# Electromagnetic Induction 

## Lab Ticket:

Each individual will write up his or her own Lab Report for this two-week experiment. You must also submit Lab Tickets individually. You are expected to discuss the plans for the lab with your partner, but we require that all written work to be done on your own.

- Week 1: A nicely polished, typed, rough draft of the Introduction, Experimental Design and Procedure sections of your lab report. We will grade these and get them back to you during week 2 of the extended projects.
- Week 2: No lab ticket. Use your spare time to work on writing your lab report. Feel free to meet with me to discuss what you have written.


## I. Introduction

Faraday's great discovery was that a change in the magnetic flux through a loop of wire induces an emf in the loop of wire. The induced emf is given by Faraday's law,

$$
\begin{equation*}
V=-\frac{d \Phi_{B}}{d t}, \tag{1}
\end{equation*}
$$

where $\Phi_{B}$ is the flux, defined by $\Phi_{B}=\vec{B} \cdot \vec{A}$, and $\vec{A}$ is the vector area of the loop (you can, as usual, picture the flux as the number of B-field lines piercing the area). Electromagnetic induction is of great practical value, particularly in its application to producing electricity through electric generators. The present experiment will be concerned with confirming Faraday's law and demonstrating the novelty that you can generate electricity simply by moving a loop through a magnetic field.

The simple form that Faraday's Law takes in equation [1] evidences how carefully modern theoretical definitions are crafted. The following section will be concerned with unwinding this simple statement into the somewhat more cluttered particularities of the real situation in the laboratory. In our experiment we will test Faraday's law by moving a thick quarter sized Neodynium magnet through a small coil of wire. As the magnet falls through the coil, the flux through the coil will change resulting in an induced emf in the coil. The induced emf can easily be observed by monitoring the voltage across the coil. We turn now to the unpacking process.

## II. The Magnetic Field of a Dipole

The basic plan of this section and the next two is as follows: first we will see that permanent magnets are well characterized by dipole magnetic fields, next we will turn to deriving an analytic form for the dipole field of a current loop, at this point we will take a brief hiatus to describe how to experimentally measure the dipole moment of a permanent magnet, finally the problem of the flux through a loop of wire due to a moving dipole field will be solved.

Ferromagnetism, as you may recall, is responsible for most permanent magnets at room temperature. The basic idea behind ferromagnetism is that the atomic structure of a ferromagnetic material is more organized than it is in most everyday substances. The precise reasons for this organization are quantum mechanical in origin and are thus beyond the scope of our present discussion. ${ }^{1}$ However, the result of this structure is clear:

[^0]in a ferromagnetic material the magnetic fields of its atomic constituents tend to line up with each other. This is in stark contrast with most materials, where the magnetic fields of the atoms point in random directions and cancel each other out. The aligned fields in a ferromagnetic material add up so that there is a strong, macroscopically observable magnetic field around the material.

What total field results from the addition of all of these tiny fields? As it turns out, the atomic magnetic fields that give rise to the macroscopic fields discussed above are essentially, dipole fields. (Recall that the magnetic field due to a loop of current is called a dipole field.) Thus, it is perhaps not surprising that the total macroscopic fields are also dipole fields, at least approximately. This is a fairly good approximation if you are "far away" from the magnet. The pictures below will hopefully convince you of the validity of this approximation. The first is a cross-sectional diagram of the dipole field due to a current loop; we will derive this field below. The second is of iron filings aligned with the magnetic field of a small bar magnet.


Note the polarity of these diagrams. It is useful to represent this polarity with a little arrow pointing from the south pole of the magnet to the north pole. This arrow is called the dipole moment $m$ and has a magnitude given by NIA, where $N$ is the number of turns in the loop, $I$ is the current through the loop, and $A$ is the area of the loop. If you are just given a loop with a current running through it, you can find the direction of the magnetic moment by applying the right hand rule. Curl your fingers in the direction of the current and your perpendicularly pointed thumb is in the direction of $m$. We turn now to deriving the magnetic field of a current loop.

Consider a small loop of current $I$ with radius $R$ situated at the origin (only one turn, $\mathrm{N}=1$ ), and intended to model our Neodynium magnet. Derive the magnetic field due to this loop along the axis going through the center of the loop at a height $z$ above the loop. Use the Biot-Savart law and take advantage of the cylindrical symmetry. ${ }^{2}$

The answer is


[^1]\[

$$
\begin{equation*}
B=\frac{\mu_{o} I R^{2}}{2\left(R^{2}+z^{2}\right)^{\frac{3}{2}}}, \tag{2}
\end{equation*}
$$

\]

where the field points along the axis and away from the loop. This result is often abbreviated in terms of the dipole moment, $m$, yielding,

$$
\begin{equation*}
B=\frac{\mu_{o} I \pi R^{2}}{2 \pi\left(R^{2}+z^{2}\right)^{\frac{3}{2}}}=\frac{\mu_{o} m}{2 \pi\left(R^{2}+z^{2}\right)^{\frac{3}{2}}}, \tag{3}
\end{equation*}
$$

where $m=I \pi R^{2}$. Note also that if we are far away from the loop, the limit that is appropriate for modeling our magnet, this expression is well approximated by

$$
\begin{equation*}
B=\frac{\mu_{o} m}{2 \pi z^{3}} . \tag{4}
\end{equation*}
$$

In this "far field" approximation the field of a dipole depends on the inverse cube of your distance from it.

## III. Measuring Dipole Moments

At this point let us pause for a moment to describe how to measure the dipole field of a permanent magnet, we will return to the theoretical model presently. The main difficulty with modeling a permanent magnet as a dipole is that we don't have any obvious method for attributing the magnet with a particular current, number of turns, or effective area. This makes calculating its dipole moment very difficult! Experimentally, however this is easily overcome. Simply measure the magnetic field at various distances along the axis through the center of the magnet. Then the dipole moment can be calculated from the slope of a plot of $B \mathrm{vs} .1 / z^{3}$.

To perform this measurement, open DataStudio and setup a Magnetic Field
Sensor. In the Magnetic Field Sensor's setup window, set the Gain to 10x and, under the Measurements tab, uncheck the Magnetic Field Strength (1x) box and check the Magnetic Field Strength (10x) box. Go to the Displays list (under the Summary list) and open a digits display by double clicking on "3.14 Digits". When you press start you will get a digital display of the current magnetic field reading. Set the field direction switch, located on top of the magnetic field sensor to "axial" and the sensitivity switch to $10 x$. Place the Neodynium magnet next to a plastic ruler. Hold the field sensor far away from the magnet and tare it. Move the sensor along the axis through the center of the magnet until you get a reading of 2 gauss. Take six or seven readings between this point and four centimeters from the magnet, recording both the distances from the magnet and the field strengths. (Note that the white dot on the end of the sensor is where the sensing element is located, so take distance measurements from this point.) If you get closer than four centimeters the far field approximation begins to break down and the field strength is no longer directly proportional to the inverse cube of the distance. Repeat the field strength measurements twice more at the same distances, remembering to tare the probe between runs. Use the average of your three measurements in the plot of $B$ vs. $1 / z^{3}$. Find the dipole moment of your magnet. Feel free to discard data points taken close to the magnet that deviate significantly from your linear fit.

## IV. The Flux through a Current Loop

At this point we have a good basic picture of the magnetic field of our magnet. Recall the plan of action: the magnet is going to be dropped through a coil of wire and then the induced voltage will be measured. To predict the induced voltage we need to calculate the flux through the loop and apply Faraday's law. To calculate the flux through
the loop we will use a couple of nifty tricks. The loop and magnet are sketched below; the origin is at the center of the magnet,


The loop is not the same loop we used above to model the magnet, the magnet is drawn in the picture. Note also that $\rho$ is quite distinct from the previous $r$. Consider the two dimensional surface constructed by sweeping the vector $\vec{\rho}$ of fixed length $\rho$ through the entire interior of our loop. This surface is a little spherical cap, like the one you would get if you cut a Ping-Pong ball into two parts by slicing it below the equator.


Now imagine closing this cap simply by filling in the loop of wire so that it is a circular plate.


According to Gauss's Law for magnetic flux the total flux through this volume is equal to zero, $\oint \vec{B} \cdot d \vec{A}=0$. Now, consider a line of flux entering through the top circular plate, this flux line either exits through this same plate in which case it contributes nothing to the total flux through this plate, or it leaves through the lower spherical cap. But this means that any flux line, which contributes to the total flux through the circular plate, contributes an equal but opposite flux to the spherical cap. Thus the total fluxes through each of the two surfaces are equal and opposite:

$$
\begin{equation*}
\Phi_{\text {plate }}=-\Phi_{\text {cap }} . \tag{5}
\end{equation*}
$$

Surprisingly, for the dipole magnetic field it turns out to be easier to calculate the flux through the spherical cap. Thus we will use equation [5] to calculate the flux through the loop's plate.

In Section II you found that the magnetic field of a permanent magnet was,

$$
\begin{equation*}
B=\frac{\mu_{o} m}{2 \pi z^{3}} \tag{6}
\end{equation*}
$$

assuming that you were in the far field and on the axis going through the magnet's center. In order to calculate the flux through the spherical cap, we have to also know what the magnetic field off of the central axis is. This is an important calculation that you will do if you take the junior level Electricity and Magnetism class. For now we will simply motivate the result. As you can see from the picture in Section II, the magnetic field of a dipole bends around the dipole. However, our clever choice of a spherical cap allows us to completely neglect this bending because it is perpendicular to the surface of the cap (don't worry if this isn't obvious). This means that we need only consider the component of the magnetic field that radiates directly away from the magnet (the so called radial component). Because we want to state the field completely in terms of the radial distance from the magnet we need to restate equation [6] in terms of the radial vector $\vec{\rho}$. But along the z -axis the radial magnitude and the $z$ distance are the same, so equation [6] becomes

$$
\begin{equation*}
B=\frac{\mu_{o} m}{2 \pi \rho^{3}} \tag{7}
\end{equation*}
$$

along the z -axis. Off of the z -axis there is a correction factor (some of the field strength is going into the bending field) that depends on the angle the radial vector makes with the z axis (see picture below):

$$
\begin{equation*}
B=\frac{\mu_{o} m}{2 \pi \rho^{3}} \cos \theta \tag{8}
\end{equation*}
$$



This picture also serves to define $\theta_{o}$ the angle with respect to the z -axis when the radial vector points to the loop. Check that equation [8] agrees with equation [7] along the z axis. Equation [8] gives us a complete description of the radial magnetic field of a dipole.

Now we can use the radial dipole field to calculate the flux $\Phi_{\text {cap }}$ through the spherical cap. To accomplish this calculation, break the cap into little rings with an angular spread of $d \theta$. These rings have width $\rho d \theta$ because they are a distance $\rho$ from the origin. The radius of the strips with respect to the z -axis is given by the projection $\rho \sin \theta$. The diagram below depicts this situation. ${ }^{3}$
${ }^{3}$ A very similar diagram and integral are presented in the discussion of the "Gravitational Force due to a Spherical Mass Distribution" contained in appendix D of your text [Giancoli, pp. A-9-11]. The more of these arguments you look at, the more intelligible they become.

The area of each ring is given by $2 \pi$ times its radius times the ring's width, $(2 \pi \rho \sin \theta) \rho d \theta$.
The dipole field times the area of the ring gives the flux through each of these rings,

$$
\begin{equation*}
d \Phi_{c a p}=\left(\frac{\mu_{o} m \cos \theta}{2 \pi \rho^{3}}\right)\left(2 \pi \rho^{2} \sin \theta\right) d \theta=\frac{\mu_{o} m}{\rho} \cos \theta \sin \theta d \theta . \tag{10}
\end{equation*}
$$

Because this formula takes the entire circumference of the ring into account we only need to integrate from the bottom of the bowl up to its rim, that is from $\theta=0$ up to $\theta=\theta_{o}$, yielding,

$$
\begin{equation*}
\Phi_{c a p}=\int d \Phi_{c a p}=\frac{\mu_{o} m}{\rho} \int_{0}^{\theta_{o}} \cos \theta \sin \theta d \theta, \tag{11}
\end{equation*}
$$

which, after integration by parts, gives us

$$
\begin{equation*}
\Phi_{c a p}=\frac{\mu_{o} m}{\rho} \frac{\sin ^{2} \theta_{o}}{2} \tag{12}
\end{equation*}
$$

Inserting this result into equation [5] and taking into account the fact that the coil has N turns instead of just one yields

$$
\begin{equation*}
\Phi_{\text {plate }}=-\frac{\mu_{o} m N \sin ^{2} \theta_{o}}{2 \rho} . \tag{13}
\end{equation*}
$$

Referring to the first diagram of this section we see that we can rewrite $\sin ^{2} \theta_{o}$ and $\rho$ in terms of the coil radius and the height of the magnet above the plate,

$$
\begin{equation*}
\Phi_{\text {plate }}=-\frac{\mu_{o} m N a^{2}}{2 \rho^{3}}=-\frac{\mu_{o} m N}{2} \frac{a^{2}}{\left(a^{2}+z^{2}\right)^{3 / 2}} . \tag{14}
\end{equation*}
$$

## V. The Induced Emf

Now, at last, we are in a position to predict the induced emf in the coil. According to Faraday's Law the induced emf is given by,

$$
\begin{equation*}
V=-\frac{d \Phi_{\text {plate }}}{d t} . \tag{15}
\end{equation*}
$$

Applying the chain rule we find that

$$
\begin{equation*}
V=-\frac{d \Phi_{\text {plate }}}{d z} \frac{d z}{d t}=-\frac{d \Phi_{\text {plate }}}{d z} v, \tag{16}
\end{equation*}
$$

where $v$ is the velocity of the falling magnet. Finally computing the derivative with respect to $z$ we find

$$
\begin{equation*}
V=-\frac{3 \mu_{o} m N a^{2} v}{2} \frac{z}{\left(a^{2}+z^{2}\right)^{5 / 2}} . \tag{17}
\end{equation*}
$$

## VI. Experimental Investigations

The hard work leading up to equation [17] is more than compensated for by the rich variety of experimental techniques you can apply to this prediction. Before releasing you to explore this fecundity on your own, let's try one guided experiment to increase your familiarity with functions of this form. In order to simplify the analysis we will setup the apparatus so that the magnet falls through the coil at a constant velocity. This is
accomplished by the use of a simple Atwood's machine. A string looped over the large pulley groove of the rotary motion sensor suspends the magnet. The weight of the magnet is balanced by a mass of equal weight on the other end of the string.

Setup DataStudio to accept measurements from a Voltage Sensor and the Rotary
Motion Sensor. Configure the voltage sensor to take measurements at 100 Hz by double clicking on its icon in the Experiment Setup window and selecting this rate in the Sensor Properties window. Double click on the icon for the rotary motion sensor and double check that its sampling rate is now at 100 Hz . Click on the "Measurements" tab at the top of the sensor properties window. Deselect the "Angular Position" variable and instead highlight the "Position" variable. Finally select the "Rotary Motion Sensor" tab from the top of the sensor properties window. Under the heading Linear Scale select the "Large Pulley (Groove)" option from the pull down menu. Now the rotary motion sensor will give you data for the linear translation of the magnet.

Practice giving the string a quick smooth tug a few times so that the magnet falls through the coil of wire at a constant velocity. When you feel your technique is good take a run of data. Open a graph of the voltage. Turn this into a graph of voltage as a function of position by dragging the Position measurement from the data summary list on the left of your screen onto the x-axis of your display. Note that a dashed box should surround the x -axis otherwise this will simply open another graph of the position as a function of time.

Now, let's examine whether this graph gives a similar result to the one that we predicted in equation [17]. Equation [17] describes an "antisymmetric" function of the position. An antisymmetric function is defined by the condition that

$$
\begin{equation*}
f(-x)=-f(x) . \tag{18}
\end{equation*}
$$

Let's check that this condition is satisfied by equation [17]

$$
\begin{align*}
V(-z) & =-\frac{3 \mu_{o} m N a^{2} v}{2} \frac{-z}{\left(a^{2}+(-z)^{2}\right)^{5 / 2}}  \tag{19}\\
& =\frac{3 \mu_{o} m N a^{2} v}{2} \frac{z}{\left(a^{2}+z^{2}\right)^{2 / 2}}=-V(z) .
\end{align*}
$$

Visually antisymmetric functions are characterized by the fact that they look identical on each side of the $y$-axis except that they are upside down from one another. The antisymmetric functions $f(x)=x, f(x)=x^{7}$, and $f(x)=\sin x$ appear below.

Does your graph look antisymmetric? This, of course, only tells us that we are on the

right track. A more rigorous test is the following: Click on the "風 $\overline{\text { Fit }} \quad \nabla$ " button that appears in the tool bar at the top of your graph window and select the User Defined fit. A dialog box with the title "User-Defined Fit" will appear in your graph window, double click on this box. This will open the Curve Fit window that appears below.


Within this window you can specify your own fitting function. This allows you to make a direct comparison of your data and your model. Enter equation [17] as the user-defined function. Make sure that the Manual button is selected. Note that DataStudio thinks of $x$ as the variable on the x -axis. So rather than entering z for the position you should enter x .

When you write out your fit you may use constants. When you enter them in your fit equation and then press Accept, they'll appear below in a table. Enter the values of your constants in the blanks labeled Initial Guess. After you've entered your values and press Accept again, DataStudio will fit your data with the equation you provide. Since you've left no free parameters the fit might not be too great, but hopefully you're getting the right shape at least.

If you run into an error with your value for $\mu_{o}$ then it's likely that DataStudio is rounding that parameter to zero since it's quite small. To remedy this click the Properties button in the User-defined fit window, select the Numeric tab, and set the number of significant figures to 6 and also unclick the Zero Threshold box.

Given that the parameters we've used (many of which are measured) may have some error associated with them it can be an interesting exercise to let DataStudio adjust the fit by modifying the parameters a little. In order to do this we need to choose which parameters to let DataStudio modify and which we're fairly certain need to remain as entered. Think about the constants you've used in your fit and determine which ones you feel might be adjustable and which ones should remain as you define them. Hint: Those with a reasonable amount of potential error in their determination ought to be left for DataStudio to adjust.

Rewrite your fit equation to incorporate actual numbers for those parameters you feel should remain fixed and leave those that you want DataStudio to tweak when performing the fit as user-defined constants. Then select the Automatic button instead of the Manual button in the User-Defined Fit window. After you press Accept, DataStudio will seek to fit your data with your fit equation and will use your Initial Guess as a starting point. It will then come up with values for the constants that provide the best fit and these will be displayed in the table next to your Initial Guess. How do these fit parameters calculated by DataStudio compare with your predicted values? Can you think of reasons why they might differ?

The one constant that you can't measure directly off the apparatus is the velocity that the magnet had. To find the velocity, open a graph of position vs. time. If the magnet was moving with a constant velocity this should be a fairly linear plot. Fit this line with a
linear function and use the slope to determine the velocity. Enter the velocity you find into your user-defined fit. How does your user-defined fit compare with the data? Try changing the values of the constants. What effect does this have on your fit? Do some of the parameters have different effects than the others? Does Faraday's Law do a good job of predicting the induced emf in a loop of wire?

You may find that the fit is very good except for the fact that it is shifted to the left or right from your data. This is merely a problem of defining your origin. To fix this problem use the "smart tool" to find where your data crosses the x -axis, call this point $x_{o}$. Replace every occurrence of x in your used defined function with ( $\mathrm{x}-x_{o}$ ).

## VII. Extreme Values and Acceleration

The modeling exercise above is only the tip of the iceberg for what you can do with this data. The two exercises below further develop our understanding of this data and this experiment.

## Maximum and Minimum Voltage

Recall the basic result from calculus that a function obtains an extreme value (or saddle point) at any point where the derivative of the function is equal to zero. Calculate
$V^{\prime}(z)=\frac{d V}{d z}$ from equation [17]. Solve the equation, $V^{\prime}(z)=0 \underline{\text { for }} \underline{z}$ to find the positions of the maximum and minimum induced voltage. What are the voltages at these points?
[Answer: $V_{ \pm}= \pm \frac{24 \mu_{o} m N v}{(5)^{\text {/2}} a^{2}}$ ]
Use the smart tool to locate the positions of the maximum and minimum voltage. Do they agree with your prediction? Use the value of the voltage at the maximum and minimum to determine the velocity of the magnet as it traveled through the loop. How does this compare with the velocity you found from the linear fit above? What is your error? What is the most likely source of this error?

## Acceleration of the Magnet

Thus far we have consider the situation in which the magnet falls through the loop of wire at a constant velocity. What if instead we were to drop the magnet through the loop, so that it experienced acceleration due to gravity? What would the expected Emf be now?
(Hint: The analysis is exactly the same except that velocity is now a function of position. All you have to do is to find $v(z)$ and plug it into equation [17].) Test this prediction experimentally. What is the error in your measurements? Where does it come from?


[^0]:    ${ }^{1}$ A slightly more in depth treatment of ferromagnetism is presented in our Giancoli text [Chapter 28-7].

[^1]:    ${ }^{2}$ If you get stuck, take a peek at p. 744 of our Giancoli text.

