin on the x axis, find E_x. At points close to the origin on the y axis, find E_y. Make suitable approximations with $x \ll \ell$ and $y \ll \ell$.

(b) Consider a small cylinder centered at the origin, with its axis along the x axis. The radius is r_0 and the length is $2x_0$. Using your results from part (a), verify that there is zero flux through the cylinder, as required by Gauss's law.

1.17 *Zero field inside a spherical shell* $**$

Consider a hollow spherical shell with uniform surface charge density. By considering the two small patches at the ends of the thin cones in Fig. 1.37, show that the electric field at any point P in the interior of the shell is zero. This then implies that the electric potential (defined in Chapter 2) is constant throughout the interior.

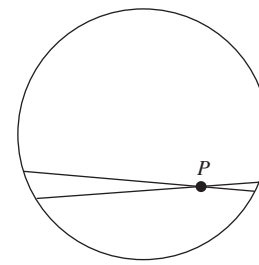


Figure 1.37.

1.18 *Fields at the surfaces* $**$

Consider the electric field at a point on the surface of (a) a sphere with radius R, (b) a cylinder with radius R whose length is infinite, and (c) a slab with thickness $2R$ whose other two dimensions are infinite. All of the objects have the same volume charge density ρ. Compare the fields in the three cases, and explain physically why the sizes take the order they do.

1.19 *Sheet on a sphere* $**$

Consider a large flat horizontal sheet with thickness x and volume charge density ρ. This sheet is tangent to a sphere with radius R and volume charge density ρ_0, as shown in Fig. 1.38. Let A be the point of tangency, and let B be the point opposite to A on the top side of the sheet. Show that the net upward electric field (from the sphere plus the sheet) at B is larger than at A if $\rho > (2/3)\rho_0$. (Assume $x \ll R$.)

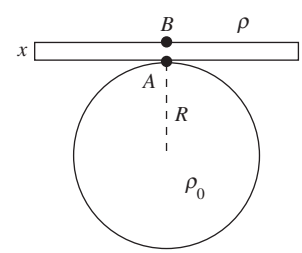


Figure 1.38.

1.20 *Thundercloud* $**$

You observe that the passage of a particular thundercloud overhead causes the vertical electric field strength in the atmosphere, measured at the ground, to rise to 3000 N/C (or V/m).

(a) How much charge does the thundercloud contain, in coulombs per square meter of horizontal area? Assume that the width of the cloud is large compared with the height above the ground.

(b) Suppose there is enough water in the thundercloud in the form of 1 mm diameter drops to make 0.25 cm of rainfall, and that it is those drops that carry the charge. How large is the electric field strength at the surface of one of the drops?

1.21 *Field in the end face* $*$

Consider a half-infinite hollow cylindrical shell (that is, one that extends to infinity in one direction) with uniform surface charge density. Show that at all points in the circular end face, the electric field is parallel to the cylinder's axis. *Hint:* Use superposition, along with what you know about the field from an infinite (in both directions) hollow cylinder.

1.22 *Field from a spherical shell, right and wrong* $**$

The electric field outside and an infinitesimal distance away from a uniformly charged spherical shell, with radius R and surface charge density σ, is given by Eq. (1.42) as σ/ϵ_0. Derive this in the following way.

(a) Slice the shell into rings (symmetrically located with respect to the point in question), and then integrate the field contributions from all the rings. You should obtain the incorrect result of $\sigma/2\epsilon_0$.

(b) Why isn't the result correct? Explain how to modify it to obtain the correct result of σ/ϵ_0. *Hint:* You could very well have performed the above integral in an effort to obtain the electric field an infinitesimal distance *inside* the shell, where we know the field is zero. Does the above integration provide a good description of what's going on for points on the shell that are very close to the point in question?

1.23 *Field near a stick* $**$

A stick with length 2ℓ has uniform linear charge density λ. Consider a point P, a distance $\eta\ell$ from the center (where $0 \leq \eta < 1$), and an infinitesimal distance away from the stick. Up close, the stick looks infinitely long, as far as the $\mathbf{E}$ component perpendicular to the stick is concerned. So we have $E_\perp = \lambda/2\pi\epsilon_0 r$. Find the $\mathbf{E}$ component parallel to the stick, $E_\parallel$. Does it approach infinity, or does it remain finite at the end of the stick?

1.24 *Potential energy of a cylinder* ***

A cylindrical volume of radius a is filled with charge of uniform density ρ. We want to know the potential energy per unit length of this cylinder of charge, that is, the work done per unit length in assembling it. Calculate this by building up the cylinder layer by layer, making use of the fact that the field outside a cylindrical distribution of charge is the same as if all the charge were located on the axis. You will find that the energy per unit length is infinite if the charges are brought in from infinity, so instead assume that they are initially distributed uniformly over a hollow cylinder with large radius R. Write your answer in terms of the charge per unit length of the cylinder, which is $\lambda = \rho\pi a^2$. (See Exercise 1.83 for a different method of solving this problem.)

1.25 *Two equal fields* **

The result of Exercise 1.78 is that the electric field at the center of a small hole in a spherical shell equals $\sigma/2\epsilon_0$. This happens to be the same as the field due to an infinite flat sheet with the same density σ. That is, at the center of the hole at the top of the spherical shell in Fig. 1.39, the field from the shell equals the field from the infinite horizontal sheet shown. (This sheet could actually be located at any height.) Demonstrate this equality by explaining why the rings on the shell and sheet that are associated with the angle θ and angular width $d\theta$ yield the same field at the top of the shell.

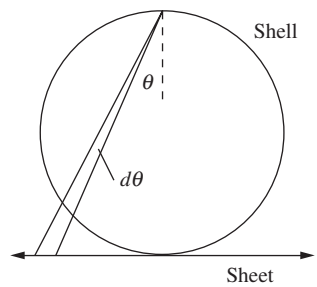


Figure 1.39.

1.26 *Stable equilibrium in electron jelly* **

The task of Exercise 1.77 is to find the equilibrium positions of two protons located inside a sphere of electron jelly with total charge $-2e$. Show that the equilibria are *stable*. That is, show that a displacement in any direction will result in a force directed back toward the equilibrium position. (There is no need to know the exact locations of the equilibria, so you can solve this problem without solving Exercise 1.77 first.)

1.27 *Uniform field in a cavity* **

A sphere has radius R_1 and uniform volume charge density ρ. A spherical cavity with radius R_2 is carved out at an arbitrary location inside the larger sphere. Show that the electric field inside the cavity is uniform (in both magnitude and direction). *Hint:* Find a vector expression for the field in the interior of a charged sphere, and then use superposition.

What are the analogous statements for the lower-dimensional analogs with cylinders and slabs? Are the statements still true?

1.28 *Average field on/in a sphere* **

(a) A point charge q is located at an arbitrary position *inside* a sphere (just an imaginary sphere in space) with radius R. Show

that the average electric field over the *surface* of the sphere is zero. *Hint:* Use an argument involving Newton's third law, along with what you know about spherical shells.

(b) If the point charge q is instead located *outside* the sphere, a distance r from the center, show that the average electric field over the *surface* of the sphere has magnitude $q/4\pi\epsilon_0 r$.

(c) Return to the case where the point charge q is located *inside* the sphere of radius R. Let the distance from the center be r. Use the above results to show that the average electric field over the entire *volume* of the sphere of radius R has magnitude $qr/4\pi\epsilon_0 R^3$ and points toward the center (if q is positive).

1.29 *Pulling two sheets apart* $**$

Two parallel sheets each have large area A and are separated by a small distance ℓ. The surface charge densities are σ and $-\sigma$. You wish to pull one of the sheets away from the other, by a small distance x. How much work does this require? Calculate this by:

(a) using the relation $W = (\text{force}) \times (\text{distance})$;

(b) calculating the increase in energy stored in the electric field. Show that these two methods give the same result.

1.30 *Force on a patch* $**$

Consider a small patch of charge that is part of a larger surface. The surface charge density is σ. If $\mathbf{E}_1$ and $\mathbf{E}_2$ are the electric fields on either side of the patch, show that the force per unit area on the patch equals $\sigma(\mathbf{E}_1 + \mathbf{E}_2)/2$. This is the result we derived in Section 1.14, for the case where the field is perpendicular to the surface. Derive it here by using the fact that the force on the patch is due to the field $\mathbf{E}^{\text{other}}$ from all the *other* charges in the system (excluding the patch), and then finding an expression for $\mathbf{E}^{\text{other}}$ in terms of $\mathbf{E}_1$ and $\mathbf{E}_2$.

1.31 *Decreasing energy?* $*$

A hollow spherical shell with radius R has charge Q uniformly distributed over it. The task of Problem 1.32 is to show that the energy stored in this system is $Q^2/8\pi\epsilon_0 R$. (You can derive this here if you want, or you can just accept it for the purposes of this problem.) Now imagine taking all of the charge and concentrating it in two point charges $Q/2$ located at diametrically opposite positions on the shell. The energy of this new system is $(Q/2)^2/4\pi\epsilon_0(2R) = Q^2/32\pi\epsilon_0 R$, which is less than the energy of the uniform spherical shell. Does this make sense? If not, where is the error in this reasoning?

1.32 *Energy of a shell* $**$
A hollow spherical shell with radius R has charge Q uniformly distributed over it. Show that the energy stored in this system is $Q^2/8\pi\epsilon_0 R$. Do this in two ways as follows.

(a) Use Eq. (1.53) to find the energy stored in the electric field.
(b) Imagine building up the shell by successively adding on infinitesimally thin shells with charge dq. Find the energy needed to add on a shell when the charge already there is q, and then integrate over q.

1.33 *Deriving the energy density* $***$
Consider the electric field of two protons a distance b apart. According to Eq. (1.53) (which we stated but did not prove), the potential energy of the system ought to be given by

$$U = \frac{\epsilon_0}{2}\int \mathbf{E}^2\,dv = \frac{\epsilon_0}{2}\int (\mathbf{E}_1+\mathbf{E}_2)^2\,dv$$
$$= \frac{\epsilon_0}{2}\int \mathbf{E}_1^2\,dv + \frac{\epsilon_0}{2}\int \mathbf{E}_2^2\,dv + \epsilon_0\int \mathbf{E}_1\cdot\mathbf{E}_2\,dv, \qquad (1.65)$$

where $\mathbf{E}_1$ is the field of one particle alone and $\mathbf{E}_2$ that of the other. The first of the three integrals on the right might be called the "electrical self-energy" of one proton; an intrinsic property of the particle, it depends on the proton's size and structure. We have always disregarded it in reckoning the potential energy of a system of charges, on the assumption that it remains constant; the same goes for the second integral. The third integral involves the distance between the charges. Evaluate this integral. This is most easily done if you set it up in spherical polar coordinates with one of the protons at the origin and the other on the polar axis, and perform the integration over r before the integration over θ. Thus, by direct calculation, you can show that the third integral has the value $e^2/4\pi\epsilon_0 b$, which we already know to be the work required to bring the two protons in from an infinite distance to positions a distance b apart. So you will have proved the correctness of Eq. (1.53) for this case, and by invoking superposition you can argue that Eq. (1.53) must then give the energy required to assemble any system of charges.

Exercises

1.34 *Aircraft carriers and specks of gold* $*$
Imagine (quite unrealistically) removing one electron from every atom in a tiny cube of gold 1 mm on a side. (Never mind how you would hold the resulting positively charged cube together.) Do the same thing with another such cube a meter away. What is the repulsive force between the two cubes? How many aircraft carriers

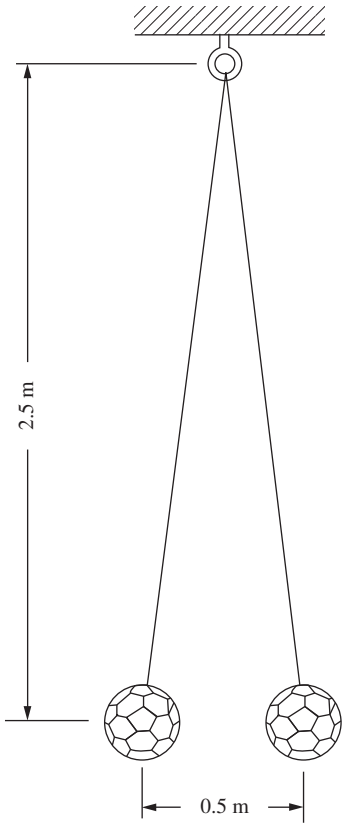


Figure 1.40.

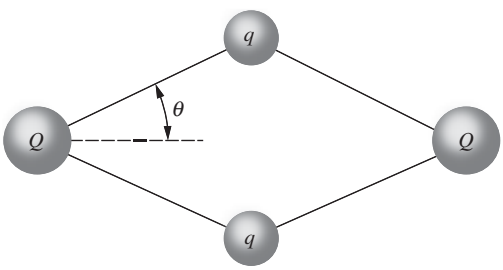


Figure 1.41.

would you need in order to have their total weight equal this force? Some data: The density of gold is 19.3 g/cm^3, and its molecular weight is 197; that is, 1 mole ($6.02 \cdot 10^{23}$) of gold atoms has a mass of 197 grams. The mass of an aircraft carrier is around 100 million kilograms.

1.35 *Balancing the weight* $*$
On the utterly unrealistic assumption that there are no other charged particles in the vicinity, at what distance below a proton would the upward force on an electron equal the electron's weight? The mass of an electron is about $9 \cdot 10^{-31}$ kg.

1.36 *Repelling volley balls* $*$
Two volley balls, mass 0.3 kg each, tethered by nylon strings and charged with an electrostatic generator, hang as shown in Fig. 1.40. What is the charge on each, assuming the charges are equal?

1.37 *Zero force at the corners* $**$
(a) At each corner of a square is a particle with charge q. Fixed at the center of the square is a point charge of opposite sign, of magnitude Q. What value must Q have to make the total force on each of the four particles zero?
(b) With Q taking on the value you just found, show that the potential energy of the system is zero, consistent with the result from Problem 1.6.

1.38 *Oscillating on a line* $**$
Two positive point charges Q are located at points $(\pm\ell, 0)$. A particle with positive charge q and mass m is initially located midway between them and is then given a tiny kick. If it is constrained to move along the line joining the two charges Q, show that it undergoes simple harmonic motion (for small oscillations), and find the frequency.

1.39 *Rhombus of charges* $**$
Four positively charged bodies, two with charge Q and two with charge q, are connected by four unstretchable strings of equal length. In the absence of external forces they assume the equilibrium configuration shown in Fig. 1.41. Show that $\tan^3\theta = q^2/Q^2$. This can be done in two ways. You could show that this relation must hold if the total force on each body, the vector sum of string tension and electrical repulsion, is zero. Or you could write out the expression for the energy U of the assembly (like Eq. (1.13) but for four charges instead of three) and minimize it.

1.40 *Zero potential energy* $**$
Find a geometrical arrangement of one proton and two electrons such that the potential energy of the system is exactly zero. How

many such arrangements are there with the three particles on the same straight line? You should find that the ratio of two of the distances involved is the golden ratio.

1.41 *Work for an octahedron* $**$

Three protons and three electrons are to be placed at the vertices of a regular octahedron of edge length a. We want to find the energy of the system, that is, the work required to assemble it starting with the particles very far apart. There are two essentially different arrangements. What is the energy of each?

1.42 *Potential energy in a one-dimensional crystal* $**$

Calculate the potential energy, per ion, for an infinite 1D ionic crystal with separation a; that is, a row of equally spaced charges of magnitude e and alternating sign. *Hint:* The power-series expansion of $\ln(1 + x)$ may be of use.

1.43 *Potential energy in a three-dimensional crystal* $**$

In the spirit of Problem 1.7, use a computer to calculate numerically the potential energy, per ion, for an infinite 3D cubic ionic crystal with separation a. In other words, derive Eq. (1.18).

1.44 *Chessboard* $**$

An infinite chessboard with squares of side s has a charge e at the center of every white square and a charge $-e$ at the center of every black square. We are interested in the work W required to transport one charge from its position on the board to an infinite distance from the board. Given that W is finite (which is plausible but not so easy to prove), do you think it is positive or negative? Calculate an approximate value for W by removing the charge from the central square of a 7×7 board. (Only nine different terms are involved in that sum.) For larger arrays you can write a program to compute the work numerically. This will give you some idea of the rate of convergence toward the value for the infinite array; see Problem 1.7.

1.45 *Zero field?* $**$

Four charges, q, $-q$, q, and $-q$, are located at equally spaced intervals on the x axis. Their x values are $-3a$, $-a$, a, and $3a$, respectively. Does there exist a point on the y axis for which the electric field is zero? If so, find the y value.

1.46 *Charges on a circular track* $**$

Suppose three positively charged particles are constrained to move on a fixed circular track. If the charges were all equal, an equilibrium arrangement would obviously be a symmetrical one with the particles spaced 120° apart around the circle. Suppose that two

of the charges are equal and the equilibrium arrangement is such that these two charges are 90° apart rather than 120°. What is the relative magnitude of the third charge?

1.47 *Field from a semicircle* ∗
A thin plastic rod bent into a semicircle of radius R has a charge Q distributed uniformly over its length. Find the electric field at the center of the semicircle.

1.48 *Maximum field from a ring* ∗∗
A charge Q is distributed uniformly around a thin ring of radius b that lies in the xy plane with its center at the origin. Locate the point on the positive z axis where the electric field is strongest.

1.49 *Maximum field from a blob* ∗∗
(a) A point charge is placed somewhere on the curve shown in Fig. 1.42. This point charge creates an electric field at the origin. Let E_y be the vertical component of this field. What shape (up to a scaling factor) should the curve take so that E_y is independent of the position of the point charge on the curve?
(b) You have a moldable material with uniform volume charge density. What shape should the material take if you want to create the largest possible electric field at a given point in space? Be sure to explain your reasoning clearly.

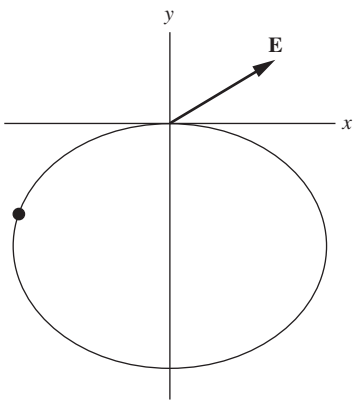


Figure 1.42.

1.50 *Field from a hemisphere* ∗∗
(a) What is the electric field at the center of a hollow hemispherical shell with radius R and uniform surface charge density σ? (This is a special case of Problem 1.12, but you can solve the present exercise much more easily from scratch, without going through all the messy integrals of Problem 1.12.)
(b) Use your result to show that the electric field at the center of a *solid* hemisphere with radius R and uniform volume charge density ρ equals $\rho R/4\epsilon_0$.

1.51 *N charges on a circle* ∗∗∗
N point charges, each with charge Q/N, are evenly distributed around a circle of radius R. What is the electric field at the location of one of the charges, due to all the others? (You can leave your answer in the form of a sum.) In the $N \to \infty$ limit, is the field infinite or finite? In the $N \to \infty$ limit, is the force on one of the charges infinite or finite?

1.52 *An equilateral triangle* ∗
Three positive charges, A, B, and C, of $3 \cdot 10^{-6}$, $2 \cdot 10^{-6}$, and $2 \cdot 10^{-6}$ coulombs, respectively, are located at the corners of an equilateral triangle of side 0.2 m.

(a) Find the magnitude in newtons of the force on each charge.
(b) Find the magnitude in newtons/coulomb of the electric field at the center of the triangle.

1.53 *Concurrent field lines* $**$
A semicircular wire with radius R has uniform charge density $-\lambda$. Show that at all points along the "axis" of the semicircle (the line through the center, perpendicular to the plane of the semicircle, as shown in Fig. 1.43), the vectors of the electric field all point toward a common point in the plane of the semicircle. Where is this point?

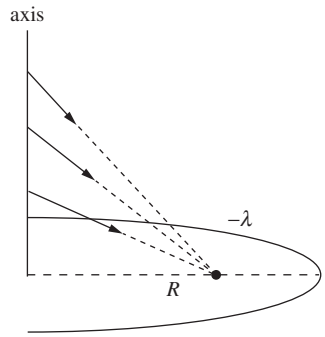


Figure 1.43.

1.54 *Semicircle and wires* $**$
(a) Two long, thin parallel rods, a distance $2b$ apart, are joined by a semicircular piece of radius b, as shown in Fig. 1.44. Charge of uniform linear density λ is deposited along the whole filament. Show that the field $\mathbf{E}$ of this charge distribution vanishes at the point C. Do this by comparing the contribution of the element at A to that of the element at B which is defined by the same values of θ and $d\theta$.
(b) Consider the analogous two-dimensional setup involving a cylinder and a hemispherical end cap, with uniform surface charge density σ. Using the result from part (a), do you think that the field at the analogous point C is directed upward, downward, or is zero? (No calculations needed!)

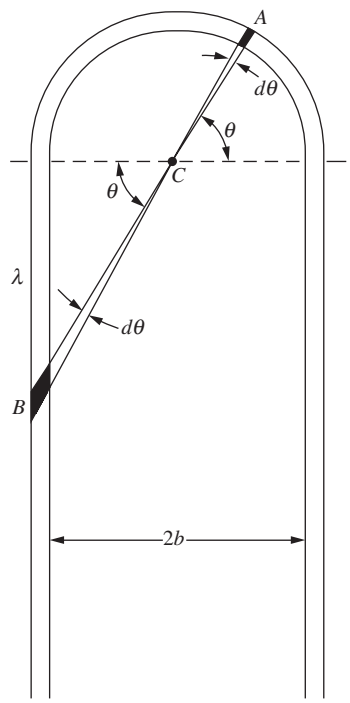


Figure 1.44.

1.55 *Field from a finite rod* $**$
A thin rod 10 cm long carries a total charge of 24 esu $= 8 \cdot 10^{-9}$ C uniformly distributed along its length. Find the strength of the electric field at each of the two points A and B located as shown in Fig. 1.45.

1.56 *Flux through a cube* $*$
(a) A point charge q is located at the center of a cube of edge d. What is the value of $\int \mathbf{E} \cdot d\mathbf{a}$ over one face of the cube?
(b) The charge q is moved to one corner of the cube. Now what is the value of the flux of $\mathbf{E}$ through each of the faces of the cube? (To make things well defined, treat the charge like a tiny sphere.)

1.57 *Escaping field lines* $**$
Charges $2q$ and $-q$ are located on the x axis at $x = 0$ and $x = a$, respectively.
(a) Find the point on the x axis where the electric field is zero, and make a rough sketch of some field lines.
(b) You should find that some of the field lines that start on the $2q$ charge end up on the $-q$ charge, while others head off to infinity. Consider the field lines that form the cutoff between these two cases. At what angle (with respect to the x axis) do

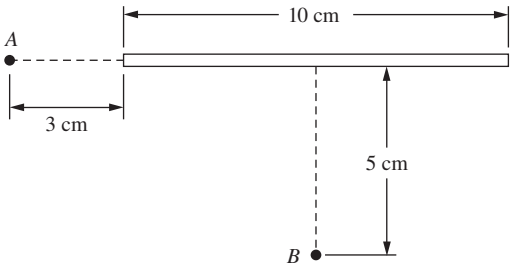


Figure 1.45.

these lines leave the $2q$ charge? *Hint:* Draw a wisely chosen Gaussian surface that mainly follows these lines.

1.58 *Gauss's law at the center of a ring* $**$

(a) A ring with radius R has total charge Q uniformly distributed around it. To leading order, find the electric field at a point along the axis of the ring, a very small distance z from the center.

(b) Consider a small cylinder centered at the center of the ring, with small radius r_0 and small height $2z_0$, with z_0 lying on either side of the plane of the ring. There is no charge in this cylinder, so the net flux through it must be zero. Using a result given in the solution to Problem 1.8, verify that this is indeed the case (to leading order in the small distances involved).

1.59 *Zero field inside a cylindrical shell* $*$

Consider a distribution of charge in the form of a hollow circular cylinder, like a long charged pipe. In the spirit of Problem 1.17, show that the electric field inside the pipe is zero.

1.60 *Field from a hollow cylinder* $*$

Consider the hollow cylinder from Exercise 1.59. Use Gauss's law to show that the field inside the pipe is zero. Also show that the field outside is the same as if the charge were all on the axis. Is either statement true for a pipe of square cross section on which the charge is distributed with uniform surface density?

1.61 *Potential energy of a sphere* $**$

A spherical volume of radius R is filled with charge of uniform density ρ. We want to know the potential energy U of this sphere of charge, that is, the work done in assembling it. In the example in Section 1.15, we calculated U by integrating the energy density of the electric field; the result was $U = (3/5)Q^2/4\pi\epsilon_0 R$. Derive U here by building up the sphere layer by layer, making use of the fact that the field outside a spherical distribution of charge is the same as if all the charge were at the center.

1.62 *Electron self-energy* $*$

At the beginning of the twentieth century the idea that the rest mass of the electron might have a purely electrical origin was very attractive, especially when the equivalence of energy and mass was revealed by special relativity. Imagine the electron as a ball of charge, of constant volume density out to some maximum radius r_0. Using the result of Exercise 1.61, set the potential energy of this system equal to mc^2 and see what you get for r_0. One defect of the model is rather obvious: nothing is provided to hold the charge together!

1.63 *Sphere and cones* $**$

(a) Consider a fixed hollow spherical shell with radius R and surface charge density σ. A particle with mass m and charge $-q$ that is initially at rest falls in from infinity. What is its speed when it reaches the center of the shell? (Assume that a tiny hole has been cut in the shell, to let the charge through.)

(b) Consider two fixed hollow conical shells (that is, ice cream cones without the ice cream) with base radius R, slant height L, and surface charge density σ, arranged as shown in Fig. 1.46. A particle with mass m and charge $-q$ that is initially at rest falls in from infinity, along the perpendicular bisector line, as shown. What is its speed when it reaches the tip of the cones? You should find that your answer relates very nicely to your answer for part (a).

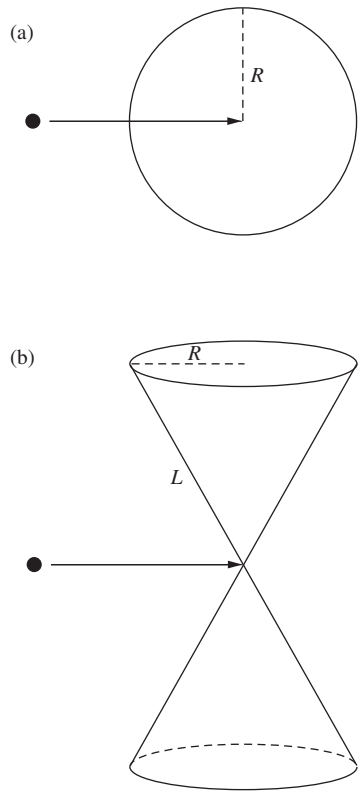


Figure 1.46.

1.64 *Field between two wires* $*$

Consider a high-voltage direct current power line that consists of two parallel conductors suspended 3 meters apart. The lines are oppositely charged. If the electric field strength halfway between them is 15,000 N/C, how much excess positive charge resides on a 1 km length of the positive conductor?

1.65 *Building a sheet from rods* $**$

An infinite uniform sheet of charge can be thought of as consisting of an infinite number of adjacent uniformly charged rods. Using the fact that the electric field from an infinite rod is $\lambda/2\pi\epsilon_0 r$, integrate over these rods to show that the field from an infinite sheet with charge density σ is $\sigma/2\epsilon_0$.

1.66 *Force between two strips* $**$

(a) The two strips of charge shown in Fig. 1.47 have width b, infinite height, and negligible thickness (in the direction perpendicular to the page). Their charge densities per unit area are $\pm\sigma$. Find the magnitude of the electric field due to one of the strips, a distance x away from it (in the plane of the page).

(b) Show that the force (per unit height) between the two strips equals $\sigma^2 b(\ln 2)/\pi\epsilon_0$. Note that this result is finite, even though you will find that the field due to a strip diverges as you get close to it.

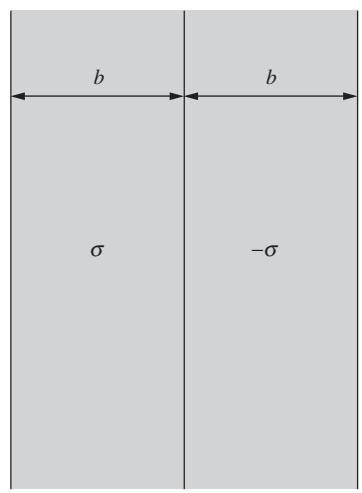


Figure 1.47.

1.67 *Field from a cylindrical shell, right and wrong* $**$

Find the electric field outside a uniformly charged hollow cylindrical shell with radius R and charge density σ, an infinitesimal distance away from it. Do this in the following way.

(a) Slice the shell into parallel infinite rods, and integrate the field contributions from all the rods. You should obtain the incorrect result of $\sigma/2\epsilon_0$.

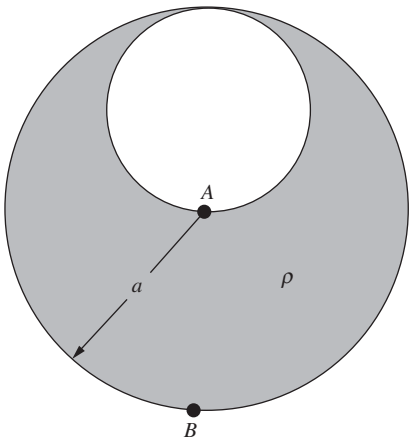


Figure 1.48.

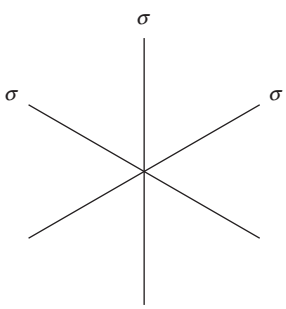


Figure 1.49.

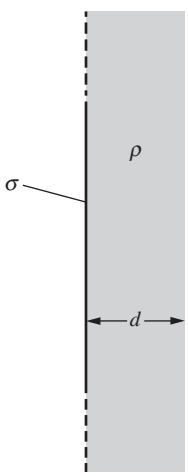


Figure 1.50.

(b) Why isn't the result correct? Explain how to modify it to obtain the correct result of σ/ϵ_0. *Hint:* You could very well have performed the above integral in an effort to obtain the electric field an infinitesimal distance *inside* the cylinder, where we know the field is zero. Does the above integration provide a good description of what's going on for points on the shell that are very close to the point in question?

1.68 *Uniform field strength* $*$
We know from the example in Section 1.11 that the electric field inside a solid sphere with uniform charge density is proportional to r. Assume instead that the charge density is not uniform, but depends only on r. What should this dependence be so that the magnitude of the field at points inside the sphere is independent of r (except right at the center, where it isn't well defined)? What should the dependence be in the analogous case where we have a cylinder instead of a sphere?

1.69 *Carved-out sphere* $**$
A sphere of radius a is filled with positive charge with uniform density ρ. Then a smaller sphere of radius $a/2$ is carved out, as shown in Fig. 1.48, and left empty. What are the direction and magnitude of the electric field at A? At B?

1.70 *Field from two sheets* $*$
Two infinite plane sheets of surface charge, with densities $3\sigma_0$ and $-2\sigma_0$, are located a distance ℓ apart, parallel to one another. Discuss the electric field of this system. Now suppose the two planes, instead of being parallel, intersect at right angles. Show what the field is like in each of the four regions into which space is thereby divided.

1.71 *Intersecting sheets* $**$
(a) Figure 1.49 shows the cross section of three infinite sheets intersecting at equal angles. The sheets all have surface charge density σ. By adding up the fields from the sheets, find the electric field at all points in space.
(b) Find the field instead by using Gauss's law. You should explain clearly why Gauss's law is in fact useful in this setup.
(c) What is the field in the analogous setup where there are N sheets instead of three? What is your answer in the $N \to \infty$ limit? This limit is related to the cylinder in Exercise 1.68.

1.72 *A plane and a slab* $**$
An infinite plane has uniform surface charge density σ. Adjacent to it is an infinite parallel layer of charge of thickness d and uniform volume charge density ρ, as shown in Fig. 1.50. All charges are fixed. Find $\mathbf{E}$ everywhere.

1.73 *Sphere in a cylinder* $**$

An infinite cylinder with uniform volume charge density ρ has its axis lying along the z axis. A sphere is carved out of the cylinder and then filled up with a material with uniform density $-\rho/2$. Assume that the center of the sphere is located on the x axis at position $x = a$. Show that inside the sphere the component of the field in the xy plane is uniform, and find its value. *Hint:* The technique used in Problem 1.27 will be helpful.

1.74 *Zero field in a sphere* $**$

In Fig. 1.51 a sphere with radius R is centered at the origin, an infinite cylinder with radius R has its axis along the z axis, and an infinite slab with thickness $2R$ lies between the planes $z = -R$ and $z = R$. The uniform volume densities of these objects are ρ_1, ρ_2, and ρ_3, respectively. The objects are superposed on top of each other; the densities add where the objects overlap. How should the three densities be related so that the electric field is zero everywhere throughout the volume of the sphere? *Hint:* Find a vector expression for the field inside each object, and then use superposition.

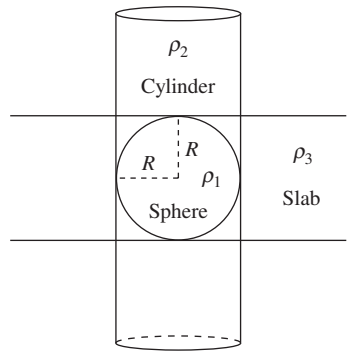


Figure 1.51.

1.75 *Ball in a sphere* $**$

We know that if a point charge q is located at radius a in the interior of a sphere with radius R and uniform volume charge density ρ, then the force on the point charge is effectively due only to the charge that is located inside radius a.

(a) Consider instead a uniform ball of charge located entirely inside a larger sphere of radius R. Let the ball's radius be b, and let its center be located at radius a in the larger sphere. Its volume charge density is such that its total charge is q. Assume that the ball is superposed on top of the sphere, so that all of the sphere's charge is still present. Can the force on the ball be obtained by treating it like a point charge and considering only the charge in the larger sphere that is inside radius a?

(b) Would the force change if we instead remove the charge in the larger sphere where the ball is? So now we are looking at the force on the ball due to the sphere with a cavity carved out, which is a more realistic scenario.

1.76 *Hydrogen atom* $**$

The neutral hydrogen atom in its normal state behaves, in some respects, like an electric charge distribution that consists of a point charge of magnitude e surrounded by a distribution of negative charge whose density is given by $\rho(r) = -Ce^{-2r/a_0}$. Here a_0 is the *Bohr radius,* $0.53 \cdot 10^{-10}$ m, and C is a constant with the value required to make the total amount of negative charge exactly e.

What is the net electric charge inside a sphere of radius a_0? What is the electric field strength at this distance from the nucleus?

1.77 *Electron jelly* $**$
Imagine a sphere of radius a filled with negative charge of uniform density, the total charge being equivalent to that of two electrons. Imbed in this jelly of negative charge two protons, and assume that, in spite of their presence, the negative charge distribution remains uniform. Where must the protons be located so that the force on each of them is zero? (This is a surprisingly realistic caricature of a hydrogen molecule; the magic that keeps the electron cloud in the molecule from collapsing around the protons is explained by quantum mechanics!)

1.78 *Hole in a shell* $**$
Figure 1.52 shows a spherical shell of charge, of radius a and surface density σ, from which a small circular piece of radius $b \ll a$ has been removed. What is the direction and magnitude of the field at the midpoint of the aperture? There are two ways to get the answer. You can integrate over the remaining charge distribution to sum the contributions of all elements to the field at the point in question. Or, remembering the superposition principle, you can think about the effect of replacing the piece removed, which itself is practically a little disk. Note the connection of this result with our discussion of the force on a surface charge – perhaps that is a third way in which you might arrive at the answer.

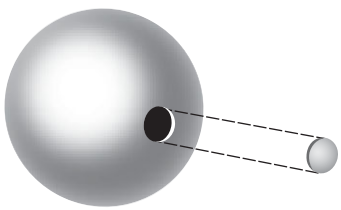


Figure 1.52.

1.79 *Forces on three sheets* $**$
Consider three charged sheets, A, B, and C. The sheets are parallel with A above B above C. On each sheet there is surface charge of uniform density: $-4 \cdot 10^{-5}\,\mathrm{C/m^2}$ on A, $7 \cdot 10^{-5}\,\mathrm{C/m^2}$ on B, and $-3 \cdot 10^{-5}\,\mathrm{C/m^2}$ on C. (The density given includes charge on both sides of the sheet.) What is the magnitude of the electrical force per unit area on each sheet? Check to see that the total force per unit area on the three sheets is zero.

1.80 *Force in a soap bubble* $**$
Like the charged rubber balloon described at the end of Section 1.14, a charged soap bubble experiences an outward electrical force on every bit of its surface. Given the total charge Q on a bubble of radius R, what is the magnitude of the resultant force tending to pull any hemispherical half of the bubble away from the other half? (Should this force divided by $2\pi R$ exceed the surface tension of the soap film, interesting behavior might be expected!)

1.81 *Energy around a sphere* $*$
A sphere of radius R has a charge Q distributed uniformly over its surface. How large a sphere contains 90 percent of the energy stored in the electrostatic field of this charge distribution?

1.82 *Energy of concentric shells* ∗

(a) Concentric spherical shells of radius a and b, with $a < b$, carry charge Q and $-Q$, respectively, each charge uniformly distributed. Find the energy stored in the electric field of this system.

(b) Calculate the stored energy in a second way: start with two neutral shells, and then gradually transfer positive charge from the outer shell to the inner shell in a spherically symmetric manner. At an intermediate stage when there is charge q on the inner shell, find the work required to transfer an additional charge dq. And then integrate over q.

1.83 *Potential energy of a cylinder* ∗∗

Problem 1.24 gives one way of calculating the energy per unit length stored in a solid cylinder with radius a and uniform volume charge density ρ. Calculate the energy here by using Eq. (1.53) to find the total energy per unit length stored in the electric field. Don't forget to include the field inside the cylinder.

You will find that the energy is infinite, so instead calculate the energy relative to the configuration where all the charge is initially distributed uniformly over a hollow cylinder with large radius R. (The field outside radius R is the same in both configurations, so it can be ignored when calculating the relative energy.) In terms of the total charge λ per unit length in the final cylinder, show that the energy per unit length can be written as $(\lambda^2/4\pi\epsilon_0)\big(1/4+\ln(R/a)\big)$.

2

The electric potential

Overview The first half of this chapter deals mainly with the *potential* associated with an electric field. The second half covers a number of mathematical topics that will be critical in our treatment of electromagnetism. The potential difference between two points is defined to be the negative *line integral* of the electric field. Equivalently, the electric field equals the negative *gradient* of the potential. Just as the electric field is the force per unit charge, the potential is the potential energy per unit charge. We give a number of examples involving the calculation of the potential due to a given charge distribution. One important example is the *dipole*, which consists of two equal and opposite charges. We will have much more to say about the applications of dipoles in Chapter 10.

Turning to mathematics, we introduce the *divergence*, which gives a measure of the flux of a vector field out of a small volume. We prove *Gauss's theorem* (or the *divergence theorem*) and then use it to write Gauss's law in differential form. The result is the first of the four equations known as *Maxwell's equations* (the subject of Chapter 9). We explicitly calculate the divergence in Cartesian coordinates. The divergence of the gradient is known as the *Laplacian* operator. Functions whose Laplacian equals zero have many important properties, one of which leads to *Earnshaw's theorem*, which states that it is impossible to construct a stable electrostatic equilibrium in empty space. We introduce the *curl*, which gives a measure of the line integral of a vector field around a small closed curve. We prove *Stokes' theorem* and explicitly calculate the curl in Cartesian coordinates. The conservative nature of a static electric

field implies that its curl is zero. See Appendix F for a discussion of the various vector operators in different coordinate systems.

2.1 Line integral of the electric field

Suppose that $\mathbf{E}$ is the field of some stationary distribution of electric charges. Let P_1 and P_2 denote two points anywhere in the field. The line integral of E between the two points is $\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{s}$, taken along some path that runs from P_1 to P_2, as shown in Fig. 2.1. This means: divide the chosen path into short segments, each segment being represented by a vector connecting its ends; take the scalar product of the path-segment vector with the field $\mathbf{E}$ at that place; add these products up for the whole path. The integral as usual is to be regarded as the limit of this sum as the segments are made shorter and more numerous without limit.

Let's consider the field of a point charge q and some paths running from point P_1 to point P_2 in that field. Two different paths are shown in Fig. 2.2. It is easy to compute the line integral of $\mathbf{E}$ along path A, which is made up of a radial segment running outward from P_1 and an arc of

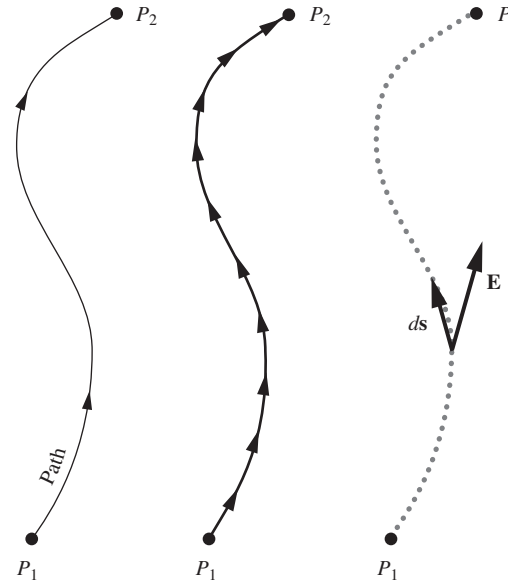


Figure 2.1.
Showing the division of the path into path elements $d\mathbf{s}$.

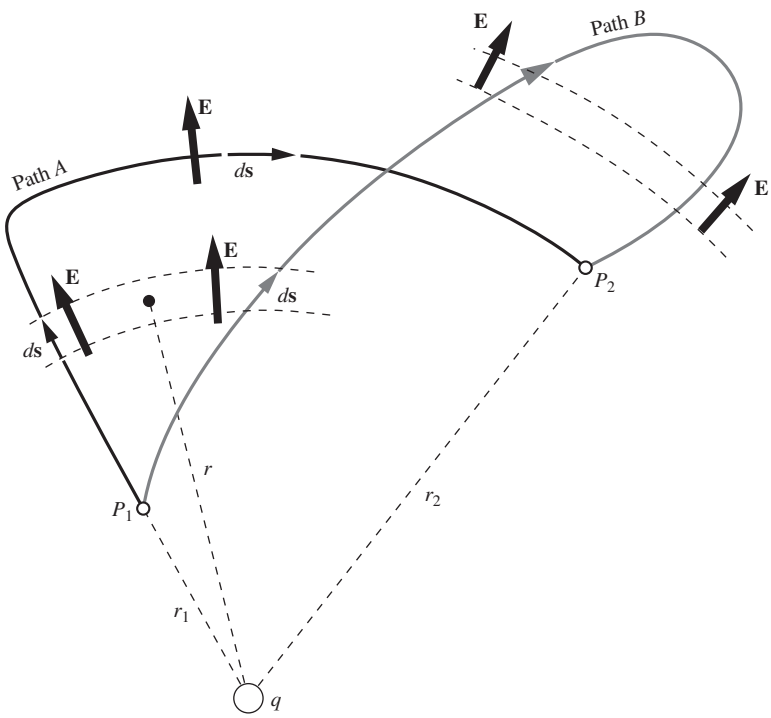


Figure 2.2.
The electric field $\mathbf{E}$ is that of a positive point charge q. The line integral of $\mathbf{E}$ from P_1 to P_2 along path A has the value $(q/4\pi\epsilon_0)(1/r_1 - 1/r_2)$. It will have exactly the same value if calculated for path B, or for any other path from P_1 to P_2.

radius r_2. Along the radial segment of path A, $\mathbf{E}$ and $d\mathbf{s}$ are parallel, the magnitude of $\mathbf{E}$ is $q/4\pi\epsilon_0 r^2$, and $\mathbf{E}\cdot d\mathbf{s}$ is simply $(q/4\pi\epsilon_0 r^2)\,ds$. Thus the line integral on that segment is

$$\int_{r_1}^{r_2} \frac{q\,dr}{4\pi\epsilon_0 r^2} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_1}-\frac{1}{r_2}\right). \tag{2.1}$$

The second leg of path A, the circular segment, gives zero because $\mathbf{E}$ is perpendicular to $d\mathbf{s}$ everywhere on that arc. The entire line integral is therefore

$$\int_{P_1}^{P_2} \mathbf{E}\cdot d\mathbf{s} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_1}-\frac{1}{r_2}\right). \tag{2.2}$$

Now look at path B. Because $\mathbf{E}$ is radial with magnitude $q/4\pi\epsilon_0 r^2$, $\mathbf{E}\cdot d\mathbf{s} = (q/4\pi\epsilon_0 r^2)\,dr$ even when $d\mathbf{s}$ is not radially oriented. The corresponding pieces of path A and path B indicated in the diagram make identical contributions to the integral. The part of path B that loops beyond r_2 makes a net contribution of zero; contributions from corresponding outgoing and incoming parts cancel. For the entire line integral, path B will give the same result as path A. As there is nothing special about path B, Eq. (2.1) must hold for *any* path running from P_1 to P_2.

Here we have essentially repeated, in different language, the argument in Section 1.5, illustrated in Fig. 1.5, concerning the work done in moving one point charge near another. But now we are interested in the total electric field produced by any distribution of charges. One more step will bring us to an important conclusion. The line integral of the sum of fields equals the sum of the line integrals of the fields calculated separately. Or, stated more carefully, if $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \cdots$, then

$$\int_{P_1}^{P_2} \mathbf{E}\cdot d\mathbf{s} = \int_{P_1}^{P_2} \mathbf{E}_1\cdot d\mathbf{s} + \int_{P_1}^{P_2} \mathbf{E}_2\cdot d\mathbf{s} + \cdots, \tag{2.3}$$

where the same path is used for all the integrations. Now any electrostatic field can be regarded as the sum of a number (possibly enormous) of point-charge fields, as expressed in Eq. (1.20) or Eq. (1.22). Therefore if the line integral from P_1 to P_2 is independent of path for each of the point-charge fields $\mathbf{E}_1, \mathbf{E}_2, \ldots$, the total field $\mathbf{E}$ must have this property:

> The line integral $\int_{P_1}^{P_2} \mathbf{E}\cdot d\mathbf{s}$ for any given electrostatic field $\mathbf{E}$ has the same value for all paths from P_1 to P_2.

The points P_2 and P_1 may coincide. In that case the paths are all closed curves, among them paths of vanishing length. This leads to the following corollary:

> The line integral $\int \mathbf{E} \cdot d\mathbf{s}$ around any closed path in an electrostatic field is zero.

By *electrostatic field* we mean, strictly speaking, the electric field of stationary charges. Later on, we shall encounter electric fields in which the line integral is *not* path-independent. Those fields will usually be associated with rapidly moving charges. For our present purposes we can say that, if the source charges are moving slowly enough, the field $\mathbf{E}$ will be such that $\int \mathbf{E} \cdot d\mathbf{s}$ is practically path-independent. Of course, if $\mathbf{E}$ itself is varying in time, the $\mathbf{E}$ in $\int \mathbf{E} \cdot d\mathbf{s}$ must be understood as the field that exists over the whole path at a given instant of time. With that understanding we can talk meaningfully about the line integral in a changing electrostatic field.

2.2 Potential difference and the potential function

Because the line integral in the electrostatic field is path-independent, we can use it to define a scalar quantity ϕ_{21}, without specifying any particular path:

$$\phi_{21} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{s}. \tag{2.4}$$

With the minus sign included here, ϕ_{21} is the work per unit charge done by an external agency in moving a positive charge from P_1 to P_2 in the field $\mathbf{E}$. (The external agency must supply a force $\mathbf{F}_{\text{ext}} = -q\mathbf{E}$ to balance the electrical force $\mathbf{F}_{\text{elec}} = q\mathbf{E}$; hence the minus sign.) Thus ϕ_{21} is a single-valued scalar function of the two positions P_1 and P_2. We call it the *electric potential difference* between the two points.

In our SI system of units, potential difference is measured in joule/coulomb. This unit has a name of its own, the *volt:*

$$1 \text{ volt} = 1 \frac{\text{joule}}{\text{coulomb}}. \tag{2.5}$$

One joule of work is required to move a charge of one coulomb through a potential difference of one volt. In the Gaussian system of units, potential difference is measured in erg/esu. This unit also has a name of its own, the statvolt ("stat" comes from "electrostatic"). As an exercise, you can use the $1\,\text{C} \approx 3 \cdot 10^9$ esu relation from Section 1.4 to show that one volt is equivalent to approximately 1/300 statvolt. These two relations are accurate to better than 0.1 percent, thanks to the accident that c is that

close to $3 \cdot 10^8$ m/s. Appendix C derives the conversion factors between all of the corresponding units in the SI and Gaussian systems. Further discussion of the exact relations between SI and Gaussian electrical units is given in Appendix E, which takes into account the definition of the meter in terms of the speed of light.

Suppose we hold P_1 fixed at some reference position. Then ϕ_{21} becomes a function of P_2 only, that is, a function of the spatial coordinates x, y, z. We can write it simply $\phi(x, y, z)$, without the subscript, if we remember that its definition still involves agreement on a reference point P_1. We can say that ϕ is the potential associated with the vector field $\mathbf{E}$. It is a scalar function of position, or a scalar field (they mean the same thing). Its value at a point is simply a number (in units of work per unit charge) and has no direction associated with it. Once the vector field $\mathbf{E}$ is given, the potential function ϕ is determined, except for an arbitrary additive constant allowed by the arbitrariness in our choice of P_1.

Example Find the potential associated with the electric field described in Fig. 2.3, the components of which are $E_x = Ky$, $E_y = Kx$, $E_z = 0$, with K a constant. This is a possible electrostatic field; we will see why in Section 2.17. Some field lines are shown.

Solution Since $E_z = 0$, the potential will be independent of z and we need consider only the xy plane. Let x_1, y_1 be the coordinates of P_1, and x_2, y_2 the coordinates of P_2. It is convenient to locate P_1 at the origin: $x_1 = 0$, $y_1 = 0$. To evaluate $-\int \mathbf{E} \cdot d\mathbf{s}$ from this reference point to a general point (x_2, y_2) it is easiest to use a path like the dashed path ABC in Fig. 2.3:

$$\phi(x_2, y_2) = -\int_{(0,0)}^{(x_2,y_2)} \mathbf{E} \cdot d\mathbf{s} = -\int_{(0,0)}^{(x_2,0)} E_x\, dx - \int_{(x_2,0)}^{(x_2,y_2)} E_y\, dy. \qquad (2.6)$$

The first of the two integrals on the right is zero because E_x is zero along the x axis. The second integration is carried out at constant x, with $E_y = Kx_2$:

$$-\int_{(x_2,0)}^{(x_2,y_2)} E_y\, dy = -\int_0^{y_2} Kx_2\, dy = -Kx_2y_2. \qquad (2.7)$$

There was nothing special about the point (x_2, y_2) so we can drop the subscripts:

$$\phi(x, y) = -Kxy \qquad (2.8)$$

for any point (x, y) in this field, with zero potential at the origin. Any constant could be added to this. That would only mean that the reference point to which zero potential is assigned had been located somewhere else.

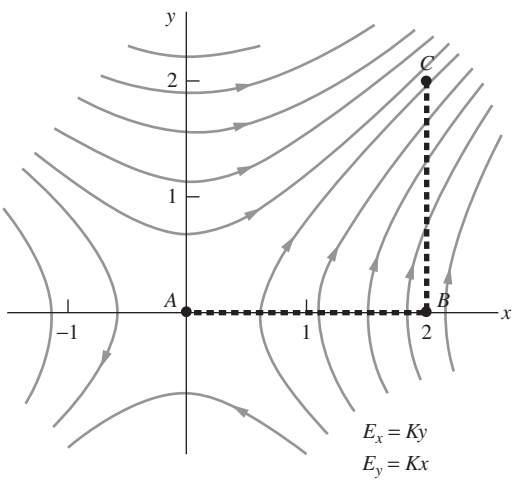


Figure 2.3.
A particular path, ABC, in the electric field $E_x = Ky$, $E_y = Kx$. Some field lines are shown.

Example (Potential due to a uniform sphere) A sphere has radius R and uniform volume charge density ρ. Use the results from the example in Section 1.11 to find the potential for all values of r, both inside and outside the sphere. Take the reference point P_1 to be infinitely far away.

Solution From the example in Section 1.11, the magnitude of the (radial) electric field inside the sphere is $E(r) = \rho r/3\epsilon_0$, and the magnitude outside is $E(r) = \rho R^3/3\epsilon_0 r^2$. Equation (2.4) tells us that the potential equals the negative of the line integral of the field, from P_1 (which we are taking to be at infinity) down to a given radius r. The potential outside the sphere is therefore

$$\phi_{\text{out}}(r) = -\int_\infty^r E(r')\,dr' = -\int_\infty^r \frac{\rho R^3}{3\epsilon_0 r'^2}\,dr' = \frac{\rho R^3}{3\epsilon_0 r}. \tag{2.9}$$

In terms of the total charge in the sphere, $Q = (4\pi R^3/3)\rho$, this potential is simply $\phi_{\text{out}}(r) = Q/4\pi\epsilon_0 r$. This is as expected, because we already knew that the potential *energy* of a charge q due to the sphere is $qQ/4\pi\epsilon_0 r$. And the potential ϕ equals the potential energy per unit charge.

To find the potential inside the sphere, we must break the integral into two pieces:

$$\begin{aligned}\phi_{\text{in}}(r) &= -\int_\infty^R E(r')\,dr' - \int_R^r E(r')\,dr' = -\int_\infty^R \frac{\rho R^3}{3\epsilon_0 r'^2}\,dr' - \int_R^r \frac{\rho r'}{3\epsilon_0}\,dr' \\ &= \frac{\rho R^3}{3\epsilon_0 R} - \frac{\rho}{6\epsilon_0}(r^2 - R^2) = \frac{\rho R^2}{2\epsilon_0} - \frac{\rho r^2}{6\epsilon_0}.\end{aligned} \tag{2.10}$$

Note that Eqs. (2.9) and (2.10) yield the same value of ϕ at the surface of the sphere, namely $\phi(R) = \rho R^2/3\epsilon_0$. So ϕ is continuous across the surface, as it should be. (The field is everywhere finite, so the line integral over an infinitesimal interval must yield an infinitesimal result.) The slope of ϕ is also continuous, because $E(r)$ (which is the negative derivative of ϕ, because ϕ is the negative integral of E) is continuous. A plot of $\phi(r)$ is shown in Fig. 2.4.

The potential at the center of the sphere is $\phi(0) = \rho R^2/2\epsilon_0$, which is $3/2$ times the value at the surface. So if you bring a charge in from infinity, it takes $2/3$ of your work to reach the surface, and then $1/3$ to go the extra distance of R to the center.

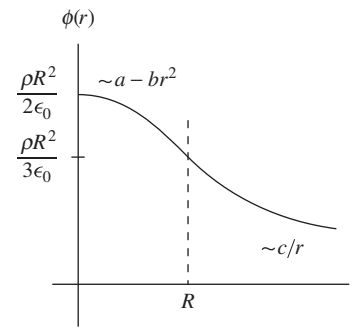


Figure 2.4.
The potential due to a uniform sphere of charge.

We must be careful not to confuse the potential ϕ associated with a given field $\mathbf{E}$ with the potential energy of a system of charges. The potential energy of a system of charges is the total work required to assemble it, starting with all the charges far apart. In Eq. (1.14), for example, we expressed U, the potential energy of the charge system in Fig. 1.6. The electric potential $\phi(x, y, z)$ associated with the field in Fig. 1.6 would be the work per unit charge required to move a unit positive test charge from some chosen reference point to the point (x, y, z) in the field of that structure of nine charges.

2.3 Gradient of a scalar function

Given the electric field, we can find the electric potential function. But we can also proceed in the other direction; from the potential we can derive the field. It appears from Eq. (2.4) that the field is in some sense the *derivative* of the potential function. To make this idea precise we introduce the *gradient* of a scalar function of position. Let $f(x, y, z)$ be

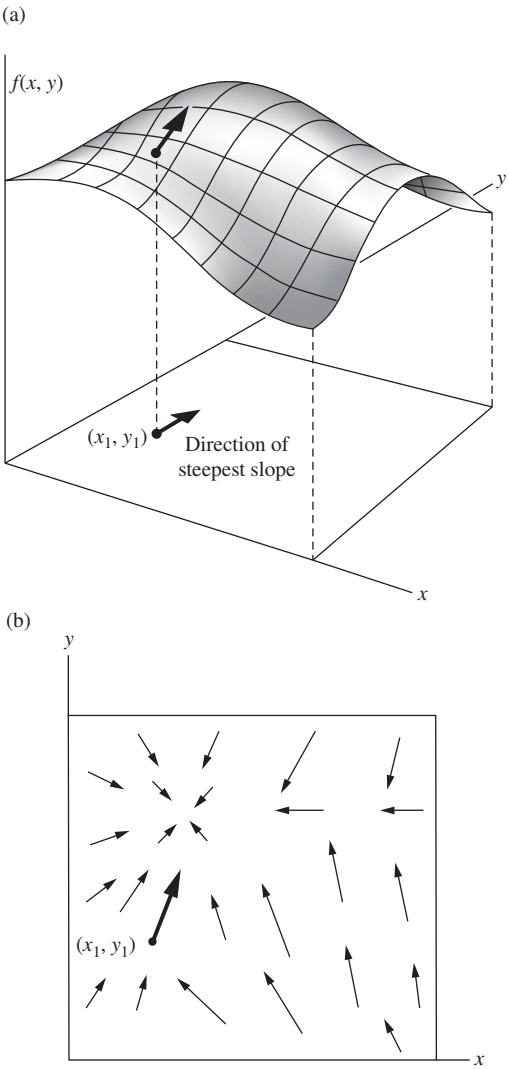


Figure 2.5.
The scalar function $f(x, y)$ is represented by the surface in (a). The arrows in (b) represent the vector function, grad f.

some continuous, differentiable function of the coordinates. With its partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ we can construct at every point in space a vector, the vector whose x, y, z components are equal to the respective partial derivatives.[1] This vector we call the *gradient* of f, written "grad f," or ∇f:

$$\nabla f \equiv \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}. \tag{2.13}$$

∇f is a vector that tells how the function f varies in the neighborhood of a point. Its x component is the partial derivative of f with respect to x, a measure of the rate of change of f as we move in the x direction. The direction of the vector ∇f at any point is the direction in which one must move from that point to find the most rapid increase in the function f. Suppose we were dealing with a function of two variables only, x and y, so that the function could be represented by a surface in three dimensions. Standing on that surface at some point, we see the surface rising in some direction, sloping downward in the opposite direction. There is a direction in which a short step will take us higher than a step of the same length in any other direction. The gradient of the function is a vector in that direction of steepest ascent, and its magnitude is the slope measured in that direction.

Figure 2.5 may help you to visualize this. Suppose some particular function of two coordinates x and y is represented by the surface $f(x, y)$ sketched in Fig. 2.5(a). At the location (x_1, y_1) the surface rises most steeply in a direction that makes an angle of about 80° with the positive x direction. The gradient of $f(x, y)$, ∇f, is a vector function of x and y. Its character is suggested in Fig. 2.5(b) by a number of vectors at various points in the two-dimensional space, including the point (x_1, y_1). The vector function ∇f defined in Eq. (2.13) is simply an extension of this idea to three-dimensional space. (Be careful not to confuse Fig. 2.5(a) with real three-dimensional xyz space; the third coordinate there is the value of the function $f(x, y)$.)

As one example of a function in three-dimensional space, suppose f is a function of r only, where r is the distance from some fixed point O. On a sphere of radius r_0 centered about O, $f = f(r_0)$ is constant. On a slightly larger sphere of radius $r_0 + dr$ it is also constant, with the value $f = f(r_0 + dr)$. If we want to make the change from $f(r_0)$ to $f(r_0 + dr)$,

[1] We remind the reader that a partial derivative with respect to x, of a function of x, y, z, written simply $\partial f/\partial x$, means the rate of change of the function with respect to x with the other variables y and z held constant. More precisely,

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}. \tag{2.11}$$

As an example, if $f = x^2yz^3$,

$$\frac{\partial f}{\partial x} = 2xyz^3, \quad \frac{\partial f}{\partial y} = x^2z^3, \quad \frac{\partial f}{\partial z} = 3x^2yz^2. \tag{2.12}$$

the shortest step we can make is to go radially (as from A to B) rather than from A to C, in Fig. 2.6. The "slope" of f is thus greatest in the radial direction, so ∇f at any point is a radially pointing vector. In fact $\nabla f = \hat{\mathbf{r}}(df/dr)$ in this case, $\hat{\mathbf{r}}$ denoting, for any point, a unit vector in the radial direction. See Section F.2 in Appendix F for further discussion of the gradient.

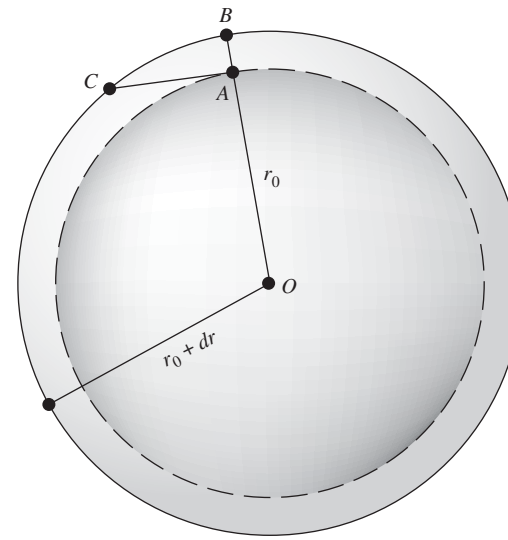


Figure 2.6.
The shortest step for a given change in f is the radial step AB, if f is a function of r only.

2.4 Derivation of the field from the potential

It is now easy to see that the relation of the scalar function f to the vector function ∇f is the same, except for a minus sign, as the relation of the potential ϕ to the field $\mathbf{E}$. Consider the value of ϕ at two nearby points, (x, y, z) and $(x + dx, y + dy, z + dz)$. The change in ϕ, going from the first point to the second, is, in first-order approximation,

$$d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz. \tag{2.14}$$

On the other hand, from the definition of ϕ in Eq. (2.4), the change can also be expressed as

$$d\phi = -\mathbf{E} \cdot d\mathbf{s}. \tag{2.15}$$

The infinitesimal vector displacement $d\mathbf{s}$ is just $\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$. Thus if we identify $\mathbf{E}$ with $-\nabla\phi$, where $\nabla\phi$ is defined via Eq. (2.13), then Eqs. (2.14) and (2.15) become identical. So the electric field is the negative of the gradient of the potential:

$$\boxed{\mathbf{E} = -\nabla\phi} \tag{2.16}$$

The minus sign came in because the electric field points from a region of greater potential toward a region of lesser potential, whereas the vector $\nabla\phi$ is defined so that it points in the direction of increasing ϕ.

To show how this works, we go back to the example of the field in Fig. 2.3. From the potential given by Eq. (2.8), $\phi = -Kxy$, we can recover the electric field we started with:

$$\mathbf{E} = -\nabla(-Kxy) = -\left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y}\right)(-Kxy) = K(\hat{\mathbf{x}}y + \hat{\mathbf{y}}x). \tag{2.17}$$

2.5 Potential of a charge distribution

We already know the potential that goes with a single point charge, because we calculated the work required to bring one charge into the neighborhood of another in Eq. (1.9). The potential at any point, in the field of an isolated point charge q, is just $q/4\pi\epsilon_0 r$, where r is the distance

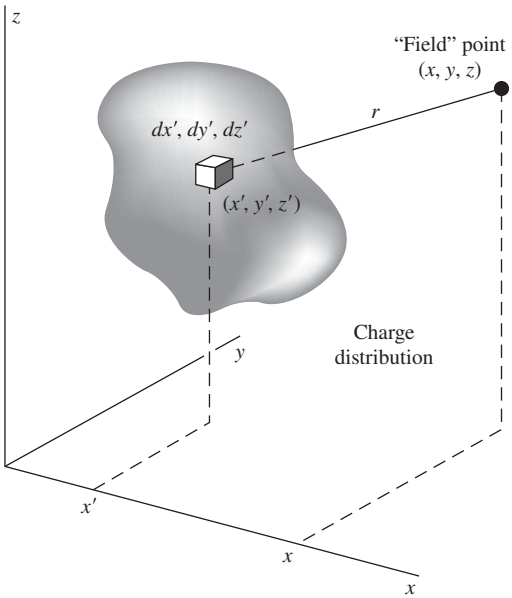


Figure 2.7.
Each element of the charge distribution $\rho(x', y', z')$ contributes to the potential ϕ at the point (x, y, z). The potential at this point is the sum of all such contributions; see Eq. (2.18).

from the point in question to the source q, and where we have assigned zero potential to points infinitely far from the source.

Superposition must work for potentials as well as fields. If we have several sources, the potential function is simply the sum of the potential functions that we would have for each of the sources present alone – *providing* we make a consistent assignment of the zero of potential in each case. If all the sources are contained in some finite region, it is always possible, and usually the simplest choice, to put zero potential at infinite distance. If we adopt this rule, the potential of any charge distribution can be specified by the integral

$$\phi(x, y, z) = \int_{\substack{\text{all} \\ \text{sources}}} \frac{\rho(x', y', z')\, dx'\, dy'\, dz'}{4\pi\epsilon_0 r}, \tag{2.18}$$

where r is the distance from the volume element $dx'\, dy'\, dz'$ to the point (x, y, z) at which the potential is being evaluated (Fig. 2.7). That is, $r = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$. Notice the difference between this and the integral giving the electric field of a charge distribution; see Eq. (1.22). Here we have r in the denominator, not r^2, and the integral is a scalar not a vector. From the scalar potential function $\phi(x, y, z)$ we can always find the electric field by taking the negative gradient of ϕ, according to Eq. (2.16).

In the case of a discrete distribution of source charges, the above integral is replaced by a sum over all the charges, indexed by i:

$$\phi(x, y, z) = \sum_{\text{all sources}} \frac{q_i}{4\pi\epsilon_0 r}, \tag{2.19}$$

where r is the distance from the charge q_i to the point (x, y, z).

Example (Potential of two point charges) Consider a very simple example, the potential of the two point charges shown in Fig. 2.8. A positive charge of 12 μC is located 3 m away from a negative charge, −6 μC. (The "μ" prefix stands for "micro," or 10^{-6}.) The potential at any point in space is the sum of the potentials due to each charge alone. The potentials for some selected points in space are given in the diagram. No vector addition is involved here, only the algebraic addition of scalar quantities. For instance, at the point on the far right, which is 6 m from the positive charge and 5 m from the negative charge, the potential has the value

$$\frac{1}{4\pi\epsilon_0}\left(\frac{12\cdot 10^{-6}\,\text{C}}{6\,\text{m}} + \frac{-6\cdot 10^{-6}\,\text{C}}{5\,\text{m}}\right) = \frac{0.8\cdot 10^{-6}\,\text{C/m}}{4\pi\epsilon_0}$$
$$= 7.2\cdot 10^{3}\,\text{J/C} = 7.2\cdot 10^{3}\,\text{V}, \tag{2.20}$$

where we have used $1/4\pi\epsilon_0 \approx 9\cdot 10^{9}\,\text{N m}^2/\text{C}^2$ (and also $1\,\text{N m} = 1\,\text{J}$). The potential approaches zero at infinite distance. It would take $7.2\cdot 10^{3}$ J of work

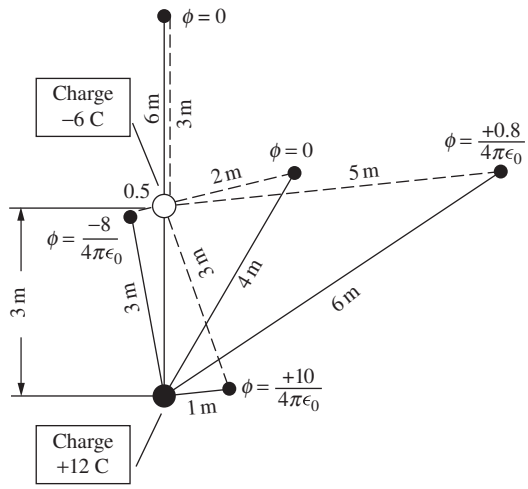


Figure 2.8.
The electric potential ϕ at various points in a system of two point charges. ϕ goes to zero at infinite distance and is given in units of volts, or joules per coulomb.

to bring a unit positive charge in from infinity to a point where $\phi = 7.2 \cdot 10^3$ V. Note that two of the points shown on the diagram have $\phi = 0$. The net work done in bringing in any charge to one of these points would be zero. You can see that there must be an infinite number of such points, forming a surface in space surrounding the negative charge. In fact, the locus of points with any particular value of ϕ is a surface – an *equipotential surface* – which would show on our two-dimensional diagram as a curve.

There is one restriction on the use of Eq. (2.18): it may not work unless all sources are confined to some finite region of space. A simple example of the difficulty that arises with charges distributed out to infinite distance is found in the long charged wire whose field **E** we studied in Section 1.12. If we attempt to carry out the integration over the charge distribution indicated in Eq. (2.18), we find that the integral diverges – we get an infinite result. No such difficulty arose in finding the electric *field* of the infinitely long wire, because the contributions of elements of the line charge to the field decrease so rapidly with distance. Evidently we had better locate the zero of potential somewhere close to home, in a system that has charges distributed out to infinity. Then it is simply a matter of calculating the difference in potential ϕ_{21}, between the general point (x, y, z) and the selected reference point, using the fundamental relation, Eq. (2.4).

Example (Potential of a long charged wire) To see how this goes in the case of the infinitely long charged wire, let us arbitrarily locate the reference point P_1 at a distance r_1 from the wire. Then to carry a charge from P_1 to

any other point P_2 at distance r_2 requires the work per unit charge, using Eq. (1.39):

$$\phi_{21} = -\int_{P_1}^{P_2} \mathbf{E}\cdot d\mathbf{s} = -\int_{r_1}^{r_2}\left(\frac{\lambda}{2\pi\epsilon_0 r}\right)dr$$
$$= -\frac{\lambda}{2\pi\epsilon_0}\ln r_2 + \frac{\lambda}{2\pi\epsilon_0}\ln r_1. \tag{2.21}$$

This shows that the electric potential for the charged wire can be taken as

$$\phi = -\frac{\lambda}{2\pi\epsilon_0}\ln r + \text{constant}. \tag{2.22}$$

The constant, $(\lambda/2\pi\epsilon_0)\ln r_1$ in this case, has no effect when we take $-\text{grad}\,\phi$ to get back to the field $\mathbf{E}$. In this case,

$$\mathbf{E} = -\nabla\phi = -\hat{\mathbf{r}}\frac{d\phi}{dr} = \frac{\lambda\hat{\mathbf{r}}}{2\pi\epsilon_0 r}. \tag{2.23}$$

2.6 Uniformly charged disk

Let us now study the electric potential and field around a uniformly charged disk. This is a charge distribution like that discussed in Section 1.13, except that it has a limited extent. The flat disk of radius a in Fig. 2.9 carries a positive charge spread over its surface with the constant density σ, in C/m^2. (This is a single sheet of charge of infinitesimal thickness, not two layers of charge, one on each side. That is, the total charge in the system is $\pi a^2\sigma$.) We shall often meet surface charge distributions in the future, especially on metallic conductors. However, the object just described is *not* a conductor; if it were, as we shall soon see, the charge could not remain uniformly distributed but would redistribute itself, crowding more toward the rim of the disk. What we have is an insulating disk, like a sheet of plastic, upon which charge has been "sprayed" so that every square meter of the disk has received, and holds fixed, the same amount of charge.

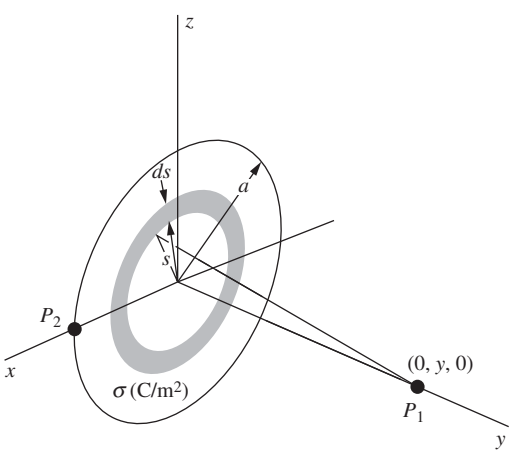


Figure 2.9.
Finding the potential at a point P_1 on the axis of a uniformly charged disk.

Example (Potential on the axis) Let us find the potential due to our uniformly charged disk, at some point P_1 on the axis of symmetry, which we have made the y axis. All charge elements in a thin, ring-shaped segment of the disk lie at the same distance from P_1. If s denotes the radius of such an annular segment and ds is its width, its area is $2\pi s\,ds$. The amount of charge it contains, dq, is therefore $dq = \sigma\,2\pi s\,ds$. Since all parts of this ring are the same distance away from P_1, namely, $r = \sqrt{y^2+s^2}$, the contribution of the ring to the potential at P_1 is $dq/4\pi\epsilon_0 r = \sigma s\,ds\big/\left(2\epsilon_0\sqrt{y^2+s^2}\right)$. To get the potential due to the whole disk, we have to integrate over all such rings:

$$\phi(0,y,0) = \int\frac{dq}{4\pi\epsilon_0 r} = \int_0^a \frac{\sigma s\,ds}{2\epsilon_0\sqrt{y^2+s^2}} = \frac{\sigma}{2\epsilon_0}\sqrt{y^2+s^2}\Big|_0^a. \tag{2.24}$$

Putting in the limits, we obtain

$$\phi(0,y,0) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{y^2+a^2} - y\right) \qquad \text{for } y > 0. \tag{2.25}$$

A minor point deserves a comment. The result we have written down in Eq. (2.25) holds for all points on the *positive* y axis. It is obvious from the physical symmetry of the system (there is no difference between one face of the disk and the other) that the potential must have the same value for negative and positive y, and this is reflected in Eq. (2.24), where only y^2 appears. But in writing Eq. (2.25) we made a choice of sign in taking the square root of y^2, with the consequence that it holds only for positive y. The correct expression for $y < 0$ is obtained by the other choice of root and is given by

$$\phi(0,y,0) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{y^2+a^2} + y\right) \qquad \text{for } y < 0. \tag{2.26}$$

In view of this, we should not be surprised to find a kink in the plot of $\phi(0,y,0)$ at $y = 0$. Indeed, the function has an abrupt change of slope there, as we see in Fig. 2.10, where we have plotted as a function of y the potential on the axis. The potential at the center of the disk is

$$\phi(0,0,0) = \frac{\sigma a}{2\epsilon_0}. \tag{2.27}$$

This much work would be required to bring a unit positive charge in from infinity, by any route, and leave it sitting at the center of the disk.

The behavior of $\phi(0,y,0)$ for very large y is interesting. For $y \gg a$ we can approximate Eq. (2.25) as follows:

$$\sqrt{y^2+a^2} - y = y\left[\left(1+\frac{a^2}{y^2}\right)^{1/2} - 1\right] = y\left[1 + \frac{1}{2}\left(\frac{a^2}{y^2}\right) + \cdots - 1\right] \approx \frac{a^2}{2y}. \tag{2.28}$$

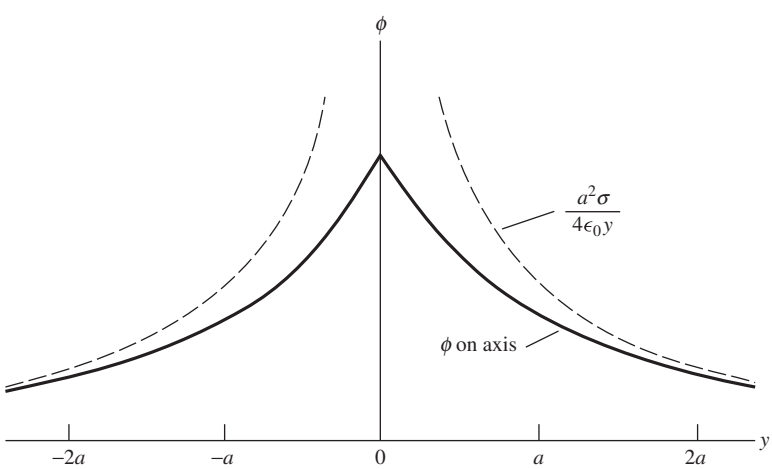


Figure 2.10.
A graph of the potential on the axis. The dashed curve is the potential of a point charge $q = \pi a^2\sigma$.

Hence

$$\phi(0, y, 0) \approx \frac{a^2\sigma}{4\epsilon_0 y} \qquad \text{for } y \gg a. \tag{2.29}$$

Now $\pi a^2\sigma$ is the total charge q on the disk, and Eq. (2.29), which can be written as $\pi a^2\sigma/4\pi\epsilon_0 y$, is just the expression for the potential due to a point charge of this magnitude. As we should expect, at a considerable distance from the disk (relative to its diameter), it doesn't matter much how the charge is shaped; only the total charge matters, in first approximation. In Fig. 2.10 we have drawn, as a dashed curve, the function $a^2\sigma/4\epsilon_0 y$. You can see that the axial potential function approaches its asymptotic form pretty quickly.

It is not quite so easy to derive the potential for general points away from the axis of symmetry, because the definite integral isn't so simple. It proves to be something called an *elliptic integral.* These functions are well known and tabulated, but there is no point in pursuing here mathematical details peculiar to a special problem. However, one further calculation, which is easy enough, may be instructive.

Example (Potential on the rim) We can find the potential at a point on the very edge of the disk, such as P_2 in Fig. 2.11. To calculate the potential at P_2 we can consider first the thin wedge of length R and angular width $d\theta$, as shown. An element of the wedge, the black patch at distance r from P_2, contains an amount of charge $dq = \sigma r\, d\theta\, dr$. Its contribution to the potential at P_2 is therefore $dq/4\pi\epsilon_0 r = \sigma\, d\theta\, dr/4\pi\epsilon_0$. The contribution of the entire wedge is then $(\sigma\, d\theta/4\pi\epsilon_0)\displaystyle\int_0^R dr = (\sigma R/4\pi\epsilon_0)\, d\theta$. Now R is $2a\cos\theta$, from the geometry of the right triangle, and the whole disk is swept out as θ ranges from $-\pi/2$ to $\pi/2$. Thus we find the potential at P_2:

$$\phi = \frac{\sigma a}{2\pi\epsilon_0}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta = \frac{\sigma a}{\pi\epsilon_0}. \tag{2.30}$$

Comparing this with the potential at the center of the disk, $\sigma a/2\epsilon_0$, we see that, as we should expect, the potential falls off from the center to the edge of the disk. The electric field, therefore, must have an *outward* component in the plane of the disk. That is why we remarked earlier that the charge, if free to move, would redistribute itself toward the rim. To put it another way, our uniformly charged disk is *not* a surface of constant potential, which any conducting surface must be unless charge is moving.[2]

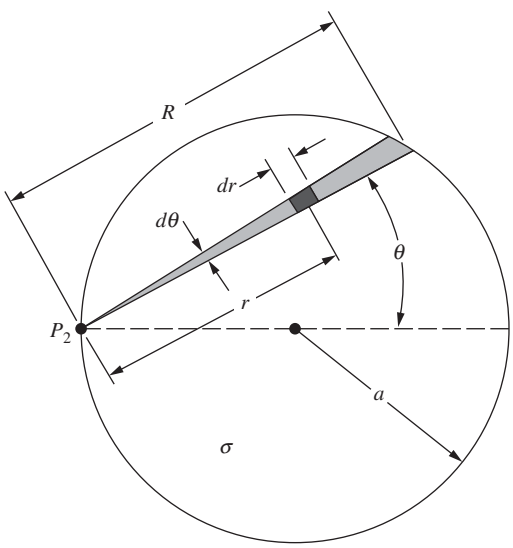


Figure 2.11.
Finding the potential at a point P_2 on the rim of a uniformly charged disk.

[2] The fact that conducting surfaces have to be equipotentials will be discussed thoroughly in Chapter 3.

Let us now examine the electric field due to the disk. For $y > 0$, the field on the symmetry axis can be computed directly from the potential function given in Eq. (2.25):

$$E_y = -\frac{\partial \phi}{\partial y} = -\frac{d}{dy}\frac{\sigma}{2\epsilon_0}\left(\sqrt{y^2+a^2} - y\right)$$
$$= \frac{\sigma}{2\epsilon_0}\left[1 - \frac{y}{\sqrt{y^2+a^2}}\right] \qquad y > 0. \tag{2.31}$$

To be sure, it is not hard to compute E_y directly from the charge distribution, for points on the axis. We can again slice the disk into concentric rings, as we did prior to Eq. (2.24). But we must remember that **E** is a vector and that only the y component survives in the present setup, whereas we did not need to worry about components when calculating the scalar function ϕ above.

As y approaches zero from the positive side, E_y approaches $\sigma/2\epsilon_0$. On the negative y side of the disk, which we shall call the back, **E** points in the other direction and its y component E_y is $-\sigma/2\epsilon_0$. This is the same as the field of an infinite sheet of charge of density σ, derived in Section 1.13. It ought to be, for at points close to the center of the disk, the presence or absence of charge out beyond the rim can't make much difference. In other words, any sheet looks infinite if viewed from close up. Indeed, E_y has the value $\sigma/2\epsilon_0$ not only at the center, but also all over the disk.

For large y, we can find an approximate expression for E_y by using a Taylor series approximation as we did in Eq. (2.28). You can show that E_y approaches $a^2\sigma/4\epsilon_0 y^2$, which can be written as $\pi a^2\sigma/4\pi\epsilon_0 y^2$. This is correctly the field due to a point charge with magnitude $\pi a^2\sigma$.

In Fig. 2.12 we show some field lines for this system and also, plotted as dashed curves, the intersections on the yz plane of the surfaces of constant potential. Near the center of the disk these are lens-like surfaces, while at distances much greater than a they approach the spherical form of equipotential surfaces around a point charge.

Figure 2.12 illustrates a general property of field lines and equipotential surfaces. A field line through any point and the equipotential surface through that point *are perpendicular to one another*, just as, on a contour map of hilly terrain, the slope is steepest at right angles to a contour of constant elevation. This must be so, because if the field at any point had a component parallel to the equipotential surface through that point, it would require work to move a test charge along a constant-potential surface.

The energy associated with this electric field could be expressed as the integral over all space of $(\epsilon_0/2)E^2\,dv$. It is equal to the work done in assembling this distribution, starting with infinitesimal charges far apart. In this particular example, as Exercise 2.56 will demonstrate, that work

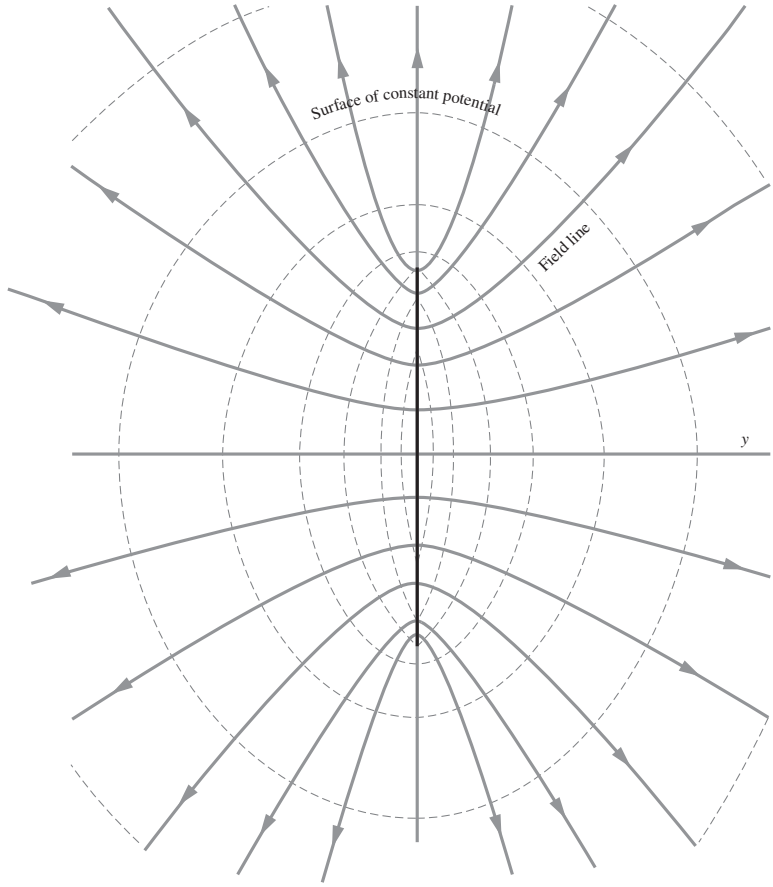


Figure 2.12.
The electric field of the uniformly charged disk. Solid curves are field lines. Dashed curves are intersections, with the plane of the figure, of surfaces of constant potential.

is not hard to calculate directly if we know the potential at the rim of a uniformly charged disk.

There is a general relation between the work U required to assemble a charge distribution $\rho(x, y, z)$ and the potential $\phi(x, y, z)$ of that distribution:

$$\boxed{U = \frac{1}{2}\int \rho\phi \, dv} \tag{2.32}$$

Equation (1.15), which gives the energy of a system of discrete point charges, could have been written in this way:

$$U = \frac{1}{2}\sum_{j=1}^{N} q_j \sum_{k\neq j} \frac{1}{4\pi\epsilon_0}\frac{q_k}{r_{jk}}. \tag{2.33}$$

The second sum is the potential at the location of the jth charge, due to all the other charges. To adapt this to a continuous distribution we merely

replace q_j with $\rho\, dv$ and the sum over j by an integral, thus obtaining Eq. (2.32).

2.7 Dipoles

Consider a setup with two equal and opposite charges $\pm q$ located at positions $\pm \ell/2$ on the y axis, as shown in Fig. 2.13. This configuration is called a *dipole*. The purpose of this section is to introduce the basics of dipoles. We save further discussion for Chapter 10, where we define the word "dipole" more precisely, derive things in more generality, and discuss examples of dipoles in actual matter. For now we just concentrate on determining the electric field and potential of a dipole. We have all of the necessary machinery at our disposal, so let's see what we can find.

We will restrict the treatment to points far away from the dipole (that is, points with $r \gg \ell$). Although it is easy enough to write down an exact expression for the potential ϕ (and hence the field $\mathbf{E} = -\nabla\phi$) at any position, the result isn't very enlightening. But when we work in the approximation of large distances, we obtain a result that, although isn't exactly correct, is in fact quite enlightening. That's how approximations work – you trade a little bit of precision for a large amount of clarity.

Our strategy will be to find the potential ϕ in polar (actually spherical) coordinates, and then take the gradient to find the electric field $\mathbf{E}$. We then determine the shape of the field-line and constant-potential curves. To make things look a little cleaner in the calculations below, we write $1/4\pi\epsilon_0$ as k in some intermediate steps.

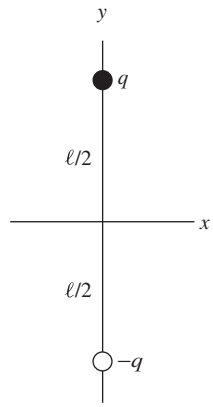


Figure 2.13.
Two equal and opposite charges form a dipole.

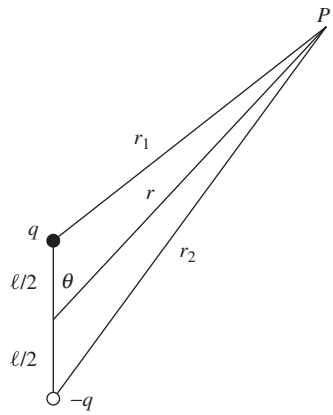


Figure 2.14.
Finding the potential ϕ at point P.

2.7.1 Calculation of ϕ and E

First note that, since the dipole setup is rotationally symmetric around the line containing the two charges, it suffices to find the potential in an arbitrary plane containing this line. We will use spherical coordinates, which reduce to polar coordinates in a plane because the angle ϕ doesn't come into play (but note that θ is measured down from the vertical axis). Consider a point P with coordinates (r, θ), as shown in Fig. 2.14. Let r_1 and r_2 be the distances from P to the two charges. Then the exact expression for the potential at P is (with $k \equiv 1/4\pi\epsilon_0$)

$$\phi_P = \frac{kq}{r_1} - \frac{kq}{r_2}. \tag{2.34}$$

If desired, the law of cosines can be used to write r_1 and r_2 in terms of r, θ, and ℓ.

Let us now derive an approximate form of this result, valid in the $r \gg \ell$ limit. One way to do this is to use the law-of-cosines expressions for r_1 and r_2; this is the route we will take in Chapter 10. But for the present purposes a simpler method suffices. In the $r \gg \ell$ limit, a closeup view of the dipole is shown in Fig. 2.15. The two lines from the charges to P are essentially parallel, so we see from the figure that the lengths of

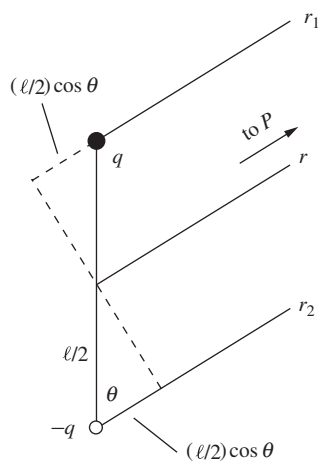


Figure 2.15.
Closeup view of Fig. 2.14.

these lines are essentially $r_1 = r - (\ell/2)\cos\theta$ and $r_2 = r + (\ell/2)\cos\theta$. Using the approximation $1/(1 \pm \epsilon) \approx 1 \mp \epsilon$, Eq. (2.34) becomes

$$\begin{aligned}\phi(r,\theta) &= \frac{kq}{r - \dfrac{\ell\cos\theta}{2}} - \frac{kq}{r + \dfrac{\ell\cos\theta}{2}} = \frac{kq}{r}\left[\frac{1}{1 - \dfrac{\ell\cos\theta}{2r}} - \frac{1}{1 + \dfrac{\ell\cos\theta}{2r}}\right] \\ &\approx \frac{kq}{r}\left[\left(1 + \frac{\ell\cos\theta}{2r}\right) - \left(1 - \frac{\ell\cos\theta}{2r}\right)\right] \\ &= \frac{kq\ell\cos\theta}{r^2} \equiv \boxed{\frac{q\ell\cos\theta}{4\pi\epsilon_0 r^2}} \equiv \frac{p\cos\theta}{4\pi\epsilon_0 r^2}, \end{aligned} \tag{2.35}$$

where $p \equiv q\ell$ is called the *dipole moment.*

There are three important things to note about this result. First, $\phi(r,\theta)$ depends on q and ℓ only through their product, $p \equiv q\ell$. This means that if we make q ten times larger and ℓ ten times smaller, the potential at a given point P stays the same (at least in the $r \gg \ell$ approximation). An *idealized dipole* or *point dipole* is one where $\ell \to 0$ and $q \to \infty$, with the product $p = q\ell$ taking on a particular finite value. In the other extreme, if we make q smaller and ℓ proportionally larger, the potential at P again stays the same. Of course, if we make ℓ too large, our $r \gg \ell$ assumption eventually breaks down.

Second, $\phi(r,\theta)$ is proportional to $1/r^2$, in contrast with the $1/r$ dependence for a point-charge potential. We will see below that the present $1/r^2$ dependence in $\phi(r,\theta)$ leads to an $\mathbf{E}$ field that falls off like $1/r^3$, in contrast with the $1/r^2$ dependence for a point-charge field. It makes sense that the potential (and field) falls off faster for a dipole, because the potentials from the two opposite point charges nearly cancel. The dipole potential is somewhat like the derivative of the point-charge potential, in that we are taking the difference of two nearby values.

Third, there is angular dependence in $\phi(r,\theta)$, in contrast with the point-charge potential. This is expected, in view of the fact that the dipole has a preferred direction along the line joining the charges, whereas a point charge has no preferred direction.

We will see in Chapter 10 that the q/r point-charge (or *monopole*) potential and the $q\ell/r^2$ dipole potential (just looking at the r dependence) are the first two pieces of what is called the *multipole expansion*. A general charge distribution also has a *quadrupole* term in the potential that goes like $q\ell^2/r^3$ (where ℓ is some length scale of the system), and an *octupole* term that goes like $q\ell^3/r^4$, and so on. These pieces have more complicated angular dependences. Two examples of quadrupole arrangements are shown in Fig. 2.16. A quadrupole is formed by placing two oppositely charged dipoles near each other, just as a dipole is formed by placing two oppositely charged monopoles near each other. The various terms in the expansion are called the *moments* of the distribution.

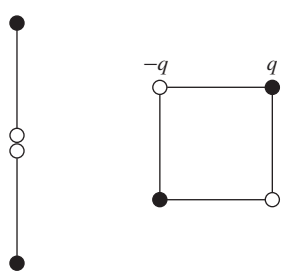


Figure 2.16.
Two possible kinds of quadrupoles.

Even the simple system of the dipole shown in Fig. 2.13 has higher terms in its multipole expansion. If you keep additional terms in the $1/(1 \pm \epsilon)$ Taylor series in Eq. (2.35), you will find that the quadrupole term is zero, but the octupole term is nonzero. It is easy to see that the terms with even powers of r are nonzero. However, in the limit of an idealized dipole ($\ell \to 0$ and $q \to \infty$, with $q\ell$ fixed), only the dipole potential survives, because the higher-order terms are suppressed by additional powers of ℓ/r.

Along the same lines, we can back up a step in the expansion and consider the monopole term. If an object has a nonzero net charge (note that our dipole does not), then the monopole potential, q/r, dominates, and all higher-order terms are comparatively negligible in the $r \gg \ell$ limit. The distribution of charge in an object determines which of the terms in the expansion is the first nonzero one, and it is this term that determines the potential (and hence field) at large distances. We label the object according to the first nonzero term; see Fig. 2.17.

Let's now find the electric field, $\mathbf{E} = -\nabla\phi$, associated with the dipole potential in Eq. (2.35). In spherical coordinates (which reduce to polar coordinates in this discussion) the gradient of ϕ is $\nabla\phi = \hat{\mathbf{r}}(\partial\phi/\partial r) + \hat{\boldsymbol{\theta}}(1/r)(\partial\phi/\partial\theta)$; see Appendix F. So the electric field is

$$\begin{aligned}
\mathbf{E}(r,\theta) &= -\hat{\mathbf{r}}\frac{\partial}{\partial r}\left(\frac{kq\ell\cos\theta}{r^2}\right) - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{kq\ell\cos\theta}{r^2}\right) \\
&= \frac{kq\ell}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) \\
&\equiv \boxed{\frac{q\ell}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)} \\
&\equiv \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right). \qquad (2.36)
\end{aligned}$$

A few field lines are shown in Fig. 2.18. Let's look at some special cases for θ. Equation (2.36) says that $\mathbf{E}$ points in the positive radial direction for $\theta = 0$ and the negative radial direction for $\theta = \pi$. These facts imply that $\mathbf{E}$ points *upward* everywhere on the y axis. Equation (2.36) also says that $\mathbf{E}$ points in the positive tangential direction for $\theta = \pi/2$ and the negative tangential direction for $\theta = 3\pi/2$. In view of the local $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ basis vectors shown in Fig. 2.18 (which vary with position, unlike the Cartesian $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ basis vectors), this means that $\mathbf{E}$ points *downward* everywhere on the x axis. We haven't drawn the lines for small r, to emphasize the fact that our results are valid only in the limit $r \gg \ell$. There *is* a field for small r, of course (and it diverges near each charge); it's just that it doesn't take the form given in Eq. (2.36).

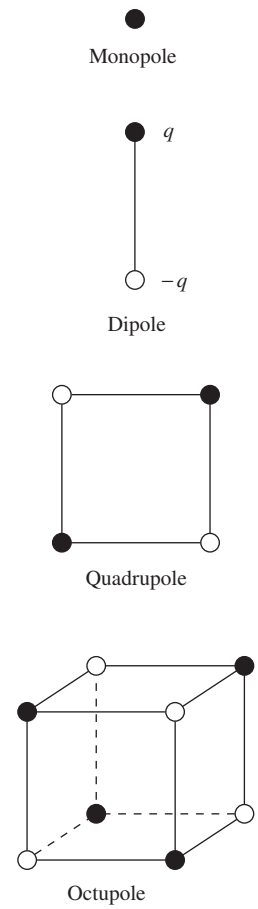


Figure 2.17.
Examples of different objects in the multipole expansion.

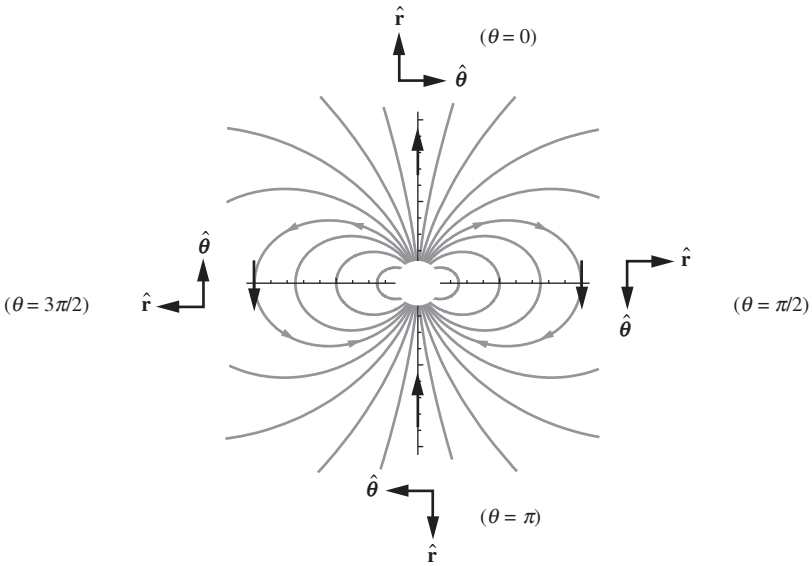


Figure 2.18.
Electric field lines for a dipole. Note that the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ basis vectors depend on position.

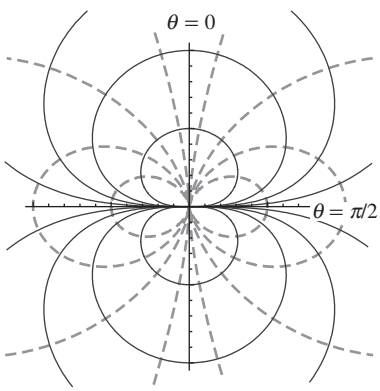


Figure 2.19.
Field lines and constant-potential curves for a dipole. The two sets of curves are orthogonal at all intersections. The solid lines show constant-ϕ curves ($r = r_0\sqrt{\cos\theta}$), and the dashed lines show $\mathbf{E}$ field lines ($r = r_0 \sin^2\theta$).

2.7.2 The shapes of the curves

Let us now be quantitative about the shape of the $\mathbf{E}$ and ϕ curves. More precisely, let us determine the equations that describe the field-line curves and the constant-potential curves. In the process we will also determine the slopes of the tangents to these curves. We know that the two classes of curves are orthogonal wherever they meet, because $\mathbf{E}$ is the (negative) gradient of ϕ, and because the gradient of a function is always perpendicular to the level-surface curves. This orthogonality is evident in Fig. 2.19. Our task now is to derive the two expressions for r given in this figure.

Let's look at ϕ first. We will find the equation for the constant-potential curves and then use this to find the slope of the tangent at any point. The equation for the curves is immediately obtained from Eq. (2.35). The set of points for which ϕ takes on the constant value ϕ_0 is given by

$$\frac{kq\ell\cos\theta}{r^2} = \phi_0 \implies r^2 = \left(\frac{kq\ell}{\phi_0}\right)\cos\theta \implies \boxed{r = r_0\sqrt{\cos\theta}} \tag{2.37}$$

where $r_0 \equiv \sqrt{kq\ell/\phi_0}$ is the radius associated with the angle $\theta = 0$. This result is valid in the upper half-plane where $-\pi/2 < \theta < \pi/2$. In the lower half-plane, both ϕ_0 and $\cos\theta$ are negative, so we need to add in some absolute-value signs. That is, $r = r_0\sqrt{|\cos\theta|}$, where $r_0 \equiv \sqrt{kq\ell/|\phi_0|}$. The constant-potential curves in Fig. 2.19 are the intersections of the constant-potential *surfaces* with the plane of the paper. These surfaces are generated by rotating the curves around the vertical axis. The curves are stretched horizontally compared with the circles described by the relation $r = r_0\cos\theta$ (which you can verify is indeed a circle).

We see that $t = RC$ is independent of A, because $R \propto 1/A$ and $C \propto A$. (Basically, if a given patch of the membrane leaks its charge on a given time scale, then putting a bunch of these patches together shouldn't change the time scale, because each patch doesn't care that there are others next to it.) Using the information given for 1 cm^2 of the membrane, we have $t = RC = (1000\,\Omega)(10^{-6}\,\text{F}) = 10^{-3}$ s.

Since $R = \rho s/A$, the resistivity is given by

$$\rho = \frac{RA}{s} = \frac{(1000\,\Omega)(10^{-4}\,\text{m}^2)}{2.7 \cdot 10^{-9}\,\text{m}} \approx 4 \cdot 10^7 \text{ ohm-m.} \tag{12.453}$$

From Fig. 4.8, this is a little more than 100 times the resistivity of pure water.

10.2 *Force on a dielectric*

(a) The equivalent capacitance of two capacitors in parallel is simply the sum of the capacitances. (The rule is opposite to that for resistors; see Problem 3.18.) The capacitance of the part with the dielectric is κ times what it would be if there were vacuum there. So the total capacitance is given by

$$\begin{aligned} C = C_1 + C_2 &= \frac{\epsilon_0 A_1}{s} + \frac{\kappa\epsilon_0 A_2}{s} \\ &= \frac{\epsilon_0 a(b-x)}{s} + \frac{\kappa\epsilon_0 a x}{s} = \frac{\epsilon_0 a}{s}\big[b + (\kappa - 1)x\big]. \end{aligned} \tag{12.454}$$

The stored energy is then

$$U = \frac{Q^2}{2C} = \frac{Q^2 s}{2\epsilon_0 a[b + (\kappa - 1)x]}. \tag{12.455}$$

Note that as x changes, the charge stays constant (by assumption), but the potential does not. So the $Q\phi/2$ and $C\phi^2/2$ forms of the energy aren't useful.

(b) The force is

$$F = -\frac{dU}{dx} = \frac{Q^2 s(\kappa - 1)}{2\epsilon_0 a[b + (\kappa - 1)x]^2}. \tag{12.456}$$

The positive sign here means that the force points in the direction of increasing x. That is, the dielectric slab is pulled into the capacitor. But it's risky to trust this sign blindly. Physically, the force points in the direction of decreasing energy. And we see from the above expression for U that the energy decreases as x increases (because $\kappa > 1$).

The force F is correctly zero if $\kappa = 1$, because in that case we don't actually have a dielectric. The $\kappa \to \infty$ limit corresponds to a conductor. In that case, both U and F are zero. Basically, all of the charge on the plates shifts to the overlap x region, and compensating charge gathers there in the dielectric, so in the end there is no field anywhere. Note that F decreases as x increases. You should think about why this is the case. *Hint:* First convince yourself why the force

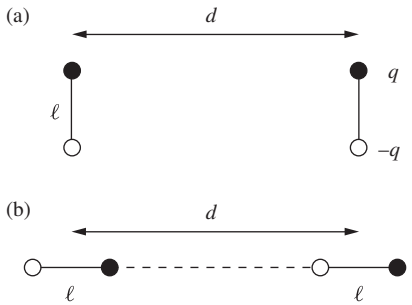


Figure 12.129.

should be proportional to the product of the charge *densities* (and not the total charges) on the two parts of the plates. And then look at Exercise 10.15.

10.3 *Energy of dipoles*

The first configuration is shown in Fig. 12.129(a). There are four relevant (non-internal) pairs of charges, so the potential energy is (with $\ell \ll d$)

$$U = \frac{1}{4\pi\epsilon_0}\left(2\cdot\frac{q^2}{d} - 2\cdot\frac{q^2}{\sqrt{d^2+\ell^2}}\right) = \frac{2q^2}{4\pi\epsilon_0 d}\left(1 - \frac{1}{\sqrt{1+\ell^2/d^2}}\right)$$
$$\approx \frac{2q^2}{4\pi\epsilon_0 d}\left(1 - \left(1 - \frac{\ell^2}{2d^2}\right)\right) = \frac{q^2\ell^2}{4\pi\epsilon_0 d^3} \equiv \frac{p^2}{4\pi\epsilon_0 d^3}, \qquad (12.457)$$

where we have used $1/\sqrt{1+\epsilon} \approx 1 - \epsilon/2$. The second configuration is shown in Fig. 12.129(b). The potential energy is now

$$\frac{1}{4\pi\epsilon_0}\left(2\cdot\frac{q^2}{d} - \frac{q^2}{d-\ell} - \frac{q^2}{d+\ell}\right) = \frac{q^2}{4\pi\epsilon_0 d}\left(2 - \frac{1}{1-\ell/d} - \frac{1}{1+\ell/d}\right)$$
$$\approx \frac{q^2}{4\pi\epsilon_0 d}\left(2 - \left(1 + \frac{\ell}{d} + \frac{\ell^2}{d^2}\right) - \left(1 - \frac{\ell}{d} + \frac{\ell^2}{d^2}\right)\right)$$
$$= \frac{q^2}{4\pi\epsilon_0 d}\left(-\frac{2\ell^2}{d^2}\right) = -\frac{p^2}{2\pi\epsilon_0 d^3}, \qquad (12.458)$$

where we have used $1/(1+\epsilon) \approx 1 - \epsilon + \epsilon^2$. Note that we needed to go to second order in the Taylor expansions here. By looking at the initial expressions for U for each setup, it is clear why the first U is positive, but not so clear why the second U is negative. However, in the limit where the dipoles nearly touch, the second U is certainly negative.

10.4 *Dipole polar components*

Remember that our convention for the angle θ is that it is measured down from the z axis in Fig. 10.6. So the radial unit vector is given by $\hat{\mathbf{r}} = \sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{z}}$. The tangential unit vector, which is perpendicular to $\hat{\mathbf{r}}$, is then given by $\hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{z}}$; this makes the dot product of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ equal to zero, and you can check that the overall sign is correct. Inverting these expressions for $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ gives

$$\hat{\mathbf{x}} = \sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}} \qquad \text{and} \qquad \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}. \qquad (12.459)$$

Therefore,

$$\begin{aligned}\mathbf{E} &= E_x\hat{\mathbf{x}} + E_z\hat{\mathbf{z}} \\ &= E_x(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}) + E_z(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) \\ &= \hat{\mathbf{r}}(E_x\sin\theta + E_z\cos\theta) + \hat{\boldsymbol{\theta}}(E_x\cos\theta - E_z\sin\theta) \\ &= \frac{p}{4\pi\epsilon_0 r^3}\Big(\hat{\mathbf{r}}\Big[(3\sin\theta\cos\theta)\sin\theta + (3\cos^2\theta - 1)\cos\theta\Big] \\ &\quad + \hat{\boldsymbol{\theta}}\Big[(3\sin\theta\cos\theta)\cos\theta - (3\cos^2\theta - 1)\sin\theta\Big]\Big). \qquad (12.460)\end{aligned}$$

Using $\sin^2\theta + \cos^2\theta = 1$ in the $\hat{\mathbf{r}}$ term, $\mathbf{E}$ quickly simplifies to

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3}\big(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\big), \tag{12.461}$$

as desired. Alternatively, E_r equals the projection of $\mathbf{E} = (E_x, E_z)$ onto $\hat{\mathbf{r}} = (\sin\theta, \cos\theta)$. Since $\hat{\mathbf{r}}$ is a unit vector, this projection equals the dot product $\mathbf{E}\cdot\hat{\mathbf{r}}$. Therefore,

$$E_r = \mathbf{E}\cdot\hat{\mathbf{r}} = (E_x, E_z)\cdot(\sin\theta, \cos\theta) = E_x\sin\theta + E_z\cos\theta, \tag{12.462}$$

in agreement with the third line in Eq. (12.460). Likewise,

$$E_\theta = \mathbf{E}\cdot\hat{\boldsymbol{\theta}} = (E_x, E_z)\cdot(\cos\theta, -\sin\theta) = E_x\cos\theta - E_z\sin\theta, \tag{12.463}$$

again in agreement with the third line in Eq. (12.460).

10.5 *Average field*

(a) From part (c) of Problem 1.28 we know that the average electric field over the volume of a sphere of radius R, due to a given charge q at radius $r < R$, has magnitude $qr/4\pi\epsilon_0 R^3$ and points toward the center (if q is positive). In vector form, this average field can be written as $-q\mathbf{r}/4\pi\epsilon_0 R^3$. If we sum this over all the charges inside the sphere, then the numerator becomes $\sum q_i\mathbf{r}_i$ (or $\int \mathbf{r}\rho\,dv$ if we have a continuous charge distribution). But this sum is, by definition, the dipole moment $\mathbf{p}$, where $\mathbf{p}$ is measured relative to the center. So the average field over the volume of the sphere is $\mathbf{E}_{\text{avg}} = -\mathbf{p}/4\pi\epsilon_0 R^3$, as desired. Note that all that matters here is the dipole moment; the monopole moment (the total charge) doesn't come into play.

(b) Since $\mathbf{E}_{\text{avg}}$ is proportional to $1/R^3$, and since volume is proportional to R^3, the total integral of $\mathbf{E}$ over the volume of a sphere is independent of R (provided that R is large enough to contain all the charges). This means that if we increase the radius by dR, we don't change the integral of $\mathbf{E}$. This implies that the average value of $\mathbf{E}$ over the *surface* of any sphere containing all the charges equals zero. (We actually already knew this from part (a) of Problem 1.28. Each individual charge yields zero average field over the surface.) A special case of this result is the centered point-dipole case in Exercise 10.25.

So for the specific case shown in Fig. 10.32(a), the average value of the field over the surface of the sphere is zero. And since the dipole moment has magnitude $p = 2q\ell$ and points upward, the result from part (a) tells us that the average value over the volume of the sphere, $\mathbf{E}_{\text{avg}} = -\mathbf{p}/4\pi\epsilon_0 R^3$, has magnitude $q\ell/2\pi\epsilon_0 R^3$ and points downward.

(c) The average value of the field over the surface of the sphere in Fig. 10.32(b) is *not* zero. From part (b) of Problem 1.28, the average field due to each charge has magnitude $q/4\pi\epsilon_0\ell^2$ and points downward. So the average field over the surface, due to both charges, has magnitude $q/2\pi\epsilon_0\ell^2$ and points downward. Since this is independent of the radius of the sphere, the average field over the volume of a sphere with $R < \ell$ also has magnitude $q/2\pi\epsilon_0\ell^2$ and points downward.

The moral of all this is that "outside" the dipole, the field points in various directions and averages out over the surface of a sphere. But "inside" the dipole, the field points generally in one direction, so the average is nonzero over the surface of a sphere.

Note that *volume* average of $\mathbf{E}$ is continuous as R crosses the $R = \ell$ cutoff between the two cases in parts (a) and (b); in both cases it has magnitude $q/2\pi\epsilon_0\ell^2$. If we multiply this by ℓ/ℓ and use $p = q\ell$, we can write it as $p/2\pi\epsilon_0\ell^3$. Multiplying by the volume $4\pi\ell^3/3$ then tells us that the total volume integral of $\mathbf{E}$, over a sphere of radius ℓ, has magnitude $2p/3\epsilon_0$ and points downward. In other words, for a fixed value of p, even the limit of an idealized dipole still has a nonzero value of $\int \mathbf{E}\, dv$, despite the fact that the only shells yielding nonzero contributions are infinitesimal ones.

10.6 *Quadrupole tensor*

Our goal is to find the potential $\phi(\mathbf{r})$ at the point $\mathbf{r} = (x_1, x_2, x_3)$. As in Section 10.2, primed coordinates will denote the position of a point in the charge distribution. The distance from $\mathbf{r}$ to a particular point $\mathbf{r}' = (x_1', x_2', x_3')$ in the distribution is

$$\begin{aligned} R &= \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2} \\ &= r\sqrt{1 + \frac{r'^2}{r^2} - \frac{2\sum x_i x_i'}{r^2}} = r\sqrt{1 + \frac{r'^2}{r^2} - \frac{2\sum \hat{x}_i x_i'}{r}}, \end{aligned} \tag{12.464}$$

where we have used $\sum x_i^2 = r^2$ and $\sum x_i'^2 = r'^2$, and where $(\hat{x}_1, \hat{x}_2, \hat{x}_3) = (x_1, x_2, x_3)/r$ is the unit vector $\hat{\mathbf{r}}$ in the $\mathbf{r}$ direction. Assuming that r' is much smaller than r, we can use the expansion $(1+\delta)^{-1/2} = 1 - \delta/2 + 3\delta^2/8 - \cdots$ to write (dropping terms of order $1/r^4$ and higher)

$$\begin{aligned} \frac{1}{R} &= \frac{1}{r}\left[1 + \frac{\sum \hat{x}_i x_i'}{r} + \frac{3\big(\sum \hat{x}_i x_i'\big)^2}{2r^2} - \frac{r'^2}{2r^2}\right] \\ &= \frac{1}{r}\left[1 + \frac{\sum \hat{x}_i x_i'}{r} + \frac{3\big(\sum \hat{x}_i x_i'\big)^2}{2r^2} - \frac{\big(\sum \hat{x}_i^2\big) r'^2}{2r^2}\right]. \end{aligned} \tag{12.465}$$

In the last term here, we have multiplied by 1 in the form of the square of the length of a unit vector, for future purposes. It is easier to understand this result for $1/R$ if we write it in terms of vectors and matrices:

$$\begin{aligned} \frac{1}{R} &= \frac{1}{r} + \frac{1}{r^2}(\hat{x}_1, \hat{x}_2, \hat{x}_3) \cdot \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \\ &\quad + \frac{1}{2r^3}(\hat{x}_1, \hat{x}_2, \hat{x}_3) \cdot \begin{pmatrix} 3x_1'^2 - r'^2 & 3x_1'x_2' & 3x_1'x_3' \\ 3x_2'x_1' & 3x_2'^2 - r'^2 & 3x_2'x_3' \\ 3x_3'x_1' & 3x_3'x_2' & 3x_3'^2 - r'^2 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix}. \end{aligned} \tag{12.466}$$

You should verify that this is equivalent to Eq. (12.465). If desired, the diagonal terms of this matrix can be written in a slightly different form. Since $r'^2 = x_1'^2 + x_2'^2 + x_3'^2$, the upper left entry equals $2x_1'^2 - x_2'^2 - x_3'^2$. Likewise for the other two diagonal entries. Note that there are only five independent entries in the matrix, because it is symmetric and has trace zero.

To obtain $\phi(\mathbf{r})$, we must compute the integral,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')dv'}{R}. \tag{12.467}$$

In other words, we must compute the volume integral of Eq. (12.466) times $\rho(\mathbf{r}')$, and then tack on a $1/4\pi\epsilon_0$. When the $1/r$ term is integrated, it simply gives q/r, where q is the total charge in the distribution. To write the other two terms in a cleaner way, define the vector $\mathbf{p}$ to be the vector whose entries are the $\rho\, dv'$ integrals of the entries in the above (x_1', x_2', x_3') vector. And likewise define the matrix $\mathbf{Q}$ to be the $\rho\, dv'$ integral of the above matrix. For example, the first component of $\mathbf{p}$ and the upper-left entry of $\mathbf{Q}$ are

$$p_1 = \int x_1'\rho(\mathbf{r}')dv' \qquad \text{and} \qquad Q_{11} = \int \left(3x_1'^2 - r'^2\right)\rho(\mathbf{r}')dv', \tag{12.468}$$

and so on. We can then write the result for the potential at an arbitrary point $\mathbf{r}$ in the compact form,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r^2} + \frac{\hat{\mathbf{r}}\cdot\mathbf{Q}\hat{\mathbf{r}}}{2r^3}\right]. \tag{12.469}$$

The advantage of Eq. (12.469) over Eq. (10.9) in the text is the following. The latter gives the correct value of ϕ at points on the z axis. However, if we want to find ϕ at another point, we must redefine θ as the angle with respect to the direction to the new point, and then recalculate all the K_i. The present result in Eq. (12.469) has the benefit that, although it involves calculating a larger number of quantities, it is valid for any choice of the point $\mathbf{r}$. The quantities q, $\mathbf{p}$, and $\mathbf{Q}$ depend *only on the distribution,* and not on the point $\mathbf{r}$ at which we want to calculate the potential. Conversely, the quantities $\hat{\mathbf{r}}$ and r in Eq. (12.469) depend only on $\mathbf{r}$ and not on the distribution. So, for a given charge distribution, we can calculate (with respect to a given set of coordinate axes) $\mathbf{p}$ and $\mathbf{Q}$ once and for all. We then simply need to plug our choice of $\mathbf{r}$ into Eq. (12.469), and this correctly gives $\phi(\mathbf{r})$ up to order $1/r^3$.

In the special case where $\mathbf{r}$ lies on the $z \equiv x_3$ axis, we have $\hat{\mathbf{r}} = (0, 0, 1)$. Since only $\hat{x}_3$ is nonzero, only Q_{33} (the lower right entry in $\mathbf{Q}$) survives in the dot product $\hat{\mathbf{r}}\cdot\mathbf{Q}\hat{\mathbf{r}}$. Furthermore, if θ is the angle of $\mathbf{r}'$ with respect to the x_3 axis, then we have $x_3' = r'\cos\theta$. So $Q_{33} = \int r'^2(3\cos^2\theta - 1)\rho\, dv'$. When the $1/2r^3$ factor in Eq. (12.469) is included, we correctly arrive at the result Eq. (10.9).

For a spherical shell, which we know has only a monopole moment, you can quickly verify that all of the entries in $\mathbf{Q}$ are zero. The off-diagonal

entries are zero from symmetry, and the diagonal elements are zero due to the example in Section 10.2 combined with the previous paragraph. Alternatively, the average value of, say, $x_1'^2$ over the surface of a sphere equals $r'^2/3$, because it has the same average value as $x_2'^2$ and $x_3'^2$, and the sum of all three averages is r'^2. If you want to get some practice with **Q**, Exercise 10.26 deals with the quadrupole arrangement in Fig. 10.5.

10.7 *Force on a dipole*

Let the dipole consist of a charge $-q$ at position $\mathbf{r}$ and a charge q at position $\mathbf{r}+\mathbf{s}$. Then the dipole vector is $\mathbf{p} = q\mathbf{s}$. If the dipole is placed in an electric field $\mathbf{E}$, the net force on it is

$$\mathbf{F} = (-q)\mathbf{E}(\mathbf{r}) + q\mathbf{E}(\mathbf{r}+\mathbf{s}). \tag{12.470}$$

The x component of this is $F_x = (-q)E_x(\mathbf{r}) + qE_x(\mathbf{r}+\mathbf{s})$. Now, the change in a function f due to a small displacement $\mathbf{s}$ is $\nabla f \cdot \mathbf{s}$, by the definition of the gradient (or at least that's one way of defining it). So we can write F_x as

$$\begin{aligned} F_x &= q\big[E_x(\mathbf{r}+\mathbf{s}) - E_x(\mathbf{r})\big] = q\nabla E_x \cdot \mathbf{s} \\ &= (q\mathbf{s}) \cdot \nabla E_x \equiv \mathbf{p} \cdot \nabla E_x, \end{aligned} \tag{12.471}$$

as desired. Likewise for the other two components.

10.8 *Force from an induced dipole*

If q is the charge of the ion, then the magnitude of the electric field of the ion at the location of the atom is $E = q/4\pi\epsilon_0 r^2$. If the polarizability of the atom is α, then the induced dipole moment of the atom is $p = \alpha E = \alpha q/4\pi\epsilon_0 r^2$. This dipole moment points along the line from the ion to the atom (see Fig. 12.130), so the magnitude of the field of the induced dipole at the location of the ion is $E_{\text{dipole}} = 2p/4\pi\epsilon_0 r^3$. The magnitude of the force on the ion is therefore

$$F = qE_{\text{dipole}} = \frac{2pq}{4\pi\epsilon_0 r^3} = \frac{2(\alpha q/4\pi\epsilon_0 r^2)q}{4\pi\epsilon_0 r^3} = \frac{2\alpha q^2}{(4\pi\epsilon_0)^2 r^5}. \tag{12.472}$$

You can quickly show that the force is attractive for either sign of q. The potential energy relative to infinity is

$$U(r) = -\int_\infty^r F(r')dr' = -\int_\infty^r -\frac{2\alpha q^2\, dr'}{(4\pi\epsilon_0)^2 r'^5} = -\frac{\alpha q^2}{2(4\pi\epsilon_0)^2 r^4}. \tag{12.473}$$

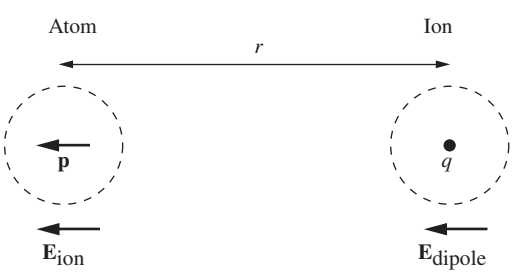


Figure 12.130.

The polarizability of sodium is given by $\alpha/4\pi\epsilon_0 = 27\cdot 10^{-30}\,\text{m}^3$. If the magnitude of the potential energy equals $|U| = 4\cdot 10^{-21}\,\text{J}$, then solving for r and setting $q = e$ gives

$$\begin{aligned} r = \left[\frac{(\alpha/4\pi\epsilon_0)q^2}{2(4\pi\epsilon_0)|U|}\right]^{1/4} &= \left[\frac{(27\cdot 10^{-30}\,\text{m}^3)(1.6\cdot 10^{-19}\,\text{C})^2}{2\cdot 4\pi\left(8.85\cdot 10^{-12}\,\frac{\text{s}^2\,\text{C}^2}{\text{kg}\,\text{m}^3}\right)(4\cdot 10^{-21}\,\text{J})}\right]^{1/4} \\ &= 9.4\cdot 10^{-10}\,\text{m}. \end{aligned} \tag{12.474}$$

If r is larger than this, then (on average) the thermal energy is sufficient to kick the ion out to infinity.

10.9 *Polarized water*

We must determine the number, n, of molecules of water per cubic centimeter. A mole of something with molecular mass M has a mass of M grams. (Equivalently, since the proton mass is $1.67 \cdot 10^{-24}$ g, it takes $1/(1.67 \cdot 10^{-24}) = 6 \cdot 10^{23}$ protons to make 1 gram, and this number is essentially Avogadro's number.) Water has a molecular weight of 18, so the number of water molecules per gram ($= \text{cm}^3$) is $n = (6 \cdot 10^{23}/\text{mole})/(18\,\text{cm}^3/\text{mole}) = 3.33 \cdot 10^{22}\,\text{cm}^{-3}$. The dipole moment of water can be written as $p = 6.13 \cdot 10^{-28}$ C-cm. Assuming the dipoles all point down, the polarization density is therefore

$$P = np = (3.33 \cdot 10^{22}\,\text{cm}^{-3})(6.13 \cdot 10^{-28}\,\text{C-cm}) = 2.04 \cdot 10^{-5}\,\text{C/cm}^2. \tag{12.475}$$

From the reasoning in Section 10.7, this is the surface charge density, σ. The number of electrons per square centimeter it corresponds to is $\sigma/e = (2.04 \cdot 10^{-5}\,\text{C/cm}^2)/(1.6 \cdot 10^{-19}\,\text{C}) = 1.3 \cdot 10^{14}\,\text{cm}^{-2}$. This is somewhat smaller than the number of surface molecules per square centimeter, which equals $n^{2/3} = 1.0 \cdot 10^{15}\,\text{cm}^{-2}$ because each edge of the $1\,\text{cm}^3$ cube is (approximately) $n^{1/3}$ molecules long.

10.10 *Tangent field lines*

Consider the Gaussian surface indicated by the heavy line in Fig. 12.131. The side part of the surface is constructed to follow the field lines, so there is no flux there. Likewise, there is no flux through the top circular face, because the field is zero outside the capacitor plates. So the only flux comes from the great circle inside the sphere. From Eq. (10.53) the field inside the sphere has the uniform value of $3\mathbf{E}_0/(2+\kappa)$. So the flux out of the Gaussian surface equals $-\pi R^2 \cdot 3E_0/(2+\kappa)$, where the minus arises because the flux is inward.

The total charge enclosed in the Gaussian surface comes from two places: the negative charge in the circle on the upper capacitor plate, and the positive charge on the upper hemisphere. The former is simply $q_{\text{cap}} = (-\sigma)\pi r^2 = (-\epsilon_0 E_0)\pi r^2$, where we have used the fact that the charge densities on the capacitor plates are what cause the uniform field E_0; hence $E_0 = \sigma/\epsilon_0$. The latter charge is just $q_{\text{sph}} = P\pi R^2$, where P is the polarization, because the top patch of the column in Fig. 10.21(a) has a charge of $P\,da$ (where da is the *horizontal* cross-sectional area), independent of the tilt angle of the actual end face. And all the da areas simply add up to the great-circle area, πR^2. (Or you could just integrate the $P\cos\theta$ surface density over the hemisphere.) Using the value of P from Eq. (10.54), Gauss's law gives

$$\begin{aligned}
\Phi &= \frac{1}{\epsilon_0}(q_{\text{cap}} + q_{\text{sph}}) \\
\Longrightarrow \quad -\pi R^2 \frac{3E_0}{\kappa+2} &= \frac{1}{\epsilon_0}\left(-\epsilon_0 E_0 \pi r^2 + 3\frac{\kappa-1}{\kappa+2}\epsilon_0 E_0 \cdot \pi R^2\right) \\
\Longrightarrow \quad -3R^2\frac{1}{\kappa+2} &= -r^2 + 3R^2\frac{\kappa-1}{\kappa+2} \\
\Longrightarrow \quad r &= R\sqrt{\frac{3\kappa}{\kappa+2}}.
\end{aligned} \tag{12.476}$$

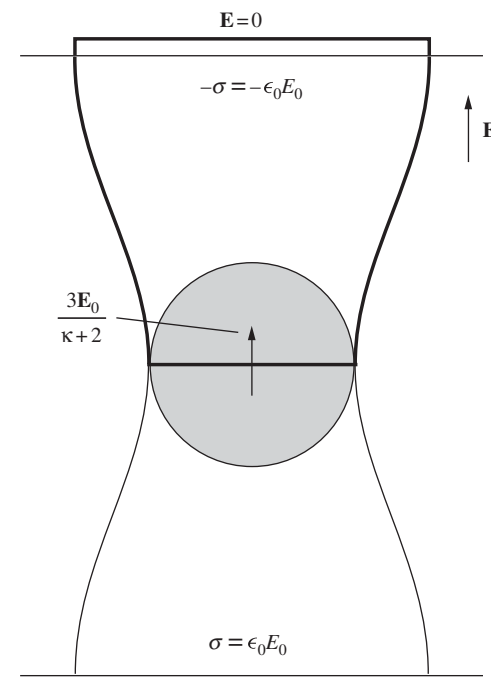


Figure 12.131.

As a check, we have $r = R$ when $\kappa = 1$. In this case, our dielectric is just vacuum, so the field remains $\mathbf{E}_0$ everywhere; the field lines are all straight. Also, we have $r = \sqrt{3}R$ when $\kappa \to \infty$. In this limit the sphere is a conductor. The factor of $\sqrt{3}$ isn't so obvious. Note that, in the case of a conductor, a field line can't actually be tangent to the surface, because field lines must always be perpendicular to the surface of a conductor. What happens is that the external field approaches zero along the equator (the zero vector is, in some sense, both parallel and perpendicular to the surface). But a tiny distance away from the equator, the field is nonzero, so it is meaningful to ask where that field line ends up on the distant capacitor plates.

10.11 *Bound charge and divergence of P*

If we take the volume integral of both sides of Eq. (10.61) and use the divergence theorem, we see that our goal is to show that $\int_S \mathbf{P} \cdot d\mathbf{a} = -q_{\text{bound}}$, where q_{bound} is the bound charge enclosed within the surface S.

Assume that the polarization $\mathbf{P}$ arises from N dipoles per unit volume, each with a dipole moment $\mathbf{p} = q\mathbf{s}$. Then $\mathbf{P} = N\mathbf{p} = Nq\mathbf{s}$. If the dipoles point in random directions, so that $\mathbf{P} = 0$, then there is no extra bound charge in a given volume. But if they are aligned, so that $\mathbf{P} \neq 0$, and if additionally $\mathbf{P}$ varies with position, then there may be a net bound charge in the volume. The reasoning is as follows.

Consider a collection of dipoles, as shown in Fig. 12.132. The vertical line represents a patch of the right-hand surface of S. How much extra negative charge is there inside S, that is, to the left of the line? If a given dipole lies entirely inside or outside S, then it contributes nothing to the net charge. But if a dipole is cut by the vertical line, then there is an extra charge of $-q$ inside S.

How many dipoles are cut by the line? Any dipole whose center lies within $s/2$ of the line gets cut by it. So the center must lie in a slab with thickness s, indicated by the shaded region in the figure. The two extreme dipole positions are indicated by the boxes. If the area of a given patch of the surface is da, then any dipole whose center lies in a slab of volume $s\,da$ will contribute a charge of $-q$ to S. Since there are N dipoles per unit volume, we see that $N(s\,da)$ dipoles are cut by the line. The extra charge associated with the patch is therefore $dq_{\text{bound}} = N(s\,da)(-q)$, which can be written as $dq_{\text{bound}} = -(Nqs)da = -P\,da$.

If a dipole is tilted at an angle θ with respect to the normal to the patch, then the volume of the relevant slab is decreased by a factor of $\cos\theta$. If we tack this factor onto P, it simply turns P into the component $P_\perp$ perpendicular to the surface. So in general the extra charge inside the volume, near a given patch with area da, equals $dq_{\text{bound}} = -P_\perp da$, which can be written as the dot product, $dq_{\text{bound}} = -\mathbf{P} \cdot d\mathbf{a}$. Integrating this over the entire surface gives the total enclosed bound charge as

$$q_{\text{bound}} = -\int \mathbf{P} \cdot d\mathbf{a}, \tag{12.477}$$

as desired.

Although we motivated this result in Section 10.11 by considering dielectrics, this problem shows (as mentioned in the text) that this result is

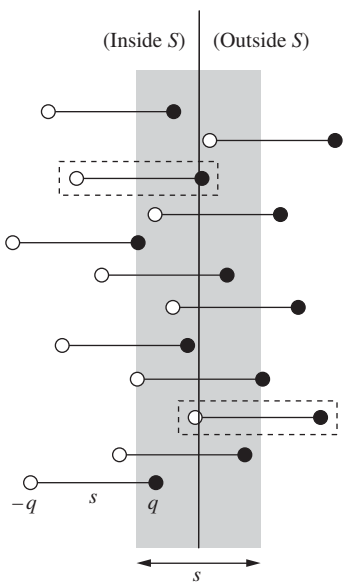


Figure 12.132.

quite independent of dielectrics. No matter how the polarization $\mathbf{P}$ comes about, the result in Eq. (12.477) is still valid. (You can manually twist the dipoles in whatever way you want, provided that $\mathbf{P}$ changes slowly on the length scale of the dipoles, so that we can talk about smooth averages.) To emphasize what we said in the text, the logical route to Eq. (10.62) is to start with Eqs. (10.59) and (10.61), both of which are universally true, and then Eq. (10.62) immediately follows. No mention has been made of dielectrics. But if we are in fact dealing with a (linear) dielectric, then $\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}$, and we can use $1 + \chi_e = \kappa$ to say that additionally

$$\epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \chi_e \epsilon_0 \mathbf{E} = \kappa \epsilon_0 \mathbf{E} \equiv \epsilon \mathbf{E}. \tag{12.478}$$

In all cases the relation $\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$ holds, but that is just a definition.

10.12 *Boundary conditions for D*

$D_\perp$ is continuous. This follows from $\operatorname{div} \mathbf{D} = \rho_{\text{free}}$; there is no free charge in the setup, so the divergence of $\mathbf{D}$ is zero. The divergence theorem then tells us that $\int \mathbf{D} \cdot d\mathbf{a} = 0$ for any closed surface. That is, there is zero flux through any surface. So if we draw a pancake-like pillbox with one face just inside the slab and one face just outside, the inward flux through one face must equal the outward flux through the other. Hence $D_\perp^{\text{in}} A = D_\perp^{\text{out}} A \Longrightarrow D_\perp^{\text{in}} = D_\perp^{\text{out}}$. That is, $D_\perp$ is continuous across the boundary.

For $D_\parallel$, we know that $E_\parallel$ is continuous across the boundary, because all we have at the boundary is a layer of bound charge, which produces no discontinuity in $E_\parallel$. So $\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$ tells us that the discontinuity in $D_\parallel$ is the same as the discontinuity in $P_\parallel$. Since $\mathbf{P} = 0$ outside, the discontinuity in $P_\parallel$ is simply $-P_\parallel^{\text{in}}$. That is, the change in $D_\parallel$ when going from inside to outside is $-P_\parallel^{\text{in}}$.

10.13 *Q for a leaky capacitor*

From Exercise 10.42, the energy density in the electric field is $\epsilon E^2/2$. And it is the same for the magnetic field, by plugging $B = \sqrt{\mu_0 \epsilon} E$ into $B^2/2\mu_0$. The total energy density is therefore ϵE^2, or $\epsilon E_0^2 \cos^2 \omega t$. But the time average of $\cos^2 \omega t$ is $1/2$, so the average energy density is $\epsilon E_0^2/2$.

The energy in the fields will decay due to ohmic resistance. To calculate this power dissipation, consider a tube of cross-sectional area A and length L. The power dissipated in this tube is

$$\begin{aligned} P &= I^2 R = (JA)^2 (\rho L/A) = J^2 \rho (AL) \\ &= (\sigma E)^2 \frac{1}{\sigma} (\text{volume}) = \sigma E^2 (\text{volume}). \end{aligned} \tag{12.479}$$

The power dissipated per unit volume is therefore σE^2. The time average of this is $\sigma E_0^2/2$. Hence

$$Q = \frac{\omega \cdot (\text{energy stored})}{\text{power loss}} = \frac{\omega(\epsilon E_0^2/2)}{\sigma E_0^2/2} = \frac{\omega \epsilon}{\sigma}, \tag{12.480}$$

as desired. From Table 4.1, the conductivity of seawater is $\sigma = 4\ (\text{ohm-m})^{-1}$. And from Fig. 10.29, the dielectric constant κ is still about 80 at a frequency of 1000 MHz (10^9 Hz). Therefore, since $\epsilon = \kappa\epsilon_0$, we have

$$Q = \frac{(2\pi \cdot 10^9 \text{s}^{-1})\left(80 \cdot 8.85 \cdot 10^{-12}\ \frac{\text{s}^2\,\text{C}^2}{\text{kg m}^3}\right)}{4\ (\text{ohm-m})^{-1}} = 1.1. \tag{12.481}$$

Since Q equals the number of radians of ωt required for the energy to decrease by a factor of $1/e$, we see that by the end of one cycle (2π radians) there is practically no energy left. The wavelength corresponding to 1000 MHz is $(c/\sqrt{\kappa})/\nu = 0.033\,\text{m}$. So microwave radar won't find submarines!

10.14 *Boundary conditions on E and B*

With no free charges or currents, the equations describing the system are

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 0, & \nabla \times \mathbf{E} &= -\partial\mathbf{B}/\partial t; \\ \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{B} &= \mu_0\, \partial\mathbf{D}/\partial t. \end{aligned} \tag{12.482}$$

The two equations involving $\mathbf{D}$ come from Eqs. (10.64) and (10.78) with ρ_{free} and $\mathbf{J}_{\text{free}}$ set equal to zero. The other two equations are two of Maxwell's equations. We can now apply the standard arguments. For the perpendicular components, we can apply the divergence theorem to the two "div" equations, with the volume chosen to be a squat pillbox, of vanishing thickness, spanning the surface. Our equations tell us that the net flux out of the volume is zero, so the perpendicular field on one side must equal the perpendicular field on the other. And for the parallel components, we can apply Stokes' theorem to the two "curl" equations, with the area chosen to be a thin rectangle, of vanishing area, spanning the surface. Our equations tell us that the line integral around the rectangle is zero, so the parallel field on one side must equal the parallel field on the other. (The finite non-zero entries on the right-hand sides of the curl equations are inconsequential, because they provide zero contribution when integrated over the area of an infinitesimally thin rectangle.) The above four equations therefore yield (with 1 and 2 labeling the two regions)

$$\begin{aligned} D_{1,\perp} &= D_{2,\perp}, & E_{1,\parallel} &= E_{2,\parallel}; \\ B_{1,\perp} &= B_{2,\perp}, & B_{1,\parallel} &= B_{2,\parallel}. \end{aligned} \tag{12.483}$$

Since $\mathbf{D} = \epsilon\mathbf{E}$ for a linear dielectric, the first of these equations gives

$$\epsilon_1 E_{1,\perp} = \epsilon_2 E_{2,\perp}. \tag{12.484}$$

So $E_\perp$ is discontinuous. But the other three components are continuous across the boundary. That is, the entire $\mathbf{B}$ field is continuous, as is the parallel component of $\mathbf{E}$.

Note that we are assuming that the materials aren't magnetic. After reading Section 11.10, you can show that in magnetic materials there is a discontinuity in $B_\parallel$.

12.11 Chapter 11

11.1 *Maxwell's equations with magnetic charge*

Maxwell's equations with only electric charge and electric current are given in Eq. (9.17). If magnetic charge existed, the last equation would have to be replaced, as discussed in Section 11.2, by $\nabla \cdot \mathbf{B} = b_1 \eta$, where η is the magnetic charge density, and b_1 is a constant that depends on how the unit of magnetic charge is chosen. With the conventional definition of the direction of $\mathbf{B}$, a positive magnetic charge would be attracted to the north pole of the earth, so it would behave like the north pole of a compass.

Magnetic charge in motion with velocity $\mathbf{v}$ would constitute a magnetic current. Let $\mathbf{K}$ be the magnetic current density. Then $\mathbf{K} = \eta \mathbf{v}$, in analogy with $\mathbf{J} = \rho \mathbf{v}$. Conservation of magnetic charge would then be expressed by the "continuity equation," $\nabla \cdot \mathbf{K} = -\partial \eta / \partial t$, in analogy with $\nabla \cdot \mathbf{J} = -\partial \rho / \partial t$.

A magnetic current would be the source of an electric field, just as an electric current is the source of a magnetic field. So we must add to the right side of the first Maxwell equation in Eq. (9.17) a term proportional to $\mathbf{K}$. (Equivalently, if we didn't add such a term, we would end up with a contradiction, similar to the one in Section 9.1, arising from the fact that $\nabla \cdot (\nabla \times \mathbf{E}) = 0$ is identically zero.) Let this new term be $b_2 \mathbf{K}$. Then we have

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + b_2 \mathbf{K}. \tag{12.485}$$

To determine the constant b_2, we can take the divergence of both sides of this equation. The left-hand side is identically zero because $\nabla \cdot (\nabla \times \mathbf{E}) = 0$, so we have (using the continuity equation)

$$\begin{aligned} 0 &= -\nabla \cdot \left(\frac{\partial \mathbf{B}}{\partial t} \right) + b_2 \nabla \cdot \mathbf{K} \\ &= -\frac{\partial}{\partial t} (\nabla \cdot \mathbf{B}) + b_2 \left(-\frac{\partial \eta}{\partial t} \right) \\ &= -\frac{\partial}{\partial t} (b_1 \eta) - b_2 \frac{\partial \eta}{\partial t} \\ &= -(b_1 + b_2) \frac{\partial \eta}{\partial t}. \end{aligned} \tag{12.486}$$

Therefore b_2 must equal $-b_1$. So the generalized Maxwell's equations take the form (with $b \equiv b_1 = -b_2$),

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} - b\mathbf{K}, \\ \nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}, \\ \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}, \\ \nabla \cdot \mathbf{B} &= b\eta. \end{aligned} \tag{12.487}$$

The constant b can be chosen arbitrarily. Two common conventions are $b = 1$ and $b = \mu_0$.

11.2 *Magnetic dipole*

If we treat the current loop like an exact dipole, then the dipole moment is $m = Ia = I\pi b^2$. Equation (11.15) gives the magnetic field at position z along the axis of the dipole as $\mu_0 m/2\pi z^3$, which here equals $\mu_0(I\pi b^2)/2\pi z^3 = \mu_0 I b^2/2z^3$.

If we treat the current loop (correctly) as a loop of finite size, then Eq. (6.53) gives the field at position z on the axis as $B_z = \mu_0 I b^2/2(z^2 + b^2)^{3/2}$. For $z \gg b$ we can ignore the b^2 term in the denominator, yielding $B_z \approx \mu_0 I b^2/2z^3$, which agrees with the above result for the idealized dipole.

The correct result is smaller than the idealized-dipole result by the factor $z^3/(z^2 + b^2)^{3/2}$. This factor approaches 1 as $z \to \infty$. It is larger than a given number η (we are concerned with $\eta = 0.99$) if

$$\frac{z^3}{(z^2+b^2)^{3/2}} > \eta \implies \frac{z^2}{z^2+b^2} > \eta^{2/3} \implies z > \frac{\eta^{1/3} b}{\sqrt{1-\eta^{2/3}}}. \tag{12.488}$$

For $\eta = 0.99$ this gives $z > (12.2)b$. You can show that if we want the factor to be larger than $1 - \epsilon$ (so $\epsilon = 0.01$ here), then to a good approximation (in the limit of small ϵ) we need $z/b > \sqrt{3/2\epsilon}$. And indeed, $\sqrt{3/2(0.01)} = \sqrt{150} = 12.2$.

11.3 *Dipole in spherical coordinates*

Using the $\nabla \times (\mathbf{A} \times \mathbf{B})$ vector identity from Appendix K, with $\mathbf{m}$ constant, we find (ignoring the $\mu_0/4\pi$ for now)

$$\mathbf{B} \propto \nabla \times \big[\mathbf{m} \times (\hat{\mathbf{r}}/r^2)\big] = \mathbf{m}\big(\nabla \cdot (\hat{\mathbf{r}}/r^2)\big) - (\mathbf{m} \cdot \nabla)(\hat{\mathbf{r}}/r^2). \tag{12.489}$$

But the divergence of $\hat{\mathbf{r}}/r^2$ is zero (except at $r = 0$), because we know that the divergence of the Coulomb field is zero; alternatively we can just use the expression for the divergence in spherical coordinates. So we are left with only the second term. Therefore, using the expression for ∇ in spherical coordinates,

$$\mathbf{B} \propto -\left(m_r \frac{\partial}{\partial r} + m_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right)\frac{\hat{\mathbf{r}}}{r^2}. \tag{12.490}$$

In the $\partial/\partial r$ term here, the vector $\hat{\mathbf{r}}$ doesn't depend on r, but r^2 does, of course, so $m_r(\partial/\partial r)(\hat{\mathbf{r}}/r^2) = -2m_r\hat{\mathbf{r}}/r^3$. In the $\partial/\partial\theta$ term, r^2 doesn't depend of θ, but the vector $\hat{\mathbf{r}}$ *does*. If we increase θ by $d\theta$, then $\hat{\mathbf{r}}$ changes direction by the angle $d\theta$. Since $\hat{\mathbf{r}}$ has length 1, it therefore picks up a component with length $d\theta$ in the $\hat{\boldsymbol{\theta}}$ direction. See Fig. F.3 in Appendix F; that figure is relevant to the oppositely defined θ in cylindrical coordinates, but the result is the same. Hence $\partial\hat{\mathbf{r}}/\partial\theta = \hat{\boldsymbol{\theta}}$. So we have $(m_\theta/r)(\partial/\partial\theta)(\hat{\mathbf{r}}/r^2) = m_\theta\hat{\boldsymbol{\theta}}/r^3$.

Finally, in Fig. 12.133 we see that the components of the fixed vector $\mathbf{m} = m\hat{\mathbf{z}}$ relative to the local $\hat{\mathbf{r}}$-$\hat{\boldsymbol{\theta}}$ basis are $m_r = m\cos\theta$, and $m_\theta = -m\sin\theta$. The negative sign here comes from the fact that $\mathbf{m}$ points

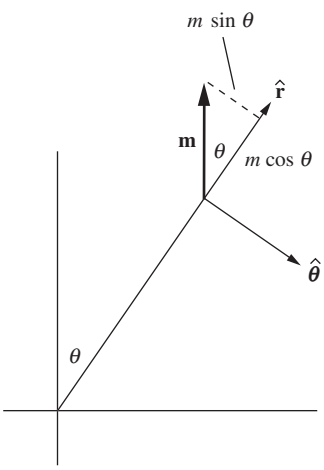


Figure 12.133.

partially in the direction of decreasing θ (at least for the right half of the sphere). Putting this all together, and bringing the $\mu_0/4\pi$ back in, gives

$$\begin{aligned}\mathbf{B} &= -\frac{\mu_0}{4\pi}\left(-2(m\cos\theta)\frac{\hat{\mathbf{r}}}{r^3} + (-m\sin\theta)\frac{\hat{\boldsymbol{\theta}}}{r^3}\right) \\ &= \hat{\mathbf{r}}\frac{\mu_0 m}{2\pi r^3}\cos\theta + \hat{\boldsymbol{\theta}}\frac{\mu_0 m}{4\pi r^3}\sin\theta,\end{aligned} \tag{12.491}$$

in agreement with Eq. (11.15).

11.4 *Force on a dipole*

(a) The expression $(\mathbf{m}\cdot\nabla)\mathbf{B}$ is shorthand for

$$(\mathbf{m}\cdot\nabla)\mathbf{B} = \left(m_x\frac{\partial}{\partial x} + m_y\frac{\partial}{\partial y} + m_z\frac{\partial}{\partial z}\right)(B_x, B_y, B_z). \tag{12.492}$$

The operator in parentheses is to be applied to each of the three components of $\mathbf{B}$, generating the three components of a vector. In the setup in Fig. 11.9 with the ring and diverging $\mathbf{B}$ field, m_z is the only nonzero component of $\mathbf{m}$. Also, B_x and B_y are identically zero on the z axis, so $\partial B_x/\partial z$ and $\partial B_y/\partial z$ are both zero (or negligibly small close to the z axis). Therefore only one of the nine possible terms in Eq. (12.492) survives, and we have

$$(\mathbf{m}\cdot\nabla)\mathbf{B} = \left(0, 0, m_z\frac{\partial B_z}{\partial z}\right), \tag{12.493}$$

as desired.

(b) The expression $\nabla(\mathbf{m}\cdot\mathbf{B})$ is shorthand for

$$\nabla(\mathbf{m}\cdot\mathbf{B}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(m_x B_x + m_y B_y + m_z B_z). \tag{12.494}$$

Each derivative acts on the whole sum in the parentheses. But again, only m_z is nonzero. Also, on the z axis, B_z doesn't depend on x or y, to first order (because, by symmetry, B_z achieves a maximum or minimum on the z axis, so the slope as a function of x and y must be zero). Hence $\partial B_z/\partial x$ and $\partial B_z/\partial y$ are both zero (or negligibly small close to the z axis). So again only one term survives and we have

$$\nabla(\mathbf{m}\cdot\mathbf{B}) = \left(0, 0, m_z\frac{\partial B_z}{\partial z}\right), \tag{12.495}$$

as desired.

(c) Let's first see what the two expressions yield for the force on the given square loop. Then we will calculate what the force actually is. The dipole moment $\mathbf{m}$ points out of the page with magnitude I(area), so we have $\mathbf{m} = \hat{\mathbf{z}}Ia^2$. Using the above expressions for $(\mathbf{m}\cdot\nabla)\mathbf{B}$ and $\nabla(\mathbf{m}\cdot\mathbf{B})$ in Eqs. (12.492) and (12.494), we obtain

$$(\mathbf{m}\cdot\nabla)\mathbf{B} = \left(0 + 0 + (Ia^2)\frac{\partial}{\partial z}\right)(0, 0, B_0 x) = (0, 0, 0) \tag{12.496}$$

and

$$\nabla(\mathbf{m}\cdot\mathbf{B}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\big(0 + 0 + (Ia^2)B_0 x\big) = (Ia^2 B_0, 0, 0). \tag{12.497}$$

We see that the first expression yields zero force on the loop, while the second yields a force of Ia^2B_0 in the positive x direction.

Let's now explicitly calculate the force. We quickly find that the net force on the top side of the square is zero (the right half cancels the left half). Likewise for the bottom side. Alternatively, the corresponding pieces of the top and bottom sides have canceling forces. So we need only look at the left and right sides. By the right-hand rule, the force on the right side is directed to the right with magnitude $IB\ell = I(B_0 a/2)(a) = IB_0 a^2/2$. The force on the left side also points to the right (both I and $\mathbf{B}$ switch sign) with the same magnitude. The total force is therefore $F = IB_0 a^2$ in the positive x direction, in agreement with Eq. (12.497). So $\nabla(\mathbf{m}\cdot\mathbf{B})$ is the correct expression for the force. (Actually, all that we've done is rule out the $(\mathbf{m}\cdot\nabla)\mathbf{B}$ force. But $\nabla(\mathbf{m}\cdot\mathbf{B})$ is in fact correct in all cases.)

11.5 *Converting χ_m*

Consider a setup in which the SI quantities are $M = 1$ amp/m and $B = 1$ tesla. Then $\chi_m = \mu_0 M/B = 4\pi\cdot 10^{-7}$. You can verify that the units do indeed cancel so that χ_m is dimensionless.

How would someone working with Gaussian units describe this setup? Since 1 amp/m equals $(3\cdot 10^9 \text{ esu/s})/(100\,\text{cm})$, this would be the value of M in Gaussian units if there weren't the extra factor of c in the definition of m. This factor reduces the value of all dipole moments m (and hence all magnetizations M) by $3\cdot 10^{10}$ cm/s. The value of M in Gaussian units is therefore

$$M = \frac{3\cdot 10^9 \text{ esu/s}}{100\,\text{cm}}\,\frac{1}{3\cdot 10^{10}\,\text{cm/s}} = 10^{-3}\,\frac{\text{esu}}{\text{cm}^2}. \tag{12.498}$$

Both of the factors of 3 here are actually 2.998, so this result is exact.

The magnetic field in Gaussian units that corresponds to 1 tesla is 10^4 gauss, so the susceptibility in Gaussian units for the given setup is

$$\chi_m = \frac{M}{B} = \frac{10^{-3}\,\text{esu/cm}^2}{10^4\,\text{gauss}} = 10^{-7}\,\frac{\text{esu}}{\text{cm}^2\,\text{gauss}} = 10^{-7}. \tag{12.499}$$

The units do indeed cancel, because the expression for the Lorentz force tells us that a gauss has the units of force per charge. So the units of χ_m are $\text{esu}^2/(\text{cm}^2\cdot\text{force})$. And these units cancel, as you can see by looking at the units in Coulomb's law. The above value of χ_m in SI units was $4\pi\cdot 10^{-7}$, which is 4π times the Gaussian value, as desired.

11.6 *Paramagnetic susceptibility of liquid oxygen*

Equation (11.20) gives the force on a magnetic moment as $F = m(\partial B_z/\partial z)$. Using the data in Table 11.1, and taking upward to be positive for all quantities, the magnetic moment of a 10^{-3} kg sample is

$$m = \frac{F}{\partial B_z/\partial z} = \frac{-7.5 \cdot 10^{-2}\ \text{N}}{-17\ \text{T/m}} = 4.4 \cdot 10^{-3}\ \text{J/T}. \tag{12.500}$$

The magnetic susceptibility is defined via $\mathbf{M} = \chi_m \mathbf{B}/\mu_0$. (The accepted $\mathbf{M} = \chi_m \mathbf{H}$ definition would give essentially the same result, because χ_m will turn out to be very small. See Exercise 11.38.) The volume of 1 gram of liquid oxygen is $V = (10^{-3}\ \text{kg})/(850\ \text{kg/m}^3) = 1.18 \cdot 10^{-6}\ \text{m}^3$. So

$$\begin{aligned} \chi_m &= \frac{M}{B/\mu_0} = \frac{(m/V)}{B/\mu_0} = \frac{m\mu_0}{BV} \\ &= \frac{(4.4 \cdot 10^{-3}\ \text{J/T})(4\pi \cdot 10^{-7}\ \text{kg m/C}^2)}{(1.8\ \text{T})(1.18 \cdot 10^{-6}\ \text{m}^3)} = 2.6 \cdot 10^{-3}. \end{aligned} \tag{12.501}$$

11.7 *Rotating shell*

For the magnetized sphere, we know from Eq. (11.55) that near the equator the surface current density is equal to M, because the sphere looks essentially like a cylinder there (the surface is parallel to $\mathbf{M}$). But away from the equator, the surface is tilted with respect to $\mathbf{M}$. From the example at the end of Section 11.8, the surface current density is given by $\mathcal{J} = M_{\parallel} \implies \mathcal{J}(\theta) = M \sin\theta$, where θ is the angle down from the top of the sphere (assuming that $\mathbf{M}$ points up).

Now consider a rotating sphere with uniform surface charge density σ. The surface current density at any point is $\mathcal{J} = \sigma v$, where $v = \omega(R\sin\theta)$ is the speed due to the rotation. Hence $\mathcal{J}(\theta) = \sigma\omega R \sin\theta$. The $\mathcal{J}(\theta)$ expressions for the magnetized and rotating spheres have the same functional dependence on θ, so they will be equal for all θ provided that $M = \sigma\omega R$.

11.8 *B inside a magnetized sphere*

(a) The field in Eq. (11.15) is obtained from the field in Eq. (10.18) by letting $p \to m$ and $\epsilon_0 \to 1/\mu_0$. If we replace all the electric dipoles p in a polarized sphere with magnetic dipoles m, then at an external point, the field from each dipole is simply multiplied by $(m/p)(\mu_0\epsilon_0)$. The integral over all the dipole fields is multiplied by this same factor, so the new magnetic field at any external point equals $(m/p)(\mu_0\epsilon_0)$ times the old electric field. We know from Section 10.9 that the old external electric field is the same as the field due to an electric dipole with strength $p_0 = (4\pi R^3/3)P$, with $P = Np$, located at the center. You can quickly check that $(m/p)(\mu_0\epsilon_0)$ times this field is the same as the magnetic field due to a magnetic dipole with strength $m_0 = (4\pi R^3/3)M$, with $M = Nm$.

(b) If $\mathbf{m}_0$ points in the z direction, then from Eq. (11.12) the Cartesian components of $\mathbf{A}$ at points (x, y, z) on the surface of the sphere are

$$\begin{aligned} A_x &= -\frac{\mu_0}{4\pi}\frac{m_0 y}{R^3} = -\mu_0 \frac{My}{3}, \\ A_y &= \frac{\mu_0}{4\pi}\frac{m_0 x}{R^3} = \mu_0 \frac{Mx}{3}, \\ A_z &= 0. \end{aligned} \tag{12.502}$$

Note that the result from Problem 11.7 then tells us that the **A** on the surface of a spinning spherical shell equals $(\mu_0\sigma\omega R/3)(-y, x, 0)$. This agrees with the **A** we found in a different manner in Problem 6.7.

Recall from Section 6.3 that A_x satisfies $\nabla^2 A_x = -\mu_0 J_x$. And similarly for A_y. But $\mathbf{J} = 0$ inside the sphere, so both A_x and A_y satisfy Laplace's equation there. By the uniqueness theorem, this means that if we can find a solution to Laplace's equation inside the sphere that satisfies the boundary conditions on the surface of the sphere, then we know that we have found *the* solution. And just as with the ϕ for the polarized sphere in Section 10.9, the solutions for A_x and A_y are easy to come by. They are simply the functions given in Eq. (12.502); their second derivatives are zero, so they each satisfy Laplace's equation. The magnetic field inside the sphere is then

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0 M}{3} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = \frac{2\mu_0 M}{3}\hat{\mathbf{z}}. \quad (12.503)$$

Like the **E** inside the polarized sphere, this **B** is uniform and points vertically. But that is where the similarities end. This **B** field points upward, whereas the old **E** field pointed downward. Additionally, the numerical factor here is $2/3$, whereas it was (negative) $1/3$ in **E**. The $2/3$ is exactly what is needed to make the component normal to the surface be continuous, and to make the tangential component have the proper discontinuity (see Exercise 11.31).

Equation (12.503), combined with the result from Problem 11.7, tells us that the field throughout the interior of a spinning spherical shell is uniform and has magnitude $2\mu_0\sigma\omega R/3$. This is consistent with the result from Problem 6.11 for the field at the center of the sphere.

11.9 *B at the north pole of a spinning sphere*

From Problem 11.7, we know that the magnetic field due to a spinning shell with radius r and uniform surface charge density σ is the same (both inside and outside) as the field due to a sphere with uniform magnetization $M_r = \sigma\omega r$. And then from Problem 11.8 we know that the external field of a magnetized sphere is that of a dipole with strength $m = (4\pi r^3/3)M_r$ located at the center. So the (radial) field at radius R outside a spinning shell with radius r (at a point located above the north pole) is

$$B = \frac{\mu_0 m}{2\pi R^3} = \frac{\mu_0}{2\pi R^3}\frac{4\pi r^3(\sigma\omega r)}{3} = \frac{2\mu_0\sigma\omega r^4}{3R^3}. \quad (12.504)$$

We can consider the solid spinning sphere to be the superposition of many spinning shells with radii ranging from $r = 0$ to $r = R$, with uniform surface charge density $\sigma = \rho\, dr$. The north pole of the solid sphere is outside all of the shells, so we can use the above dipole form of B for every shell. The total field at the north pole (that is, at radius R) is therefore

$$B = \int_0^R \frac{2\mu_0(\rho\, dr)\omega r^4}{3R^3} = \frac{2\mu_0\rho\omega R^2}{15}. \quad (12.505)$$

This field is $2/5$ as large as the field at the center of the sphere; see Exercise 11.32. In terms of the total charge $Q = (4\pi R^3/3)\rho$, we can write B as $B = \mu_0 \omega Q/10\pi R$.

11.10 *Surface current on a cube*

Equation (11.55) gives the surface current density as $\mathcal{J} = M$. Since the units of magnetization (J/Tm^3) can also be written as A/m, we have $\mathcal{J} = 4.8 \cdot 10^5$ A/m. This current density spans a ribbon that is $\ell = 0.05$ m wide, so the current is $I = \mathcal{J}\ell = (4.8 \cdot 10^5 \text{ A/m})(0.05 \text{ m}) = 24{,}000$ A.

The dipole moment of the cube is

$$m = MV = (4.8 \cdot 10^5 \text{ J T}^{-1} \text{ m}^{-3})(0.05 \text{ m})^3 = 60 \text{ J/T}. \tag{12.506}$$

The field at a distance of 2 meters, along the axis, is given by Eq. (11.15) as

$$B = \frac{\mu_0 m}{2\pi r^3} = \frac{(4\pi \cdot 10^{-7} \text{ kg m/C}^2)(60 \text{ J/T})}{2\pi (2 \text{ m})^3} = 1.5 \cdot 10^{-6} \text{ T}, \tag{12.507}$$

or 0.015 gauss. This is about 30 times smaller than the earth's field of ≈ 0.5 gauss, so it wouldn't disturb a compass much.

11.11 *An iron torus*

From Fig. 11.32, a B field of 1.2 tesla requires an H field of about 120 A/m. Consider the line integral $\int \mathbf{H} \cdot d\boldsymbol{l}$ around the "middle" circle of the solenoid, with diameter 11 cm. If I is the current in the wire, then $NI = 20I$ is the free current enclosed by our circular loop. Therefore,

$$\int \mathbf{H} \cdot d\boldsymbol{l} = I_{\text{free}} \implies (120 \text{ A/m}) \cdot \pi (0.11 \text{ m}) = 20I \implies I = 2.1 \text{ A}. \tag{12.508}$$

A

Differences between SI and Gaussian units

In this appendix we discuss the differences between the SI and Gaussian systems of units. First, we will look at the units in each system, and then we will talk about the clear and not so clear ways in which they differ.

A.1 SI units

Consider the SI system, which is the one we use in this book. The four main SI units that we deal with are the meter (m), kilogram (kg), second (s), and coulomb (C). The coulomb actually isn't a fundamental SI unit; it is defined in terms of the ampere (A), which is a measure of current (charge per time). The coulomb is a derived unit, defined to be 1 ampere-second.

The reason why the ampere, and not the coulomb, is the fundamental unit involving charge is one of historical practicality. It is relatively easy to measure current via a galvanometer (see Section 7.1). More crudely, a current can be determined by measuring the magnetic force that two pieces of a current-carrying wire in a circuit exert on each other (see Fig. 6.4). Once we determine the current that flows onto an object during a given time, we can then determine the charge on the object. On the other hand, although it is possible to measure charge directly via the force that two equally charged objects exert on each other (imagine two balls hanging from strings, repelling each other, as in Exercise 1.36), the setup is a bit cumbersome. Furthermore, it tells us only what the *product* of the charges is, in the event that they aren't equal. The point is that it is

easy to measure current by hooking up an ammeter (the main component of which is a galvanometer) to a circuit.[1]

The exact definition of an ampere is: if two parallel wires carrying equal currents are separated by 1 meter, and if the force per meter on one wire, due to the entirety of the other wire, is $2 \cdot 10^{-7}$ newtons, then the current in each wire is 1 ampere. The power of 10 here is an arbitrary historical artifact, as is the factor of 2. This force is quite small, but by decreasing the separation the effect can be measured accurately enough with the setup shown in Fig. 6.4.

Having defined the ampere in this manner, and then having defined the coulomb as 1 ampere-second (which happens to correspond to the negative of the charge of about $6.24 \cdot 10^{18}$ electrons), a reasonable thing to do, at least in theory, is to find the force between two 1 coulomb charges located, say, 1 meter apart. Since the value of 1 coulomb has been fixed by the definition of the ampere, this force takes on a particular value. We are not free to adjust it by tweaking any definitions. It happens to be about $9 \cdot 10^9$ newtons – a seemingly arbitrary number, but in fact related to the speed of light. (It has the numerical value of $c^2/10^7$; we see why in Section 6.1.) This (rather large) number therefore appears out in front of Coulomb's law. We could label this constant with one letter, such as "k," but for various reasons it is labeled as $1/4\pi\epsilon_0$, with $\epsilon_0 = 8.85 \cdot 10^{-12}\,\mathrm{C^2\,s^2\,kg^{-1}\,m^{-3}}$. These units are what are needed to make the right-hand side of Coulomb's law, $F = (1/4\pi\epsilon_0)q_1q_2/r^2$, have units of newtons (namely $\mathrm{kg\,m\,s^{-2}}$). In terms of the fundamental ampere unit, the units of ϵ_0 are $\mathrm{A^2\,s^4\,kg^{-1}\,m^{-3}}$.

The upshot of all this is that because we made the choice to define current via the Lorentz force (specifically, the magnetic part of the Lorentz force) between two wires carrying current I, the Coulomb force between two objects of charge q ends up being a number that we just have to accept. We can make the pre-factor be a nice simple number in either one of these force laws, but not both.[2] The SI system gives preference to the Lorentz force, due to the historical matters of practicality mentioned above.

It turns out that there are actually seven fundamental units in the SI system. They are listed in Table A.1. The candela isn't relevant to our study of electromagnetism, and the mole and kelvin come up only occasionally. So for our purposes the SI system consists of essentially just the first four units.

[1] If we know the capacitance of an object, then we *can* easily measure the charge on it by measuring the voltage with a voltmeter. But the main component of a voltmeter is again a galvanometer, so the process still reduces to measuring a current.

[2] The Biot–Savart law, which allows us to calculate the magnetic field that appears in the Lorentz force, contains what appears to be a messy pre-factor, namely $\mu_0/4\pi$. But since μ_0 is defined to be exactly $4\pi \cdot 10^{-7}\,\mathrm{kg\,m/C^2}$, this pre-factor takes on the simple value of $10^{-7}\,\mathrm{kg\,m/C^2}$.

Table A.1.
SI base units

Quantity	Name	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Thermodynamic temperature	kelvin	K
Amount of substance	mole	mol
Luminous intensity	candela	cd

A.2 Gaussian units

What do the units look like in the Gaussian system? As with the SI system, the last three of the above units (or their analogs) rarely come up, so we will ignore them. The first two units are the centimeter and gram. These differ from the SI units simply by a few powers of 10, so it is easy to convert from one system to the other. The third unit, the second, is the same in both systems.

The fourth unit, that of charge, is where the two systems fundamentally diverge. The Gaussian unit of charge is the esu (short for "electrostatic unit"), which isn't related to the coulomb by a simple power of 10. The reason for this non-simple relation is that the coulomb and esu are defined in different ways. The coulomb is a derivative unit of the ampere (which is defined via the Lorentz force) as we saw above, whereas the esu is defined via the Coulomb force. In particular, it is defined so that Coulomb's law,

$$\mathbf{F} = k\frac{q_1 q_2 \hat{\mathbf{r}}}{r^2}, \tag{A.1}$$

takes on a very simple form with $k = 1$. The price to pay for this simple form of the Coulomb force is the not as simple form of the Lorentz force between two current-carrying wires (although it isn't so bad; like the Coulomb force in SI units, it just involves a factor of c^2; see Eq. (6.16)). This is the opposite of the situation with the SI system, where the Lorentz force is the "nice" one. Again, in each system we are free to define things so that one, but not both, of the Lorentz force and Coulomb force takes on a nice form.

A.3 Main differences between the systems

In Section A.2 we glossed over what turns out to be the most important difference between the SI and Gaussian systems. In the SI system, the constant in Coulomb's law,

$$k_{\text{SI}} \equiv \frac{1}{4\pi\epsilon_0} = 8.988 \cdot 10^9 \ \frac{\text{N m}^2}{\text{C}^2}, \tag{A.2}$$

has nontrivial dimensions, whereas in the Gaussian system the constant

$$k_G = 1 \tag{A.3}$$

is *dimensionless*. We aren't just being sloppy and forgetting to write the units; k is simply the number 1. Although the first thing that may strike you about the two k constants is the large difference in their numerical values, this difference is fairly inconsequential. It simply changes the numerical size of various quantities. The truly fundamental and critical difference is that k_{SI} has units whereas k_G does not. We could, of course, imagine a system of units where $k = 1$ dyne-cm^2/esu^2. This definition would parallel the units of k_{SI}, with the only difference being the numerical value. But this is not what the Gaussian system does.

The reason why the dimensionlessness of k_G creates such a profound difference between the two systems is that it allows us to *solve for the esu in terms of other Gaussian units.* In particular, from looking at the units in Coulomb's law, we can write (using 1 dyne $= 1\,\text{g}\cdot\text{cm/s}^2$)

$$\text{dyne} = (\textit{dimensionless}) \cdot \frac{\text{esu}^2}{\text{cm}^2} \quad \Longrightarrow \quad \text{esu} = \sqrt{\frac{\text{g}\cdot\text{cm}^3}{\text{s}^2}}. \tag{A.4}$$

The esu is therefore not a fundamental unit. It can be expressed in terms of the gram, centimeter, and second. In contrast, the SI unit of charge, the coulomb, cannot be similarly expressed. Since k_{SI} has units of $\text{N}\,\text{m}^2/\text{C}^2$, the C's (and everything else) cancel in Coulomb's law, and we can't solve for C in terms of other units.

For our purposes, therefore, the SI system has four fundamental units (m, kg, s, A), whereas the Gaussian system has only three (cm, g, s). We will talk more about this below, but first let us summarize the three main differences between the SI and Gaussian systems. We state them in order of increasing importance.

(1) The SI system uses kilograms and meters, whereas the Gaussian system uses grams and centimeters. This is the most trivial of the three differences, because all it does is introduce some easily dealt with powers of 10.

(2) The SI unit of charge (the coulomb) is defined via the ampere, which in turn is defined in terms of the force between current-carrying wires. The Gaussian unit of charge (the esu) is defined directly in terms of Coulomb's law. This latter definition is the reason why Coulomb's law takes on a nicer form in Gaussian units. The differences between the two systems now involve more than simple powers of 10. However, although these differences can sometimes be a hassle, they aren't terribly significant. They are just numbers – no different from powers of 10, except a little messier. All of the conversions you might need to use are listed in Appendix C.

(3) In Gaussian units, the k in Coulomb's law is chosen to be *dimensionless*, whereas in SI units the k (which involves ϵ_0) has units.[3] The result is that the esu can be expressed in terms of other Gaussian units, whereas the analogous statement is not true for the coulomb. This is the most important difference between the two systems.

A.4 Three units versus four

Let us now discuss in more detail the issue of the number of units in each system. The Gaussian system has one fewer because the esu can be expressed in terms of other units via Eq. (A.4). This has implications with regard to checking units at the end of a calculation. In short, less information is gained when checking units in the Gaussian system, because the charge information is lost when the esu is written in terms of the other units. Consider the following example.

In SI units the electric field due to a sheet of charge is given in Eq. (1.40) as $\sigma/2\epsilon_0$. In Gaussian units the field is $2\pi\sigma$. Recalling the units of ϵ_0 in Eq. (1.3), the units of the SI field are $\text{kg m C}^{-1}\text{s}^{-2}$ (or $\text{kg m A}^{-1}\text{s}^{-3}$ if you want to write it in terms of amperes, but we use coulombs here to show analogies with the esu). This correctly has dimensions of (force)/(charge). The units of the Gaussian $2\pi\sigma$ field are simply esu/cm^2, but since the esu is given by Eq. (A.4), the units are $\text{g}^{1/2}\text{cm}^{-1/2}\,\text{s}^{-1}$. These are the true Gaussian units of the electric field when written in terms of fundamental units.

Now let's say that two students working in the Gaussian system are given a problem where the task is to find the electric field due to a thin sheet with charge density σ, mass m, volume V, moving with a nonrelativistic speed v. The first student realizes that most of this information is irrelevant and solves the problem correctly, obtaining the answer of $2\pi\sigma$ (ignoring relativistic corrections). The second student royally messes things up and obtains an answer of $\sigma^3 V m^{-1} v^{-2}$. Since the fundamental Gaussian units of σ are $\text{g}^{1/2}\,\text{cm}^{-1/2}\,\text{s}^{-1}$, the units of this answer are

$$\frac{\sigma^3 V}{mv^2} \longrightarrow \frac{\left(\text{g}^{1/2}\,\text{cm}^{-1/2}\,\text{s}^{-1}\right)^3 (\text{cm})^3}{(\text{g})(\text{cm/s})^2} = \frac{\text{g}^{1/2}}{\text{cm}^{1/2}\,\text{s}}, \tag{A.5}$$

which are the correct Gaussian units of electric field that we found above. More generally, in view of Eq. (A.4) we see that any answer with the units of $(\text{g}^{1/2}\,\text{cm}^{-1/2}\,\text{s}^{-1})\left(\text{esu g}^{-1/2}\,\text{cm}^{-3/2}\,\text{s}\right)^n$ has the correct units for the field. The present example has $n = 3$.

There are, of course, also many ways to obtain incorrect answers in the SI system that just happen by luck to have the correct units. Correctness of the units doesn't guarantee correctness of the answer. But the

[3] To draw a more accurate analogy: in SI units the defining equation for the ampere (from which the coulomb is derived) contains the *dimensionful* constant μ_0 in the force between two wires.

point is that because the charge information is swept under the rug in Gaussian units, we have at our disposal the information of only three fundamental units instead of four. Compared with the SI system, there is therefore a larger class of incorrect answers in the Gaussian system that have the correct units.

A.5 The definition of B

Another difference between the SI and Gaussian systems of units is the way in which the magnetic field is defined. In SI units the Lorentz force (or rather the magnetic part of it) is $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, whereas in Gaussian units it is $\mathbf{F} = (q/c)\mathbf{v} \times \mathbf{B}$. This means that wherever a B appears in an SI expression, a B/c appears in the corresponding Gaussian expression (among other possible modifications). Or equivalently, a Gaussian B turns into an SI cB. This difference, however, is a trivial definitional one and has nothing to do with the far more important difference discussed above, where the esu can be expressed in terms of other Gaussian units.

In the Gaussian system, E and B have the same dimensions. In the SI system they do not; the dimensions of E are velocity times the dimensions of B. In this sense the Gaussian definition of B is more natural, because it makes sense for two quantities to have the same dimensions if they are related by a Lorentz transformation, as the **E** and **B** fields are; see Eq. (6.76) for the SI case and Eq. (6.77) for the Gaussian case. After all, the Lorentz transformation tells us that the **E** and **B** fields are simply different ways of looking at the same field, depending on the frame of reference. However, having a "c**B**" instead of a "**B**" in the SI Lorentz transformation can't be so bad, because x and t are also related by a Lorentz transformation, and they don't have the same dimensions (the direct correspondence is between x and ct). Likewise for p and E (where the direct correspondence is between pc and E). At any rate, this issue stems from the arbitrary choice of whether a factor of c is included in the expression for the Lorentz force. One can easily imagine an SI-type system (where charge is a distinct unit) in which the Lorentz force takes the form $\mathbf{F} = q\mathbf{E} + (q/c)\mathbf{v} \times \mathbf{B}$, yielding the same dimensions for E and B.

A.6 Rationalized units

You might wonder why there are factors of 4π in the SI versions of Coulomb's law and the Biot–Savart law; see Eqs. (1.4) and (6.49). These expressions would certainly look a bit less cluttered without these factors. The reason is that the presence of 4π's in these laws leads to the absence of such factors in Maxwell's equations. And for various reasons it is deemed more important to have Maxwell's equations be the "clean" ones without the 4π factors. The procedure of inserting 4π into Coulomb's law and the Biot–Savart law, in order to keep them out of Maxwell's equations, is called "rationalizing" the units. Of course, for

people concerned more with applications of Coulomb's law than with Maxwell's equations, this procedure might look like a step in the wrong direction. But since Maxwell's equations are the more fundamental equations, there is logic in this convention.

It is easy to see why the presence of 4π factors in Coulomb's law and the Biot–Savart law leads to the absence of 4π factors in Gauss's law and Ampère's law, which are equivalent to two of Maxwell's equations (or actually one and a half; Ampère's law is supplemented with another term). In the case of Gauss's law, the absence of the 4π basically boils down to the area of a sphere being $4\pi r^2$ (see the derivation in Section 1.10). In the case of Ampère's law, the absence of the 4π is a consequence of the reasoning in Sections 6.3 and 6.4, which again boils down to the area of a sphere being $4\pi r^2$ (because Eq. (6.44) was written down by analogy with Eq. (6.30)). Or more directly: the $1/4\pi$ in the Biot–Savart law turns into a $1/2\pi$ in the field from an infinite straight wire (see Eq. (6.6)), and this 2π is then canceled when we take the line integral around a circle with circumference $2\pi r$.

If there were no factors of 4π in Coulomb's law or the Biot–Savart law, then there *would* be factors of 4π in Maxwell's equations. This is exactly what happens in the Gaussian system, where the "curl $\mathbf{B}$" and "div $\mathbf{E}$" Maxwell equations each involve a 4π; see Eq. (9.20). Note, however, that one can easily imagine a Gaussian-type system (that is, one where the pre-factor in Coulomb's law is dimensionless) that has factors of 4π in Coulomb's law and the Biot–Savart law, and none in Maxwell's equations. This is the case in a variation of Gaussian units called Heaviside–Lorentz units.

B

SI units of common quantities

We begin this appendix with the definitions of all of the derived SI units relevant to electromagnetism (for example, the joule, ohm, etc.). We then list the units of all of the main quantities that appear in this book (basically, anything that has earned the right to be labeled with its own letter).

In SI units the ampere is the fundamental unit involving charge. The coulomb is a derived unit, being defined as one ampere-second. However, since most people find it more natural to think in terms of charge than current, we treat the coulomb as the fundamental unit in this appendix. The ampere is then defined as one coulomb/second.

For each of the main quantities listed, we give the units in terms of the fundamental units (m, kg, s, C, and occasionally K), and then also in terms of other derived units in certain forms that come up often. For example, the units of electric field are $\mathrm{kg\,m\,C^{-1}s^{-2}}$, but they are also newtons/coulomb and volts/meter.

The various derived units are as follows:

$$\text{newton (N)} = \frac{\mathrm{kg\,m}}{\mathrm{s^2}}$$

$$\text{joule (J)} = \text{newton-meter} = \frac{\mathrm{kg\,m^2}}{\mathrm{s^2}}$$

$$\text{ampere (A)} = \frac{\text{coulomb}}{\text{second}} = \frac{\mathrm{C}}{\mathrm{s}}$$

$$\text{volt (V)} = \frac{\text{joule}}{\text{coulomb}} = \frac{\mathrm{kg\,m^2}}{\mathrm{C\,s^2}}$$

$$\text{farad (F)} = \frac{\text{coulomb}}{\text{volt}} = \frac{\text{C}^2\,\text{s}^2}{\text{kg}\,\text{m}^2}$$

$$\text{ohm } (\Omega) = \frac{\text{volt}}{\text{ampere}} = \frac{\text{kg}\,\text{m}^2}{\text{C}^2\,\text{s}}$$

$$\text{watt (W)} = \frac{\text{joule}}{\text{second}} = \frac{\text{kg}\,\text{m}^2}{\text{s}^3}$$

$$\text{tesla (T)} = \frac{\text{newton}}{\text{coulomb} \cdot \text{meter/second}} = \frac{\text{kg}}{\text{C}\,\text{s}}$$

$$\text{henry (H)} = \frac{\text{volt}}{\text{ampere/second}} = \frac{\text{kg}\,\text{m}^2}{\text{C}^2}$$

The main quantities are listed by chapter.

Chapter 1

$$\text{charge } q\text{: C}$$

$$k \text{ in Coulomb's law: } \frac{\text{kg}\,\text{m}^3}{\text{C}^2\,\text{s}^2} = \frac{\text{N}\,\text{m}^2}{\text{C}^2}$$

$$\epsilon_0\text{: } \frac{\text{C}^2\,\text{s}^2}{\text{kg}\,\text{m}^3} = \frac{\text{C}^2}{\text{N}\,\text{m}^2} = \frac{\text{C}}{\text{V}\,\text{m}} = \frac{\text{F}}{\text{m}}$$

$$E \text{ field (force per charge): } \frac{\text{kg}\,\text{m}}{\text{C}\,\text{s}^2} = \frac{\text{N}}{\text{C}} = \frac{\text{V}}{\text{m}}$$

$$\text{flux } \Phi \text{ } (E \text{ field times area}): \frac{\text{kg}\,\text{m}^3}{\text{C}\,\text{s}^2} = \frac{\text{N}\,\text{m}^2}{\text{C}} = \text{V}\,\text{m}$$

$$\text{charge density } \lambda, \sigma, \rho\text{: } \frac{\text{C}}{\text{m}}, \frac{\text{C}}{\text{m}^2}, \frac{\text{C}}{\text{m}^3}$$

Chapter 2

$$\text{potential } \phi \text{ (energy per charge): } \frac{\text{kg}\,\text{m}^2}{\text{C}\,\text{s}^2} = \frac{\text{J}}{\text{C}} = \text{V}$$

$$\text{dipole moment } p\text{: C}\,\text{m}$$

Chapter 3

$$\text{capacitance } C \text{ (charge per potential): } \frac{\text{C}^2\,\text{s}^2}{\text{kg}\,\text{m}^2} = \frac{\text{C}}{\text{V}} = \text{F}$$

Chapter 4

current I (charge per time): $\dfrac{\mathrm{C}}{\mathrm{s}} = \mathrm{A}$

current density J (current per area): $\dfrac{\mathrm{C}}{\mathrm{m}^2\,\mathrm{s}} = \dfrac{\mathrm{A}}{\mathrm{m}^2}$

conductivity σ (current density per field): $\dfrac{\mathrm{C}^2\,\mathrm{s}}{\mathrm{kg}\,\mathrm{m}^3} = \dfrac{1}{\Omega\,\mathrm{m}}$

resistivity ρ (field per current density): $\dfrac{\mathrm{kg}\,\mathrm{m}^3}{\mathrm{C}^2\,\mathrm{s}} = \Omega\,\mathrm{m}$

resistance R (voltage per current): $\dfrac{\mathrm{kg}\,\mathrm{m}^2}{\mathrm{C}^2\,\mathrm{s}} = \dfrac{\mathrm{V}}{\mathrm{A}} = \Omega$

power P (energy per time): $\dfrac{\mathrm{kg}\,\mathrm{m}^2}{\mathrm{s}^3} = \dfrac{\mathrm{J}}{\mathrm{s}} = \mathrm{W}$

Chapter 5

speed of light c: $\dfrac{\mathrm{m}}{\mathrm{s}}$

Chapter 6

B field (force per charge-velocity): $\dfrac{\mathrm{kg}}{\mathrm{C}\,\mathrm{s}} = \mathrm{T}$

μ_0 : $\dfrac{\mathrm{kg}\,\mathrm{m}}{\mathrm{C}^2} = \dfrac{\mathrm{T}\,\mathrm{m}}{\mathrm{A}}$

vector potential A: $\dfrac{\mathrm{kg}\,\mathrm{m}}{\mathrm{C}\,\mathrm{s}} = \mathrm{T}\,\mathrm{m}$

surface current density $\mathcal{J}$ (current per length): $\dfrac{\mathrm{C}}{\mathrm{m}\,\mathrm{s}} = \dfrac{\mathrm{A}}{\mathrm{m}}$

Chapter 7

electromotive force $\mathcal{E}$: $\dfrac{\mathrm{kg}\,\mathrm{m}^2}{\mathrm{C}\,\mathrm{s}^2} = \dfrac{\mathrm{J}}{\mathrm{C}} = \mathrm{A}\,\Omega = \mathrm{V}$

flux Φ (B field times area): $\dfrac{\mathrm{kg}\,\mathrm{m}^2}{\mathrm{C}\,\mathrm{s}} = \mathrm{T}\,\mathrm{m}^2$

inductance M, L: $\dfrac{\mathrm{kg}\,\mathrm{m}^2}{\mathrm{C}^2} = \dfrac{\mathrm{V}\,\mathrm{s}}{\mathrm{A}} = \mathrm{H}$

Chapter 8

frequency ω: $\dfrac{1}{\mathrm{s}}$

quality factor Q: 1 (*dimensionless*)

phase ϕ: 1 (*dimensionless*)

admittance Y (current per voltage): $\dfrac{\mathrm{C}^2\,\mathrm{s}}{\mathrm{kg\,m}^2} = \dfrac{\mathrm{A}}{\mathrm{V}} = \dfrac{1}{\Omega}$

impedance Z (voltage per current): $\dfrac{\mathrm{kg\,m}^2}{\mathrm{C}^2\,\mathrm{s}} = \dfrac{\mathrm{V}}{\mathrm{A}} = \Omega$

Chapter 9

power density S (power per area): $\dfrac{\mathrm{kg}}{\mathrm{s}^3} = \dfrac{\mathrm{J}}{\mathrm{m}^2\,\mathrm{s}} = \dfrac{\mathrm{W}}{\mathrm{m}^2}$

Chapter 10

dielectric constant κ: 1 (*dimensionless*)

dipole moment p: $\mathrm{C\,m}$

torque N: $\dfrac{\mathrm{kg\,m}^2}{\mathrm{s}^2} = \mathrm{N\,m}$

atomic polarizability $\alpha/4\pi\epsilon_0$: m^3

polarization density P: $\dfrac{\mathrm{C}}{\mathrm{m}^2}$

electric susceptibility χ_e: 1 (*dimensionless*)

permittivity ϵ: $\dfrac{\mathrm{C}^2\,\mathrm{s}^2}{\mathrm{kg\,m}^3} = \dfrac{\mathrm{C}^2}{\mathrm{N\,m}^2}$

displacement vector D: $\dfrac{\mathrm{C}}{\mathrm{m}^2}$

temperature T: K

Boltzmann's constant k: $\dfrac{\mathrm{kg\,m}^2}{\mathrm{s}^2\,\mathrm{K}} = \dfrac{\mathrm{J}}{\mathrm{K}}$

Chapter 11

magnetic moment m: $\dfrac{\mathrm{C\,m^2}}{\mathrm{s}} = \mathrm{A\,m^2} = \dfrac{\mathrm{J}}{\mathrm{T}}$

angular momentum L: $\dfrac{\mathrm{kg\,m^2}}{\mathrm{s}}$

Planck's constant h: $\dfrac{\mathrm{kg\,m^2}}{\mathrm{s}} = \mathrm{J\,s}$

magnetization M (m per volume): $\dfrac{\mathrm{C}}{\mathrm{m\,s}} = \dfrac{\mathrm{A}}{\mathrm{m}} = \dfrac{\mathrm{J}}{\mathrm{T\,m^3}}$

magnetic susceptibility χ_m: 1 (*dimensionless*)

H field: $\dfrac{\mathrm{C}}{\mathrm{m\,s}} = \dfrac{\mathrm{A}}{\mathrm{m}}$

permeability μ: $\dfrac{\mathrm{kg\,m}}{\mathrm{C^2}} = \dfrac{\mathrm{T\,m}}{\mathrm{A}}$

C

Unit conversions

In this appendix we list, and then derive, the main unit conversions between the SI and Gaussian systems. As you will see below, many of the conversions involve simple plug-and-chug calculations involving conversions that are already known. However, a few of them (charge, B field, H field) require a little more thought, because the relevant quantities have different definitions in the two systems.

C.1 Conversions

Except for the first five (nonelectrical) conversions below, we technically shouldn't be using "=" signs, because they suggest that the units in the two systems are actually the same, up to a numerical factor. This is not the case. All of the electrical relations involve charge in one way or another, and a coulomb cannot be expressed in terms of an esu. This is a consequence of the fact that the esu is defined in terms of the other Gaussian units; see Appendix A for a discussion of how the coulomb and esu differ. The proper way to express, say, the sixth relation below is "1 coulomb is *equivalent* to $3 \cdot 10^9$ esu." But we'll generally just use the "=" sign, and you'll know what we mean.

The "[3]" in the following relations stands for the "2.998" that appears in the speed of light, $c = 2.998 \cdot 10^8$ m/s. The coulomb-esu discussion below explains how this arises.

time: 1 second = 1 second

length: 1 meter = 10^2 centimeter

mass: 1 kilogram = 10^3 gram

$$
\begin{aligned}
\textit{force:} \quad & 1 \text{ newton} = 10^5 \text{ dyne} \\
\textit{energy:} \quad & 1 \text{ joule} = 10^7 \text{ erg} \\
\textit{charge:} \quad & 1 \text{ coulomb} = [3] \cdot 10^9 \text{ esu} \\
\textit{E potential:} \quad & 1 \text{ volt} = \frac{1}{[3] \cdot 10^2} \text{ statvolt} \\
\textit{E field:} \quad & 1 \text{ volt/meter} = \frac{1}{[3] \cdot 10^4} \text{ statvolt/cm} \\
\textit{capacitance:} \quad & 1 \text{ farad} = [3]^2 \cdot 10^{11} \text{ cm} \\
\textit{resistance:} \quad & 1 \text{ ohm} = \frac{1}{[3]^2 \cdot 10^{11}} \text{ s/cm} \\
\textit{resistivity:} \quad & 1 \text{ ohm-meter} = \frac{1}{[3]^2 \cdot 10^9} \text{ s} \\
\textit{inductance:} \quad & 1 \text{ henry} = \frac{1}{[3]^2 \cdot 10^{11}} \text{ s}^2/\text{cm} \\
\textit{B field:} \quad & 1 \text{ tesla} = 10^4 \text{ gauss} \\
\textit{H field:} \quad & 1 \text{ amp/meter} = 4\pi \cdot 10^{-3} \text{ oersted}
\end{aligned}
$$

C.2 Derivations

C.2.1 Force: newton vs. dyne

$$1 \text{ newton} = 1 \frac{\text{kg m}}{\text{s}^2} = \frac{(1000 \text{ g})(100 \text{ cm})}{\text{s}^2} = 10^5 \frac{\text{g cm}}{\text{s}^2} = 10^5 \text{ dynes}. \tag{C.1}$$

C.2.2 Energy: joule vs. erg

$$1 \text{ joule} = 1 \frac{\text{kg m}^2}{\text{s}^2} = \frac{(1000 \text{ g})(100 \text{ cm})^2}{\text{s}^2} = 10^7 \frac{\text{g cm}^2}{\text{s}^2} = 10^7 \text{ ergs}. \tag{C.2}$$

C.2.3 Charge: coulomb vs. esu

From Eqs. (1.1) and (1.2), two charges of 1 coulomb separated by a distance of 1 m exert a force on each other equal to $8.988 \cdot 10^9$ N $\approx 9 \cdot 10^9$ N, or equivalently $9 \cdot 10^{14}$ dynes. How would someone working in Gaussian units describe this situation? In Gaussian units, Coulomb's law gives the force simply as q^2/r^2. The separation is 100 cm, so if 1 coulomb equals N esu (with N to be determined), the $9 \cdot 10^{14}$ dyne force between the charges can be expressed as

$$9 \cdot 10^{14} \text{ dyne} = \frac{(N \text{ esu})^2}{(100 \text{ cm})^2} \implies N^2 = 9 \cdot 10^{18} \implies N = 3 \cdot 10^9. \tag{C.3}$$

So 1 coulomb equals $3 \cdot 10^9$ esu. If we had used the more exact value of k in Eq. (1.2), the "3" in this result would have been replaced by $\sqrt{8.988} = 2.998$, which is precisely the 2.998 that appears in the speed of light, $c = 2.998 \cdot 10^8$ m/s. The reason for this is the following.

If you follow through the above derivation while keeping things in terms of $k \equiv 1/4\pi\epsilon_0$, you will see that the number $3 \cdot 10^9$ is actually $\sqrt{\{k\} \cdot 10^5 \cdot 10^4}$, where we have put the braces around k to signify that it is just the number $8.988 \cdot 10^9$ without the SI units. (The factors of 10^5 and 10^4 come from the conversions to dynes and centimeters, respectively.) But we know from Eq. (6.8) that $\epsilon_0 = 1/\mu_0 c^2$, so we have $k = \mu_0 c^2/4\pi$. Furthermore, the numerical value of μ_0 is $\{\mu_0\} = 4\pi \cdot 10^{-7}$, so the numerical value of k is $\{k\} = \{c\}^2 \cdot 10^{-7}$. Therefore, the number N that appears in Eq. (C.3) is really

$$N = \sqrt{\{k\} \cdot 10^9} = \sqrt{(\{c\}^2 \cdot 10^{-7})10^9} = \{c\} \cdot 10 = 2.998 \cdot 10^9 \equiv [3] \cdot 10^9. \tag{C.4}$$

C.2.4 Potential: volt vs. statvolt

$$1 \text{ volt} = 1\,\frac{\text{J}}{\text{C}} = \frac{10^7 \text{ erg}}{[3] \cdot 10^9 \text{ esu}} = \frac{1}{[3] \cdot 10^2}\,\frac{\text{erg}}{\text{esu}} = \frac{1}{[3] \cdot 10^2} \text{ statvolt}. \tag{C.5}$$

C.2.5 Electric field: volt/meter vs. statvolt/centimeter

$$1\,\frac{\text{volt}}{\text{meter}} = \frac{\dfrac{1}{[3] \cdot 10^2} \text{ statvolt}}{100\,\text{cm}} = \frac{1}{[3] \cdot 10^4}\,\frac{\text{statvolt}}{\text{cm}}. \tag{C.6}$$

C.2.6 Capacitance: farad vs. centimeter

$$1 \text{ farad} = 1\,\frac{\text{C}}{\text{V}} = \frac{[3] \cdot 10^9 \text{ esu}}{\dfrac{1}{[3] \cdot 10^2} \text{ statvolt}} = [3]^2 \cdot 10^{11}\,\frac{\text{esu}}{\text{statvolt}}. \tag{C.7}$$

We can alternatively write these Gaussian units as centimeters. This is true because 1 statvolt = 1 esu/cm (because the potential from a point charge is q/r), so 1 esu/statvolt = 1 cm. We therefore have

$$1 \text{ farad} = [3]^2 \cdot 10^{11} \text{ cm}. \tag{C.8}$$

C.2.7 Resistance: ohm vs. second/centimeter

$$\begin{aligned} 1 \text{ ohm} &= 1\,\frac{\text{V}}{\text{A}} = 1\,\frac{\text{V}}{\text{C/s}} = \frac{\dfrac{1}{[3] \cdot 10^2} \text{ statvolt}}{[3] \cdot 10^9 \text{ esu/s}} = \frac{1}{[3]^2 \cdot 10^{11}}\,\frac{\text{s}}{\text{esu/statvolt}} \\ &= \frac{1}{[3]^2 \cdot 10^{11}}\,\frac{\text{s}}{\text{cm}}, \end{aligned} \tag{C.9}$$

where we have used 1 esu/statvolt = 1 cm.

C.2.8 Resistivity: ohm-meter vs. second

$$1 \text{ ohm-meter} = \left(\frac{1}{[3]^2 \cdot 10^{11}} \frac{\text{s}}{\text{cm}}\right)(100\,\text{cm}) = \frac{1}{[3]^2 \cdot 10^9}\,\text{s}. \tag{C.10}$$

C.2.9 Inductance: henry vs. second^2/centimeter

$$\begin{aligned} 1 \text{ henry} &= 1\,\frac{\text{V}}{\text{A/s}} = 1\,\frac{\text{V}}{\text{C/s}^2} = \frac{\dfrac{1}{[3]\cdot 10^2}\,\text{statvolt}}{[3]\cdot 10^9\,\text{esu/s}^2} \\ &= \frac{1}{[3]^2 \cdot 10^{11}}\,\frac{\text{s}^2}{\text{esu/statvolt}} = \frac{1}{[3]^2 \cdot 10^{11}}\,\frac{\text{s}^2}{\text{cm}}, \end{aligned} \tag{C.11}$$

where we have used 1 esu/statvolt = 1 cm.

C.2.10 Magnetic field B: tesla vs. gauss

Consider a setup in which a charge of 1 C travels at 1 m/s in a direction perpendicular to a magnetic field with strength 1 tesla. Equation (6.1) tells us that the force on the charge is 1 newton. Let us express this fact in terms of the Gaussian force relation in Eq. (6.9), which involves a factor of c. We know that $1\,\text{N} = 10^5$ dyne and $1\,\text{C} = [3]\cdot 10^9$ esu. If we let 1 tesla $= N$ gauss, then the way that Eq. (6.9) describes the given situation is

$$10^5 \text{ dyne} = \frac{[3]\cdot 10^9\,\text{esu}}{[3]\cdot 10^{10}\,\text{cm/s}}\left(100\,\frac{\text{cm}}{\text{s}}\right)(N\,\text{gauss}). \tag{C.12}$$

Since 1 gauss equals 1 dyne/esu, all the units cancel (as they must), and we end up with $N = 10^4$, as desired. This is an exact result because the two factors of [3] cancel.

C.2.11 Magnetic field H: ampere/meter vs. oersted

The H field is defined differently in the two systems (there is a μ_0 in the SI definition), so we have to be careful. Consider a B field of 1 tesla in vacuum. What H field does this B field correspond to in each system? In the Gaussian system, B is 10^4 gauss. But in Gaussian units $H = B$ in vacuum, so $H = 10^4$ oersted, because an oersted and a gauss are equivalent units. In the SI system we have (you should verify these units)

$$H = \frac{B}{\mu_0} = \frac{1\,\text{tesla}}{4\pi\cdot 10^{-7}\,\text{kg m/C}^2} = \frac{10^7}{4\pi}\,\frac{\text{A}}{\text{m}}. \tag{C.13}$$

Since this is equivalent to 10^4 oersted, we arrive at 1 amp/meter $= 4\pi\cdot 10^{-3}$ oersted. Going the other way, 1 oersted equals roughly 80 amp/meter.

D

SI and Gaussian formulas

The following pages provide a list of all the main results in this book, in both SI and Gaussian units. After looking at a few of the corresponding formulas, you will discover that transforming from SI units to Gaussian units involves one or more of the three types of conversions discussed below.

Of course, even if a formula takes exactly the same form in the two systems of units, it says two entirely different things. For example, the formula relating force and electric field is the same in both systems: $\mathbf{F} = q\mathbf{E}$. But in SI units this equation says that a charge of 1 coulomb placed in an electric field of 1 volt/meter feels a force of 1 newton, whereas in Gaussian units it says that a charge of 1 esu placed in an electric field of 1 statvolt/centimeter feels a force of 1 dyne. When we say that two formulas are the "same," we mean that they look the same on the page, even though the various letters mean different things in the two systems.

The three basic types of conversions from SI to Gaussian units are given in Sections D.1 to D.3. We then list the formulas in Section D.4 by chapter.

D.1 Eliminating ϵ_0 and μ_0

Our starting point in this book was Coulomb's law in Eq. (1.4). The SI expression for this law contains the factor $1/4\pi\epsilon_0$, whereas the Gaussian expression has no factor (or rather just a 1). To convert from SI units to Gaussian units, we therefore need to set $4\pi\epsilon_0 = 1$, or equivalently $\epsilon_0 = 1/4\pi$ (along with possibly some other changes, as we will see below). That is, we need to erase all factors of $4\pi\epsilon_0$ that appear, or equivalently replace all ϵ_0's with $1/4\pi$'s. In many formulas this change

is all that is needed. A few examples are: Gauss's law, Eq. (1.31) in the list in Section D.4;[1] the field due to a line or sheet, Eqs. (1.39) and (1.40); the energy in an electric field, Eq. (1.53); and the capacitance of a sphere or parallel plates, Eqs. (3.10) and (3.15).

A corollary of the $\epsilon_0 \to 1/4\pi$ rule is the $\mu_0 \to 4\pi/c^2$ rule. We introduced μ_0 in Chapter 6 via the definition $\mu_0 \equiv 1/\epsilon_0 c^2$, so if we replace ϵ_0 with $1/4\pi$, we must also replace μ_0 with $4\pi/c^2$. An example of this $\mu_0 \to 4\pi/c^2$ rule is the force between two current-carrying wires, Eq. (6.15).

It is also possible to use these rules to convert formulas in the other direction, from Gaussian units to SI units, although the process isn't quite as simple. The conversion must (at least for conversions where only ϵ_0 and μ_0 are relevant) involve multiplying by some power of $4\pi\epsilon_0$ (or equivalently $4\pi/\mu_0 c^2$). And there is only one power that will make the units of the resulting SI expression correct, because ϵ_0 has units, namely $\mathrm{C^2\,s^2\,kg^{-1}\,m^{-3}}$. For example, the Gaussian expression for the field due to a sheet of charge is $2\pi\sigma$ in Eq. (1.40) in the list below, so the SI expression must take the form of $2\pi\sigma(4\pi\epsilon_0)^n$. You can quickly show that $2\pi\sigma(4\pi\epsilon_0)^{-1} = \sigma/2\epsilon_0$ has the correct units of electric field (it suffices to look at the power of any one of the four units: kg, m, s, C).

D.2 Changing B to B/c

If all quantities were defined in the same way in the two systems of units (up to factors of $4\pi\epsilon_0$ and $4\pi/\mu_0 c^2$), then the above rules involving ϵ_0 and μ_0 would be sufficient for converting from SI units to Gaussian units. But unfortunately certain quantities are defined differently in the two systems, so we can't convert from one system to the other without knowing what these arbitrary definitions are.

The most notable example of differing definitions is the magnetic field. In SI units the Lorentz force (or rather the magnetic part of it) is $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, while in Gaussian units it is $\mathbf{F} = (q/c)\mathbf{v} \times \mathbf{B}$. To convert from an SI formula to a Gaussian formula, we therefore need to replace every B with a B/c (and likewise for the vector potential A). An example of this is the B field from an infinite wire, Eq. (6.6). In SI units we have $B = \mu_0 I/2\pi r$. Applying our rules for μ_0 and B, the Gaussian B field is obtained as follows:

$$B = \frac{\mu_0 I}{2\pi r} \quad \longrightarrow \quad \left(\frac{B}{c}\right) = \left(\frac{4\pi}{c^2}\right)\frac{I}{2\pi r} \quad \Longrightarrow \quad B = \frac{2I}{rc}, \tag{D.1}$$

which is the correct result. Other examples involving the $B \to B/c$ rule include Ampère's law, Eqs. (6.19) and (6.25); the Lorentz transformations, Eq. (6.76); and the energy in a magnetic field, Eq. (7.79).

[1] The "double" equations in the list in Section D.4, where the SI and Gaussian formulas are presented side by side, are labeled according to the equation number that the SI formula has in the text.

D.3 Other definitional differences

The above two conversion procedures are sufficient for all formulas up to and including Chapter 9. However, in Chapters 10 and 11 we encounter a number of new quantities (χ_e, **D**, **H**, etc.), and many of these quantities are defined differently in the two systems of units,[2] mainly due to historical reasons. For example, after using the $\epsilon_0 \to 1/4\pi$ rule in Eq. (10.41), we see that we need to replace χ_e by $4\pi\chi_e$ in going from SI to Gaussian units. The Gaussian expression is then given by

$$\chi_e = \frac{P}{\epsilon_0 E} \longrightarrow (4\pi\chi_e) = \left(\frac{4\pi}{1}\right)\frac{P}{E} \Longrightarrow \chi_e = \frac{P}{E}, \tag{D.2}$$

which is correct. This $\chi_e \to 4\pi\chi_e$ rule is consistent with Eq. (10.42). Similarly, Eq. (10.63) shows that **D** is replaced by $\mathbf{D}/4\pi$.

On the magnetic side of things, a few examples are the following. Equation (11.9) shows that **m** (and hence **M**) is replaced by $c\mathbf{m}$ when going from SI to Gaussian units (because $\mathbf{m} = I\mathbf{a} \to c\mathbf{m} = I\mathbf{a} \Rightarrow \mathbf{m} = I\mathbf{a}/c$, which is the correct Gaussian expression). Also, Eqs. (11.69) and (11.70) show that **H** is replaced by $(c/4\pi)\mathbf{H}$. Let's check that Eq. (11.68) is consistent with these rules. The SI expression for **H** is converted to Gaussian as follows:

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \longrightarrow \left(\frac{c}{4\pi}\mathbf{H}\right) = \left(\frac{c^2}{4\pi}\right)\left(\frac{\mathbf{B}}{c}\right) - (c\mathbf{M})$$

$$\Longrightarrow \quad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}, \tag{D.3}$$

which is the correct Gaussian expression. Although it is possible to remember all the different rules and then convert things at will, there are so many differing definitions in Chapters 10 and 11 that it is probably easiest to look up each formula as you need it. But for Chapters 1–9, you can get a lot of mileage out of the first two rules above, namely (1) $\epsilon_0 \to 1/4\pi$, $\mu_0 \to 4\pi/c^2$, and (2) $B \to B/c$.

D.4 The formulas

In the pages that follow, the SI formula is given first, followed by the Gaussian equivalent.

[2] The preceding case with B is simply another one of these differences, but we have chosen to discuss it separately because the B field appears so much more often in this book than other such quantities.

Chapter 1

Coulomb's law (1.4):	$\mathbf{F} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_1 q_2 \hat{\mathbf{r}}}{r^2}$	$\mathbf{F} = \dfrac{q_1 q_2 \hat{\mathbf{r}}}{r^2}$
potential energy (1.9):	$U = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_1 q_2}{r}$	$U = \dfrac{q_1 q_2}{r}$
electric field (1.20):	$\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q\hat{\mathbf{r}}}{r^2}$	$\mathbf{E} = \dfrac{q\hat{\mathbf{r}}}{r^2}$
force and field (1.21):	$\mathbf{F} = q\mathbf{E}$	(same)
flux (1.26):	$\Phi = \displaystyle\int \mathbf{E}\cdot d\mathbf{a}$	(same)
Gauss's law (1.31):	$\displaystyle\int \mathbf{E}\cdot d\mathbf{a} = \frac{q}{\epsilon_0}$	$\displaystyle\int \mathbf{E}\cdot d\mathbf{a} = 4\pi q$
field due to line (1.39):	$E_r = \dfrac{\lambda}{2\pi\epsilon_0 r}$	$E_r = \dfrac{2\lambda}{r}$
field due to sheet (1.40):	$E = \dfrac{\sigma}{2\epsilon_0}$	$E = 2\pi\sigma$
$\Delta\mathbf{E}$ across sheet (1.41):	$\Delta\mathbf{E} = \dfrac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$	$\Delta\mathbf{E} = 4\pi\sigma\hat{\mathbf{n}}$
field near shell (1.42):	$E_r = \dfrac{\sigma}{\epsilon_0}$	$E_r = 4\pi\sigma$
F/(area) on sheet (1.49):	$\dfrac{F}{A} = \dfrac{1}{2}(E_1 + E_2)\sigma$	(same)
energy in E field (1.53):	$U = \dfrac{\epsilon_0}{2}\displaystyle\int E^2\, dv$	$U = \dfrac{1}{8\pi}\displaystyle\int E^2\, dv$

Chapter 2

electric potential (2.4):	$\phi = -\displaystyle\int \mathbf{E}\cdot d\mathbf{s}$	(same)
field and potential (2.16):	$\mathbf{E} = -\nabla\phi$	(same)
potential and density (2.18):	$\phi = \displaystyle\int \frac{\rho\, dv}{4\pi\epsilon_0 r}$	$\phi = \displaystyle\int \frac{\rho\, dv}{r}$
potential energy (2.32):	$U = \dfrac{1}{2}\displaystyle\int \rho\phi\, dv$	(same)
dipole potential (2.35):	$\phi = \dfrac{q\ell\cos\theta}{4\pi\epsilon_0 r^2}$	$\phi = \dfrac{q\ell\cos\theta}{r^2}$
dipole moment (2.35):	$p = q\ell$	(same)

dipole field (2.36):	$\mathbf{E} = \dfrac{q\ell}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$	$\mathbf{E} = \dfrac{q\ell}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$
divergence theorem (2.49):	$\displaystyle\int_S \mathbf{F}\cdot d\mathbf{a} = \int_V \operatorname{div}\mathbf{F}\,dv$	(same)
$\mathbf{E}$ and ρ (2.52):	$\operatorname{div}\mathbf{E} = \dfrac{\rho}{\epsilon_0}$	$\operatorname{div}\mathbf{E} = 4\pi\rho$
$\mathbf{E}$ and ϕ (2.70):	$\operatorname{div}\mathbf{E} = -\nabla^2\phi$	(same)
ϕ and ρ (2.72):	$\nabla^2\phi = -\dfrac{\rho}{\epsilon_0}$	$\nabla^2\phi = -4\pi\rho$
Stokes' theorem (2.83):	$\displaystyle\int_C \mathbf{F}\cdot d\mathbf{s} = \int_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{a}$	(same)

Chapter 3

charge and capacitance (3.7):	$Q = C\phi$	(same)
sphere C (3.10):	$C = 4\pi\epsilon_0 a$	$C = a$
parallel-plate C (3.15):	$C = \dfrac{\epsilon_0 A}{s}$	$C = \dfrac{A}{4\pi s}$
energy in capacitor (3.29):	$U = \dfrac{1}{2}C\phi^2$	(same)

Chapter 4

current, current density (4.7):	$I = \displaystyle\int \mathbf{J}\cdot d\mathbf{a}$	(same)
$\mathbf{J}$ and ρ (4.10):	$\operatorname{div}\mathbf{J} = -\dfrac{\partial\rho}{\partial t}$	(same)
conductivity (4.11):	$\mathbf{J} = \sigma\mathbf{E}$	(same)
Ohm's law (4.12):	$V = IR$	(same)
resistivity (4.16):	$\mathbf{J} = \left(\dfrac{1}{\rho}\right)\mathbf{E}$	(same)
resistance, resistivity (4.17):	$R = \dfrac{\rho L}{A}$	(same)
power (4.31):	$P = IV = I^2R$	(same)
R, C time constant (4.43):	$\tau = RC$	(same)

Chapter 5

Lorentz force (5.1):	$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$	$\mathbf{F} = q\mathbf{E} + \dfrac{q}{c}\mathbf{v} \times \mathbf{B}$
charge in a region (5.2):	$Q = \epsilon_0 \displaystyle\int \mathbf{E} \cdot d\mathbf{a}$	$Q = \dfrac{1}{4\pi} \displaystyle\int \mathbf{E} \cdot d\mathbf{a}$
E transformations (5.7):	$E'_\parallel = E_\parallel, \;\; E'_\perp = \gamma E_\perp$	(same)
E from moving Q (5.15):	$E' = \dfrac{Q}{4\pi\epsilon_0 r'^2} \dfrac{1-\beta^2}{(1-\beta^2 \sin^2\theta')^{3/2}}$	$E' = \dfrac{Q}{r'^2} \dfrac{1-\beta^2}{(1-\beta^2 \sin^2\theta')^{3/2}}$
F transformations (5.17):	$\dfrac{dp_\parallel}{dt} = \dfrac{dp'_\parallel}{dt'}, \;\; \dfrac{dp_\perp}{dt} = \dfrac{1}{\gamma}\dfrac{dp'_\perp}{dt'}$	(same)
F from current (5.28):	$F_y = \dfrac{qv_x I}{2\pi\epsilon_0 r c^2}$	$F_y = \dfrac{2qv_x I}{rc^2}$

Chapter 6

B due to wire (6.3), (6.6):	$\mathbf{B} = \hat{\mathbf{z}}\dfrac{I}{2\pi\epsilon_0 r c^2} = \hat{\mathbf{z}}\dfrac{\mu_0 I}{2\pi r}$	$\mathbf{B} = \hat{\mathbf{z}}\dfrac{2I}{rc}$
speed of light (6.8):	$c^2 = \dfrac{1}{\mu_0\epsilon_0}$	(no analog)
F on a wire (6.14):	$F = IBl$	$F = \dfrac{IBl}{c}$
F between wires (6.15):	$F = \dfrac{\mu_0 I_1 I_2 l}{2\pi r}$	$F = \dfrac{2I_1 I_2 l}{c^2 r}$
Ampère's law (6.19):	$\displaystyle\int \mathbf{B} \cdot d\mathbf{s} = \mu_0 I$	$\displaystyle\int \mathbf{B} \cdot d\mathbf{s} = \dfrac{4\pi}{c} I$
(differential form) (6.25):	$\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J}$	$\operatorname{curl} \mathbf{B} = \dfrac{4\pi}{c}\mathbf{J}$
vector potential (6.32):	$\mathbf{B} = \operatorname{curl} \mathbf{A}$	(same)
$\mathbf{A}$ and $\mathbf{J}$ (6.44):	$\mathbf{A} = \dfrac{\mu_0}{4\pi} \displaystyle\int \dfrac{\mathbf{J}\, dv}{r}$	$\mathbf{A} = \dfrac{1}{c} \displaystyle\int \dfrac{\mathbf{J}\, dv}{r}$
Biot–Savart law (6.49):	$d\mathbf{B} = \dfrac{\mu_0 I}{4\pi} \dfrac{d\boldsymbol{l} \times \hat{\mathbf{r}}}{r^2}$	$d\mathbf{B} = \dfrac{I}{c} \dfrac{d\boldsymbol{l} \times \hat{\mathbf{r}}}{r^2}$
B in solenoid (6.57):	$B_z = \mu_0 n I$	$B_z = \dfrac{4\pi n I}{c}$
ΔB across sheet (6.58):	$\Delta B = \mu_0 \mathcal{J}$	$\Delta B = \dfrac{4\pi \mathcal{J}}{c}$
$F/$(area) on sheet (6.63):	$\dfrac{F}{A} = \dfrac{(B_z^+)^2 - (B_z^-)^2}{2\mu_0}$	$\dfrac{F}{A} = \dfrac{(B_z^+)^2 - (B_z^-)^2}{8\pi}$

E, **B** transforms (6.76):	$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}$	(same)
	$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$	(same)
	$\mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}_{\perp} + \boldsymbol{\beta} \times c\mathbf{B}_{\perp}\right)$	$\mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B}_{\perp}\right)$
	$c\mathbf{B}'_{\perp} = \gamma\left(c\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E}_{\perp}\right)$	$\mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E}_{\perp}\right)$
Hall $\mathbf{E}_t$ field (6.84):	$\mathbf{E}_t = \dfrac{-\mathbf{J} \times \mathbf{B}}{nq}$	$\mathbf{E}_t = \dfrac{-\mathbf{J} \times \mathbf{B}}{nqc}$

Chapter 7

electromotive force (7.5):	$\mathcal{E} = \dfrac{1}{q}\displaystyle\int \mathbf{f} \cdot d\mathbf{s}$	(same)
Faraday's law (7.26):	$\mathcal{E} = -\dfrac{d\Phi}{dt}$	$\mathcal{E} = -\dfrac{1}{c}\dfrac{d\Phi}{dt}$
(differential form) (7.31):	$\operatorname{curl}\mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\operatorname{curl}\mathbf{E} = -\dfrac{1}{c}\dfrac{\partial \mathbf{B}}{\partial t}$
mutual inductance (7.37), (7.38):	$\mathcal{E}_{21} = -M_{21}\dfrac{dI_1}{dt}$	(same)
self-inductance (7.57), (7.58):	$\mathcal{E}_{11} = -L_1\dfrac{dI_1}{dt}$	(same)
L of toroid (7.62):	$L = \dfrac{\mu_0 N^2 h}{2\pi}\ln\left(\dfrac{b}{a}\right)$	$L = \dfrac{2N^2 h}{c^2}\ln\left(\dfrac{b}{a}\right)$
R, L time constant (7.69):	$\tau = L/R$	(same)
energy in inductor (7.74):	$U = \dfrac{1}{2}LI^2$	(same)
energy in B field (7.79):	$U = \dfrac{1}{2\mu_0}\displaystyle\int B^2\, dv$	$U = \dfrac{1}{8\pi}\displaystyle\int B^2\, dv$

Chapter 8

RLC time constant (8.8):	$\tau = \dfrac{1}{\alpha} = \dfrac{2L}{R}$	(same)
RLC frequency (8.9):	$\omega = \sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}$	(same)
Q factor (8.12):	$Q = \omega \cdot \dfrac{\text{energy}}{\text{power}}$	(same)
I_0 for series RLC (8.38):	$I_0 = \dfrac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$	(same)

ϕ for series RLC (8.39):	$\tan\phi = \dfrac{1}{R\omega C} - \dfrac{\omega L}{R}$	(same)
resonant ω (8.41):	$\omega_0 = \dfrac{1}{\sqrt{LC}}$	(same)
width of I curve (8.45):	$\dfrac{2\lvert\Delta\omega\rvert}{\omega_0} = \dfrac{1}{Q}$	(same)
admittance (8.61):	$\tilde{I} = Y\tilde{V}$	(same)
impedance (8.62):	$\tilde{V} = Z\tilde{I}$	(same)
impedances (Table 8.1):	R, $i\omega L$, $-i/\omega C$	(same)
average power in R (8.81):	$\overline{P}_R = \dfrac{V_{\text{rms}}^2}{R}$	(same)
average power (general) (8.85):	$\overline{P} = V_{\text{rms}} I_{\text{rms}} \cos\phi$	(same)

Chapter 9

displacement current (9.15):	$\mathbf{J}_{\text{d}} = \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$	$\mathbf{J}_{\text{d}} = \dfrac{1}{4\pi} \dfrac{\partial \mathbf{E}}{\partial t}$
Maxwell's equations (9.17):	$\operatorname{curl} \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\operatorname{curl} \mathbf{E} = -\dfrac{1}{c} \dfrac{\partial \mathbf{B}}{\partial t}$
	$\operatorname{curl} \mathbf{B} = \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$	$\operatorname{curl} \mathbf{B} = \dfrac{1}{c} \dfrac{\partial \mathbf{E}}{\partial t} + \dfrac{4\pi}{c} \mathbf{J}$
	$\operatorname{div} \mathbf{E} = \dfrac{\rho}{\epsilon_0}$	$\operatorname{div} \mathbf{E} = 4\pi\rho$
	$\operatorname{div} \mathbf{B} = 0$	(same)
speed of wave (9.26), (9.27):	$v = \dfrac{1}{\sqrt{\mu_0 \epsilon_0}} = c$	$v = c$
E, B amplitudes (9.26), (9.27):	$E_0 = \dfrac{B_0}{\sqrt{\mu_0 \epsilon_0}} = cB_0$	$E_0 = B_0$
power density (9.34):	$S = \epsilon_0 \overline{E^2} c$	$S = \dfrac{\overline{E^2} c}{4\pi}$
Poynting vector (9.42):	$\mathbf{S} = \dfrac{\mathbf{E} \times \mathbf{B}}{\mu_0}$	$\mathbf{S} = \dfrac{c}{4\pi} \mathbf{E} \times \mathbf{B}$
invariant 1 (9.51):	$\mathbf{E}' \cdot \mathbf{B}' = \mathbf{E} \cdot \mathbf{B}$	(same)
invariant 2 (9.51):	$E'^2 - c^2 B'^2 = E^2 - c^2 B^2$	$E'^2 - B'^2 = E^2 - B^2$

Chapter 10

dielectric constant (10.3):	$\kappa = Q/Q_0$	(same)
dipole moment (10.13):	$\mathbf{p} = \int \mathbf{r}'\rho\, dv'$	(same)
dipole potential (10.14):	$\phi(\mathbf{r}) = \dfrac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2}$	$\phi(\mathbf{r}) = \dfrac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r^2}$
dipole (E_r, E_θ) (10.18):	$\dfrac{p}{4\pi\epsilon_0 r^3}(2\cos\theta, \sin\theta)$	$\dfrac{p}{r^3}(2\cos\theta, \sin\theta)$
torque on dipole (10.21):	$\mathbf{N} = \mathbf{p}\times\mathbf{E}$	(same)
force on dipole (10.26):	$F_x = \mathbf{p}\cdot\operatorname{grad} E_x$	(same)
polarizability (10.29):	$\mathbf{p} = \alpha\mathbf{E}$	(same)
polarization density (10.31):	$\mathbf{P} = \mathbf{p}N$	(same)
ϕ due to column (10.34):	$\phi = \dfrac{P\,da}{4\pi\epsilon_0}\left(\dfrac{1}{r_2} - \dfrac{1}{r_1}\right)$	$\phi = P\,da\left(\dfrac{1}{r_2} - \dfrac{1}{r_1}\right)$
surface density (10.35):	$\sigma = P$	(same)
average field (10.37):	$\langle\mathbf{E}\rangle = -\dfrac{\mathbf{P}}{\epsilon_0}$	$\langle\mathbf{E}\rangle = -4\pi\mathbf{P}$
susceptibility (10.41):	$\chi_e = \dfrac{P}{\epsilon_0 E}$	$\chi_e = \dfrac{P}{E}$
χ_e and κ (10.42):	$\chi_e = \kappa - 1$	$\chi_e = \dfrac{\kappa - 1}{4\pi}$
$\mathbf{E}$ in polar sphere (10.47):	$\mathbf{E}_{\text{in}} = -\dfrac{\mathbf{P}}{3\epsilon_0}$	$\mathbf{E}_{\text{in}} = -\dfrac{4\pi\mathbf{P}}{3}$
permittivity (10.56):	$\epsilon = \kappa\epsilon_0$	(no analog)
$\mathbf{P}$ divergence (10.61):	$\operatorname{div}\mathbf{P} = -\rho_{\text{bound}}$	(same)
displacement $\mathbf{D}$ (10.63):	$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$	$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$
$\mathbf{D}$ divergence (10.64):	$\operatorname{div}\mathbf{D} = \rho_{\text{free}}$	$\operatorname{div}\mathbf{D} = 4\pi\rho_{\text{free}}$
$\mathbf{D}$ for linear (10.65):	$\mathbf{D} = \epsilon\mathbf{E}$	$\mathbf{D} = \kappa\mathbf{E}$
χ_e for weak E (10.73):	$\chi_e \approx \dfrac{Np^2}{\epsilon_0 kT}$	$\chi_e \approx \dfrac{Np^2}{kT}$
bound current $\mathbf{J}$ (10.74):	$\mathbf{J}_{\text{bound}} = \dfrac{\partial\mathbf{P}}{\partial t}$	(same)
curl of $\mathbf{B}$ (10.78):	$\operatorname{curl}\mathbf{B} = \mu_0\dfrac{\partial\mathbf{D}}{\partial t} + \mu_0\mathbf{J}$	$\operatorname{curl}\mathbf{B} = \dfrac{1}{c}\dfrac{\partial\mathbf{D}}{\partial t} + \dfrac{4\pi}{c}\mathbf{J}$

speed of wave (10.83):	$v = \dfrac{c}{\sqrt{\kappa}}$	(same)
E, B amplitudes (10.83):	$E_0 = \dfrac{cB_0}{\sqrt{\kappa}} = vB_0$	$E_0 = \dfrac{B_0}{\sqrt{\kappa}}$

Chapter 11

dipole moment (11.9):	$\mathbf{m} = I\mathbf{a}$	$\mathbf{m} = \dfrac{I\mathbf{a}}{c}$
vector potential (11.10):	$\mathbf{A} = \dfrac{\mu_0}{4\pi}\dfrac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$	$\mathbf{A} = \dfrac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$
dipole (B_r, B_θ) (11.15):	$\dfrac{\mu_0 m}{4\pi r^3}(2\cos\theta, \sin\theta)$	$\dfrac{m}{r^3}(2\cos\theta, \sin\theta)$
force on dipole (11.23):	$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$	(same)
orbital $\mathbf{m}$ for e (11.29):	$\mathbf{m} = \dfrac{-e}{2m_e}\mathbf{L}$	$\mathbf{m} = \dfrac{-e}{2m_e c}\mathbf{L}$
polarizability (11.41):	$\dfrac{\Delta m}{B} = -\dfrac{e^2r^2}{4m_e}$	$\dfrac{\Delta m}{B} = -\dfrac{e^2r^2}{4m_e c^2}$
torque on dipole (11.47):	$\mathbf{N} = \mathbf{m}\times\mathbf{B}$	(same)
polarization density (11.51):	$\mathbf{M} = \dfrac{\mathbf{m}}{\text{volume}}$	(same)
susceptibility χ_m (11.52):	$\mathbf{M} = \chi_m\dfrac{\mathbf{B}}{\mu_0}$	$\mathbf{M} = \chi_m\mathbf{B}$
χ_{pm} for weak B (11.53):	$\chi_{pm} \approx \dfrac{\mu_0 N m^2}{kT}$	$\chi_{pm} \approx \dfrac{Nm^2}{kT}$
surface density $\mathcal{J}$ (11.55):	$\mathcal{J} = M$	$\mathcal{J} = Mc$
volume density $\mathbf{J}$ (11.56):	$\mathbf{J} = \operatorname{curl}\mathbf{M}$	$\mathbf{J} = c\ \operatorname{curl}\mathbf{M}$
$\mathbf{H}$ field (11.68):	$\mathbf{H} = \dfrac{\mathbf{B}}{\mu_0} - \mathbf{M}$	$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$
curl of $\mathbf{H}$ (11.69):	$\operatorname{curl}\mathbf{H} = \mathbf{J}_{\text{free}}$	$\operatorname{curl}\mathbf{H} = \dfrac{4\pi}{c}\mathbf{J}_{\text{free}}$
(integrated form) (11.70):	$\displaystyle\int \mathbf{H}\cdot d\boldsymbol{l} = I_{\text{free}}$	$\displaystyle\int \mathbf{H}\cdot d\boldsymbol{l} = \frac{4\pi}{c}I_{\text{free}}$
χ_m (accepted def.) (11.72):	$\mathbf{M} = \chi_m\mathbf{H}$	(same)
permeability (11.74):	$\mu = \mu_0(1+\chi_m)$	$\mu = 1 + 4\pi\chi_m$
$\mathbf{B}$ and $\mathbf{H}$ (11.74):	$\mathbf{B} = \mu\mathbf{H}$	(same)

Appendix H

tangential E_θ (H.3):	$E_\theta = \dfrac{qa\sin\theta}{4\pi\epsilon_0 c^2 R}$	$E_\theta = \dfrac{qa\sin\theta}{c^2 R}$
power (H.7):	$P_{\text{rad}} = \dfrac{q^2a^2}{6\pi\epsilon_0 c^3}$	$P_{\text{rad}} = \dfrac{2q^2a^2}{3c^3}$

E

Exact relations among SI and Gaussian units

In 1983 the General Conference on Weights and Measures officially redefined the meter as the distance that light travels in vacuum during a time interval of 1/299,792,458 of a second. The second is defined in terms of a certain atomic frequency in a way that does not concern us here. The nine-digit integer was chosen to make the assigned value of c agree with the most accurate measured value to well within the uncertainty in the latter. Henceforth the velocity of light is, *by definition*, 299,792,458 meters/second. An experiment in which the passage of a light pulse from point A to point B is timed is to be regarded as a measurement of the distance from A to B, not a measurement of the speed of light.

While this step has no immediate practical consequences, it does bring a welcome simplification of the exact relations connecting various electromagnetic units. As we learn in Chapter 9, Maxwell's equations for the vacuum fields, formulated in SI units, have a solution in the form of a traveling wave with velocity $c = (\mu_0\epsilon_0)^{-1/2}$. The SI constant μ_0 has always been defined exactly as $4\pi \cdot 10^{-7}\,\mathrm{kg\,m/C^2}$, whereas the value of ϵ_0 has depended on the experimentally determined value of the speed of light, any refinement of which called for adjustment of the value of ϵ_0. But now ϵ_0 acquires a permanent and perfectly precise value of its own, through the requirement that

$$(\mu_0\epsilon_0)^{-1/2} = 299{,}792{,}458 \text{ meters/second.} \tag{E.1}$$

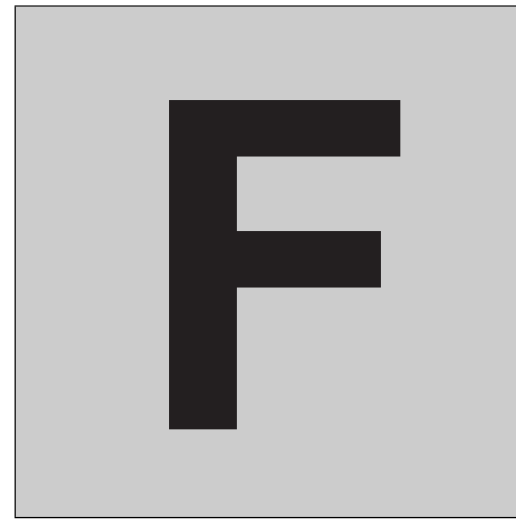
In the Gaussian system no such question arises. Wherever c is involved, it appears in plain view, and all other quantities are defined exactly, beginning with the electrostatic unit of charge, the esu, whose definition by Coulomb's law involves no arbitrary factor.

With the adoption of Eq. (E.1) in consequence of the redefinition of the meter, the relations among the units in the systems we have been using can be stated with unlimited precision. These relations are listed at the beginning of Appendix C for the principal quantities we deal with. In the list the symbol [3] stands for the precise decimal 2.99792458.

The exact numbers are uninteresting and for our work quite unnecessary. It is sheer luck that [3] happens to be so close to 3, an accidental consequence of the length of the meter and the second. When 0.1 percent accuracy is good enough we need only remember that "300 volts is a statvolt" and "$3 \cdot 10^9$ esu is a coulomb." Much less precisely, but still within 12 percent, a capacitance of 1 cm is equivalent to 1 picofarad.

An important SI constant is $(\mu_0/\epsilon_0)^{1/2}$, which is a resistance in ohms. Since $\epsilon_0 = 1/\mu_0 c^2$, this resistance equals $\mu_0 c$. Using the exact values of μ_0 and c, we find $(\mu_0/\epsilon_0)^{1/2} = 40\pi \cdot [3]$ ohms ≈ 376.73 ohms. One tends to remember it, and even refer to it, as "377 ohms." It is the ratio of the electric field strength E, in volts/meter, in a plane wave in vacuum, to the strength, in amperes/meter, of the accompanying magnetic field H. For this reason the constant $(\mu_0/\epsilon_0)^{1/2}$ is sometimes denoted by Z_0 and called, rather cryptically, the *impedance of the vacuum.* In a plane wave in vacuum in which E_{rms} is the rms electric field in volts/meter, the mean density of power transmitted, in watts/m^2, is E_{rms}^2/Z_0.

The logical relation of the SI electrical units to one another now takes on a slightly different aspect. Before the redefinition of the meter, it was customary to designate one of the electrical units as *primary,* in this sense: its precise value could, at least in principle, be established by a procedure involving the SI mechanical and metrical units only. Thus the ampere, to which this role has usually been assigned, was defined in terms of the force in newtons between parallel currents, using the relation in Eq. (6.15). This was possible because the constant μ_0 in that relation has the precise value $4\pi \cdot 10^{-7}$ kg m/C^2. Then with the ampere as the primary electrical unit, the coulomb was defined precisely as 1 ampere-second. The coulomb itself, owing to the presence of ϵ_0 in Coulomb's law, was not eligible to serve as the primary unit. Now with ϵ_0 as well as μ_0 assigned an exact numerical value, the system can be built up with any unit as the starting point. All quantities are in this sense on an equal footing, and the choice of a primary unit loses its significance. Never a very interesting question anyway, it can now be relegated to history.



Curvilinear coordinates

We begin this appendix by listing the main vector operators (gradient, divergence, curl, Laplacian) in Cartesian, cylindrical, and spherical coordinates. We then talk a little about each operator – define things, derive a few results, give some examples, etc. You will note that some of the expressions below are rather scary looking. However, you won't have to use their full forms in this book. In the applications that come up, invariably only one or two of the terms in the expressions are nonzero.

F.1 Vector operators

F.1.1 Cartesian coordinates

$$
\begin{aligned}
d\mathbf{s} &= dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}, \\
\nabla &= \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}, \\
\nabla f &= \frac{\partial f}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{z}}, \\
\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}, \\
\nabla \times \mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}, \\
\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.
\end{aligned}
\tag{F.1}
$$

F.1.2 Cylindrical coordinates

$$
\begin{aligned}
d\mathbf{s} &= dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + dz\,\hat{\mathbf{z}},\\
\nabla &= \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z},\\
\nabla f &= \frac{\partial f}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{z}},\\
\nabla\cdot\mathbf{A} &= \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z},\\
\nabla\times\mathbf{A} &= \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\theta}}\\
&\quad + \frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\hat{\mathbf{z}},\\
\nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}.
\end{aligned}
\tag{F.2}
$$

F.1.3 Spherical coordinates

$$
\begin{aligned}
d\mathbf{s} &= dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}},\\
\nabla &= \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi},\\
\nabla f &= \frac{\partial f}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}},\\
\nabla\cdot\mathbf{A} &= \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi},\\
\nabla\times\mathbf{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial (A_\phi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial (rA_\phi)}{\partial r}\right)\hat{\boldsymbol{\theta}},\\
&\quad + \frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\hat{\boldsymbol{\phi}},\\
\nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}.
\end{aligned}
\tag{F.3}
$$

F.2 Gradient

The gradient produces a vector from a scalar. The gradient of a function f, written as ∇f or $\text{grad}\, f$, may be defined[1] as the vector with the

[1] We used a different definition in Section 2.3, but we will show below that the two definitions are equivalent.

property that the change in f brought about by a small change $d\mathbf{s}$ in position is

$$df = \nabla f \cdot d\mathbf{s}. \tag{F.4}$$

The vector ∇f depends on position; there is a different gradient vector associated with each point in the parameter space.

You might wonder whether a vector that satisfies Eq. (F.4) actually exists. We are claiming that if f is a function of, say, three variables, then at every point in space there exists a *unique* vector, ∇f, such that for *any* small displacement $d\mathbf{s}$ from a given point, the change in f equals $\nabla f \cdot d\mathbf{s}$. It is not immediately obvious why a single vector gets the job done for all possible displacements $d\mathbf{s}$ from a given point. But the existence of such a vector can be demonstrated in two ways. First, we can explicitly construct ∇f; we will do this below in Eq. (F.5). Second, any (well-behaved) function looks like a linear function up close, and for a linear function a vector ∇f satisfying Eq. (F.4) does indeed exist. We will explain why in what follows. However, before addressing this issue, let us note an important property of the gradient.

From the definition in Eq. (F.4), it immediately follows (as mentioned in Section 2.3) that ∇f points in the direction of steepest ascent of f. This is true because we can write the dot product $\nabla f \cdot d\mathbf{s}$ as $|\nabla f||d\mathbf{s}| \cos\theta$, where θ is the angle between the vector ∇f and the vector $d\mathbf{s}$. So for a given length of the vector $d\mathbf{s}$, this dot product is maximized when $\theta = 0$. We therefore want the displacement $d\mathbf{s}$ to point in the direction of ∇f, if we want to produce the maximum change in f.

If we consider the more easily visualizable case of a function of two variables, the function can be represented by a surface above the xy plane. This surface is locally planar; that is, a sufficiently small bug walking around on it would think it is a (generally tilted) flat plane. If we look at the direction of steepest ascent in the local plane, and then project this line onto the xy plane, the resulting line is the direction of ∇f; see Fig. 2.5. The function f is constant along the direction perpendicular to ∇f. The magnitude of ∇f equals the change in f per unit distance in the parameter space, in the direction of ∇f. Equivalently, if we restrict the parameter space to the one-dimensional line in the direction of steepest ascent, then the gradient is simply the standard single-variable derivative in that direction.

We could alternatively work "backwards" and define the gradient as the vector that points in the direction (in the parameter space) of steepest ascent, with its magnitude equal to the rate of change in that direction. It then follows that the general change in f, for *any* displacement $d\mathbf{s}$ in the parameter space, is given by Eq. (F.4). This is true because the dot product picks out the component of $d\mathbf{s}$ along the direction of ∇f. This component causes a change in f, whereas the orthogonal component does not.

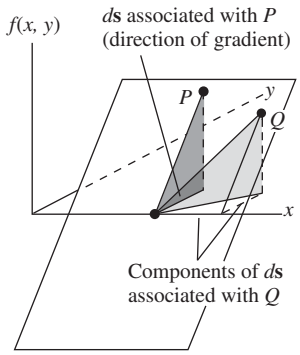


Figure F.1.
Only the component of $d\mathbf{s}$ in the direction of the gradient causes a change in f.

Figure F.1 shows how this works in the case of a function of two variables. We have assumed for simplicity that the local plane representing the surface of the function intersects the xy plane along the x axis. (We can always translate and rotate the coordinate system so that this is true at a given point.) The gradient then points in the y direction. The point P shown lies in the direction straight up the plane from the given point. The projection of this direction onto the xy plane lies along the gradient. The point Q is associated with a $d\mathbf{s}$ interval that doesn't lie along the gradient in the xy plane. This $d\mathbf{s}$ can be broken up into an interval along the x axis, which causes no change in f, plus an interval in the y direction, or equivalently the direction of the gradient, which causes the change in f up to the point Q.

The preceding two paragraphs explain why the vector ∇f defined by Eq. (F.4) does in fact exist; any well-behaved function is locally linear, and a unique vector ∇f at each point will get the job done in Eq. (F.4) if f is linear. But as mentioned above, we can also demonstrate the existence of such a vector by simply constructing it. Let's calculate the gradient in Cartesian coordinates, and then in spherical coordinates.

F.2.1 Cartesian gradient

In Cartesian coordinates, a general change in f for small displacements can be written as $df = (\partial f/\partial x)dx + (\partial f/\partial y)dy + (\partial f/\partial z)dz$. This is just the start of the Taylor series in three variables. The interval $d\mathbf{s}$ is simply (dx, dy, dz), so if we want $\nabla f \cdot d\mathbf{s}$ to be equal to df, we need

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \equiv \frac{\partial f}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{z}}, \tag{F.5}$$

in agreement with the ∇f expression in Eq. (F.1). In Section 2.3 we took Eq. (F.5) as the definition of the gradient and then discussed its other properties.

F.2.2 Spherical gradient

In spherical coordinates, a general change in f is given by $df = (\partial f/\partial r)dr + (\partial f/\partial \theta)d\theta + (\partial f/\partial \phi)d\phi$. However, the interval $d\mathbf{s}$ takes a more involved form compared with the Cartesian $d\mathbf{s}$. It is

$$d\mathbf{s} = (dr, r\,d\theta, r\sin\theta\,d\phi) \equiv dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}. \tag{F.6}$$

If we want $\nabla f \cdot d\mathbf{s}$ to be equal to df, then we need

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\right) \equiv \frac{\partial f}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}}, \tag{F.7}$$

in agreement with Eq. (F.3).

We see that the extra factors (compared with the Cartesian case) in the denominators of the gradient come from the coefficients of the unit vectors in the expression for $d\mathbf{s}$. Similarly, the form of the gradient in cylindrical coordinates in Eq. (F.2) can be traced to the fact that the interval $d\mathbf{s}$ equals $dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + dz\,\hat{\mathbf{z}}$. Since the extra factors that appear in $d\mathbf{s}$ show up in the denominators of the ∇-operator terms, and since the ∇ operator determines all of the other vector operators, we see that every result in this appendix can be traced back to the form of $d\mathbf{s}$ in the different coordinate systems. For example, the big scary expression listed in Eq. (F.3) for the curl in spherical coordinates is a direct consequence of the $d\mathbf{s} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$ interval.

Note that the consideration of units tells us that there must be a factor of r in the denominators in the $\partial f/\partial\theta$ and $\partial f/\partial\phi$ terms in the spherical gradient, and in the $\partial f/\partial\theta$ term in the cylindrical gradient.

F.3 Divergence

The divergence produces a scalar from a vector. The divergence of a vector function was defined in Eq. (2.47) as the net flux out of a given small volume, divided by the volume. In Section 2.10 we derived the form of the divergence in Cartesian coordinates, and it turned out to be the dot product of the ∇ operator with a vector $\mathbf{A}$, that is, $\nabla \cdot \mathbf{A}$. We use the same method here to derive the form in cylindrical coordinates. We then give a second, more mechanical, derivation. A third derivation is left for Exercise F.2.

F.3.1 Cylindrical divergence, first method

Consider the small volume that is generated by taking the region in the r-θ plane shown in Fig. F.2 and sweeping it through a span of z values from a particular z up to $z + \Delta z$ (the $\hat{\mathbf{z}}$ axis points out of the page). Let's first look at the flux of a vector field $\mathbf{A}$ through the two faces perpendicular to the $\hat{\mathbf{z}}$ direction. As in Section 2.10, only the z component of $\mathbf{A}$ is relevant to the flux through these faces. In the limit of a small volume, the area of these faces is $r\,\Delta r\,\Delta\theta$. The inward flux through the bottom face equals $A_z(z)\,r\,\Delta r\,\Delta\theta$, and the outward flux through the top face equals $A_z(z+\Delta z)\,r\,\Delta r\,\Delta\theta$. We have suppressed the r and θ arguments of A_z for simplicity, and we have chosen points at the midpoints of the faces, as in Fig. 2.22. The net outward flux is therefore

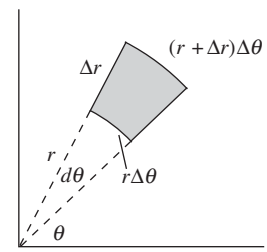


Figure F.2.
A small region in the r-θ plane.

$$\begin{aligned}
\Phi_{z\text{ faces}} &= A_z(z+\Delta z)\,r\,\Delta r\,\Delta\theta - A_z(z)\,r\,\Delta r\,\Delta\theta \\
&= \left(\frac{A_z(z+\Delta z) - A_z(z)}{\Delta z}\right) r\,\Delta r\,\Delta\,\theta\,\Delta z \\
&= \frac{\partial A_z}{\partial z}\,r\,\Delta r\,\Delta\theta\,\Delta z. \qquad \text{(F.8)}
\end{aligned}$$

Upon dividing this net outward flux by the volume $r\,\Delta r\,\Delta\theta\,\Delta z$, we obtain $\partial A_z/\partial z$, in agreement with the third term in $\nabla \cdot \mathbf{A}$ in Eq. (F.2). This was exactly the same argument we used in Section 2.10. The z coordinate in cylindrical coordinates is, after all, basically a Cartesian coordinate. However, things get more interesting with the r coordinate.

Consider the flux through the two faces (represented by the curved lines in Fig. F.2) that are perpendicular to the $\hat{\mathbf{r}}$ direction. The key point to realize is that *the areas of these two faces are not equal.* The upper right one is larger. So the difference in flux through these faces depends not only on the value of A_r, but also on the area. The inward flux through the lower left face equals $A_r(r)\big[r\,\Delta\theta\,\Delta z\big]$, and the outward flux through the upper right face equals $A_r(r+\Delta r)\big[(r+\Delta r)\,\Delta\theta\,\Delta z\big]$. As above, we have suppressed the θ and z arguments for simplicity, and we have chosen points at the midpoints of the faces. The net outward flux is therefore

$$\begin{aligned}
\Phi_{r \text{ faces}} &= (r+\Delta r)A_r(r+\Delta r)\,\Delta\theta\,\Delta z - rA_r(r)\,\Delta\theta\,\Delta z \\
&= \left(\frac{(r+\Delta r)A_r(r+\Delta r) - rA_r(r)}{\Delta r}\right)\Delta r\,\Delta\theta\,\Delta z \\
&= \frac{\partial(rA_r)}{\partial r}\,\Delta r\,\Delta\theta\,\Delta z. \qquad \text{(F.9)}
\end{aligned}$$

Upon dividing this net outward flux by the volume $r\,\Delta r\,\Delta\theta\,\Delta z$, we have a leftover r in the denominator, so we obtain $(1/r)\big(\partial(rA_r)/\partial r\big)$, in agreement with the first term in Eq. (F.2).

For the last two faces, the ones perpendicular to the $\hat{\boldsymbol{\theta}}$ direction, we don't have to worry about different areas, so we quickly obtain

$$\begin{aligned}
\Phi_{\theta \text{ faces}} &= A_\theta(\theta+\Delta\theta)\,\Delta r\,\Delta z - A_\theta(\theta)\,\Delta r\,\Delta z \\
&= \left(\frac{A_\theta(\theta+\Delta\theta) - A_\theta(\theta)}{\Delta\theta}\right)\Delta r\,\Delta\theta\,\Delta z \\
&= \frac{\partial A_\theta}{\partial\theta}\,\Delta r\,\Delta\theta\,\Delta z. \qquad \text{(F.10)}
\end{aligned}$$

Upon dividing this net outward flux by the volume $r\,\Delta r\,\Delta\theta\,\Delta z$, we again have a leftover r in the denominator, so we obtain $(1/r)(\partial A_\theta/\partial\theta)$, in agreement with the second term in Eq. (F.2).

If you like this sort of calculation, you can repeat this derivation for the case of spherical coordinates. However, it's actually not too hard to derive the general form of the divergence for *any* set of coordinates; see Exercise F.3. You can then check that this general formula reduces properly for spherical coordinates.

F.3.2 Cylindrical divergence, second method

Let's determine the divergence in cylindrical coordinates by explicitly calculating the dot product,

$$\nabla \cdot \mathbf{A} = \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\theta}}A_\theta + \hat{\mathbf{z}}A_z\right). \tag{F.11}$$

At first glance, it appears that $\nabla \cdot \mathbf{A}$ doesn't produce the form of the divergence given in Eq. (F.2). The second two terms work out, but it seems like the first term should simply be $\partial A_r/\partial r$ instead of $(1/r)\big(\partial(rA_r)/\partial r\big)$. However, the dot product does indeed correctly yield the latter term, because we must remember that, in contrast with Cartesian coordinates, *in cylindrical coordinates the unit vectors themselves depend on position.* This means that in Eq. (F.11) the derivatives in the ∇ operator also act on the unit vectors in $\mathbf{A}$. This issue doesn't come up in Cartesian coordinates because $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are fixed vectors, but that is more the exception than the rule. Writing $\mathbf{A}$ in the abbreviated form (A_r, A_θ, A_z) tends to hide important information. The full expression for $\mathbf{A}$ is $\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\theta}}A_\theta + \hat{\mathbf{z}}A_z$. There are six quantities here (three vectors and three components), and if any of these quantities vary with the coordinates, then these variations cause $\mathbf{A}$ to change. The derivatives of the unit vectors that are nonzero are

$$\frac{\partial \hat{\mathbf{r}}}{\partial \theta} = \hat{\boldsymbol{\theta}} \qquad \text{and} \qquad \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} = -\hat{\mathbf{r}}. \tag{F.12}$$

To demonstrate these relations, we can look at what happens to $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ if we rotate them through an angle $d\theta$. Since the unit vectors have length 1, we see from Fig. F.3 that $\hat{\mathbf{r}}$ picks up a component of length $d\theta$ in the $\hat{\boldsymbol{\theta}}$ direction, and $\hat{\boldsymbol{\theta}}$ picks up a component of length $d\theta$ in the $-\hat{\mathbf{r}}$ direction. The other seven of the nine possible derivatives are zero because none of the unit vectors depends on r or z, and furthermore $\hat{\mathbf{z}}$ doesn't depend on θ.

Due to the orthogonality of the unit vectors, we quickly see that, in addition to the three "corresponding" terms that survive in Eq. (F.11), one more term is nonzero:

$$\hat{\boldsymbol{\theta}} \cdot \frac{1}{r}\frac{\partial}{\partial \theta}(\hat{\mathbf{r}}A_r) = \hat{\boldsymbol{\theta}} \cdot \frac{1}{r}\left(\frac{\partial \hat{\mathbf{r}}}{\partial \theta}A_r + \hat{\mathbf{r}}\frac{\partial A_r}{\partial \theta}\right) = \hat{\boldsymbol{\theta}} \cdot \frac{1}{r}\hat{\boldsymbol{\theta}}A_r + 0 = \frac{A_r}{r}. \tag{F.13}$$

Equation (F.11) therefore becomes

$$\nabla \cdot \mathbf{A} = \frac{\partial A_r}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} + \frac{A_r}{r}. \tag{F.14}$$

The sum of the first and last terms here can be rewritten as the first term in $\nabla \cdot \mathbf{A}$ in Eq. (F.2), as desired.

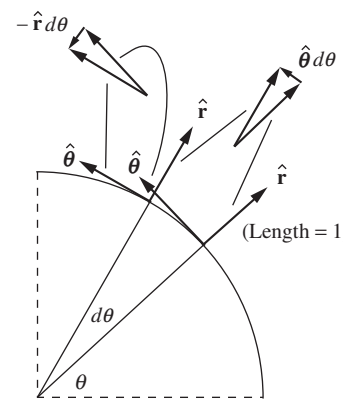


Figure F.3.
How the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ unit vectors depend on θ.

F.4 Curl

The curl produces a vector from a vector. The curl of a vector function was defined in Eq. (2.80) as the net circulation around a given small area, divided by the area. (The three possible orientations of the area yield the three components.) In Section 2.16 we derived the form of the curl in Cartesian coordinates, and it turned out to be the cross product of the ∇ operator with the vector $\mathbf{A}$, that is, $\nabla \times \mathbf{A}$. We'll use the same method here to derive the form in cylindrical coordinates, after which we derive it a second way, analogous to the above second method for the divergence. Actually, we'll calculate just the z component; this should make the procedure clear. As an exercise you can calculate the other two components.

F.4.1 Cylindrical curl, first method

The z component of $\nabla \times \mathbf{A}$ is found by looking at the circulation around a small area in the r-θ plane (or more generally, in some plane parallel to the r-θ plane). Consider the upper right and lower left (curved) edges in Fig. F.2. Following the strategy in Section 2.16, the counterclockwise line integral along the upper right edge equals $A_\theta(r+\Delta r)\big[(r+\Delta r)\,\Delta\theta\big]$, and the counterclockwise line integral along the lower left edge equals $-A_\theta(r)\big[r\,\Delta\theta\big]$. We have suppressed the θ and z arguments for simplicity, and we have chosen points at the midpoints of the edges. Note that we have correctly incorporated the fact that the upper right edge is longer than the lower left edge (the same issue that came up in the above calculation of the divergence). The net circulation along these two edges is

$$\begin{aligned} C_{\theta \text{ sides}} &= (r+\Delta r)A_\theta(r+\Delta r)\,\Delta\theta - rA_\theta(r)\,\Delta\theta \\ &= \left(\frac{(r+\Delta r)A_\theta(r+\Delta r) - rA_\theta(r)}{\Delta r}\right)\Delta r\,\Delta\theta \\ &= \frac{\partial(rA_\theta)}{\partial r}\,\Delta r\,\Delta\theta. \end{aligned} \tag{F.15}$$

Upon dividing this circulation by the area $r\,\Delta r\,\Delta\theta$, we have a leftover r in the denominator, so we obtain $(1/r)\big(\partial(rA_\theta)/\partial r\big)$, in agreement with the first of the two terms in the z component of $\nabla \times \mathbf{A}$ in Eq. (F.2).

Now consider the upper left and lower right (straight) edges. The counterclockwise line integral along the upper left edge equals $-A_r(\theta+\Delta\theta)\,\Delta r$, and the counterclockwise line integral along the lower right edge equals $A_r(\theta)\,\Delta r$. The net circulation along these two edges is

$$\begin{aligned} C_{r \text{ sides}} &= -A_r(\theta+\Delta\theta)\,\Delta r + A_r(\theta)\,\Delta r \\ &= -\left(\frac{A_r(\theta+\Delta\theta) - A_r(\theta)}{\Delta\theta}\right)\Delta r\,\Delta\theta \\ &= -\frac{\partial A_r}{\partial\theta}\,\Delta r\,\Delta\theta. \end{aligned} \tag{F.16}$$

Upon dividing this circulation by the area $r\,\Delta r\,\Delta\theta$, we again have a left-over r in the denominator, so we obtain $-(1/r)(\partial A_r/\partial\theta)$, in agreement with Eq. (F.2).

F.4.2 Cylindrical curl, second method

Our goal is to calculate the cross product,

$$\nabla\times\mathbf{A}=\left(\hat{\mathbf{r}}\frac{\partial}{\partial r}+\hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\times\left(\hat{\mathbf{r}}A_r+\hat{\boldsymbol{\theta}}A_\theta+\hat{\mathbf{z}}A_z\right), \tag{F.17}$$

while remembering that some of the unit vectors depend on the coordinates according to Eq. (F.12). As above, we'll look at just the z component. This component arises from terms of the form $\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}$ or $\hat{\boldsymbol{\theta}}\times\hat{\mathbf{r}}$. In addition to the two obvious terms of this form, we also have the one involving $\hat{\boldsymbol{\theta}}\times(\partial\hat{\boldsymbol{\theta}}/\partial\theta)$, which from Eq. (F.12) equals $\hat{\boldsymbol{\theta}}\times(-\hat{\mathbf{r}})=\hat{\mathbf{z}}$. The complete z component of the cross product is therefore

$$\begin{aligned}(\nabla\times\mathbf{A})_z&=\hat{\mathbf{r}}\times\frac{\partial(\hat{\boldsymbol{\theta}}A_\theta)}{\partial r}+\hat{\boldsymbol{\theta}}\times\frac{1}{r}\frac{\partial(\hat{\mathbf{r}}A_r)}{\partial\theta}+\hat{\boldsymbol{\theta}}\times\frac{1}{r}\frac{\partial(\hat{\boldsymbol{\theta}}A_\theta)}{\partial\theta}\\&=\hat{\mathbf{z}}\left(\frac{\partial A_\theta}{\partial r}-\frac{1}{r}\frac{\partial A_r}{\partial\theta}+\frac{A_\theta}{r}\right).\end{aligned} \tag{F.18}$$

The sum of the first and last terms here can be rewritten as the first term in the z component of $\nabla\times\mathbf{A}$ in Eq. (F.2), as desired.

F.5 Laplacian

The Laplacian produces a scalar from a scalar. The Laplacian of a function f (written as $\nabla^2 f$ or $\nabla\cdot\nabla f$) is defined as the divergence of the gradient of f. Its physical significance is that it gives a measure of how the average value of f over the surface of a sphere compares with the value of f at the center of the sphere. Let's be quantitative about this.

Consider the average value of a function f over the surface of a sphere of radius r. Call it $f_{\text{avg},r}$. If we choose the origin of our coordinate system to be the center of the sphere, then $f_{\text{avg},r}$ can be written as (with A being the area of the sphere)

$$f_{\text{avg},r}=\frac{1}{A}\int f\,dA=\frac{1}{4\pi r^2}\int f\,r^2 d\Omega=\frac{1}{4\pi}\int f\,d\Omega, \tag{F.19}$$

where $d\Omega=\sin\theta\,d\theta\,d\phi$ is the solid-angle element. We are able to take the r^2 outside the integral and cancel it because r is constant over the sphere. This expression for $f_{\text{avg},r}$ is no surprise, of course, because the integral of $d\Omega$ over the whole sphere is 4π. But let us now take the d/dr derivative of both sides of Eq. (F.19), which will alow us to invoke

the divergence theorem. On the right-hand side, the integration doesn't involve r, so we can bring the derivative inside the integral. This yields (using $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$)

$$\frac{df_{\text{avg},r}}{dr} = \frac{1}{4\pi} \int \frac{\partial f}{\partial r}\, d\Omega = \frac{1}{4\pi} \int \hat{\mathbf{r}} \frac{\partial f}{\partial r} \cdot \hat{\mathbf{r}}\, d\Omega = \frac{1}{4\pi r^2} \int \hat{\mathbf{r}} \frac{\partial f}{\partial r} \cdot \hat{\mathbf{r}}\, r^2 d\Omega. \tag{F.20}$$

(Again, we are able to bring the r^2 inside the integral because r is constant over the sphere.) But $\hat{\mathbf{r}}\, r^2 d\Omega$ is just the vector area element of the sphere, $d\mathbf{a}$. And $\hat{\mathbf{r}}(\partial f/\partial r)$ is the $\hat{\mathbf{r}}$ component of ∇f in spherical coordinates. The other components of ∇f give zero when dotted with $d\mathbf{a}$, so we can write

$$\frac{df_{\text{avg},r}}{dr} = \frac{1}{4\pi r^2} \int \nabla f \cdot d\mathbf{a}. \tag{F.21}$$

The divergence theorem turns this into

$$\frac{df_{\text{avg},r}}{dr} = \frac{1}{4\pi r^2} \int \nabla \cdot \nabla f\, dV \implies \boxed{\frac{df_{\text{avg},r}}{dr} = \frac{1}{4\pi r^2} \int \nabla^2 f\, dV} \tag{F.22}$$

There are two useful corollaries of this result. First, if $\nabla^2 f = 0$ everywhere, then $df_{\text{avg},r}/dr = 0$ for all r. In other words, the average value of f over the surface of a sphere doesn't change as the sphere grows (while keeping the same center). So all spheres centered at a given point have the same average value of f. In particular, they have the same average value that an infinitesimal sphere has. But the average value over an infinitesimal sphere is simply the value at the center. Therefore, if $\nabla^2 f = 0$, then the average value of f over the surface of a sphere (of *any* size) equals the value at the center:

$$\nabla^2 f = 0 \implies f_{\text{avg},r} = f_{\text{center}}. \tag{F.23}$$

This is the result we introduced in Section 2.12 and proved for the special case of the electrostatic potential ϕ.

Second, we can derive an expression for how f changes, for *small* values of r. Up to this point, all of our results have been exact. We will now work in the small-r approximation. In this limit we can say that $\nabla^2 f$ is essentially constant throughout the interior of the sphere (assuming that f is well-enough behaved). So its value everywhere is essentially the value at the center. The volume integral in Eq. (F.22) then equals $(4\pi r^3/3)(\nabla^2 f)_{\text{center}}$, and we have

$$\frac{df_{\text{avg},r}}{dr} = \frac{1}{4\pi r^2} \frac{4\pi r^3}{3} (\nabla^2 f)_{\text{center}} \implies \frac{df_{\text{avg},r}}{dr} = \frac{r}{3} (\nabla^2 f)_{\text{center}}. \tag{F.24}$$

Since $(\nabla^2 f)_{\text{center}}$ is a constant, we can quickly integrate both sides of this relation to obtain

$$f_{\text{avg},r} = f_{\text{center}} + \frac{r^2}{6}(\nabla^2 f)_{\text{center}} \qquad \text{(for small } r\text{)}, \tag{F.25}$$

where the constant of integration has been chosen to give equality at $r = 0$. We see that the average value of f over a (small) sphere grows quadratically, with the quadratic coefficient being $1/6$ times the value of the Laplacian at the center.

Let's check this result for the function $f(r, \theta, \phi) = r^2$, or equivalently $f(x, y, z) = x^2 + y^2 + z^2$. By using either Eq. (F.1) or Eq. (F.3) we obtain $\nabla^2 f = 6$. If our sphere is centered at the origin, then Eq. (F.25) gives $f_{\text{avg},r} = 0 + (r^2/6)(6) = r^2$, which is correct because f takes on the constant value of r^2 over the sphere. In this simple case, the result is exact for all r.

F.5.1 Cylindrical Laplacian

Let's explicitly calculate the Laplacian in cylindrical coordinates by calculating the divergence of the gradient of f. As we've seen in a few cases above, we must be careful to take into account the position dependence of some of the unit vectors. We have

$$\nabla \cdot \nabla f = \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}\right). \tag{F.26}$$

In addition to the three "corresponding" terms, we also have the term involving $\hat{\boldsymbol{\theta}} \cdot (\partial\, \hat{\mathbf{r}}/\partial\theta)$, which from Eq. (F.12) equals $\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = 1$. So this fourth term reduces to $(1/r)(\partial f/\partial r)$. The Laplacian is therefore

$$\begin{aligned} \nabla^2 f &= \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) + \frac{1}{r}\frac{\partial f}{\partial r} \\ &= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} + \frac{1}{r}\frac{\partial f}{\partial r}. \end{aligned} \tag{F.27}$$

The sum of the first and last terms here can be rewritten as the first term in the $\nabla^2 f$ expression in Eq. (F.2), as desired.

Exercises

F.1 *Divergence using two systems* $**$

(a) The vector $\mathbf{A} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$ in Cartesian coordinates equals the vector $\mathbf{A} = r\,\hat{\mathbf{r}}$ in cylindrical coordinates. Calculate $\nabla \cdot \mathbf{A}$ in both Cartesian and cylindrical coordinates, and verify that the results are equal.

(b) Repeat (a) for the vector $\mathbf{A} = x\,\hat{\mathbf{x}} + 2y\,\hat{\mathbf{y}}$. You will need to find the cylindrical components of $\mathbf{A}$, which you can do by using $\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$ and $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$. Alternatively,

you can project $\mathbf{A}$ onto the unit vectors, $\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta$ and $\hat{\boldsymbol{\theta}} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta$.

F.2 *Cylindrical divergence* ***

Calculate the divergence in cylindrical coordinates in the following way. We know that the divergence in Cartesian coordinates is $\nabla \cdot \mathbf{A} = \partial A_x/\partial x + \partial A_y/\partial y + \partial A_z/\partial z$. To rewrite this in terms of cylindrical coordinates, show that the Cartesian derivative operators can be written as (the $\partial/\partial z$ derivative stays the same)

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}, \\ \frac{\partial}{\partial y} &= \sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}, \end{aligned} \tag{F.28}$$

and that the components of $\mathbf{A}$ can be written as (A_z stays the same)

$$\begin{aligned} A_x &= A_r\cos\theta - A_\theta\sin\theta, \\ A_y &= A_r\sin\theta + A_\theta\cos\theta. \end{aligned} \tag{F.29}$$

Then explicitly calculate $\nabla \cdot \mathbf{A} = \partial A_x/\partial x + \partial A_y/\partial y + \partial A_z/\partial z$. It gets to be a big mess, but it simplifies in the end.

F.3 *General expression for divergence* ***

Let $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, $\hat{\mathbf{x}}_3$ be the (not necessarily Cartesian) basis vectors of a coordinate system. For example, in spherical coordinates these vectors are $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$. Note that the $d\mathbf{s}$ line elements listed at the beginning of this appendix all take the form of

$$d\mathbf{s} = f_1\,dx_1\,\hat{\mathbf{x}}_1 + f_2\,dx_2\,\hat{\mathbf{x}}_2 + f_3\,dx_3\,\hat{\mathbf{x}}_3, \tag{F.30}$$

where the f factors are (possibly trivial) functions of the coordinates. For example, in Cartesian coordinates, f_1, f_2, f_3 are $1, 1, 1$; in cylindrical coordinates they are $1, r, 1$; and in spherical coordinates they are $1, r, r\sin\theta$. As we saw in Section F.2, these values of f determine the form of ∇ (the f factors simply end up in the denominators), so they determine *everything* about the various vector operators. Show, by applying the first method we used in Section F.3, that the general expression for the divergence is

$$\nabla \cdot \mathbf{A} = \frac{1}{f_1 f_2 f_3}\left[\frac{\partial(f_2 f_3 A_1)}{\partial x_1} + \frac{\partial(f_1 f_3 A_2)}{\partial x_2} + \frac{\partial(f_1 f_2 A_3)}{\partial x_3}\right]. \tag{F.31}$$

Verify that this gives the correct result in the case of spherical coordinates. (The general expression for the curl can be found in a similar way.)

F.4 *Laplacian using two systems* $**$

(a) The function $f = x^2 + y^2$ in Cartesian coordinates equals the function $f = r^2$ in cylindrical coordinates. Calculate $\nabla^2 f$ in both Cartesian and cylindrical coordinates, and verify that the results are equal.

(b) Repeat (a) for the function $f = x^4 + y^4$. You will need to determine what f looks like in cylindrical coordinates.

F.5 *"Sphere" averages in one and two dimensions* $**$

Equation (F.25) holds for a function f in 3D space, but analogous results also hold in 2D space (where the "sphere" is a circle bounding a disk) and in 1D space (where the "sphere" is two points bounding a line segment). Derive those results. Although it is possible to be a little more economical in the calculations by stripping off some dimensions at the start, derive the results in a 3D manner exactly analogous to the way we derived Eq. (F.25). For the 2D case, the relevant volume is a cylinder, with f having no dependence on z. For the 1D case, the relevant volume is a rectangular slab, with f having no dependence on y or z. The 1D result should look familiar from the standard 1D Taylor series.

F.6 *Average over a cube* $***$

By using the second-order Taylor expansion for a function of three Cartesian coordinates, show that the average value of a function f over the surface of a cube of side 2ℓ (with edges parallel to the coordinate axes) is

$$f_{\text{avg}} = f_{\text{center}} + \frac{5\ell^2}{18}\,(\nabla^2 f)_{\text{center}}. \tag{F.32}$$

You should convince yourself why the factor of $5/18$ here is correctly larger than the $1/6$ in Eq. (F.25) and smaller than $(\sqrt{3})^2/6$.

G

A short review of special relativity

G.1 Foundations of relativity

We assume that the reader has already been introduced to special relativity. Here we shall review the principal ideas and formulas that are used in the text beginning in Chapter 5. Most essential is the concept of an inertial frame of reference for space-time events and the transformation of the coordinates of an event from one inertial frame to another.

A frame of reference is a coordinate system laid out with measuring rods and provided with clocks. Clocks are everywhere. When something happens at a certain place, the time of its occurrence is read from a clock that was at, and stays at, that place. That is, time is measured by a *local* clock that is *stationary* in the frame. The clocks belonging to the frame are all *synchronized.* One way to accomplish this (not the only way) was described by Einstein in his great paper of 1905. Light signals are used. From a point A, at time t_A, a short pulse of light is sent out toward a remote point B. It arrives at B at the time t_B, as read on a clock at B, and is immediately reflected back toward A, where it arrives at t'_A. If $t_B = (t_A + t'_A)/2$, the clocks at A and B are synchronized. If not, one of them requires adjustment. In this way, all clocks in the frame can be synchronized. Note that the job of observers in this procedure is merely to record local clock readings for subsequent comparison.

An *event* is located in space and time by its coordinates x, y, z, t in some chosen reference frame. The event might be the passage of a particle at time t_1, through the space point (x_1, y_1, z_1). The history of the particle's motion is a sequence of such events. Suppose the sequence has the special property that $x = v_x t$, $y = v_y t$, $z = v_z t$, at every time t, with v_x, v_y, and v_z constant. That describes motion in a straight line at

constant speed with respect to this frame. An *inertial frame of reference* is a frame in which an isolated body, free from external influences, moves in this way. An inertial frame, in other words, is one in which Newton's first law is obeyed. Behind all of this, including the synchronization of clocks, are two assumptions about empty space: it is *homogeneous* (that is, all locations in space are equivalent) and it is *isotropic* (that is, all directions in space are equivalent).

Two frames, let us call them F and F', can differ in several ways. One can simply be displaced with respect to the other, the origin of coordinates in F' being fixed at a point in F that is not at the F coordinate origin. Or the axes in F' might not be parallel to the axes in F. As for the timing of events, if F and F' are not moving with respect to one another, a clock stationary in F is stationary also in F'. In that case we can set all F' clocks to agree with the F clocks and then ignore the distinction. Differences in frame location and frame orientation only have no interesting consequences if space is homogeneous and isotropic. Suppose now that the origin of frame F' is *moving* relative to the origin of frame F. The description of a sequence of events by coordinate values and clock times in F can differ from the description of the same events by space coordinate values in F' and times measured by clocks in F'. How must the two descriptions be related? In answering that we shall be concerned only with the case in which F is an inertial frame and F' is a frame that is moving relative to F at constant velocity and without rotating. In that case F' is also an inertial frame.

Special relativity is based on the postulate that physical phenomena observed in different inertial frames of reference appear to obey exactly the same laws. In that respect one frame is as good as another; no frame is unique. If true, this relativity postulate is enough to determine the way a description of events in one frame is related to the description in a different frame of the same events. In that relation there appears a universal speed, the same in all frames, whose value must be found by experiment. Sometimes added as a second postulate is the statement that a measurement of the velocity of light in any frame of reference gives the same result whether the light's source is stationary in that frame or not. One may regard this as a statement about the nature of light rather than an independent postulate. It asserts that electromagnetic waves in fact travel with the limiting speed implied by the relativity postulate. The deductions from the relativity postulate, expressed in the formulas of special relativity, have been precisely verified by countless experiments. Nothing in physics rests on a firmer foundation.

G.2 Lorentz transformations

Consider two events, A and B, observed in an inertial frame F. *Observed,* in this usage, is short for "whose space-time coordinates are determined with the measuring rods and clocks of frame F." (Remember

that our observers are equipped merely with pencil and paper, and we must post an observer at the location of every event!) The displacement of one event from the other is given by the four numbers

$$x_B - x_A, \quad y_B - y_A, \quad z_B - z_A, \quad t_B - t_A. \tag{G.1}$$

The *same two events* could have been located by giving their coordinates in some other frame F'. Suppose F' is moving with respect to F in the manner indicated in Fig. G.1. The spatial axes of F' remain parallel to those in F, while, as seen from F, the frame F' moves with speed v in the positive x direction. This is a special case, obviously, but it contains most of the interesting physics.

Event A, as observed in F', occurred at x'_A, y'_A, z'_A, t'_A, the last of these numbers being the reading of a clock belonging to (that is, *stationary in*) F'. The space-time displacement, or *interval* between events A and B in F', is not the same as in F. Its components are related to those in F by the *Lorentz transformation*,

$$\begin{aligned} x'_B - x'_A &= \gamma(x_B - x_A) - \beta\gamma c(t_B - t_A), \\ y'_B - y'_A &= y_B - y_A, \\ z'_B - z'_A &= z_B - z_A, \\ t'_B - t'_A &= \gamma(t_B - t_A) - \beta\gamma(x_B - x_A)/c. \end{aligned} \tag{G.2}$$

In these equations c is the speed of light, $\beta = v/c$, and $\gamma = 1/\sqrt{1-\beta^2}$. The inverse transformation has a similar appearance – as it should if no frame is unique. It can be obtained from Eq. (G.2) simply by exchanging primed and unprimed symbols and reversing the sign of β, as you can verify by explicitly solving for the quantities $x_B - x_A$ and $t_B - t_A$.

Two events A and B are *simultaneous* in F if $t_B - t_A = 0$. But that does not make $t'_B - t'_A = 0$ unless $x_B = x_A$. Thus events that are simultaneous in one inertial frame may not be so in another. Do not confuse this fundamental "relativity of simultaneity" with the obvious fact that an observer not equally distant from two simultaneous explosions will receive light flashes from them at different times. The times t'_A and t'_B are recorded by *local* clocks at each event, clocks stationary in F' that have previously been perfectly synchronized.

Consider a rod stationary in F' that is parallel to the x' axis and extends from x'_A to x'_B. Its length in F' is just $x'_B - x'_A$. The rod's length as measured in frame F is the distance $x_B - x_A$ between two points in the frame F that its ends pass *simultaneously* according to clocks in F. For these two events, then, $t_B - t_A = 0$. With this condition the first of the Lorentz transformation equations above gives us at once

$$x_B - x_A = (x'_B - x'_A)/\gamma. \tag{G.3}$$

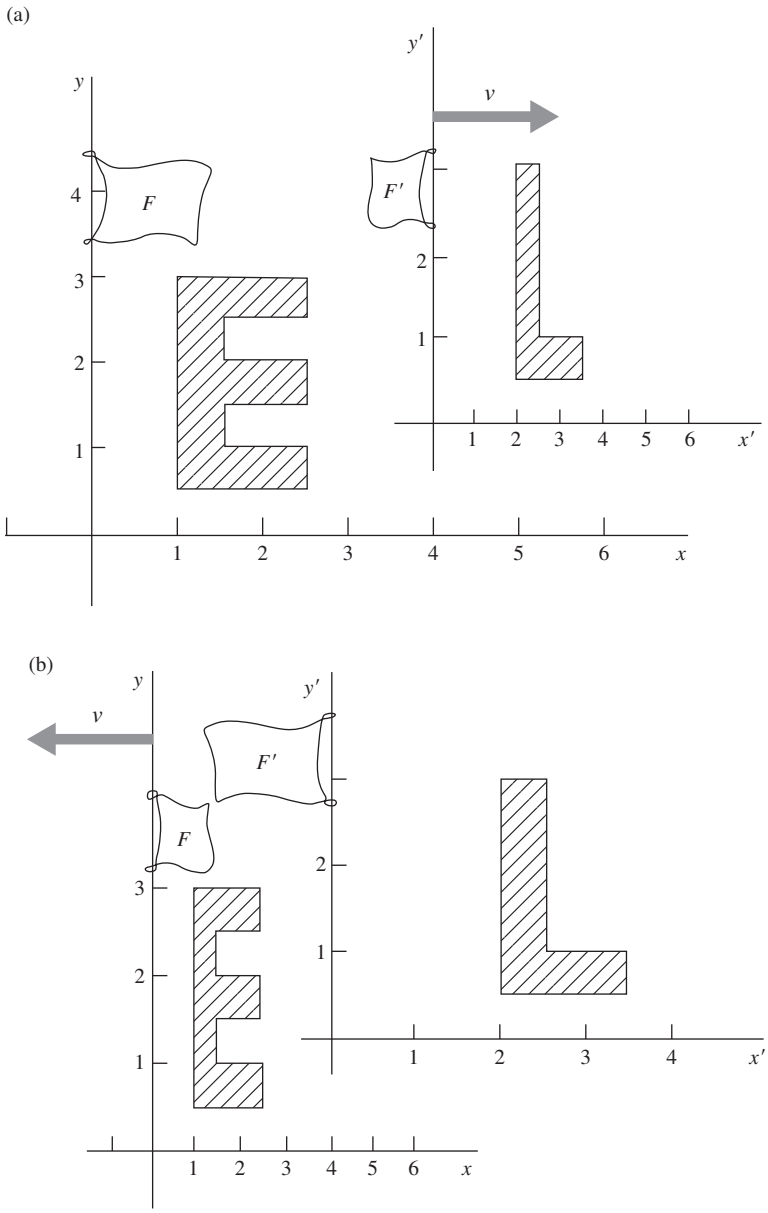


Figure G.1.
Two frames moving with relative speed v. The "E" is stationary in frame F. The "L" is stationary in frame F'. In this example $\beta = v/c = 0.866$; $\gamma = 2$. (a) Where everything was, as determined by observers in F at a particular instant of time t according to clocks in F. (b) Where everything was, as determined by observers in F' at a particular instant of time t' according to clocks in F'.

Question: Suppose the clocks in the two frames happened to be set so that the left edge of the E touched the left edge of the L at $t = 0$ according to a local clock in F, and at $t' = 0$ according to a local clock in F'. Let the distances be in feet and take c as 1 foot/nanosecond. What is the reading t of all the F clocks in (a)? What is the reading t' of all the F' clocks in (b)?

Answer: $t = 4.62$ nanoseconds; $t' = 4.04$ nanoseconds. If you don't agree, study the example again.

This is the famous *Lorentz contraction.* Loosely stated, lengths between fixed points in F', if parallel to the relative velocity of the frames, are judged by observers in F to be shorter by the factor $1/\gamma$. This statement remains true if F' and F are interchanged. Lengths perpendicular to the relative velocity measure the same in the two frames.

Consider one of the clocks in F'. It is moving with speed v through the frame F. Let us record as t'_A its reading as it passes one of our local clocks in F; the local clock reads at that moment t_A. Later this moving clock passes another F clock. At that event the local F clock reads t_B, and the reading of the moving clock is recorded as t'_B. The two events are separated in the F frame by a distance $x_B - x_A = v(t_B - t_A)$. Substituting this into the fourth equation of the Lorentz transformation, Eq. (G.2), we obtain

$$t'_B - t'_A = \gamma(t_B - t_A)(1 - \beta^2) = (t_B - t_A)/\gamma. \tag{G.4}$$

According to the moving clock, less time has elapsed between the two events than is indicated by the stationary clocks in F. This is the *time dilation* that figures in the "twin paradox." It has been verified in many experiments, including one in which an atomic clock was flown around the world.

Remembering that "moving clocks run slow, by the factor $1/\gamma$," and that "moving graph paper is shortened parallel to its motion by the factor $1/\gamma$," you can often figure out the consequences of a Lorentz transformation without writing out the equations. This behavior, it must be emphasized, is not a peculiar physical property of our clocks and paper, but is intrinsic in space and time measurement under the relativity postulate.

G.3 Velocity addition

The formula for the addition of velocities, which we use in Chapter 5, is easily derived from the Lorentz transformation equations. Suppose an object is moving in the positive x direction in frame F with velocity u_x. What is its velocity in the frame F'? To simplify matters let the moving object pass the origin at $t = 0$. Then its position in F at any time t is simply $x = u_x t$. To simplify further, let the space and time origins of F and F' coincide. Then the first and last of the Lorentz transformation equations become

$$x' = \gamma x - \beta\gamma ct \quad \text{and} \quad t' = \gamma t - \beta\gamma x/c. \tag{G.5}$$

By substituting $u_x t$ for x on the right side of each equation, and dividing the first by the second, we get

$$\frac{x'}{t'} = \frac{u_x - \beta c}{1 - \beta u_x/c}. \tag{G.6}$$

On the left we have the velocity of the object in the F' frame, u'_x. The formula is usually written with v instead of βc.

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2}. \tag{G.7}$$

By solving Eq. (G.7) for u_x you can verify that the inverse is

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}\,, \tag{G.8}$$

and that in no case will these relations lead to a velocity, either u_x or u'_x, larger than c. As with the inverse Lorentz transformation, you can also obtain Eq. (G.8) from Eq. (G.7) simply by exchanging primed and unprimed symbols and reversing the sign of v.

A velocity component perpendicular to v, the relative velocity of the frames, transforms differently, of course. Analogous to Eq. (G.5), the second and last of the Lorentz transformation equations are

$$y' = y \quad \text{and} \quad t' = \gamma t - \beta\gamma x/c. \tag{G.9}$$

If we have $x = u_x t$ and $y = u_y t$ in frame F (in general the object can be moving diagonally), then we can substitute these into Eq. (G.9) and divide the first equation by the second to obtain

$$\frac{y'}{t'} = \frac{u_y}{\gamma(1 - \beta u_x/c)} \quad \Longrightarrow \quad u'_y = \frac{u_y}{\gamma(1 - u_x v/c^2)}\,. \tag{G.10}$$

In the special case where $u_x = 0$ (which means that the velocity points in the y direction in frame F), we have $u'_y = u_y/\gamma$. That is, the y speed is slower in the frame F' where the object is flying by diagonally. In the special case where $u_x = v$ (which means that the object travels along with the F' frame, as far as the x direction is concerned), you can show that Eq. (G.10) reduces to $u'_y = \gamma u_y \Longrightarrow u_y = u'_y/\gamma$. This makes sense; the object has $u'_x = 0$, so this result is analogous to the preceding $u'_y = u_y/\gamma$ result for the $u_x = 0$ case. In effect we have simply switched the primed and unprimed labels. These special cases can also be derived directly from time dilation.

G.4 Energy, momentum, force

A dynamical consequence of special relativity can be stated as follows. Consider a particle moving with velocity $\mathbf{u}$ in an inertial frame F. We find that energy and momentum are conserved in the interactions of this particle with others if we attribute to the particle a momentum and an energy given by

$$\mathbf{p} = \gamma m_0 \mathbf{u} \quad \text{and} \quad E = \gamma m_0 c^2, \tag{G.11}$$

where m_0 is a constant characteristic of that particle. We call m_0 the *rest mass* (or just *the mass*) of the particle. It could have been determined in a frame in which the particle is moving so slowly that Newtonian mechanics applies – for instance, by bouncing the particle against some standard mass. The factor γ multiplying m_0 is $(1 - u^2/c^2)^{-1/2}$, where u is the speed of the particle as observed in our frame F.

Given **p** and E, the momentum and energy of a particle as observed in F, what is the momentum of that particle, and its energy, as observed in another frame F'? As before, we assume F' is moving in the positive x direction, with speed v, as seen from F. The transformation turns out to be this:

$$\begin{aligned} p'_x &= \gamma p_x - \beta\gamma E/c, \\ p'_y &= p_y, \\ p'_z &= p_z, \\ E' &= \gamma E - \beta\gamma c p_x. \end{aligned} \tag{G.12}$$

Note that βc is here the relative velocity of the two frames, as it was in Eq. (G.2), not the particle velocity.

Compare this transformation with Eq. (G.2). The resemblance would be perfect if we considered cp instead of p in Eq. (G.12), and ct rather than t in Eq. (G.2). A set of four quantities that transform in this way is called a *four-vector*.

The meaning of *force* is rate of change of momentum. The force acting on an object is simply $d\mathbf{p}/dt$, where **p** is the object's momentum in the chosen frame of reference and t is measured by clocks in that frame. To find how forces transform, consider a particle of mass m_0 initially at rest at the origin in frame F upon which a force f acts for a short time Δt. We want to find the rate of change of momentum dp'/dt', observed in a frame F'. As before, we shall let F' move in the x direction as seen from F. Consider first the effect of the force component f_x. In time Δt, p_x will increase from zero to $f_x\Delta t$, while x increases by

$$\Delta x = \frac{1}{2}\left(\frac{f_x}{m_0}\right)(\Delta t)^2, \tag{G.13}$$

and the particle's energy increases by $\Delta E = (f_x\Delta t)^2/2m_0$; this is the kinetic energy it acquires, as observed in F. (The particle's speed in F is still so slight that Newtonian mechanics applies there.) Using the first of Eqs. (G.12) we find the change in p'_x:

$$\Delta p'_x = \gamma\,\Delta p_x - \beta\gamma\,\Delta E/c, \tag{G.14}$$

and using the fourth of Eqs. (G.2) gives

$$\Delta t' = \gamma\,\Delta t - \beta\gamma\,\Delta x/c. \tag{G.15}$$

Now both ΔE and Δx are proportional to $(\Delta t)^2$, so when we take the limit $\Delta t \to 0$, the last term in each of these equations will drop out, giving

$$\frac{dp'_x}{dt'} = \lim_{\Delta t'\to 0}\frac{\Delta p'_x}{\Delta t'} = \frac{\gamma\,(f_x\,\Delta t)}{\gamma\,\Delta t} = f_x. \tag{G.16}$$

Conclusion: the force component *parallel* to the relative frame motion has the same value in the moving frame as in the rest frame of the particle.

A transverse force component behaves differently. In frame F, $\Delta p_y = f_y \Delta t$. But now $\Delta p'_y = \Delta p_y$, and $\Delta t' = \gamma \Delta t$, so we get

$$\frac{dp'_y}{dt'} = \frac{f_y \Delta t}{\gamma \Delta t} = \frac{f_y}{\gamma}\,. \tag{G.17}$$

A force component perpendicular to the relative frame motion, observed in F', is *smaller* by the factor $1/\gamma$ than the value determined by observers in the rest frame of the particle.

The transformation of a force from F' to some other moving frame F'' would be a little more complicated. We can always work it out, if we have to, by transforming to the rest frame of the particle and then back to the other moving frame.

We conclude our review with a remark about Lorentz invariance. If you square both sides of Eq. (G.12) and remember that $\gamma^2 - \beta^2\gamma^2 = 1$, you can easily show that

$$c^2(p'^2_x + p'^2_y + p'^2_z) - E'^2 = c^2(p_x^2 + p_y^2 + p_z^2) - E^2. \tag{G.18}$$

Evidently this quantity $c^2p^2 - E^2$ is *not changed* by a Lorentz transformation. It is often called the *invariant four-momentum* (even though it has dimensions of energy squared). It has the same value in every frame of reference, including the particle's rest frame. In the rest frame the particle's momentum is zero and its energy E is just m_0c^2. The invariant four-momentum is therefore $-m_0^2c^4$. It follows that in any other frame

$$E^2 = c^2p^2 + m_0^2c^4. \tag{G.19}$$

The invariant constructed in the same way with Eq. (G.2) is

$$(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2 - c^2(t_B - t_A)^2. \tag{G.20}$$

Two events, A and B, for which this quantity is positive are said to have a *spacelike* separation. It is always possible to find a frame in which they are simultaneous. If the invariant is negative, the events have a *timelike* separation. In that case a frame exists in which they occur at different times, but at the same place. If this "invariant interval" is zero, the two events can be connected by a flash of light.

H

Radiation by an accelerated charge

A particle with charge q has been moving in a straight line at constant speed v_0 for a long time. It runs into something, let us imagine, and in a short period of constant deceleration, of duration τ, the particle is brought to rest. The graph of velocity versus time in Fig. H.1 describes its motion. What must the electric field of this particle look like after that? Figure H.2 shows how to derive it.

We shall assume that v_0 is small compared with c. Let $t = 0$ be the instant the deceleration began, and let $x = 0$ be the position of the particle at that instant. By the time the particle has completely stopped it will have moved a little farther on, to $x = v_0\tau/2$. That distance, indicated in Fig. H.2, is small compared with the other distances that will be involved.

We now examine the electric field at a time $t = T \gg \tau$. Observers farther away from the origin than $R = cT$ cannot have learned that the particle was decelerated. Throughout that region, region I in Fig. H.2, the field must be that of a charge that has been moving *and is still moving* at the constant speed v_0. That field, as we discovered in Section 5.7, appears to emanate from the present position of the charge, which for an observer anywhere in region I is the point $x = v_0T$ on the x axis. That is where the particle would be now if it hadn't been decelerated. On the other hand, for any observer whose distance from the origin is less than $c(T - \tau)$, that is to say, for any observer in region II, the field is that of a charge at rest close to the origin (actually at $x = v_0\tau/2$).

What must the field be like in the transition region, the spherical shell of thickness $c\tau$? Gauss's law provides the key. A field line such as AB lies on a cone around the x axis that includes a certain amount of flux from the charge q. If CD makes the same angle θ with the axis,

the cone on which it lies includes that same amount of flux. (Because v_0 is small, the relativistic compression of field lines visible in Fig. 5.15 and Fig. 5.19 is here negligible.) Hence AB and CD must be parts of the same field line, connected by a segment BC. This tells us the *direction* of the field $\mathbf{E}$ within the shell; it is the direction of the line segment BC. This field $\mathbf{E}$ within the shell has both a radial component E_r and a transverse component E_θ. From the geometry of the figure their ratio is easily found:

$$\frac{E_\theta}{E_r} = \frac{v_0 T \sin\theta}{c\tau}. \tag{H.1}$$

Now E_r must have the same value within the shell thickness that it does in region II near B. (Gauss's law again!) Therefore $E_r = q/4\pi\epsilon_0 R^2 = q/4\pi\epsilon_0 c^2 T^2$, and substituting this into Eq. (H.1) we obtain

$$E_\theta = \frac{v_0 T \sin\theta}{c\tau} E_r = \frac{q v_0 \sin\theta}{4\pi\epsilon_0 c^3 T \tau}. \tag{H.2}$$

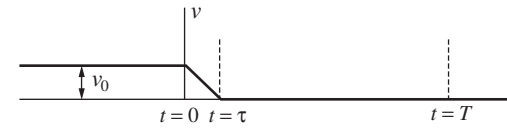


Figure H.1.
Velocity-time diagram for a particle that traveled at constant speed v_0 until $t = 0$. It then experienced a constant negative acceleration of magnitude $a = v_0/\tau$, which brought it to rest at time $t = \tau$. We assume v_0 is small compared with c.

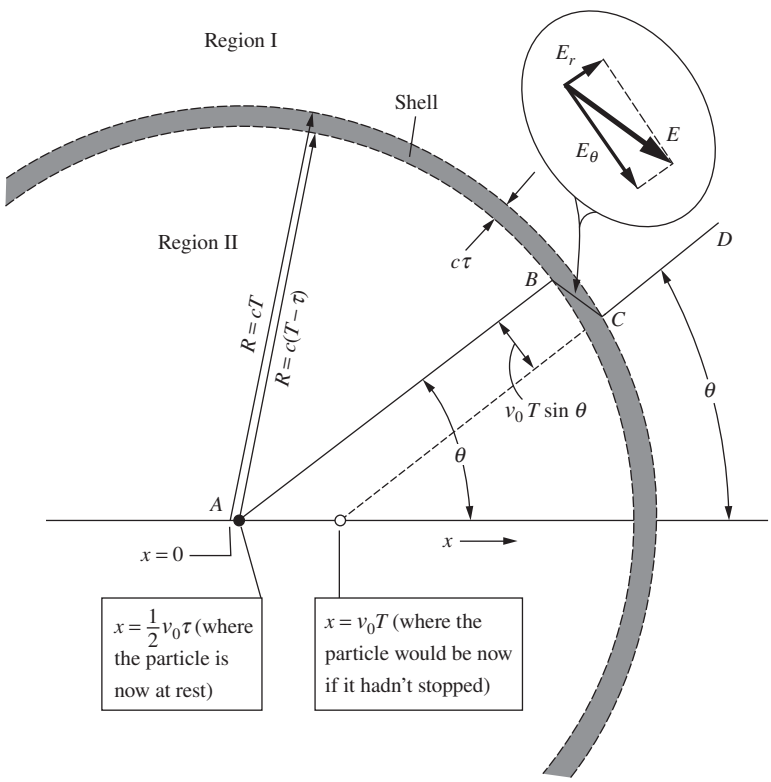


Figure H.2.
Space diagram for the instant $t = T \gg \tau$, a long time after the particle has stopped. For observers in region I, the field must be that of a charge located at the position $x = v_0 T$; for observers in region II, it is that of a particle at rest close to the origin. The transition region is a shell of thickness $c\tau$.

But $v_0/\tau = a$, the magnitude of the (negative) acceleration, and $cT = R$, so our result can be written as follows:

$$\boxed{E_\theta = \frac{qa\sin\theta}{4\pi\epsilon_0 c^2 R}} \tag{H.3}$$

A remarkable fact is here revealed: E_θ is proportional to $1/R$, *not* to $1/R^2$! As time goes on and R increases, the transverse field E_θ will eventually become very much stronger than E_r. Accompanying this transverse (that is, perpendicular to **R**) electric field will be a magnetic field of strength E_θ/c perpendicular to both **R** and **E**. This is a general property of an electromagnetic wave, explained in Chapter 9.

Let us calculate the energy stored in the transverse electric field above, in the whole spherical shell. The energy density is

$$\frac{\epsilon_0 E_\theta^2}{2} = \frac{q^2 a^2 \sin^2\theta}{32\pi^2\epsilon_0 R^2 c^4}. \tag{H.4}$$

The volume of the shell is $4\pi R^2 c\tau$, and the average value of $\sin^2\theta$ over a sphere[1] is $2/3$. The total energy of the transverse electric field is therefore

$$\frac{2}{3}\,4\pi R^2 c\tau\,\frac{q^2a^2}{32\pi^2\epsilon_0 R^2 c^4} = \frac{q^2a^2\tau}{12\pi\epsilon_0 c^3}. \tag{H.5}$$

To this we must add an equal amount (see Section 9.6.1) for the energy stored in the transverse magnetic field:

$$\text{Total energy in transverse electromagnetic field} = \frac{q^2a^2\tau}{6\pi\epsilon_0 c^3}. \tag{H.6}$$

The radius R has canceled out. This amount of energy simply travels outward, undiminished, with speed c from the site of the deceleration. Since τ is the duration of the deceleration, and is also the duration of the electromagnetic pulse a distant observer measures, we can say that the *power* radiated during the acceleration process was

$$\boxed{P_{\text{rad}} = \frac{q^2a^2}{6\pi\epsilon_0 c^3}} \tag{H.7}$$

As it is the square of the instantaneous acceleration that appears in Eq. (H.7), it doesn't matter whether a is positive or negative. Of course it ought not to, for stopping in one inertial frame could be starting in

[1] Our polar axis in Fig. H.2 is the x axis: $\cos^2\theta = x^2/R^2$. With a bar denoting an average over the sphere, $\overline{x^2} = \overline{y^2} = \overline{z^2} = R^2/3$. Hence $\overline{\cos^2\theta} = 1/3$, and $\overline{\sin^2\theta} = 1 - \overline{\cos^2\theta} = 2/3$. Or you can just do an integral; the area of a circular strip around the x axis is proportional to $\sin\theta$, so you end up integrating $\sin^3\theta$.

another. Speaking of different frames, P_{rad} itself turns out to be Lorentz-invariant, which is sometimes very handy. That is because P_{rad} is *energy/time*, and energy transforms like time, each being the fourth component of a four-vector, as noted in Appendix G.

We have here a more general result than we might have expected. Equation (H.7) correctly gives the instantaneous rate of radiation of energy by a charged particle moving with variable acceleration – for instance, a particle vibrating in simple harmonic motion. It applies to a wide variety of radiating systems from radio antennas to atoms and nuclei.

Exercises

H.1 *Ratio of energies* $*$

An electron moving initially at constant (nonrelativistic) speed v is brought to rest with uniform deceleration a lasting for a time $t = v/a$. Compare the electromagnetic energy radiated during the deceleration with the electron's initial kinetic energy. Express the ratio in terms of two lengths, the distance light travels in time t and the classical electron radius r_0, defined as $e^2/4\pi\epsilon_0 mc^2$.

H.2 *Simple harmonic moton* $**$

An elastically bound electron vibrates in simple harmonic motion at frequency ω with amplitude A.

(a) Find the average rate of loss of energy by radiation.

(b) If no energy is supplied to make up the loss, how long will it take for the oscillator's energy to fall to $1/e$ of its initial value? (Answer: $6\pi\epsilon_0 mc^3/e^2\omega^2$.)

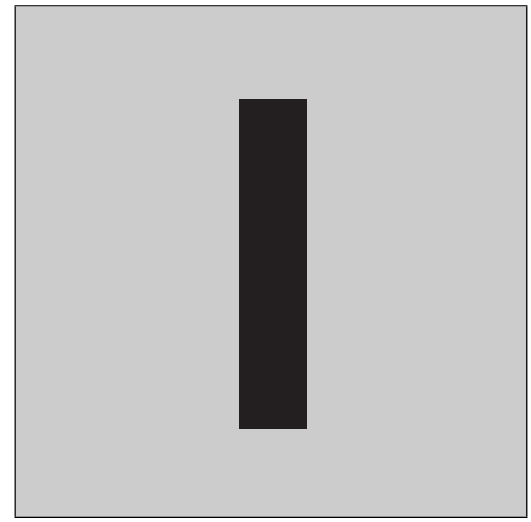
H.3 *Thompson scattering* $**$

A plane electromagnetic wave with frequency ω and electric field amplitude E_0 is incident on an isolated electron. In the resulting sinusoidal oscillation of the electron the maximum acceleration is $E_0 e/m$ (the maximum force divided by m). How much power is radiated by this oscillating charge, averaged over many cycles? (Note that it is independent of the frequency ω.) Divide this average radiated power by $\epsilon_0 E_0^2 c/2$, the average power density (power per unit area of wavefront) in the incident wave. This gives a constant σ with the dimensions of area, called a *scattering cross section.* The energy radiated, or scattered, by the electron, and thus lost from the plane wave, is equivalent to that falling on an area σ. (The case here considered, involving a free electron moving nonrelativistically, is often called *Thomson scattering* after J. J. Thomson, the discoverer of the electron, who first calculated it.)

H.4 *Synchrotron radiation* $**$

Our master formula, Eq. (H.7), is useful for relativistically moving particles, even though we assumed $v_0 \ll c$ in the derivation.

All we have to do is transform to an inertial frame F' in which the particle in question is, at least temporarily, moving slowly, apply Eq. (H.7) in that frame, then transform back to any frame we choose. Consider a highly relativistic electron ($\gamma \gg 1$) moving perpendicular to a magnetic field **B**. It is continually accelerated perpendicular to the field, and must radiate. At what rate does it lose energy? To answer this, transform to a frame F' moving momentarily along with the electron, find E' in that frame, and P'_{rad}. Now show that, because power is (energy)/(time), $P_{\text{rad}} = P'_{\text{rad}}$. This radiation is generally called *synchrotron radiation.* (Answer: $P_{\text{rad}} = \gamma^2 e^4 B^2 / 6\pi\epsilon_0 m^2 c$.)



Superconductivity

The metal lead is a moderately good conductor at room temperature. Its resistivity, like that of other pure metals, varies approximately in proportion to the absolute temperature. As a lead wire is cooled to 15 K its resistance falls to about 1/20 of its value at room temperature, and the resistance continues to decrease as the temperature is lowered further. But as the temperature 7.22 K is passed, there occurs without forewarning a startling change: the electrical resistance of the lead wire vanishes! So small does it become that a current flowing in a closed ring of lead wire colder than 7.22 K – a current that would ordinarily die out in much less than a microsecond – will flow for *years* without measurably decreasing. This phenomenon has been directly demonstrated. Other experiments indicate that such a current could persist for billions of years. One can hardly quibble with the flat statement that the resistivity is zero. Evidently something quite different from ordinary electrical conduction occurs in lead below 7.22 K. We call it *superconductivity.*

Superconductivity was discovered in 1911 by the great Dutch low-temperature experimenter Kamerlingh Onnes. He observed it first in mercury, for which the critical temperature is 4.16 K. Since then hundreds of elements, alloys, and compounds have been found to become superconductors. Their individual critical temperatures range from roughly a millikelvin up to the highest yet discovered, 138 K. Curiously, among the elements that do *not* become superconducting are some of the best normal conductors such as silver, copper, and the alkali metals.

Superconductivity is essentially a quantum-mechanical phenomenon, and a rather subtle one at that. The freely flowing electric current consists of electrons in perfectly orderly motion. Like the motion of an electron in an atom, this electron flow is immune to small disturbances – and for

a similar reason: a finite amount of energy would be required to make any change in the state of motion. It is something like the situation in an insulator in which all the levels in the valence band are occupied and separated by an energy gap from the higher energy levels in the conduction band. But unlike electrons filling the valence band, which must in total give exactly zero net flow, the lowest energy state of the superconducting electrons can have a net electron velocity, hence current flow, in some direction. Why should such a strange state become possible below a certain critical temperature? We can't explain that here.[1] It involves the interaction of the conduction electrons not only with each other, but also with the whole lattice of positive ions through which they are moving. That is why different substances can have different critical temperatures, and why some substances are expected to remain normal conductors right down to absolute zero.

In the physics of superconductivity, magnetic fields are even more important than you might expect. We must state at once that the phenomena of superconductivity *in no way* violate Maxwell's equations. Thus the persistent current that can flow in a ring of superconducting wire is a direct consequence of Faraday's law of induction, given that the resistance of the ring is really zero. For if we start with a certain amount of flux Φ_0 threading the ring, then because $\int \mathbf{E} \cdot d\mathbf{s}$ around the ring remains always zero (otherwise there would be infinite current due to the zero resistance), $d\Phi/dt$ must be zero. The flux cannot change; the current I in the ring will automatically assume whatever value is necessary to maintain the flux at Φ_0. Figure I.1 outlines a simple demonstration of this, and shows how a persistent current can be established in an isolated superconducting circuit.

Superconductors can be divided into two types. In Type 1 superconductors, the magnetic field inside the material itself (except very near the surface) is always zero. That is *not* a consequence of Maxwell's equations, but a property of the superconducting state, as fundamental, and once as baffling, a puzzle as the absence of resistance. The condition $\mathbf{B} = 0$ inside the bulk of a Type 1 superconductor is automatically maintained by currents flowing in a thin surface layer. In Type 2 superconductors, quantized magnetic flux tubes may exist for a certain range of temperature and external magnetic field. These tubes are surrounded by vortices of current (essentially little solenoids) which allow the magnetic field to be zero in the rest of the material. Outside the flux tubes the material is superconducting.

A strong magnetic field destroys superconductivity, although Type 2 superconductors generally can tolerate much larger magnetic fields than

[1] The abrupt emergence of a state of order at a certain critical temperature reminds us of the spontaneous alignment of electron spins that occurs in iron below its Curie temperature (mentioned in Section 11.11). Such *cooperative* phenomena always involve a large number of mutually interacting particles. A more familiar cooperative phenomenon is the freezing of water, also characterized by a well-defined critical temperature.

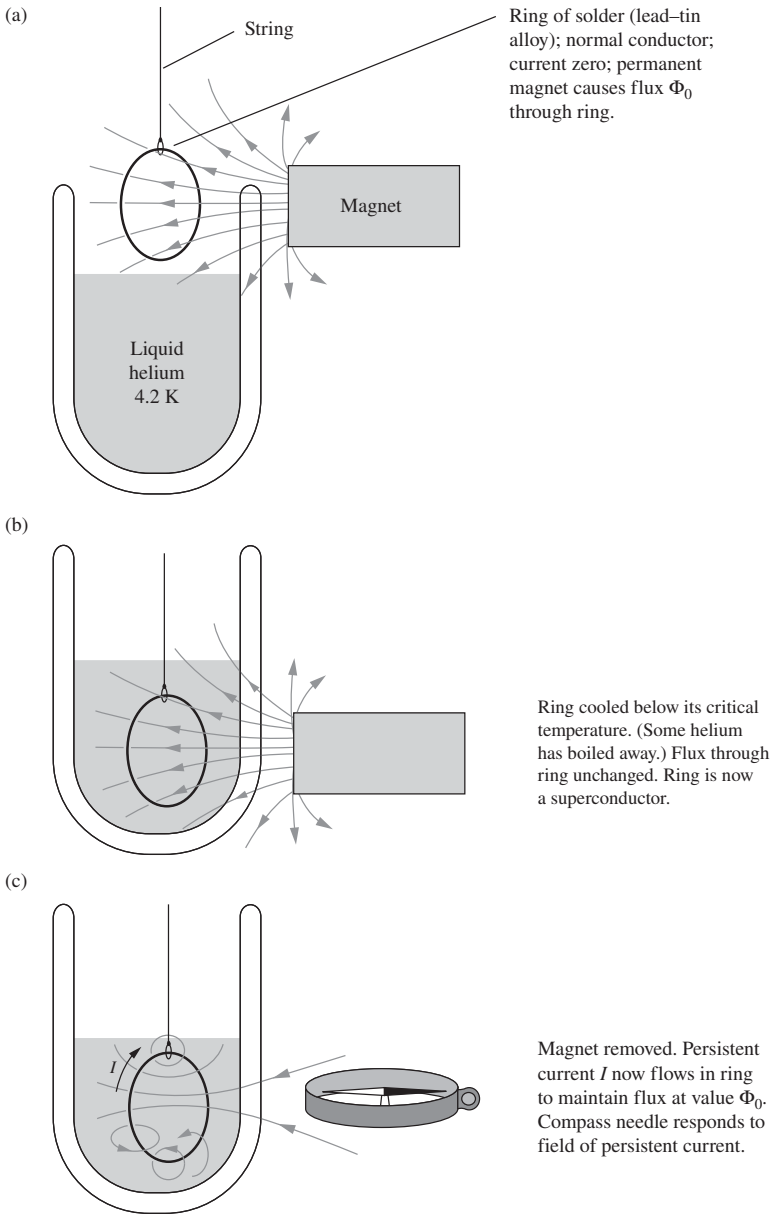


Figure I.1.
Establishing a persistent current in a superconducting ring. The ring is made of ordinary solder, a lead–tin alloy. (a) The ring, not yet cooled, is a normal conductor with ohmic resistance. Bringing up the permanent magnet will induce a current in the ring that will quickly die out, leaving the magnetic flux from the magnet, in amount Φ, passing through the ring. (b) The helium bath is raised without altering the relative position of the ring and the permanent magnet. The ring, now cooled below its critical temperature, is a superconductor with resistance zero. (c) The magnet is removed. The flux through the zero resistance ring cannot change. It is maintained at the value Φ by a current in the ring that will flow as long as the ring remains below the critical temperature. The magnetic field of the persistent current can be demonstrated with the compass.

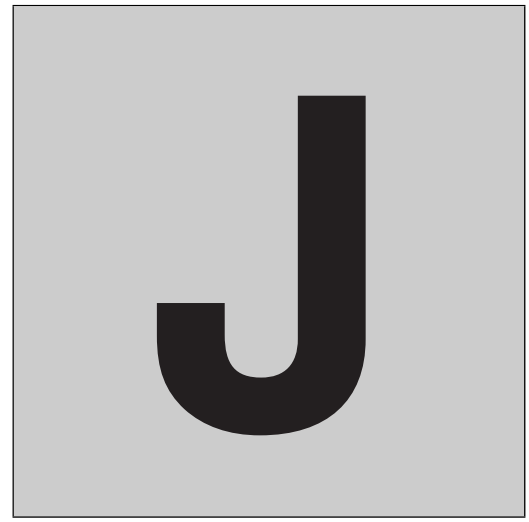
Type 1. None of the superconductors known before 1957 could stand more than a few hundred gauss. That discouraged practical applications of zero-resistance conductors. One could not pass a large current through a superconducting wire because the magnetic field of the current itself would destroy the superconducting state. But then a number of Type 2 superconductors were discovered that could preserve zero resistance in fields up to 10 tesla or more. A widely used Type 2 superconductor is

an alloy of niobium and tin that has a critical temperature of 18 K and if cooled to 4 K remains superconducting in fields up to 25 tesla. Type 2 superconducting solenoids are now common that produce steady magnetic fields of 20 tesla without any cost in power other than that incident to their refrigeration. Uses of superconductors include magnetic resonance imaging (MRI) machines (which are based on the physics discussed in Appendix J) and particle accelerators. There are also good prospects for the widespread use of superconductors in large electrical machinery, maglev trains, and the long-distance transmission of electrical energy.

In addition to the critical magnetic field, the critical temperature is also a factor in determining the large-scale utility of a superconductor. In particular, a critical temperature higher than 77 K allows relatively cheap cooling with liquid nitrogen (as opposed to liquid helium at 4 K). Prior to 1986, the highest known critical temperature was 23 K. Then a new type of superconductor (a copper oxide, or *cuprate*) was observed with a critical temperature of 30 K. The record critical temperature was soon pushed to 138 K. These superconductors are called *high-temperature superconductors.* Unfortunately, although they are cheaper to cool, their utility is limited because they tend to be brittle and hence difficult to shape into wires. However, in 2008 a new family of high-temperature superconductors was discovered, with iron as a common element. This family is more ductile than cuprates, but the highest known critical temperature is 55 K. The hope is that this will eventually cross the 77 K threshold.

The mechanism that leads to high-temperature superconductivity is more complex than the mechanism for low-temperature superconductivity. In contrast with the well-established BCS theory (named after Bardeen, Cooper, and Schrieffer; formulated in 1957) for low-temperature superconductors, a complete theory of high-temperature superconductors does not yet exist. All known high-temperature superconductors are Type 2, but not all Type 2 superconductors are high-temperature. Indeed, *low*-temperature Type 2 superconductors (being both ductile and tolerant of large magnetic fields) are the ones presently used in MRI machines and other large-scale applications.

At the other end of the scale, the quantum physics of superconductivity makes possible electrical measurements of unprecedented sensitivity and accuracy – including the standardization of the volt in terms of an easily measured oscillation frequency. To the physicist, superconductivity is a fascinating large-scale manifestation of quantum mechanics. We can trace the permanent magnetism of the magnet in Fig. I.1 down to the intrinsic magnetic moment of a spinning electron – a kind of supercurrent in a circuit less than 10^{-10} m in size. The ring of solder wire with the persistent current flowing in it is, in some sense, like a gigantic atom, the motion of its associated electrons, numerous as they are, marshaled into the perfectly ordered behavior of a single quantum state.



Magnetic resonance

The electron has angular momentum of spin, **J**. Its magnitude is always the same, $h/4\pi$, or $5.273 \cdot 10^{-35}\,\mathrm{kg\,m^2/s}$. Associated with the axis of spin is a magnetic dipole moment μ of magnitude $0.9285 \cdot 10^{-23}$ joule/tesla (see Section 11.6). An electron in a magnetic field experiences a torque tending to align the magnetic dipole in the field direction. It responds like any rapidly spinning gyroscope: instead of lining up with the field, the spin axis *precesses* around the field direction. Let us see why any spinning magnet does this. In Fig. J.1 the magnetic moment $\boldsymbol{\mu}$ is shown pointing opposite to the angular momentum **J**, as it would for a negatively charged body like an electron. The magnetic field **B** (the field of some solenoid or magnet not shown) causes a torque equal to $\boldsymbol{\mu} \times \mathbf{B}$. This torque is a vector in the negative $\hat{\mathbf{x}}$ direction at the time of our picture. Its magnitude is given by Eq. (11.48); it is $\mu B \sin\theta$. In a short time Δt, the torque adds to the angular momentum of our top a vector increment $\Delta\mathbf{J}$ in the direction of the torque vector and of magnitude $\mu B \sin\theta\, \Delta t$. The horizontal component of **J**, in magnitude $J \sin\theta$, is thereby rotated through a small angle $\Delta\psi$ given by

$$\Delta\psi = \frac{\Delta J}{J \sin\theta} = \frac{\mu B\, \Delta t}{J}\,. \tag{J.1}$$

As this continues, the upper end of the vector **J** will simply move around the circle with constant angular velocity ω_p:

$$\omega_\mathrm{p} = \frac{\Delta\psi}{\Delta t} = \frac{\mu B}{J}\,. \tag{J.2}$$

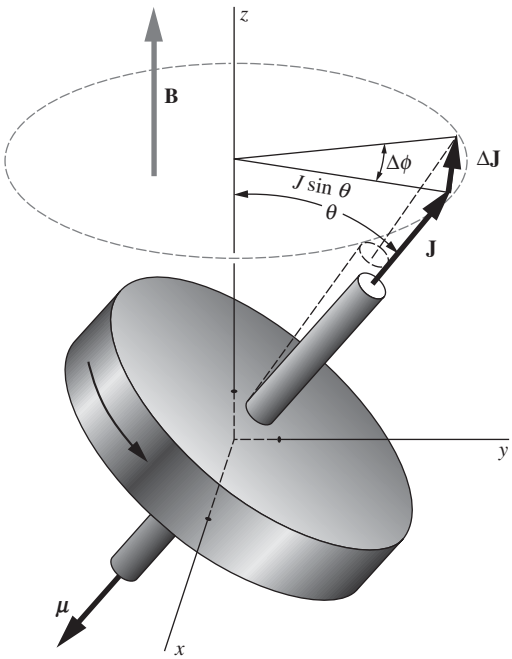
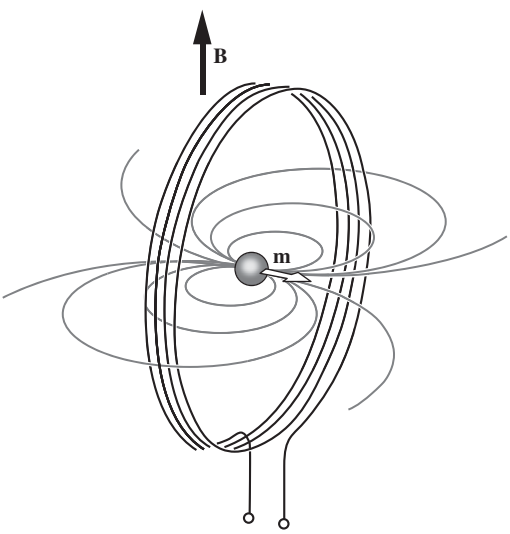


Figure J.1.
The precession of a magnetic top in an external field. The angular momentum of spin **J** and the magnetic dipole moment $\boldsymbol{\mu}$ are oppositely directed, as they would be for a negatively charged rotor.



This is the rate of precession of the axis of spin. Note that it is the same for any angle of tip; $\sin\theta$ has canceled out.

For the electron, μ/J has the value $1.761 \cdot 10^{11}$ s^{-1}tesla^{-1}. In a field of 1 gauss (10^{-4} tesla) the spin vector precesses at $1.761 \cdot 10^{7}$ radians/s, or $2.80 \cdot 10^{6}$ revolutions per second. The proton has exactly the same intrinsic spin angular momentum as the electron, $h/4\pi$, but the associated magnetic moment is smaller. That is to be expected since the mass of the proton is 1836 times the mass of the electron; as in the case of orbital angular momentum (see Eq. (11.29)), the magnetic moment of an elementary particle with spin ought to be inversely proportional to its mass, other things being equal. Actually the proton's magnetic moment is $1.411 \cdot 10^{-26}$ joule/tesla, only about 660 times smaller than the electron moment, which shows that the proton is in some way a composite particle. In a field of 1 gauss the proton spin precesses at 4258 revolutions per second. About 40 percent of the stable atomic nuclei have intrinsic angular momenta and associated magnetic dipole moments.

We can detect the precession of magnetic dipole moments through their influence on an electric circuit. Imagine a proton in a magnetic field B, with its spin axis perpendicular to the field, and surrounded by a small coil of wire, as in Fig. J.2. The precession of the proton causes some alternating flux through the coil, as would the end-over-end rotation of a little bar magnet. A voltage alternating at the precession frequency will be induced in the coil. As you might expect, the voltage thus induced by a single proton would be much too feeble to detect. But it is easy to provide more protons – 1 cm^3 of water contains about $7 \cdot 10^{22}$ protons (we're concerned with the two hydrogen atoms in each water molecule), and all of them will precess at the same frequency. Unfortunately they will not all be pointing in the same direction at the same instant. In fact, their spin axes and magnetic moments will be distributed so uniformly over all possible directions that their fields will very nearly cancel one another. But not quite, if we introduce another step. If we apply a strong magnetic field B to water, for several seconds there will develop a slight excess of proton moments pointing in the direction of **B**, the direction they energetically favor. The fractional excess will be $\mu B/kT$ in order of magnitude, as in ordinary paramagnetism. It may be no more than one in a million, but these uncanceled moments, if they are now caused to precess in our coil, will induce an observable signal.

A simple method for observing nuclear spin precession in weak fields, such as the earth's field, is described in Fig. J.3. Many other schemes are used to observe the spin precession of electrons and of

Figure J.2.
A precessing magnetic dipole moment at the center of a coil causes a periodic change in the flux through the coil, inducing an alternating electromotive force in the coil. Note that the flux from the dipole **m** that links the coil is that which loops around outside it. See Exercise J.1.

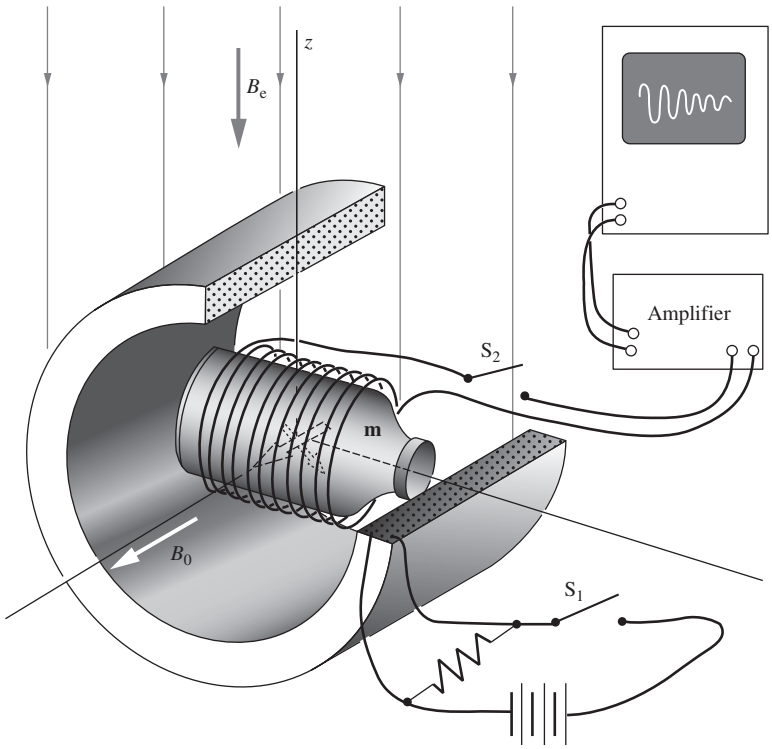


Figure J.3.
Apparatus for observing proton spin precession in the earth's field B_e. A bottle of water is surrounded by two orthogonal coils. With switch S_2 open and switch S_1 closed, the large solenoid creates a strong magnetic field B_0. As in ordinary paramagnetism (Section 11.6), the energy is lowered if the dipoles point in the direction of the field, but thermal agitation causes disorder. Our dipoles here are the protons (hydrogen nuclei) in the molecules of water. When thermal equilibrium has been attained, which in this case takes several seconds, the magnetization is what you would get by lining up with the magnetic field the small fraction $\mu B_0/kT$ of all the proton moments. We now switch off the strong field B_0 and close switch S_2 to connect the coil around the bottle to the amplifier. The magnetic moment $\mathbf{m}$ now precesses in the xy plane around the remaining, relatively weak, magnetic field B_e, with precession frequency given by Eq. (J.2). The alternating y component of the rotating vector $\mathbf{m}$ induces an alternating voltage in the coil which can be amplified and observed. From its frequency, B_e can be very precisely determined. This signal itself will die away in a few seconds as thermal agitation destroys the magnetization the strong field B_0 had brought about. Magnetic resonance magnetometers of this and other types are used by geophysicists to explore the earth's field, and even by archaeologists to locate buried artifacts.

nuclei. They generally involve a combination of a steady magnetic field and oscillating magnetic fields with frequency in the neighborhood of ω_p. For electron spins (*electron paramagnetic resonance,* or EPR) the frequencies are typically several thousand megahertz, while for nuclear spins (*nuclear magnetic resonance,* or NMR) they are several tens of megahertz. The exact frequency of precession, or resonance, in a given applied field can be slightly shifted by magnetic interactions within a molecule. This has made NMR, in particular, useful in chemistry. The position of a proton in a complex molecule can often be deduced from the small shift in its precession frequency.

Magnetic fields easily penetrate ordinary nonmagnetic materials, and that includes alternating magnetic fields if their frequency or the electric conductivity of the material is not too great. A steady field of 2000 gauss applied to the bottle of water in our example would cause any proton polarization to precess at a frequency of $8.516 \cdot 10^6$ revolutions per second. The field of the precessing moments would induce a signal of 8.516 MHz frequency in the coil outside the bottle. This applies as well to the human body, which, viewed as a dielectric, is simply an assembly of more or less watery objects. In *NMR imaging* (or *magnetic resonance imaging*, *MRI*) the interior of the body is mapped by means of nuclear magnetic resonance. The concentration of hydrogen atoms at a

particular location is revealed by the radiofrequency signal induced in an external coil by the precessing protons. The location of the source within the body can be inferred from the precise frequency of the signal if the steady field B, which determines the frequency according to Eq. (J.2), varies spatially with a known gradient.

Exercises

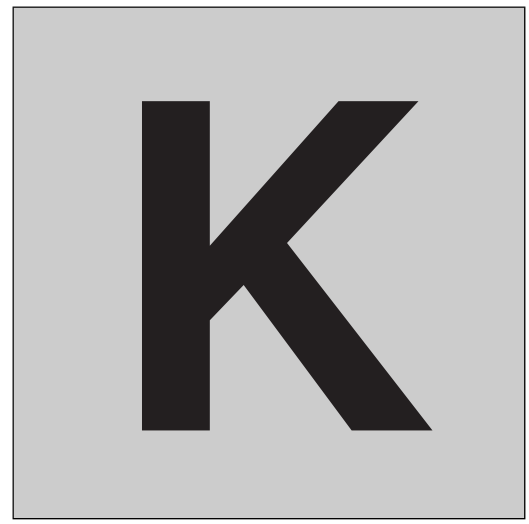
J.1 *Emf from a proton* $**$

At the center of the four-turn coil of radius a in Fig. J.2 is a single proton, precessing at angular rate ω_p. Derive a formula for the amplitude of the induced alternating electromotive force in the coil, given that the proton moment is $1.411 \cdot 10^{-26}$ joule/tesla.

J.2 *Emf from a bottle* $***$

(a) If the bottle in Fig. J.3 contains 200 cm^3 of H_2O at room temperature, and if the field B_0 is 1000 gauss, how large is the net magnetic moment $\mathbf{m}$?

(b) Using the result of Exercise J.1, make a rough estimate of the signal voltage available from a coil of 500 turns and 4 cm radius when the field strength B_e is 0.4 gauss.



Helpful formulas/facts

K.1 Fundamental constants

speed of light	c	$2.998 \cdot 10^{8}$ m/s
elementary charge	e	$1.602 \cdot 10^{-19}$ C
		$4.803 \cdot 10^{-10}$ esu
electron mass	m_{e}	$9.109 \cdot 10^{-31}$ kg
proton mass	m_{p}	$1.673 \cdot 10^{-27}$ kg
Avogadro's number	N_{A}	$6.022 \cdot 10^{-23}$ mole^{-1}
Boltzmann constant	k	$1.381 \cdot 10^{-23}$ J/K
Planck constant	h	$6.626 \cdot 10^{-34}$ J s
gravitational constant	G	$6.674 \cdot 10^{-11}$ m^3/(kg s^2)
electron magnetic moment	μ_{e}	$9.285 \cdot 10^{-24}$ J/T
proton magnetic moment	μ_{p}	$1.411 \cdot 10^{-26}$ J/T
permittivity of free space	ϵ_0	$8.854 \cdot 10^{-12}$ C^2/(N m^2)
permeability of free space	μ_0	$1.257 \cdot 10^{-6}$ T m/A

The exact numerical value of μ_0 is $4\pi \cdot 10^{-7}$ (by definition).

The exact numerical value of ϵ_0 is $(4\pi \cdot [3]^2 \cdot 10^9)^{-1}$, where $[3] \equiv 2.99792458$ (see Appendix E).

K.2 Integral table

$$\int \frac{dx}{x^2+r^2} = \frac{1}{r}\tan^{-1}\left(\frac{x}{r}\right) \tag{K.1}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \tag{K.2}$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \ln\left(x+\sqrt{x^2-1}\right) \tag{K.3}$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(\sqrt{x^2+a^2}+x\right) \tag{K.4}$$

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2(a^2+x^2)^{1/2}} \tag{K.5}$$

$$\int \ln x\,dx = x\ln x - x \tag{K.6}$$

$$\int x^n \ln\left(\frac{a}{x}\right) dx = \frac{x^{n+1}}{(n+1)^2} + \frac{x^{n+1}}{n+1}\ln\left(\frac{a}{x}\right) \tag{K.7}$$

$$\int xe^{-x}\,dx = -(x+1)e^{-x} \tag{K.8}$$

$$\int x^2 e^{-x}\,dx = -(x^2+2x+2)e^{-x} \tag{K.9}$$

$$\int \sin^3 x\,dx = -\cos x + \frac{\cos^3 x}{3} \tag{K.10}$$

$$\int \cos^3 x\,dx = \sin x - \frac{\sin^3 x}{3} \tag{K.11}$$

$$\int \frac{dx}{\cos x} = \ln\left(\frac{1+\sin x}{\cos x}\right) \tag{K.12}$$

$$\int \frac{dx}{\sin x} = \ln\left(\frac{1-\cos x}{\sin x}\right) \tag{K.13}$$

$$\int \frac{\cos x\,dx}{(1-a^2\cos^2 x)^{3/2}} = \frac{\sin x}{(1-a^2)\sqrt{1-a^2\cos^2 x}} \tag{K.14}$$

$$\int \frac{\sin x\,dx}{(1-a^2\sin^2 x)^{3/2}} = \frac{-\cos x}{(1-a^2)\sqrt{1-a^2\sin^2 x}} \tag{K.15}$$

$$\int \frac{\cos x\,dx}{\left(1-b^2\sin^2(x-a)\right)^{3/2}} = \frac{(2-b^2)\sin x + b^2\sin(2a-x)}{2(1-b^2)\sqrt{1-b^2\sin^2(a-x)}} \tag{K.16}$$

$$\int \frac{\sin x(a\cos x - b)\,dx}{(a^2+b^2-2ab\cos x)^{3/2}} = \frac{-a+b\cos x}{b^2\sqrt{a^2+b^2-2ab\cos x}} \tag{K.17}$$

K.3 Vector identities

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \times (\nabla f) = 0$$

$$\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} + (\nabla f) \times \mathbf{A}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

K.4 Taylor series

The general form of a Taylor series is

$$f(x_0 + x) = f(x_0) + f'(x_0)x + \frac{f''(x_0)}{2!}x^2 + \frac{f'''(x_0)}{3!}x^3 + \cdots . \quad \text{(K.18)}$$

This equality can be verified by taking successive derivatives and then setting $x = 0$. For example, taking the first derivative and then setting $x = 0$ gives $f'(x_0)$ on the left, and also $f'(x_0)$ on the right, because the first term is a constant and gives zero when differentiated, the second term gives $f'(x_0)$, and all the rest of the terms give zero once we set $x = 0$ because they all contain at least one power of x. Likewise, if we take the second derivative of each side and then set $x = 0$, we obtain $f''(x_0)$ on both sides. And so on for all derivatives. Therefore, since the two functions on each side of Eq. (K.18) are equal at $x = 0$ and also have their nth derivatives equal at $x = 0$ for all n, they must in fact be the same function (assuming that they are nicely behaved functions, which we generally assume in physics).

Some specific Taylor series that come up often are listed below; they are all expanded around $x_0 = 0$. We use these series countless times throughout this book when checking how expressions behave in the limit of some small quantity. The series are all derivable via Eq. (K.18), but sometimes there are quicker ways of obtaining them. For example, Eq. (K.20) is most easily obtained by taking the derivative of Eq. (K.19), which itself is simply the sum of a geometric series.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \text{(K.19)}$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \quad \text{(K.20)}$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \quad \text{(K.21)}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{K.22}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{K.23}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{K.24}$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots \tag{K.25}$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} + \cdots \tag{K.26}$$

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots \tag{K.27}$$

K.5 Complex numbers

The imaginary number i is defined to be the number for which $i^2 = -1$. (Of course, $-i$ also has its square equal to -1.) A general complex number z with both real and imaginary parts can be written in the form $a + bi$, where a and b are real numbers. Such a number can be described by the point (a, b) in the complex plane, with the x and y axes being the real and imaginary axes, respectively.

The most important formula involving complex numbers is

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{K.28}$$

This can quickly be proved by writing out the Taylor series for both sides. Using Eq. (K.22), the first, third, fifth, etc. terms on the left-hand side of Eq. (K.28) are real, and from Eq. (K.23) their sum is $\cos\theta$. Similarly, the second, fourth, sixth, etc. terms are imaginary, and from Eq. (K.24) their sum is $i\sin\theta$. Writing it all out, we have

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \\ &= \cos\theta + i\sin\theta, \end{aligned} \tag{K.29}$$

as desired.

Letting $\theta \to -\theta$ in Eq. (K.28) yields $e^{-i\theta} = \cos\theta - i\sin\theta$. Combining this with Eq. (K.28) allows us to solve for $\cos\theta$ and $\sin\theta$ in terms of the complex exponentials:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{K.30}$$

A complex number z described by the Cartesian coordinates (a, b) in the complex plane can also be described by the polar coordinates (r, θ). The radius r and angle θ are given by the usual relation between Cartesian and polar coordinates (see Fig. K.1),

$$r = \sqrt{a^2 + b^2} \qquad \text{and} \qquad \theta = \tan^{-1}(b/a). \tag{K.31}$$

Using Eq. (K.28), we can write z in *polar form* as

$$a + bi = (r\cos\theta) + (r\sin\theta)i = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{K.32}$$

We see that the quantity in the exponent (excluding the i) equals the angle of the vector in the complex plane.

The *complex conjugate* of z, denoted by z^* (or by $\bar{z}$), is defined to be $z^* \equiv a - bi$, or equivalently $z^* \equiv re^{-i\theta}$. It is obtained by reflecting the Cartesian point (a, b) across the real axis. Note that either of these expressions for z^* implies that r can be written as $r = \sqrt{zz^*}$. The radius r is known as the *magnitude* or *absolute value* of z, and is commonly denoted by $|z|$. The complex conjugate of a product is the product of the complex conjugates, that is, $(z_1 z_2)^* = z_1^* z_2^*$. You can quickly verify this by writing z_1 and z_2 in polar form. The Cartesian form works too, but that takes a little longer. The same result holds for the quotient of two complex
numbers.

As an example of the use of Eq. (K.28), we can quickly derive the double-angle formulas for sine and cosine. We have

$$\begin{aligned} \cos 2\theta + i\sin 2\theta = e^{i2\theta} &= \left(e^{i\theta}\right)^2 = (\cos\theta + i\sin\theta)^2 \\ &= (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta). \end{aligned} \tag{K.33}$$

Equating the real parts of the expressions on either end of this equation gives $\cos 2\theta = \cos^2\theta - \sin^2\theta$. And equating the imaginary parts gives $\sin 2\theta = 2\sin\theta\cos\theta$. This method easily generalizes to other trig sum formulas.

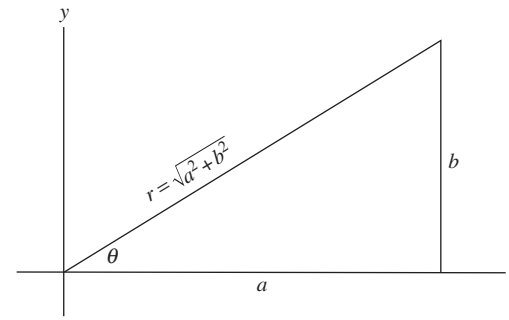


Figure K.1.
Cartesian and polar coordinates in the complex plane.

K.6 Trigonometric identities

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta \tag{K.34}$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{K.35}$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{K.36}$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \tag{K.37}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}, \qquad \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \tag{K.38}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta} \tag{K.39}$$

The hyperbolic trig functions are defined by analogy with Eq. (K.30), with the i's omitted:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2} \tag{K.40}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{K.41}$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \frac{d}{dx}\sinh x = \cosh x \tag{K.42}$$

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Index

Addition of velocities, relativistic, 808–809
additivity of interactions, 10, 13
admittance, 408–414
Alnico V, *B-H* curve for, 569
alternating current, 394–418
  representation by complex number, 406–408
alternating-current circuit, 405–414
  power and energy in, 415–418
alternating electromotive force, 395
alternator, 371
aluminum, doping of silicon with, 203–204
ammeter, 224
ammonia molecule, dipole moment of, 483
ampere (unit of current), 178, 283, 762–763, 790
Ampère, Andre-Marie, 2, 236, 238, 259, 531
Ampère's law, 288
  differential form, 291
amplitude modulation (AM), 455
Andrews, M., 640
angular momentum
  conservation of, in changing magnetic field, 580
  of electron spin, 546–547
  orbital, relation to magnetic moment, 541
  precession of, 822–823
anode of vacuum diode, 181
antimatter, 3
antineutron, 3
antiproton, 3
Assis, A. K. T., 263
atom, electric current in, 540
atomic polarizability, 480–482
aurora borealis, 318
Auty, R. P., 505

B, magnetic field, 239, 278
  and **M**, and **H** inside magnetized cylinder, 565
bacteria, magnetic, 571, 580
battery, lead–sulfuric acid, 209–212
B-H curve, 569–570
Biot–Savart law, 298, 435
Bitter plates, 320
Blakemore, R. P., 580
Bloomfield, L. A., 35
Bohr radius a_0, 55, 481, 544
Boltzmann factor, 202
Boltzmann's constant k, 202, 503
Boos, F. L., 460
boost converter, 372
Bose, S. K., 305
bound and free charge, 497–498
  arbitrariness of the distinction, 506–507
bound-charge current, 505–507
bound-charge density, 498
bound currents, 559–560
boundary of dielectric, change in **E** at, 494–495
boundary-value problem, 132, 151–153
bridge network, 208, 233

capacitance, 141–147
  of cell membrane, 513
  coefficients of, 148
  of prolate spheroid, 171
  units of, 142
    illustrated, 145
capacitor, 141–147
  dielectric-filled, 489–492
  energy stored in, 149–151
  parallel-plate, 143–144, 467

capacitor (cont.)
uses of, 153
vacuum, 467
capacitor plate, force on, 151, 162
carbon monoxide molecule, dipole moment of, 483
cartesian coordinates, 791
cassette tape, 570
cathode of vacuum diode, 181
Cavendish, Henry, 11
centimeter (as unit of capacitance), 145
CH_3OH (methanol) molecule, dipole moment of, 483
charge
electric, *see* electric charge
magnetic, absence of, 529
in motion, *see* moving charge
charge density, linear, 28
charge distribution
cylindrical, field of, 83
electric, 20–22
moments of, 74, 471–474
spherical, field of, 26–28
on a surface, 29
charged balloon, 32
charged disk, 68–71
field lines and equipotentials of, 72
potential of, 69
charged wire
potential of, 68
circuit breaker, 320
circuit element, 205
circuits
LR, 366–367
RC, 215–216
RLC, 389, 398, 410
alternating-current, 394–418
direct-current, 204–207
equivalent, 206
resonant, 388–394
circulation, 90
Clausius–Mossotti relation, 502
CO (carbon monoxide) molecule, dipole moment of, 483
coefficients
of capacitance, 148
of potential, 148
coil
cylindrical (solenoid), magnetic field of, 300–303, 338
toroidal
energy stored in, 369
inductance of, 364
Cole, R. H., 505
comets, 454
compass needle, 239
complex exponential solutions, 402–405
complex-number representation of alternating current, 406–408
complex numbers, review of, 828–829
conduction, electrical, 181–204
ionic, 189–195
in metals, 198–200
in semiconductors, 200–204
conduction band, 201–202
conductivity, electrical, 182–188
anisotropic, 182
of metals, 198–200
units for, 182
of various materials, 188, 195–197
conductors, electrical, 125–141
charged, system of, 128
properties of, 129
spherical, field around, 131
conformal mapping, 151
conservation of electric charge, 4–5, 180–181
distinguished from charge invariance, 242
conservative forces, 12
continuity equation, 181
copper, resistivity of, 188, 196–197
copper chloride, paramagnetism of, 526
corona discharge, 37
coulomb (SI unit of charge), 8, 762
relation to esu, 9
Coulomb, Charles de, 10
Coulomb's law, 7–11, 259
tests of, 10–11
Crandall, R. E., 11
Crawford, F. S., 378
critical damping, 394
Crosignani, B., 590
cross product (vector product) of two vectors, 238
Curie, Pierre, 566
Curie point, 566
curl, 90–99, 798–799
in Cartesian coordinates, 93–95, 100
physical meaning of, 95
curlmeter, 96
current density **J**, 177–180
current loop
magnetic dipole moment of, 534
magnetic field of, 531–535
torque on, 547
current ring, magnetic field of, 299
current sheet, 303–306
magnetic field of, 303–304
currents
alternating, 394–418
bound and free, 559–560
bound-charge, 505–507
displacement, 433–436
electric, *see* electric currents
fluctuations of, random, 195
curvilinear coordinates, 791–801
cylinder, magnetized, compared with cylinder polarized, 557
cylindrical coordinates, 792

damped harmonic oscillator, 389
damped sinusoidal oscillation, 392
damping of resonant circuit, 388–394
critical, 394
Davis, L., Jr., 11
decay of proton, 6
decay time for earth's magnetic field, 386
deer, flying, 102
"del" notation, 83, 95, 100
detergent, 510
deuterium molecule, 242
Di Porto, P., 590
diamagnetic substances, 526
diamagnetism, 527, 540, 546
of electron orbits, 545
diamond
crystal structure of, 200
wide band gap of, 203
dielectric constant κ, 468
of various substances, 469
dielectric sphere in uniform field, 495–496
dielectrics, 467–471

diode, 219
  silicon junction, 229
  vacuum, 181
dipole
  comparison of electric and magnetic, 535–536
  electric, *see* electric dipole
  magnetic, *see* magnetic dipole
dipole moment
  electric, *see* electric dipole moment
  magnetic, *see* magnetic dipole moment
disk
  conducting, field of, 140
  charged, 68–72
displacement, electric, **D**, 499, 560–561
displacement current, 433–436
distribution of electric charge, 20–22
divergence, 78–79, 795–797
  in Cartesian coordinates, 81–83, 100
divergence theorem, 79–80, 100
domains, magnetic, 567
doorbell, 321
doping of silicon, 203–204
dot product of two vectors, 12
dynamic random access memory (DRAM), 153
dynamo, 379, 386
dyne (Gaussian unit of force), 8

ϵ_0, permittivity of free space, 8
Earnshaw's theorem, 87
earth's magnetic field, 280, 577
  decay time of, 386
  possible source of, 380
eddy-current braking, 370
Edison, Thomas, 419
Einstein, Albert, 2, 236, 281, 314
electret, 558
electric charge, 1–11, 242
  additivity of, 10, 13
  conservation of, 4–5, 180–181
  distribution of, 20–22
  free and bound, 497–498, 506–507
  fundamental quantum of, 8
  invariance of, 241–243
  quantization of, 5–7, 242
  sign of, 4
electric currents, 177–189
  and charge conservation, 180–181
  energy dissipation in flow of, 207–208
  parallel, force between, 283
  variable
    in capacitors and resistors, 215–216
    in inductors and resistors, 366–367
electric dipole
  potential and field of, 73–77, 474–476
  torque and force on, in external field, 477–478
electric dipole moment, 74, 473, 475
  induced, 479–482
  permanent, 482–483
electric displacement **D**, 499, 560–561
electric eels, 219
electric field
  definition of, 17
  in different reference frames, 243–246
  of dipole, 75, 476
  of Earth, 36
  energy stored in, 33
  of flat sheet of charge, 29
  flux of, 22–26
    Gauss's law, 23–26
  inside hollow conductor, 134
  of line charge, 28
  line integral of, 59–61
  macroscopic, 488–489
  in matter, spatial average of, 487
  microscopic, 488
  of point charge with constant velocity, 247–251
  relation to ϕ and ρ, 89
  transformation of, 245, 310
  units of, 17
  visualization of, 18–20
electric field lines, 18, 19, 71, 72, 76–77
electric generator, 370
electric guitar, 370
electric potential, *see* potential, electric
electric quadrupole moment, 74, 473
electric susceptibility χ_e, 490, 501, 503
electrical breakdown, 36, 100
electrical conduction, *see* conduction, electrical
electrical conductivity, *see* conductivity, electrical
electrical conductors, *see* conductors, electrical
electrical insulators, 125–126
electrical potential energy, 13–16
  of a system of charges, 33, 63
electrical shielding, 135
electrodynamic tether, 369
electromagnet, 320
  design of, 584
electromagnetic field components, transformation of, 310
electromagnetic force, range of, 11
electromagnetic induction, 343–357
electromagnetic wave, 254, 438–453
  in dielectric, 507–509
  in different reference frames, 452–453
  energy transport by, 446–452
  general properties of, 440–441
  reflection of, 445, 447, 521
  standing, 442–446
  traveling pulse, 441
electromotive force, 209–211, 347, 357
  alternating, 395
electron, 3, 5, 6, 198–204, 540–549
  charge of, 8
  magnetic moment of, 547
  valence, 200
electron motion, wave aspect of, 199
electron orbit, 540–545
  diamagnetism of, 545
  magnetic moment of, 540–541
electron paramagnetic resonance (EPR), 823
electron radius, classical, 52, 545
electron spin, 546–549
  angular momentum of, 546–547
electronic paper, 37
electrostatic field, 61, *see also* electric field
  equilibrium in, 88
electrostatic unit (esu) of charge, 8, 765
energy, *see also* potential energy, electrical
  in alternating-current circuit, 415–418
  dissipation of, in resistor, 207–208
  electrical, of ionic crystal, 14–16
  stored
    in capacitor, 150
    in electric field, 33
    in inductor, 368
    in magnetic field, 369
  of system of charges, 11–14
energy gap, 201
equilibrium of charged particle, 88

equipotential surfaces, 71, 131
in field of conducting disk, 140
in field of dipole, 76
in field of uniformly charged disk, 72
equivalence of inertial frames, 237, 805
equivalent circuit, 206
for voltaic cell, 211
esu (electrostatic unit), 8, 765

Faller, J. E., 11
farad (unit of capacitance), 142
Faraday, Michael, 2, 236, 314
discovery of induction by, 343–345
reconstruction of experiment by, 384
Waterloo Bridge experiment by, 380
Faraday's law of induction, 356–357
ferrofluid, 572
ferromagnetic substances, 526
ferromagnetism, 527, 565–568
Feynman, R. P., 37, 539
field
electric, *see* electric field
magnetic, *see* magnetic field
meaning of, 245
Fisher, L. H., 348
fluctuations of current, random, 195
flux
of electric field, definition of, 22–26
magnetic, 348–351
flux tube, 349, 351
force components, Lorentz transformation of, 810–811
application of, 255–257
force(s)
between parallel currents, 283
on capacitor plate, 151, 162
conservative, 12
on electric dipole, 478
electromotive, 209–211, 347, 357, 395
with finite range, 88
on layer of charge, 30–32, 46
magnetic, 237–239
on magnetic dipole, 535–539
on moving charged particle, 255–267, 278
Foster's theorem, 224
Frankel, R. B., 580
Franklin, Benjamin, 10, 516, 529
free and bound charge, 497–498
arbitrariness of the distinction, 506–507
free currents, 559–560
frequency modulation (FM), 455
Friedberg, R., 639
fundamental constants, 825
fuse, 219

Galili, I., 452, 464
Galvani, Luigi, 209, 236
galvanic currents, 236
galvanometer, 224, 344
Gauss, C. F., 286
gauss (unit of magnetic field strength), 282
Gaussian units, 762–768
Gauss's law, 23–26, 80, 88
applications of, 26–30, 88, 243–245, 254, 262, 266, 488, 812
and fields in a dielectric, 497–498
Gauss's theorem, 79–80, 100
gecko, 510
generator, electric, 370
germanium, 202
conductivity of, 195
crystal structure of, 200
resistivity of, 188
Gilbert, William, 236
Goihbarg, E., 452, 464
golden ratio, 49, 168, 231, 665
Goldhaber, A. S., 11
Good, R. H., 157, 639, 640
gradient, 63–65, 792–795
graphite
anisotropic conductivity of, 183
diamagnetism of, 546
gravitation, 3, 10, 28, 39, 163
gravitational field and Gauss's law, 25
Gray, Stephen, 125
Griffiths, D. J., 298, 640
ground-fault circuit interrupter (GFCI), 371
gyromagnetic ratio, 541

H, magnetic field, 560–565
and **B**, and **M** inside magnetized cylinder, 565
relation to free current, 560, 561
H_2O molecule, dipole moment of, 483
hadron, 6

Hall, E. H., 317
Hall effect, 314–317
hard disk, 571
harmonic functions, 87, 152
harmonic oscillator, 389
HCl (hydrogen chloride) molecule, dipole moment of, 482, 483
Heald, M. A., 298
helical coil, magnetic field of, 302
helicopters, static charge on, 102
helium atom, neutrality of, 241
helix, handedness of, 279
Henry, Joseph, 361
henry (SI unit of inductance), 361
Hertz, Heinrich, 236, 281, 314, 394
hertz (unit of frequency), 394
Hill, H. A., 11
hole, 201
Hughes, V. W., 5
hybrid car, 371
hydrogen atom
charge distribution in, 479
polarizability of, 481
hydrogen chloride molecule, dipole moment of, 482, 483
hydrogen ions, 189
hydrogen molecule, 5, 242
hydrogen negative ion, 328
hyperbolic functions, 830
hysteresis, magnetic, 569

ice, dielectric constant of, 505
ignition system coil, 372
image charge, 136–140
for a spherical shell, 159
impedance, 408–414
index of refraction, 509
inductance
mutual, 359–364
reciprocity theorem for, 362–364
self-, 364–366
circuit containing, 366–367
induction
electromagnetic, 343–357
Faraday's law of, 356–357
inductive reactance, 396
insulators, electrical, 125–126
integral table, 826
internal resistance of electrolytic cell, 210

interstellar magnetic field, 286, 386
invariance of charge, 241–243
  distinguished from charge conservation, 242
  evidence for, 241
ionic crystal, energy of, 14–16
ions, 189–198
  in air, 190
  in gases, 190
  in water, 189–190
iron, *B-H* curve for, 569

Jackson, J. D., 189, 452
Jefimenko, O., 188
junction, silicon diode, 229
junkyard magnet, 321

Karlsruhe, University of, 394
King, J. G., 5
Kirchhoff's loop rule, 207, 359
Kirchhoff's rules, 206, 212

Laplace's equation, 86–88, 132–134
Laplacian operator, 85–86, 799–801
lead
  superconductivity of, 197
  resistivity of, 197
lead–sulfuric acid cell, 209–212
Leighton, R. B., 37, 539
Lenz's law, 351
Leyden jar, 516
Li, Y., 640
Liénard–Wiechert potential, 707
light, velocity of, definition of, 789
light-emitting diode, 220
lightning, 37
lightning rod, 153
line charge density, 28
line integral
  of electric field, 59–61
  of magnetic field, 287–291
linear dielectric, 490
linear physical system, 148
liquid oxygen, paramagnetism of, 525, 526, 548
lodestone (magnetite), 236, 526, 527, 565, 570
long straight wire, magnetic field of, 280
loop of a network, 207
Lorentz, H. A., 2, 236
Lorentz contraction, 261, 807
Lorentz force, 278
Lorentz invariants, 465, 811
Lorentz transformation
  applications of, 247–248, 255–257
  of electric and magnetic field components, 310
  of force components, 810–811
  of momentum and energy, 810
  of space-time coordinates, 806
LR circuits, 366–367
  time constant of, 367

μ_0, permeability of free space, 281
M, magnetization, 550
  and **B**, and **H** inside magnetized cylinder, 565
macroscopic description of matter, 470
macroscopic electric field in matter, 488–489
maglev train, 321
magnetic bottle, 318
magnetic charge, absence of, 529
magnetic dipole
  field of, 534–535
    compared with electric dipole field, 535
  force on, 535–539
  torque on, 547
  vector potential of, 531–534
magnetic dipole moment
  of current loop, 534
  of electron orbit, 540–541
  associated with electron spin, 547
magnetic domains, 567
magnetic field, 238, 278, *see also* earth's magnetic field
  of current loop, 531–535
  of current ring, 299
  of current sheet, 303–304
  of Earth, 373
  energy stored in, 368–369
  of helical coil, 302
  interstellar, 386
  line integral of, 287–291
  of long straight wire, 280
  of solenoid (coil), 300–303, 338
  transformation of, 310
magnetic field **B**, *see* **B**, magnetic field
magnetic field **H**, *see* **H**, magnetic field
magnetic flux, 348–350
magnetic forces, 237–239
magnetic monopole, 529
magnetic permeability μ, 563
magnetic polarizability of electron orbit, 544
magnetic pressure, 306
magnetic susceptibility χ_m, 550, 563
magnetite (lodestone), 236, 526, 527, 565, 570
magnetization **M**, *see* **M**, magnetization
magnetogyric ratio, 541
magnetohydrodynamics, 306
magnetomechanical ratio, orbital, 541
magnetron, 419
Mania, A. J., 263
Marcus, A., 188
mass spectrometer, 317
Maxwell, James Clerk, 2, 11, 141, 236, 436
Maxwell's equations, 436–438
Mermin, N. D., 237
metal detector, 370
methane, structure and polarizability of, 481
methanol molecule, dipole moment of, 483
method of images, *see* image charge
microphone
  condenser, 154
  dynamic, 371
microscopic description of matter, 470
microscopic electric field in matter, 488
microwave background radiation, 454
microwave oven, 419, 510
mine-shaft problem, 601
moments of charge distribution, 74, 471–474
momentum, *see* angular momentum
motor, electric, 319
moving charge
  force on, 255–267, 278
  interaction with other moving charges, 259–267
  measurement of, 239–240
multipole expansion, 74, 472
muon, trajectory in magnetized iron, 582
mutual inductance, 359–364
  reciprocity theorem for, 362–364

Nan-Xian, C., 643
network
  alternating current, 405–414
  bridge, 208, 233
  direct-current, 205–207
  ladder, 231

neurons, 102
neutron, 3, 6
Newton, Isaac, 27
NH_3 (ammonia) molecule, dipole moment of, 483
nickel, Curie point of, 566
nickel sulfate, paramagnetism of, 526
Nieto, M. M., 11
niobium, 819
nitric oxide, paramagnetism of, 548
node of a network, 207
north pole, definition of, 280, 529
n-type semiconductor, 203–204
nuclear magnetic resonance (NMR), 823
nucleon, 39
nucleus, atomic, 3

octupole moment, 74, 473
O'Dell, S. L., 546
Oersted, Hans Christian, 236–237, 259, 331
oersted (unit of field *H*), 775
ohm (SI unit of resistance), 186
ohmmeter, 232
Ohm's law, 181–183, 193
  breakdown of, 198
  deviations from, in metals, 200
Onnes, H. K., 817
orbital magnetic moment, 540–541
oscillator, harmonic, 389
oxygen, negative molecular ion, 190

Page, L., 237
paint sprayer, electrostatic, 37
pair creation, 4
parallel currents, force between, 283
Parallel-plate capacitor, 144
  filled with dielectric, 467, 489–492
parallel *RLC* circuit, 410
paramagnetic substances, 526
paramagnetism, 527, 540, 548
partial derivative, definition of, 64
permanent magnet, field of, 557–559
permeability, magnetic, μ, 563
permittivity, ϵ, 497
pH value of hydrogen ion concentration, 189
phase angle in alternating-current circuit, 402, 404, 409
phosphorous, doping of silicon with, 203
photocopiers, 37
photon, 4, 460
photovoltaic effect, 220
picofarad (unit of capacitance), 142
piezoelectric effect, 511
pion, 34
Planck, Max, 2
Planck's constant *h*, 546
p–*n* junction, 219
point charge, 21
  accelerated, radiation by, 812–815
  moving with constant velocity, 247–251
  near conducting disk, 139
  starting or stopping, 251–255
Poisson's equation, 86, 89
polar molecules, dipole moments of, 483
polarizability
  magnetic, of electron orbit, 544
  of various atoms, 481–482
polarization, frequency dependence of, 504
polarization density **P**, 484, 498, 501, 503
polarized matter, 483–489
polarized sphere, electric field of, 492–495
pollination, by bees, 509
positron, 3
potential
  coefficients of, 148
  electric, ϕ, 61–73, 86–89
    of charged disk, 69
    of charged wire, 67
    derivation of field from, 65
    of electric dipole, 73–74, 475
    of two point charges, 66
  vector, 293–296
    of current loop, 531–534
potential energy, electrical, 13–16
  of a system of charges, 33, 63
power
  in alternating-current circuit, 415–418
  dissipated in resistor, 208
  radiated by accelerated charge, 814
power adapter, 420
power-factor correction, 420
Poynting vector, 448–452
precession of magnetic top, 821, 822
Press, F., 380
Priestly, Joseph, 10
proton, 3
  decay of, 6
  and electron charge equality, 5
  magnetic moment of, 822
p-type semiconductor, 203–204

Q, of resonant circuit, 392, 402
quadrupole
  moment, 74, 473
  tensor, 514
quantization of charge, 5–7, 242
quantum electrodynamics, 2
quark, 6, 35
quartz clock, 511

radiation by accelerated charge, 812–815
radio frequency identification (RFID) tags, 454
railgun, 319
random fluctuations of current, 195
range of electromagnetic force, 11
rare-earth magnets, 573
rationalized units, 767
RC circuit, 215–216
  time constant of, 216
reactance, inductive, 396
reciprocity theorem for mutual inductance, 362–364
recombination of ions, 190
refractive index, 509
regenerative braking, 371
relaxation method, 153, 174
relaxation of field in conductor, 217
relaxation time, 217
relay, electric, 320
remanence, magnetic, 569
resistance, electrical, 183–187
resistances in parallel and in series, 206
resistivity, 186
  of various materials, 188, 196
resistor, 205, 207
resonance, 418
resonant circuit, 388–394
  damping of, 391–394
    critical, 394
  energy transfer in, 392
resonant frequency, 400
retarded potential, 329
Roberts, D., 211

Rodrigues, W. A. Jr., 263
Romer, R. H., 359
Rowland, Henry, 259, 314, 315, 317
Rowland's experiment, 315
RLC circuit
  parallel, 410
  series, 389, 398

Sands, M., 37, 539
saturation magnetization, 565
scalar product of two vectors, 12
Scott, G. K., 305
seawater, resistivity of, 188, 190, 196
second (as Gaussian unit of resistivity), 187
self-energy of elementary particles, 35
self-inductance, 364–366
  circuit containing, 366–367
semiconductors, 126, 195, 200–204
  n-type, 203–204
  p-type, 203, 204
Semon, M. D., 296
series *RLC* circuit, 389, 398
shake flashlight, 370
sheets of charge, moving, electric field of, 243–245
shielding, electrical, 135
SI units, 762–768
  derived, 769
Siever, R., 380
silicon, 195, 200–204
  band gap in, 201
  crystal structure of, 200
slope detection, 455
smoke detector, 219
Smyth, C. P., 505
sodium and chlorine ions in water, 190
sodium chloride crystal
  diamagnetism of, 526
  electrical potential energy of, 14–16
  free and bound charge in, 507
sodium metal, conductivity of, 198–199
solar cells, 220
solenoid (coil), magnetic field of, 300–303, 338
speakers, 321
spherical coordinates, 792
spin of electron, 546–549
sprites, 219
St. Elmo's fire, 37
standing wave, electromagnetic, 442–446
Starfish Prime, 318
statvolt (Gaussian unit of electric potential), 61
Stokes' theorem, 92–93, 100
storage battery, lead-sulfuric acid, 209–212
supercapacitor, 154
superconductivity, 197, 817–820
superposition, principle of, 10
  applications of, 25, 147, 207, 245, 301, 442, 490, 492
surface charge
  on current-carrying wire, 188–189, 263, 452
  density, 129
  distribution, 29
surface current density, 303
surface integral, definition of, 23
surfaces, equipotential, *see* equipotential surfaces
surfactant, 510
susceptibility
  electric χ_e, 490, 501, 503
  magnetic χ_m, 550, 563
symmetry argument, 21
synchrotron radiation, 815

Taylor, J. R., 296
Taylor series, 827–828
television set, 318
temperature, effect of
  on alignment of electron spins, 548–549
  on alignment of polar molecules, 503
  on conductivity, 195–197
Tesla, Nikola, 286, 419
tesla (SI unit of magnetic field strength), 280
Thévenin's theorem, 213–215, 225
three-phase power, 419
torque
  on current loop, 332, 547
  on electric dipole, 477, 478
  transatlantic telegraph, 227
transatlantic telegraph cable, 217
transformation, *see* Lorentz transformation
transformer, 372
transistor, 220
triboelectric effect, 36
trigonometric identities, 829

uniqueness theorem, 132–133
units, SI and Gaussian, 762–768
  conversions, 774–777, 789–790
  formulas, 778–788

vacuum capacitor, 467
valence band, 201–204
valence electrons, 200
Van Allen belts, 318
Van de Graaff generator, 182, 209, 211
van der Waals force, 510
Varney, R. N., 348
vector identities, 827
vector potential, 293–296
  of current loop, 531–534
vector product (cross product) of two vectors, 238
volt (SI unit of electric potential), 61
Volta, Alessandro, 209, 236
Voltaic cell, 209
  equivalent circuit for, 211
voltmeter, 224

Waage, H. M., 530
Walker, J., 35
War of Currents, 419
water
  dielectric constant of, 505
  ions in, 189–190
  pure, resistivity of, 188, 196
water molecule, dipole moment of, 483
watt (SI unit of power), 208
wave, electromagnetic, *see* electromagnetic wave
weber (SI unit of magnetic flux), 357
Whittaker, E., 500
Williams, E. R., 11
wire
  charged, potential of, 67
  magnetic field of, 280
work, by magnetic force, 572

Zia, R. K. P., 546

Derived units

$$\text{newton (N)} = \frac{\text{kg m}}{\text{s}^2}$$

$$\text{joule (J)} = \text{newton-meter} = \frac{\text{kg m}^2}{\text{s}^2}$$

$$\text{ampere (A)} = \frac{\text{coulomb}}{\text{second}} = \frac{\text{C}}{\text{s}}$$

$$\text{volt (V)} = \frac{\text{joule}}{\text{coulomb}} = \frac{\text{kg m}^2}{\text{C s}^2}$$

$$\text{farad (F)} = \frac{\text{coulomb}}{\text{volt}} = \frac{\text{C}^2\,\text{s}^2}{\text{kg m}^2}$$

$$\text{ohm } (\Omega) = \frac{\text{volt}}{\text{ampere}} = \frac{\text{kg m}^2}{\text{C}^2\,\text{s}}$$

$$\text{watt (W)} = \frac{\text{joule}}{\text{second}} = \frac{\text{kg m}^2}{\text{s}^3}$$

$$\text{tesla (T)} = \frac{\text{newton}}{\text{coulomb} \cdot \text{meter/second}} = \frac{\text{kg}}{\text{C s}}$$

$$\text{henry (H)} = \frac{\text{volt}}{\text{ampere/second}} = \frac{\text{kg m}^2}{\text{C}^2}$$

Fundamental constants

speed of light	c	$2.998 \cdot 10^{8}$ m/s
elementary charge	e	$1.602 \cdot 10^{-19}$ C
		$4.803 \cdot 10^{-10}$ esu
electron mass	m_{e}	$9.109 \cdot 10^{-31}$ kg
proton mass	m_{p}	$1.673 \cdot 10^{-27}$ kg
Avogadro's number	N_{A}	$6.022 \cdot 10^{-23}$ mole^{-1}
Boltzmann constant	k	$1.381 \cdot 10^{-23}$ J/K
Planck constant	h	$6.626 \cdot 10^{-34}$ J s
gravitational constant	G	$6.674 \cdot 10^{-11}$ m^3/(kg s^2)
electron magnetic moment	μ_{e}	$9.285 \cdot 10^{-24}$ J/T
proton magnetic moment	μ_{p}	$1.411 \cdot 10^{-26}$ J/T
permittivity of free space	ϵ_0	$8.854 \cdot 10^{-12}$ C^2 s^2/(kg m^3)
permeability of free space	μ_0	$1.257 \cdot 10^{-6}$ kg m/C^2

Maxwell's equations

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\operatorname{curl} \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$

$$\operatorname{div} \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\operatorname{div} \mathbf{B} = 0$$

Divergence theorem

$$\int_{\text{surface}} \mathbf{F} \cdot d\mathbf{a} = \int_{\text{volume}} \operatorname{div} \mathbf{F}\, dv$$

Stokes' theorem

$$\int_{\text{curve}} \mathbf{A} \cdot d\mathbf{s} = \int_{\text{surface}} \operatorname{curl} \mathbf{A} \cdot d\mathbf{a}$$

Gradient theorem

$$\phi_2 - \phi_1 = \int_{\text{curve}} \operatorname{grad} \phi \cdot d\mathbf{s}$$

Vector operators

Cartesian coordinates

$$d\mathbf{s} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$$

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

$$\nabla f = \frac{\partial f}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{z}}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Cylindrical coordinates

$$d\mathbf{s} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + dz\,\hat{\mathbf{z}}$$

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

$$\nabla f = \frac{\partial f}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{z}}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\hat{\mathbf{z}}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical coordinates

$$d\mathbf{s} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$$

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\nabla f = \frac{\partial f}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left(\frac{\partial (A_\phi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial (rA_\phi)}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\hat{\boldsymbol{\phi}}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$