Electrodynamics II

Université de Genève, printemps 2018

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Preamble

This is a translation of lecture notes, originally prepared in French for the course electrodynamics II at the University of Geneva aimed at students of the second year of their Bachelor's degree in physics. The subjects covered are taken mostly from the following sources:

- J. D. Jackson: "*Classical Electrodynamics*" The classic reference with all the details but without any pity on the reader. The best choice to buy for those who plan to continue with subjects involving the classical theory of electrodynamics or its applications.
- A. Zangwill: " *Modern Electrodynamics*": a very modern book on this old subject. This reference is more or less a more pedagogical version of Jackson in the sense that it offers a complete range of subjects as well as a number of advanced topics. A very good choice for the level of this course.
- Landau & Liftshitz: "*Classical theory of fields*": another classic text. This reference takes a rather theoretical point of view and treats certain aspects of the subject with a great deal of elegance. Also contains a treatment of the general theory of relativity. The series of L & L also contains a second volume, treating electrodynamics of continuous media.
- David Tong: "*Electromagnetism*" (Lecture Notes, University of Cambridge). Course for second and third year students of the Bachelor's degree at Cambridge. Contains some advanced topics towards the end, including some interesting further details concerning applications of electrodynamics to condensed matter physics.

I would like to thank Nathaniel Avish as well as Gabrielle Guttormsen for their help in producing this translated version from the original French.

Prof. Julian Sonner, Genève (February 2018)

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Maxwell's Equations

> [...]the only laws of matter are those which our minds must fabricate, and the only laws of mind are fabricated for it by matter.
>
> James Clerk Maxwell

1.1 Introduction

All physical phenomena in our Universe can be reduced to the action of four forces, called fundamental forces. Gravity, the dominant force at large scales, explains the dynamics of the solar system, galaxies and even the entire universe. The strong nuclear force, dominant on microscopic scales, is responsible for the link between quarks and gluons and between protons and neutrons (and other "baryons"), the weak nuclear force is equally important. Microscopic scales, is responsible for the disintegration of nuclei and force between neutrinos (and other "leptons"). The fourth force, the force of Coulomb-Lorentz describes the force undergone by a charge q moving with a speed $\mathbf{v}$ in a electric field $\mathbf{E}(t, \mathbf{x})$ as well as magnetic $\mathbf{B}(t, \mathbf{x})$,

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) . \tag{1.1}$$

Here we use Gaussian units which are better suited for theoretical considerations and consequently often found in the more advanced literature like quantum electrodynamics. We discuss the units more generally below, and in particular the relationship of Gaussian units with the international system (SI) of units. In Gaussian units a single constant appears in Maxwell equations, the velocity of light (in a vacuum)

$$c = 299792458 \text{ ms}^{-1} .$$

In this first chapter we discuss the basic quantities: charge, current, fields and their units.

1.2 Charge and Current

The orgine of the term "electric" is the greek word for amber, $\eta\lambda\epsilon\kappa\tau\rho o\nu$; electrostatic properties of which were already appreciated by the scientists of the Hellenic period. Modern physics explains these phenomena by the transfer of an intrinsic property of matter, which we call electrical charge. For the formulation of the electrodynamics in terms of the fields consider charge as a continuous substance, whereas, in reality, the charge can only be present in multiple [1], $q = \mathbb{Z}e$, with the fundamental,

$$e = 1.60217733(49) \times 10^{-19} C .$$

The charge of an electron is $q = -1 \times e$.

1.2.1 Charge density

For most applications it is more convenient to use a description of the charge in terms of the charge density

$$\rho(t, \mathbf{x})$$

Which varies continuously according to space and time. The charge contained in a space region V is therefore given by the integral

$$Q = \int_V d^3x \rho(t, \mathbf{x}) . \tag{1.2}$$

[1] The quarks and the gluons, which make up the nuclei, carry a fractional charge, but they are never detected in isolation. Any free charge satisfies the condition $q = \mathbb{Z}e$.

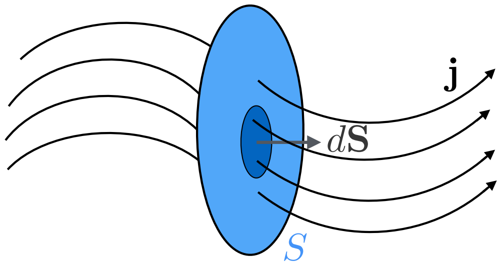

Figure 1.1: The current I through S is given by the integral through the surface S of the projection of the current density $\mathbf{j}$ on the surface element $d\mathbf{S}$. The current is represented by flux lines, using the analogy with a fluid.

A point charge at $\mathbf{x_0}$ can be written with the help of the Delta Dirac function $\rho(t, \mathbf{x}) = q\delta(\mathbf{x} - \mathbf{x_0})$. A group of point charges q_k at positions $\mathbf{x}_k$ with a charge density:

$$\rho(t, \mathbf{x}) = \sum_{k=1}^{N} q_k \delta(\mathbf{x} - \mathbf{x}_k(t)) \,. \tag{1.3}$$

Note the position of each charge can depend on time.

1.2.2 Current Density

We have already indicated that the charge density (1.2.1), (1.3) depends on time, for example, because of the movement of a point charge following a trajectory $\mathbf{x}_k(t)$. In general, moving charges generate a current $I(t)$, with a current density $\mathbf{j}(t, \mathbf{x})$. For the formal definition consider a surface, S, which charges or a density of charges pass through. If $\hat{\mathbf{n}}$ is the unit vector perpendicular to the area element dS and so $d\mathbf{S} = \hat{\mathbf{n}} dS$. The current through a surface S is given by the integral

$$I = \frac{dQ}{dt} = \int_S \mathbf{j} \cdot d\mathbf{S} \,. \tag{1.4}$$

By analogy with a continuous fluid of the current can be characterized as a flow of charges. If one expresses this in terms of the local speed of the charge $\mathbf{v}(t, \mathbf{x})$, the current could be expressed as

$$\mathbf{j} = \rho \mathbf{v} \,. \tag{1.5}$$

1.2.3 Charge conservation

We shall now derive an equation which expresses a fundamental law of nature: every process known in physics conserves charge. For example, a photon, the neutral particle of light, can decay into two charged particles, an electron and a positron. However, the total charge before and after the disintegration remains the same. This is the case for any other process known in physics as a consequence of a symmetry principle, called gauge symmetry[2] that we will see later in this course. The current can be written as an integral over the volume V,

$$I = \int_V \nabla \cdot \mathbf{j}\, d^3x \,, \tag{1.6}$$

The following is the application of the divergence theorem to (1.4). since the vector $\hat{\mathbf{n}}$ is directed towards the outside of the volume V, We have that the charge contained in the volume V *decreases* according to

$$-\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho d^3x = -\int_V \frac{\partial \rho}{\partial t} \,. \tag{1.7}$$

Using the second expression of the current (1.6) one deduces the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \,. \tag{1.8}$$

This equation expresses the fact that charge can not be created or destroyed. Each change in the total charge Q contained in a volume V can be traced back to a flux of current through the surface S bounding V, in other words, the charges which come from the interior of the volume V to the outside and vice versa.

Consider now a collection of charges, one can express the charge density as in the equation (1.3). One can use the continuity equation to deduce the current density in the case of a distribution of point charges. One has that

$$\frac{\partial \rho}{\partial t} = -\sum_{k=1}^{N} q_k \mathbf{v}_k \cdot \nabla \delta(\mathbf{x} - \mathbf{x}_k(t)) = -\nabla \sum_k q_k \mathbf{v}_k \delta(\mathbf{x} - \mathbf{x}_k(t)) \,. \tag{1.9}$$

[2] This is the theoretical point of view. Experimentally no disintegration of a charged particle of neutral particles has ever been observed. More precisely, the time limitation of the disintegration of an electron in neutral particles is greater than 10^{24} years (Belli et al. 1999).

The last step is not applicable for a constant speed. Comparison with the continuity equation (1.8), allows one to deduce

$$\mathbf{j} = \sum_{k=1}^{N} q_k \mathrm{v}_k \delta(\mathbf{x} - \mathbf{x}_k(t))\,. \tag{1.10}$$

This expression for the current density is, however, more generally true, since the charged medium can be considered as being incompressible (isochoric), i.e. it satisfies the relationship $\nabla \cdot \mathbf{v}_k = 0$.

1.2.4 Force density

One can write an equation for the force that acts on a charge distribution $\rho(t, \mathbf{x})$ and current $\mathbf{j}(t, \mathbf{x})$ in a volume V with the integral

$$\mathbf{F} = \int_V \mathbf{f} d^3x \tag{1.11}$$

with the force density equal to

$$\mathbf{f} = \rho\mathbf{E} + \frac{\mathbf{j}}{c} \times \mathbf{B}\,. \tag{1.12}$$

1.3 Maxwell's Equations

The laws describing all the phenomena discussed in this course form a system of four di erential linear equations, called the Maxwell Equations (in the vacuum),

$$\nabla \cdot \mathbf{E} = 4\pi\rho\,, \tag{1.13}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{1.14}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.15}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}\,, \tag{1.16}$$

in connection with the Coulomb-Lorentz force (1.1)

1.3.1 Electrostatic Consequences

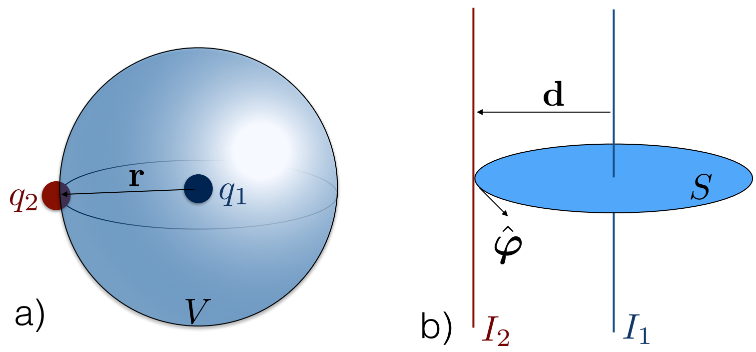

Figure 1.2: A) Electrical field of a point charge at rest at the origin and force felt by a second charge (Coulomb's law). B) The magnetic field of a stationary current and the force felt by the second charge (Biot-Savart's law).

In static fields [3] $\mathbf{E}$ and $\mathbf{B}$ become independent. To find the effects of the electric field, it is therefore sufficient to solve the first two equations (1.13) and (1.14) with $\frac{\partial}{\partial t}\mathbf{B} = 0$. Considering a single point charge q_1 at rest at the origin, the density of charge is written

$$\rho = q_1 \delta(\mathbf{x}) \,.$$

One integrates the two parts of the equation (1.13) over a spherical volume centered at the origin of radius R. By symmetry the electric field at each point is directed along the radial direction, and depends only on the radius $\mathbf{E} = E(r)\hat{\mathbf{r}}$. One easily finds

$$\mathbf{E} = \frac{q_1}{r^2}\hat{\mathbf{r}} \,, \tag{1.17}$$

Which is called *loi de Coulomb*. The force felt by a second charge is written

$$\mathbf{F} = q_2\mathbf{E} = \frac{q_2 q_1}{r^2}\hat{\mathbf{r}} \,. \tag{1.18}$$

[3]Strictly speaking one calls $\mathbf{B}$ *magnetic induction*, the term *magnetic field* is reserved for the $\mathbf{H}$ field (look at chapter 4).

Note that the second equation(1.14)

$$\nabla \times \mathbf{E} = 0$$

Implies that the field $\mathbf{E}$ could be written as

$$\mathbf{E}(\mathbf{x}) = \nabla\varphi(\mathbf{x}) \tag{1.19}$$

as a function of the scalar potential φ. From the above two equations one deduces that

$$\varphi = -\frac{q_1}{r}$$

Plus a constant that we neglect. Note that this is consistent with the equation (1.13) and the charge density (1.3.1), if

$$\nabla^2\left(-\frac{1}{4\pi r}\right) = \delta(\mathbf{x})$$

or, by translation

$$\nabla^2\left(-\frac{1}{4\pi\left|\mathbf{x}-\mathbf{x}_0\right|}\right) = \delta(\mathbf{x}-\mathbf{x}_0)\,. \tag{1.20}$$

This relationship will prove suitable for the following part.

1.3.2 Magnetostatique consequences

To find the magnetic fields it suffices to solve the last two of Maxwell's equations (1.15) and (1.16). For a single charge in linear motion the current density is written as

$$\mathbf{j} = q_1\mathbf{v}\delta(\mathbf{x}-\mathbf{x}(t)) := \mathbf{j}_1\delta(\mathbf{x}-\mathbf{x}(t))$$

Now one integrates the two parts of the equation (1.16) over the surface S (like in figure 1.2b). The right hand side gives, by definition (1.4) the current I_1 and for the left-hand side we use the Stokes theorem. We then find

$$\mathbf{B} = \frac{2I_1}{cd}\hat{\theta} \tag{1.21}$$

where d is the distance perpendicular to the trajectory of the charge and $\hat{\theta}$ is the unit vector parallel to the contour C bounding the surface S (Figure 1.2b).

The force density felt by a second current distribution at a distance d is given (1.12) by the integral over

$$d\mathbf{f} = \frac{d\mathbf{j}_2}{c} \times \mathbf{B}\,.$$

If we take for the second distributions a stationary current along a straight wire parallel to the first one, we obtain the force felt by a length L_2 of wire

$$\mathbf{F} = \int \delta(\mathbf{x} - \mathbf{x}_2)\frac{\mathbf{j}_2}{c} \times \hat{\theta}\, dV = -\frac{2I_1 I_2 L_2}{cd}\hat{\mathbf{d}}\,. \tag{1.22}$$

This is sometimes called *Biot-Savart's law.* The sign in front of the last part of the right hand side is negative, and therefore the force is attractive if currents I_1 and I_2 have the same sign and repulsive if the signs are opposite.

One now turns to a discussion of units in the theory of electromagnetism. Expressions for electrostatic as well as magnetostatic force will prove very useful because they represent a point of reference from which one can easily understand the different choices of units.

1.4 Units

The equations of physics can be expressed in arbitrary units, provided that each choice of units is consistent. It is thus possible to employ units according to the requirements of the situation on has in mind, such as, for example, engineering applications or the quantum physics of particles. On the other hand, *dimensions* are not arbitrary. The difference between dimensions and units is the abstract version of the difference between “length” and “meter”. The first expresses that a physical quantity has the dimensions of length, for example a distance between two points. The second expresses this distance in certain units, in this case, the meter. One could have chosen the centimeter, which does not change the fact that it is a quantity of length dimension, but changes the numerical value of this distance.

Here we discuss the different systems of units, but mainly the international system of units (SI), used in the course of electrodynamics in the first year, and the system of Gaussian units (cgs) used in this course. We also discuss what characterizes a consistent choice of units by considering the example of Maxwell's theory. A system of units begins with a choice of basic units, such as length, mass, time, and so on. The other units, called derived units, can be expressed by dimensional analysis in terms of the basic units. For example, if base units are chosen as units of length, mass and time, under Newton's second law,

$$F = ma\,, \qquad [\text{force}] = \left[\frac{m\ell}{t^2}\right]\,. \tag{1.23}$$

It follows from the discussion of dimensions versus units, that we must also distinguish between the choice of basic units, i.e. which dimensional quantities we (arbitrarily) decide to be "basic", and their concrete definition. That is to say that one is free to choose units for the same basic quantity, which differ only by their quantitative value, such as the centimeter and the inch for a quantity of dimension length. There is therefore a great deal of choice for the various systems of units and, especially in the case of electromagnetism, there are many heated debates on these choices. Without getting into these debates, we will now define the Gaussian units and their relationship with the international system of units.

1.4.1 Gaußian Units

The Gaussian system is an example of a "cgs" system, indicating that the basic units are length ℓ (in cm), mass m (in grams) and time t (in seconds). The international system adds these three the unit of current, the ampere, as a fourth base unit. It is perfectly consistent to declare that the unit of current is a basic unit. For example, the ampere can be defined by measuring the mass of silver deposited per unit of time on the cathode of a silver nitrate voltameter, which, in fact, was the case historically. In this way the ampere is, in this way, defined by a reproducible experiment, independent of the other units.

The international system also uses different quantitative choices for basic units, where they coincide with the Gaussian sysem (like the meter for length and kilogram for the mass). For this reasonthe old name of this unit system was MKSA (Meter, Kilogram, Seconds, Ampere). The different basic units for these two systems are given in the table 1.1.

Let us now discuss electromagnetic units more concretely. For this it is convenient to consider the forces themselves. The Coulomb's law (1.17) gives an expression for the electric force acting on two charges q_1 and q_2

$$F_e = k_e \frac{q_1 q_2}{r^2} = q_1 E \tag{1.24}$$

With a constant k_e[4] to be determined. The last member of this formula defines the (modulus of) the electric field, E, at a radial distance r of a charge q_2, as being

$$E = k_e \frac{q_2}{r^2} \,. \tag{1.25}$$

The force per unit length acting on a wire with current I_2, due to a second wire of length L_1 with a current I_1, situated at a distance d, is given by (1.22)

$$\frac{dF_m}{dL} = k_m \frac{2I_1 I_2}{d} = I_1 B \,. \tag{1.26}$$

This relation involves the constant k_m to be determined later[5]. As before, the last member of the equation above defines the (modulus of) the magnetic field as

$$B = k_m \beta \frac{2I_1}{d} \tag{1.27}$$

These two relations (1.24) and (1.26) above allow us to draw the conclusion that the ratio of k_e and k_m is a speed

$$\left[\frac{k_e}{k_m} \right] = \left[\frac{\ell^2}{t^2} \right] \,. \tag{1.28}$$

We shall see later that this speed is equal to the speed of light in a vacuum

$$\frac{k_e}{k_m} = c^2 \,. \tag{1.29}$$

Indeed this must be determined by comparing the predictions of Maxwell's equations written with the constants k_e and k_m, with experiment, as we are going to

[4] In Gaussian units $k_e = 1$, But we are looking for the most general form here.
[5] In Gaussian units $k_m = c^{-1}$, but one looks for the most general form here.

do a little later. Note also in equation (1.27) the presence of a second dimensional constant, β. It determines the ratio

$$\left[\frac{E}{B}\right] = \left[\frac{\ell}{t\beta}\right] \tag{1.30}$$

and thus allows us to choose the relative units of the electric and magnetic field.

Let us now revisit Maxwell's equations, without prejudice about units. It is clear that all equations must be written with dimensional constants so that the units of each term of a given equation are the same. So,

$$\nabla \cdot \mathbf{E} = 4\pi k_e \rho \,, \tag{1.31}$$

$$\nabla \times \mathbf{E} = -k\frac{\partial \mathbf{B}}{\partial t} \tag{1.32}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.33}$$

$$\nabla \times \mathbf{B} = \frac{k_m \beta}{k_e}\frac{\partial \mathbf{E}}{\partial t} + 4\pi k_m \beta \mathbf{j} \,, \tag{1.34}$$

The constants k_e, k_m and β are chosen in such a way that the static equations (1.24) and (1.26) above are reproduced, and that β has dimentions of speed. In a vaccum, the equations (1.32) and (1.34) lead to

$$\nabla^2 \mathbf{E} - k\beta \frac{k_m}{k_e}\frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{1.35}$$

and similiarly for $\mathbf{B}$ field. These equations describe the propagation of electromagnetic waves in vacuo. Experiment tells us that these propagate at the speed of light, in other words

$$\frac{k_e}{k_m k \beta} = c^2 \,, \tag{1.36}$$

From which we conclude that $k = \beta^{-1}$ (using the equation (1.29)).

Different systems of units make different choices for the constants k_e, k_m, β and k. However, not all of these constants can be chosen arbitrarily. By virtue of the equations (1.29) and (1.36) only two of them are independent and thus free.

The Gaussian system, employed in this course, makes the choice of having the same units for the fields $\mathbf{E}$ and $\mathbf{B}$, as well as $k_m = c^{-2}$ and $\beta = c$. Constants in

System	k_e	k_m	β	k
Gauss	1	$c^{-2}\ (t^2\ell^{-2})$	$c\ (\ell t^{-1})$	$c^{-1}\ (t^1\ell^{-1})$
SI	$\frac{1}{4\pi\epsilon_0} = 10^{-7}c^2$	$\frac{\mu_0}{4\pi} := 10^{-7}\ (t^2\ell^{-2})$	1	1

Table 1.1: The dimensions are given in parentheses after the numerical values.

different unit systems appear in the tables 1.1.

1.5 A brief reminder of vector calculus

In the theory of electromagnetism we almost always rely on vector calculus. Here we recall some definitions as well as theorems frequently encountered in the rest of the course.

Einstein summation convention

Let $\mathbf{A} \in \mathbb{R}^3$ be a vector. We denote its components in a base $\hat{\mathbf{e}}^i$ $(i = 1, 2, 3)$ by

$$\mathbf{A} \to A_i \qquad (i = 1, 2, 3)\,. \tag{1.37}$$

Let $\mathbf{B} \in \mathbb{R}^3$ be a second vector. The scalar product of $\mathbf{A}$ and $\mathbf{B}$ and its components takes the form

$$\mathbf{A} \cdot \mathbf{B} = \sum_{i=1}^{3} A_i B_i := A_i B_i\,. \tag{1.38}$$

The last member of this equation introduces the Einstein summation convention. It is understood that the sum is taken on any two repeated indices, as the two i above. Let us now consider the vector product of $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \to C_i = \sum_{j,k=1}^{3} \epsilon_{ijk} A_j B_k := \epsilon_{ijk} A_j B_k \tag{1.39}$$

with the Levi-Civita symbol, defined as

$$\epsilon_{ijk} := \begin{cases} 1 & ijk = \text{even} \quad \text{permutation} \quad \text{of } 123 \\ -1 & ijk = \text{odd} \quad \text{permutation} \quad \text{of } 123 \\ 0 & i = j\,, \quad \text{or} \quad j = k\,, \quad \text{or} \quad i = k \end{cases} \tag{1.40}$$

Note also that we again used the Einstein summation convention in the last member of equation (1.39), involving the summation on the two paris of repeated indices, that is to say j and k.

Gradient, divergence and curl

Let $\lambda(\mathbf{x})$ be a scalar function. The gradient of $\lambda(\mathbf{x})$ is defined as

$$\text{grad}\lambda(\mathbf{x}) := \nabla\lambda(\mathbf{x}) \to [\text{grad}\lambda(\mathbf{x})]_i = \frac{\partial\lambda}{\partial x_i} \tag{1.41}$$

Let $\mathbf{A}(\mathbf{x})$ be a vector field. The divergence of $\mathbf{A}(\mathbf{x})$ is

$$\text{div}\mathbf{A}(\mathbf{x}) := \nabla \cdot \mathbf{A}(\mathbf{x}) \to [\text{div}\mathbf{A}(\mathbf{x})]_i = \frac{\partial A_j}{\partial x_j}\,, \tag{1.42}$$

while the curl of $\mathbf{A}(\mathbf{x})$ is given by

$$\text{rot}\mathbf{A}(\mathbf{x}) := \nabla \times \mathbf{A}(\mathbf{x}) \to [\text{rot}\mathbf{A}(\mathbf{x})]_i = \epsilon_{ijk}\frac{\partial A_k}{\partial x_j} \tag{1.43}$$

Gauss' Theorem

Let V be a volume bounded by a surface $S = \partial V$. Let $\hat{\mathbf{n}}$ be a unit vector perpendicular to the surface S and outward pointing. It follows that

$$\int_V \nabla \cdot \mathbf{A} dS = \oint_S \mathbf{A} \cdot \hat{\mathbf{n}} dS = \oint_S \mathbf{A} \cdot d\mathbf{S}\,, \tag{1.44}$$

where the last member of this equation defines the vector element of area $d\mathbf{S} := \hat{\mathbf{n}} dS$.

Stokes' Theorem

Let S be a surface bounded by the countour $C = \partial S$. Let $\hat{\mathbf{t}}$ be a unit vector perpendicular to the surface S and $\hat{\mathbf{l}}$ the unit vector parallel to the curve C. It follows that

$$\int_S (\nabla \times \mathbf{A}) \cdot \hat{\mathbf{t}} dS = \oint_C \mathbf{A} \cdot d\mathbf{l} \tag{1.45}$$

1.6 The structure of Maxwell's equations

1.6.1 Potentials & gauge invariance

[Jackson 6.2, 6.3]
It is often convenient to introduce *potentials*, in terms of which Maxwell's equations are reduced to a system of two differential equations of second order, mutually coupled. Our starting point is the fact that any vector field of zero divergence can be expressed as a curl of another vector field. The equation $\nabla \cdot \mathbf{B} = 0$ thus allows us to introduce the vector potential $\mathbf{A}$ satisfying

$$\mathbf{B} = \nabla \times \mathbf{A}\,. \tag{1.46}$$

The advantage of this definition is that the homogeneous equation (1.15) is satisfied identically by working with the potential $\mathbf{A}$. Faraday's law (1.14) is now written

$$\nabla \times \left(\mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) = 0. \tag{1.47}$$

Any field of vanishing curl can be expressed as the gradient of a scalar function, which allows us to introduce the scalar potential Φ through the relationship

$$\mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi \tag{1.48}$$

so that

$$\mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}\,. \tag{1.49}$$

According to the definition (1.49), the equation (1.14) is identically satisfied. We have thus succeeded in reformulating Maxwell's equations using the potentials $\mathbf{A}$ and Φ with the advantage that the homogeneous equations are satisfied identically. It remains to rewrite the inhomogeneous equations in terms of the potentials, which leads to the system

$$\begin{aligned}\nabla^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) &= -4\pi\rho \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right) &= -\frac{4\pi}{c}\mathbf{j}\,. \end{aligned} \tag{1.50}$$

Since the curl of a gradient is zero, one is always free to add the gradient of a scalar function to the potential $\mathbf{A}$ without any change in the magnetic field. It is therefore possible to carry out a transformation

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \nabla\lambda \tag{1.51}$$

for any scalar function λ. For the field $\mathbf{E}$ to also remaine unchanged, one is obliged to transform the scalar potential with the help of the same function λ

$$\Phi \longrightarrow \Phi' = \Phi - \frac{1}{c}\frac{\partial\lambda}{\partial t}\,. \tag{1.52}$$

The freedom to redefine potential (1.51), (1.52) is referred to as gauge symmetry and is used as a starting point for the definition of the theory of electromagnetism in more sophisticated treatments. Here we use it in order to simplify the remaining equations, namely the inhomogeneous equations (1.50).

Lorenz Gauge, Coulomb Gauge

We limit ourselves to a set of two gauge choices, commonly used in the literature.

1. Lorenz Gauge[6]: If the potentials are chosen so as to satisfy the relation

$$\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0\,, \tag{1.53}$$

[6] This is not an error: Ludvig Lorenz (1829 - 1891) of the gauge is not Hendrik Lorentz (1853 - 1928) of the eponymous transformation.

the equations (1.50) take a symmetrical form

$$\begin{aligned}\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} &= -4\pi\rho \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\frac{4\pi}{c}\mathbf{j}\,. \end{aligned} \tag{1.54}$$

The left-hand sides of these equations take the form of wave equations. This gauge has the advantage that the two equations are completely decoupled.

2. Coulomb Gauge: If one choses the potential $\mathbf{A}$ so as to satify the relationship

$$\nabla\cdot\mathbf{A} = 0\,, \tag{1.55}$$

the equations (1.50) take the form

$$\begin{aligned}\nabla^2\Phi &= -4\pi\rho \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial}{\partial t}\nabla\Phi\,. \end{aligned} \tag{1.56}$$

If we look at the first equation we see that it has the same form as in electrostatics, which gives rise to the Coulomb potential. It can therefore be solved immediately, using what we know already about the Coulomb potential. This is the origin of the name of this choice of gauge.

It is possible to demonstrate (see exercice sheet) that these two gauge choices are consistent in the sense that one can always find a function λ which transforms a given potential into the desired gauge and that once selected, the form of gauge is kept at all times under evolution of the fields.

1.6.2 Boundary Conditions

Boundary conditions for the $\mathbf{E}$ and $\mathbf{B}$ fields can be determined from the basic equations (1.13 -1.16). We start with Gauss' Law (1.13). If one integrates the left hand side (1.13) over a volume V, as indicated in Figure 1.3, one finds, using the

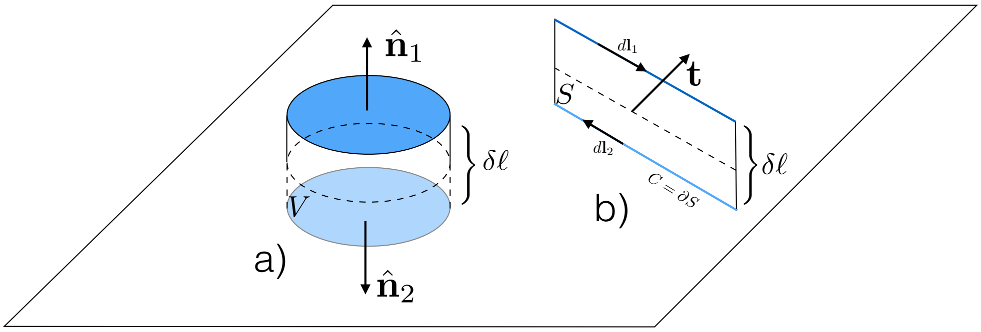

Figure 1.3: Interface $\mathcal{I}$ with a surface charge density σ and surface current density $\mathbf{K}$. The unit vector at the interface is given by $\hat{\mathbf{n}}_1$. a) little volume V over the interface $\mathcal{I}$ with the small thickness $\delta\ell$. b) Small loop over the interface. The curve C bounds the surface S of the loop and has as the normal vector unit $\hat{\mathbf{t}}$.

Gauss' theorem

$$\int_V \nabla \cdot \mathbf{E}\, dV = \int_{S_1} \mathbf{E}_1 \cdot \hat{\mathbf{n}}_1 dS + \int_{S_2} \mathbf{E}_2 \cdot \hat{\mathbf{n}}_2 dS + \mathcal{O}\left(\delta\ell\right) ,$$

where the contribution of the white part in Figure 1.3a) has the order of magnitude $\mathcal{O}\left(\delta\ell\right)$ and can be neglected in the limit $\delta\ell \to 0$ that we take now. In this limit, the integral is reduced to

$$\int_S \left(\mathbf{E}_1 - \mathbf{E}_2\right) \cdot \hat{\mathbf{n}}_1 dS .$$

We denote the fields on either side of the interface by $\mathbf{E}_{1,2}$. The minus sign results from the relation $\hat{\mathbf{n}}_2 = -\hat{\mathbf{n}}_1$. In the same limit one gets the right-hand side of (1.13)

$$\lim_{\delta\ell \to 0} \int 4\pi\rho\, dV = \int_S 4\pi\sigma\, dS$$

with a surface charge density σ, if such a charge exists in the problem at hand. It can be deduced that

$$\left(\mathbf{E}_1 - \mathbf{E}_2\right) \cdot \hat{\mathbf{n}}_1 = 4\pi\sigma . \tag{1.57}$$

Using the small contour C in the Figure 1.3b We will now look at equation (1.16), namely Ampere's Law. The unit vector $\hat{\mathbf{l}}$ parallel to the upper part of the contour is written $\hat{\mathbf{l}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$. According to Stokes' theorem, the left-hand side of Ampère's law is written

$$\int_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \int_{C_1} \mathbf{B}_1 \cdot (\hat{\mathbf{t}} \times \hat{\mathbf{n}}) - \int_{C_2} \mathbf{B}_2 \cdot (\hat{\mathbf{t}} \times \hat{\mathbf{n}}) + \mathcal{O}\left(\delta\ell\right) \,,$$

while the right-hand side, in the limit $\delta\ell \to 0$, reduces to

$$\lim_{\delta\ell \to 0} \left(\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{j} \right) = \frac{4\pi}{c} \mathbf{K}$$

with a surface current density $\mathbf{K}$. In that limit, the contribution of the electric field is zero, and one concludes that

$$\hat{\mathbf{t}} \cdot [\hat{\mathbf{n}}_1 \times (\mathbf{B}_1 - \mathbf{B}_2)] = \frac{4\pi}{c} \mathbf{K} \,.$$

Since this must be true for any orientation of the small loop, i.e. for any vector $\hat{\mathbf{t}}$ in the interface, it is true that

$$\hat{\mathbf{n}}_1 \times (\mathbf{B}_1 - \mathbf{B}_2) = \frac{4\pi}{c} \mathbf{K} \,. \tag{1.58}$$

Using the same kind of manipulations for Maxwell's other equations, one easily finds the boundary conditions

$$\begin{aligned}
\hat{\mathbf{n}}_1 \cdot (\mathbf{E}_1 - \mathbf{E}_2) &= 4\pi\sigma \,, \\
\hat{\mathbf{n}}_1 \cdot (\mathbf{B}_1 - \mathbf{B}_2) &= 0 \\
\hat{\mathbf{n}}_1 \times (\mathbf{E}_1 - \mathbf{E}_2) &= 0 \\
\hat{\mathbf{n}}_1 \times (\mathbf{B}_1 - \mathbf{B}_2) &= \frac{4\pi}{c} \mathbf{K}
\end{aligned}$$

These conditions and their generalizations for so-called macroscopic fields (in later Section 3.1.4) will play a crucial role in our investigation of the propagation of electromagnetic waves as well as in many other applications.

1.6.3 Conservation Laws

We are already familiar with the notion that a quantity can be preserved by the dynamics of a physical system, such as the momentum of a particle in classical mechanics. The quantities which are conserved during the evolution of a system give useful constraints on the form of the phenomena realized in this system. Here we develop conservation laws for the field-particle system in classical electrodynamic theory. Any conserved quantity arises as a consequence of symmetry, according to the Noether's theorem, and the case of electromagnetism is noexception. However, we do not develop this point of view in this course, however, it can be found, for example, in the excellent book by Landau and Lifshitz.

Energy

The instantaneous work accomplished by a charge moving with velocity $\mathbf{v}$ under the influence of a force $\mathbf{F}$ is equal to $\mathbf{F} \cdot \mathbf{v}$. In electrodynamics the force is given by the Coulomb-Lorentz Law (1.12). We can therefore write for a charge distribution contained in a volume V

$$\frac{dW}{dt} = \int_V \left(\rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B} \right) \cdot \mathbf{v} \, dV = \int_V \mathbf{j} \cdot \mathbf{E} \, dV \,. \tag{1.59}$$

In order to transform this into an expression of conservation of energy we use Maxwell's equations as well as some standard relations of vector calculus.

First one eliminates $\mathbf{j}$ using Ampere's law (1.16). This leads to

$$\frac{dW}{dt} = \frac{1}{4\pi} \int_V dV \left[c \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} \right] \cdot \mathbf{E} \tag{1.60}$$

For the term including the rotational of $\mathbf{B}$ one uses

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}) = -\mathbf{B} \cdot \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot (\nabla \times \mathbf{B}) \tag{1.61}$$

Where the last equality follows from Faraday's law (1.14). In combination with

(1.59) we then find

$$\frac{d}{dt}\int_V \frac{1}{8\pi}\left(E^2+B^2\right)\, dV + \int_V \nabla\cdot\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\right) = -\int_V \mathbf{j}\cdot\mathbf{E}\, dV\,. \tag{1.62}$$

We see that the integral of the first term on the left-hand side is reduced to the energy of the **E** and **B** fields in the static case. It is natural to interpret the expression here as the generalization of this notion to non-static situations. The following development will establish that this interpretation makes sense. Equation (1.62) is sometimes called Poynting's theorem. Since this equation is satisfied for any volume V we deduce the following interpretations:

1. If we consider a region without charge or current, the Poynting theorem takes the familiar form
$$\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = 0 \tag{1.63}$$
with the definitions
$$\begin{aligned} u &= \frac{E^2+B^2}{8\pi} && \text{energy density} \\ \mathbf{S} &= \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} && \text{Poynting's vector} \end{aligned}$$
In analogy with the equation of continuity of charge (1.8) the expression here expresses the fact that the energy due to the **E** and **B** fields contained in the region V can escape only by the vector flux **S** through the surface S bounding the region. The dimensions of **S** are
$$[\mathbf{S}] = \left[\frac{\text{energy}}{\text{volume}}\times\text{speed}\right]$$
or, in other words, a current of energy density, which is consistent with the given interpretation.

2. In the presence of charges and currents (of charges) the relation tells us that
$$\frac{dU}{dt} + \int_{\partial V}\mathbf{S}\cdot\hat{\mathbf{n}}dS = -\int_V \mathbf{j}\cdot\mathbf{E}\, dV\,. \tag{1.64}$$
The right hand side represents the mechanical work performed *on* [7] the

[7] The instantaneous work performed by the charge distribution is given by the integral of $+\mathbf{j}\cdot\mathbf{E}$

charges contained in the region V *by* the fields. The interpretation of the Poynting theorem is thus the following: the energy of the $\mathbf{E}$ and $\mathbf{B}$ fields in a region V may decrease either because of the flow of energy current due to Poynting's vector $\mathbf{S}$ coming out of the region, or by the transfer field energy into mechanical energy of the charges. The local (differential) form of energy conservation is

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j} \cdot \mathbf{E} \,. \tag{1.65}$$

Momentum

Now look again at the expression for the force acting on a charge distribution

$$\mathbf{F} = \int_V \left(\rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B} \right) dV = \int_V \mathbf{f} \, dV \,. \tag{1.66}$$

We know that the force is related to the momentum through

$$\frac{d\mathbf{P}_{\mathrm{meca}}}{dt} = \mathbf{F} \,. \tag{1.67}$$

As before, we wish to rewrite this equation by using the fields themselves without reference to charges or currents. This can be accomplished as follows. We start by eliminating ρ and $\mathbf{j}$ using Maxwell's equations, namely equation (1.13) for ρ and the equation (1.15) for $\mathbf{j}$. The force density then becomes the expression

$$\rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B} = \frac{1}{4\pi} \left[\mathbf{E}(\nabla \cdot \mathbf{E}) + \frac{1}{c} \mathbf{B} \times \frac{\partial \mathbf{E}}{\partial t} - \mathbf{B} \times (\nabla \times \mathbf{B}) \right] . \tag{1.68}$$

The strategy is now similar to the one we have followed for energy conservation. The second term in square brackets can be manipulated as follows

$$\mathbf{B} \times \frac{\partial \mathbf{E}}{\partial t} = -\frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} \tag{1.69}$$

with the positive sign.

in order to bring out Poynting's vector. Then, we eliminate the derivative $\partial_t \mathbf{B}$ using the equation (1.14), which leads to

$$\frac{d\mathbf{p}_{\mathrm{meca}}}{dt}+\frac{\partial}{\partial t}\left(\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\right)=\frac{1}{4\pi}\left[\mathbf{E}(\nabla\cdot\mathbf{E})+\mathbf{B}(\nabla\cdot\mathbf{B})-\mathbf{E}\times(\nabla\times\mathbf{E})-\mathbf{B}\times(\nabla\times\mathbf{B})\right]. \tag{1.70}$$

One have written $\mathbf{p}_{\mathrm{meca}}$ for the momentum density, which after integration gives the momentum, $\mathbf{P}_{\mathrm{meca}}$. For reasons of symmetry we have added zero in the form $0 = \nabla\cdot\mathbf{B}$ in this relationship. To interpret what we have derived it will be convenient to write the above equation in the form of an integral in a similar way to Poynting's theorem (1.62). Obviously, the terms appearing with the spatial derivative (∇) should be rewritten as a total divergence. Let us first look at the terms involving the electric field

$$\begin{aligned}\mathbf{E}(\nabla\cdot\mathbf{E})-\mathbf{E}\times(\nabla\times\mathbf{E}) &= -\nabla\frac{E^2}{2}+(\nabla\cdot\mathbf{E})\mathbf{E}+\mathbf{E}(\nabla\cdot\mathbf{E})\\ &=\nabla\cdot\left(-\mathbf{1}\frac{E^2}{2}+\mathbf{EE}\right).\end{aligned}$$

In the last line we introduced the so-called dyadic notation, with the components of the unit dyad

$$\mathbf{1}\to\mathbf{1}_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases} \tag{1.71}$$

The components of the dyadic product between two electric fields are given by

$$\mathbf{EE}\to E_iE_j\,. \tag{1.72}$$

An analogous expression is also satisfied by the field $\mathbf{B}$ replacing $\mathbf{E}\to\mathbf{B}$ and so (1.70) turns into

$$\frac{d\mathbf{p}_{\mathrm{meca}}}{dt}+\frac{\partial}{\partial t}\left(\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\right)=-\nabla\cdot\left(\mathbf{1}\frac{E^2+B^2}{8\pi}-\frac{\mathbf{EE}+\mathbf{BB}}{4\pi}\right). \tag{1.73}$$

Let us now turn to the interpretation of this equation.

1. First, it follows that the electromagnetic field has its own momentum density

$$\mathbf{g}=\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}=\mathbf{p}_{\mathrm{EM}} \tag{1.74}$$

So the left hand side of (1.73) the interpretation of the total change in momentum, i.e. the momentum of the particles (the mechanical part) as well as that of the fields (the EM part). We notice the close connection with Poynting's vector[8]

$$\mathbf{S} = c^2\mathbf{g}\,. \tag{1.75}$$

2. In integral form we have

$$\begin{aligned}\frac{d}{dt}\left(\mathbf{P}_{\text{meca}} + \mathbf{P}_{\text{EM}}\right) &= -\int_V \nabla\cdot\left(\mathbf{1}\frac{E^2+B^2}{8\pi} - \frac{\mathbf{EE}+\mathbf{BB}}{4\pi}\right) dV \\ &= -\int_S \hat{\mathbf{n}}\cdot\mathbf{T}dS\,,\end{aligned}$$

where $\mathbf{T}$ is the Maxwell stress tensor with components

$$\mathbf{T} \to T_{ij} = \delta_{ij}u - \frac{E_iE_j + B_iB_j}{4\pi}\,, \tag{1.76}$$

and $\hat{\mathbf{n}}$ is, as usual, the unit vector perpendicular to the surface S bounding the volume V, directed towards the outside. We then deduce that the quantity

$$\hat{\mathbf{n}}\cdot\mathbf{T} \to n_iT_{ij} \tag{1.77}$$

Represents a flux of momentum through the surface S. Note that this is a vectorial quantity, which is consistent with the link with momentum, which itself is a vector. In other words, the quantity n_iT_{ij} represents the flux of the j^{th} component of the momentum across the surface S. This is also why we used a tensorial quantity (dyadic) for the momentum flux: a scalar quantity (like the charge or energy) has a vector flux, whereas a vector quantity like the momentum has a tensor tensor (=dyadic) flux. Loosely speaking, in each

[8]We have found a relation between the flux $\mathbf{S}$ of the energy u contained in the electromagnetic field and its momentum $\mathbf{g}$. This link is given by (1.75). Dimensional analysis of this equation gives the structure

$$\text{energy density} \times \text{speed} = c^2(\text{mass density} \times \text{speed})$$

which is equivalent to the relation

$$E = mc^2\,.$$

It is obviously the famous equation of restricted relativity discovered by Einstein by considering Maxwell's equations (*Zur Elektrodynamik bewegter Körper*). More detail in chapter 2 case we just add one more index to the density in order to get flux.

Angular momentum

For a mechanical system the derivative of the angular momentum, $\mathbf{L}_{\text{mech}}$ can be written as

$$\frac{d\mathbf{L}_{\text{méca}}}{dt} = \int_V \mathbf{x} \times \mathbf{f}\, dV\,. \tag{1.78}$$

The integrand appearing in this expression has components

$$\mathbf{x} \times \mathbf{f} \to [\mathbf{x} \times \mathbf{f}]_i = \epsilon_{ijk} x_j f_k \tag{1.79}$$

Let us recall now that equation (1.73) expresses the force density of a system of fields and particles. We would like to use the conservation law associated with the momentum (1.73) for a derivation of the conservation of angular momentum, as defined above. It is clear that the contributions of the fields must again be taken into account, which will give us a definition of the angular momentum of the fields. We begin by writing the conservation of momentum in components

$$\frac{\partial g_k}{\partial t} + \frac{\partial T_{lk}}{\partial x_l} + f_k = 0\,. \tag{1.80}$$

Then we multiply it x_j, which leads to

$$\frac{\partial (x_j g_k)}{\partial t} + \frac{\partial (x_l T_{lk})}{\partial x_j} - T_{jk} + x_j f_k = 0\,. \tag{1.81}$$

Here we used the fact that

$$\frac{\partial x_l}{\partial x_j} = \delta_{lj}\,. \tag{1.82}$$

Finally we multiply(1.81) by ϵ_{ijk}And take the sum on the repeated indices. We find

$$\frac{\partial (\epsilon_{ijk} x_j g_k)}{\partial t} + \frac{\partial (\epsilon_{ijk} x_l T_{lk})}{\partial x_j} + \epsilon_{ijk} x_j f_k = 0\,. \tag{1.83}$$

The last step is true thanks to the symmetry of the tensor $T_{ij} = T_{ji}$, in other words,

$$\epsilon_{ijk} T_{jk} = 0\,. \tag{1.84}$$

Let us now turn to the interpretation of the relation we have just derived.

1. We deduce the interpretations

$$\begin{aligned} \mathcal{L}_{\text{EM}} &= \mathbf{x} \times \mathbf{g} && \text{density of angular momentum} \\ \mathcal{K}_{ij} &= \epsilon_{jkl} x_k T_{il} && \text{tensor of angular momentum flux} \end{aligned}$$

   The density $\mathcal{L}_{\text{EM}}$ gives, after integration over the volume V, the total angular momentum of the fields

$$\mathbf{L}_{\text{EM}} = \int_V \mathcal{L}_{\text{EM}} \, dV \,. \tag{1.85}$$

2. With the above identifications we can write the law of conservation of total angular momentum. The latter follows from an integral of (1.83) over the volume V and takes the form

$$\frac{d(\mathbf{L}_{\text{méca}} + \mathbf{L}_{\text{EM}})}{dt} = -\int_S \hat{\mathbf{n}} \cdot \mathcal{K} dS \,, \tag{1.86}$$

   where the right-hand side follows from an application of the Gauss' theorem.

All conservation laws found in this chapter can be written in a single equation, using a single object, the energy-momentum tensor. In order to define and use this object we must first introduce the concepts and the notation of special relativity. We will discuss this in chapter 2.

1.6.4 Momentum of localized field configurations

Consider a localized field configuration, that is, a collection of fields contained in a finite region of space V that propagate as a function of time (see Figure 1.4). We assume that there is only radiation, and in particular no particles or other mechanical systems. In other words, we want to describe a localized momentum of radiation with, by hypothesis, finite energy and momentum. For this purpose, it will be useful to define the 'center of energy'

$$\langle \mathbf{x} \rangle := \frac{1}{E} \int_V \mathbf{x} u dV \tag{1.87}$$

where $E = \int_V u dV$, i.e. the total energy, is conserved since we assume that $\int_{\partial V} \mathbf{S} \cdot \hat{\mathbf{n}} dS = 0$. Of course this quantity corresponds to the center of energy of our pulse

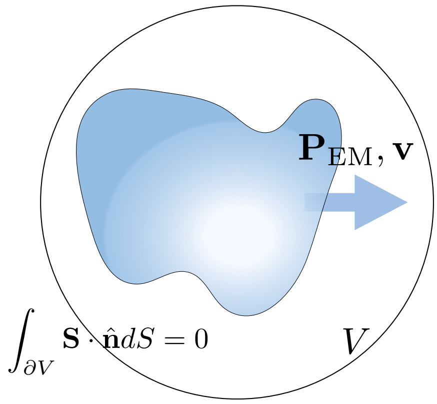

Figure 1.4: Localized configuration of electric and magnetic fields. The volume V at each instant contains all the fields, and consequently the integral of Poynting's vector on the ∂V is zero.

of radiation and thus serves as a good description of its motion. According to Poynting's theorem(1.76) one has

$$\begin{aligned} E\frac{d\langle \mathbf{x} \rangle}{dt} &= E\left\langle \frac{\partial \mathbf{x}}{\partial t} \right\rangle + \int_V \mathbf{x}\frac{\partial u}{\partial t} dV \\ &= E\left\langle \frac{\partial \mathbf{x}}{\partial t} \right\rangle - \int_V \mathbf{x}(\nabla \cdot \mathbf{S}) dV \\ &= E\left\langle \frac{\partial \mathbf{x}}{\partial t} \right\rangle - \int_V \mathbf{S} dV \,. \end{aligned} \tag{1.88}$$

The last equality results from the integration by parts of $\int_V \mathbf{x}(\nabla \cdot \mathbf{S}) dV$ which leads us to evaluate the term

$$[(\nabla \mathbf{x}) \cdot \mathbf{S}]_i \to \frac{\partial x_j}{\partial x_i} S_i = \delta_{ij} S_j = S_i \tag{1.89}$$

Technical aside: here we see an example in which it is obviously much simpler to use component notation. Indeed, the term $\nabla \mathbf{x}$ is the dyadic with components $\frac{\partial x_i}{\partial x_j}$.

By using (1.75), the last part of the equation (1.88) can be expressed as

$$\frac{E}{c^2}\left\langle\frac{\partial\mathbf{x}}{\partial t}\right\rangle = \int_V \mathbf{p}_{\mathrm{EM}} dV = \mathbf{P}_{\mathrm{EM}}\,. \tag{1.90}$$

Yet the left hand side can be interpreted as the propagation velocity of the radiation momentum (times E/c^2). Yet the left part can be interpreted as the propagation velocity of the pulse of radiation

$$\mathbf{P}_{\mathrm{EM}} = \frac{E}{c^2}\langle\mathbf{v}\rangle \tag{1.91}$$

that is to say, the momentum of a pulse of electromagnetic radiation is equal to the product of E/c^2 times the momentum speed.