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Electromagnetic solitary waves in magnetized plasmas

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A Hamiltonian formulation, in terms of a non-canonical Poisson bracket, is presented for a nonlinear fluid system that includes reduced magnetohydrodynamics and the Hasegawa–Mima equation as limiting cases. The single-helicity and axisymmetric versions possess three nonlinear Casimir invariants, from which a generalized potential can be constructed. Variation of the generalized potential yields a description of exact nonlinear stationary states. The new equilibria, allowing for plasma flow as well as partial electron adiabaticity, are distinct from those found in conventional magnetohydrodynamic theory. They differ from electrostatic stationary states in containing plasma current and magnetic field excitation. One class of steady-state solutions is shown to provide a simple electromagnetic generalization of drift-solitary waves.

1. Introduction

Solitary wave phenomena have a demonstrated importance in fluid dynamics, for example with regard to oceanic or atmospheric currents (Flierl *et al.* 1979). The possible significance of such nonlinear coherent motions in plasma physics, although less well established, is drawing increased theoretical attention. In particular, solitary drift waves (large-amplitude, coherent electrostatic disturbances in a non-uniform magnetized plasma) have been extensively studied (Larichev & Reznik 1976; Meiss & Horton 1983; Pavlenko & Petviashvili 1984; Mikhailovskii *et al.* 1984).

The present work extends the analysis of solitary drift waves to the electromagnetic case: we allow for large-amplitude perturbations of both the electrostatic and magnetic fields. Such extension is motivated by the fact that disturbances observed in tokamak experiments, including those that are relatively coherent, often involve significant magnetic excitation. Furthermore, the electromagnetic generalization of drift solitary waves has intrinsic interest.

A second, closely related, purpose of our work also extends previous theory. We show that the Hamiltonian formulation (Morrison & Hazeltine 1984) of reduced magnetohydrodynamics (RMHD) (Strauss 1976, 1977) can be generalized to include non-magnetohydrodynamical effects, such as electron adiabaticity. Thus we present a generalized Poisson bracket, $\{F, G\}$, where F and G are functionals of the field variables, in terms of which the dynamical equations can be expressed in Hamiltonian form. The bracket formalism facilitates the identification of various integral invariants, simplifies the derivation of solitary wave equations, and provides insight into nonlinear stability issues (Holm *et al.* 1985; Hazeltine *et al.* 1984*b*).

Both extensions exploit a reduced inclusive system (Hazeltine 1983) that has been shown to include the physics of both RMHD and the Charney–Hasegawa–Mima equation (CHM) (Hasegawa & Mima 1977; Charney 1948). Recall that the CHM equation, which first occurred in ocean current investigations (Charney 1948), is the conventional starting point for studies of electrostic drift-wave nonlinearity. The inclusive system (an electromagnetic generalization of CHM that also incorporates non-fluid effects into RMHD) provides a natural setting for the study of generalized Hamiltonian plasma dynamics and generalized solitary wave behaviour.

In § 2 we briefly review the physics contained in the reduced inclusive system, and introduce our basic notation. § 3 presents the Hamiltonian expression of this system, considers its nonlinear significance, and identifies the key integral invariants. Equations describing a particular class of nonlinear solutions (the electromagnetic solitary waves) are derived in § 4.

2. Reduced inclusive system

2.1. *Notation*

The nonlinear system analysed here was first derived in tokamak geometry, following procedures similar to those of RMHD. However, because the plasma beta is assumed small, toroidal curvature effects do not enter and the system is also pertinent in slab geometry. It uses three appropriately normalized fields: ϕ, the electrostatic potential; ψ, the parallel component of the magnetic vector potential, or poloidal magnetic flux; and χ, the plasma density perturbation. For the explicit normalizations, as well as details of the derivation, the reader is referred to Hazeltine (1983). The following introductory comments are intended to clarify the physical interpretation of the system.

The equations are expressed in terms of effectively Cartesian co-ordinates (x, y, z); in the tokamak case, z can be identified with the toroidal angle, while x and y are co-ordinates in the poloidal cross-section. The basic ordering parameter is denoted by

$$\epsilon \ll 1$$

and can be identified with the tokamak inverse aspect ratio. This parameter appears in the normalized magnetic field, given by

$$\mathbf{B} = \hat{z}(1-\epsilon x) - \epsilon\hat{z}\times\nabla_\perp\psi + O(\epsilon^2), \tag{1}$$

where

$$\nabla_\perp = \hat{x}\,\partial/\partial x + \hat{y}\,\partial/\partial y$$

is the (normalized) gradient in the poloidal plane. Equation (1) expresses the presumed dominance of the vacuum magnetic field, which is purely toroidal.

A geometrical assumption, complementary to (1), is that the 'longitudinal' or toroidal scale length is relatively large: $\partial/\partial z = O(\epsilon)$. Then, for any scalar S,

$$\nabla S = \nabla_\perp S + O(\epsilon),$$

and (1) implies

$$\mathbf{B}\,.\,\nabla S = \epsilon[\partial S/\partial z - \hat{z}\,.\,\nabla_\perp\psi\times\nabla_\perp S] + O(\epsilon^2). \tag{2}$$

A more compact expression of (2) uses the conventional Poisson bracket

$$[f, g] = \hat{z}\,.\,\nabla_\perp f\times\nabla_\perp g, \tag{3}$$

and the definition

$$\nabla_\parallel f \equiv \partial f/\partial z - [\psi, f]. \tag{4}$$

Thus $\mathbf{B}\,.\,\nabla S = \epsilon\nabla_\parallel S + O(\epsilon^2)$. Note that $\nabla_\parallel$ is a nonlinear parallel gradient, involving the total poloidal flux, ψ, including non-equilibrium contributions.

The RMHD time-scale is measured with respect to $a/\epsilon v_A$, where a is a poloidal length and v_A is the Alfvén speed. It corresponds to shear-Alfvén evolution and yields $\partial/\partial t = O(\epsilon)$, making the transverse electric field predominantly electrostatic:

$$\mathbf{E} = -\epsilon\nabla_\perp\phi + O(\epsilon^2). \tag{5}$$

However, because of the long parallel scale length, the electrostatic and electromagnetic terms make comparable contributions to $E_\parallel$:

$$E_\parallel = -\epsilon^2(\nabla_\parallel\phi + \partial\psi/\partial t) + O(\epsilon^3). \tag{6}$$

Equation (5) shows that the perpendicular drift velocity,

$$\mathbf{v}_E = B^{-2}\mathbf{E}\times\mathbf{B} = \epsilon\hat{z}\times\nabla_\perp\phi + O(\epsilon^2), \tag{7}$$

is primarily electrostatic, whence

$$\mathbf{v}_E\,.\,\nabla S = \epsilon[\phi, S] + O(\epsilon^2), \tag{8}$$

for an arbitrary function S. The ϵ factors in (5)–(8) reflect the relatively slow (on the scale of compressional Alfvén waves) motions of interest.

Like RMHD and CHM, the inclusive system assumes weak variation of the plasma density, n. That is, $n \approx n_c$, where n_c is a temporal and spatial constant. The field variable χ measures the departure of n from n_c; it is treated linearly in the sense that χ^2 and higher-order terms are neglected in the equations of motion. A prominent example of such linearization concerns the adiabatic or Maxwell–Boltzmann limit,

$$\ln n = e\Phi/T_e, \tag{9}$$

where e is the electronic charge, Φ is the unnormalized electrostatic potential, and T_e is the electron temperature measured in units of energy. The relation

$$\alpha\chi = \phi, \tag{10}$$

where α is a constant, corresponds to the linearization of (9) for $n \approx n_c$. Because α measures the (squared) ratio of ion-acoustic gyroradius to transverse scale length, we call it the scale-length parameter. (By 'ion-acoustic gyroradius' we refer to the thermal Larmor radius of ions at the electron temperature.)

Equation (9) or (10) corresponds physically to the instantaneous equilibration of electrons with a slowly evolving potential; this limit is interesting because, while assumed to hold universally in CHM, it is entirely inaccessible to RMHD. The system presently being described is inclusive essentially because it neither forbids, nor insists upon, equation (10).

Finally we introduce the abbreviations,

$$J = \nabla_\perp^2 \psi, \quad U = \nabla_\perp^2 \phi. \tag{11}$$

It is clear from (1) that J measures the parallel current density, $J \propto -\hat{z}.\nabla \times \mathbf{B}$. Similarly, (7) shows that $U \propto \hat{z}.\nabla \times \mathbf{v}_E$ measures the parallel fluid vorticity.

2.2. *Dynamical equations*

The inclusive system consists of three equations: a shear-Alfvén law,

$$\partial U/\partial t + [\phi, U] + \nabla_\parallel J = 0; \tag{12}$$

a generalized Ohm's law,

$$\partial \psi/\partial t + \nabla_\parallel \phi = \eta J + \alpha \nabla_\parallel \chi; \tag{13}$$

and an electron conservation law,

$$\partial \chi/\partial t + [\phi, \chi] + \nabla_\parallel J = 0. \tag{14}$$

Equation (12) describes vorticity evolution in a manner identical to RMHD; it is derived by neglecting $O(\epsilon^3)$ terms in the parallel curl of the magnetohydrodynamical equation of motion. Curvature terms are absent because the plasma pressure is assumed to be $O(\epsilon^2)$.

Equation (13) equates the parallel electric field (recall (6)) to the sum of the Ohmic parallel current and the parallel pressure gradient ('Hall term'). Because the model is isothermal, the pressure gradient is proportional to $\nabla\chi$. The parameter η is a normalized measure of collisional resistivity. Most of our subsequent analysis is restricted to dissipationless evolution: η will be neglected.

The second term in (14) represents perpendicular advection of electrons, as can be seen from (8). The third term gives the divergence of electron parallel flow, which is proportional to the current because ion parallel flow is presumed small:

$$\mathbf{B}.\mathbf{V}_i \ll \mathbf{B}.\mathbf{V}_e. \tag{15}$$

The divergence of the perpendicular electron flow is of higher order in ϵ.

We next briefly review the relation between (12)–(14) and other nonlinear systems, following Hazeltine (1983).

Low-beta RMHD is recovered from the inclusive system by neglecting α in (13), as is appropriate for perturbations with scale lengths much larger than the ion-acoustic gyroradius. Because χ is decoupled in this limit, low-beta RMHD involves only two coupled fields. This system is commonly used to simulate nonlinear kink and tearing evolution in tokamaks.

To obtain CHM, one assumes the electrons to be adiabatic,

$$\alpha\chi = \phi, \tag{16}$$

as in (10). Then (14) implies $\nabla_{\parallel} J = -\partial\chi/\partial t$, and (12) becomes

$$\partial U/\partial t + [\phi, U] = \alpha^{-1}\partial\phi/\partial t, \tag{17}$$

which is the CHM equation for the electrostatic field.

Equilibrium magnetic shear enters the inclusive system through the parallel gradient terms; it is evidently missing in the CHM case, equation (17). Meiss & Horton (1983) adopt a different generalization of (17), which is electrostatic in the sense that ψ is presumed given. They retain the effects of shear, even in the adiabatic limit, by including ion parallel flow terms, unlike our equation (15).

Other electromagnetic models, differing from the present model by the inclusion of ion parallel dynamics and various finite gyroradius effects, have also been derived (Drake & Antonsen 1984; Hazeltine *et al.* 1984*a*); such descriptions necessarily involve additional field variables.

3. Hamiltonian formulation

3.1. *Generalized Poisson brackets*

In this section we present a Poisson bracket acting on functionals of the fields that allows the dynamical equations to be expressed in Hamiltonian form. Such a description provides access to the theoretical technology of Hamiltonian dynamics: variational principles, mapping and transformation theory, and so on (Morrison 1982; Marsden & Morrison 1984). The Poisson bracket is especially valuable in the study of conservation laws, or integral invariants (Morrison & Hazeltine 1984). A certain class of such invariants, described below, plays a critical role in treatments of linear and nonlinear stability (Holm *et al.* 1985; Hazeltine *et al.* 1984*b*).

For notational simplicity we represent the three independent field variables by a vector, $\boldsymbol{\xi}(x,t)$, with $(\xi_1, \xi_2, \xi_3) = (U, \psi, \chi)$; notice that $U = \nabla_\perp^2 \phi$, rather than ϕ itself, is chosen to measure the electrostatic field. The following argument is also simplified by assuming that $\boldsymbol{\xi}(x,t)$, together with its gradients, vanishes for sufficiently large $|\mathbf{x}|$. In other words, we consider the infinite-space problem, with homogeneous boundary conditions.

Because the ξ_i do not occur in conjugate pairs, the formulation derived here is not canonical. Instead it uses the non-canonical Poisson-bracket version of Hamiltonian mechanics: a generalized bracket acting on functionals of the ξ_i will allow the field equations to be expressed as

$$\partial\boldsymbol{\xi}/\partial t = \{\boldsymbol{\xi}, H\}. \tag{18}$$

Here $H(\boldsymbol{\xi})$ is the Hamiltonian functional, whose explicit form is given below, and the bracket, $\{\,,\,\}$, satisfies four requirements:

(i) it is linear in both its arguments,

$$\{E, F+G\} = \{E,F\} + \{E,G\},$$

etc.;

(ii) it is antisymmetric,

$$\{F, G\} = -\{G, F\}; \tag{19}$$

(iii) it is a derivation, in the sense that

$$\{E, FG\} = F\{E, G\} + \{E, F\} G; \tag{20}$$

(iv) it satisfies the Jacobi identity,

$$\{E, \{F, G\}\} + \{F, \{G, E\}\} + \{G, \{E, F\}\} = 0. \tag{21}$$

Here, of course, E, F and G are arbitrary functionals.

The bracket $\{\ ,\ \}$ will be called the 'outer' bracket, distinguishing it from the 'inner' bracket, $[\ ,\ ]$, of (3), which acts on the ξ_i. (The reason for this nomenclature will become apparent presently.) As suggested by properties (i)–(iv), the outer bracket is a natural extension of the inner one, from the space of functions to that functionals.

Before making the outer bracket explicit, we introduce two notational conventions. Integrals over the infinite spatial domain are abbreviated by defining

$$\langle f \rangle \equiv \int d\mathbf{x} f.$$

Then the functional derivative, $\delta F/\delta \xi_i$, of any functional $F(\xi_i)$ is given by

$$dF(\xi_i + \epsilon w)/d\epsilon|_{\epsilon=0} \equiv \langle w\, \delta F/\delta \xi_i \rangle. \tag{22}$$

We frequently abbreviate this derivative with a subscript, e.g. $F_U \equiv \delta F/\delta U$. The function $w(\mathbf{x})$ in (22) is arbitrary (provided surface terms can be neglected).

3.2. *Explicit formulation*

The energy functional for (12)–(14) is

$$H = \tfrac{1}{2}\langle |\nabla_\perp \psi|^2 + |\nabla_\perp \phi|^2 + \alpha \chi^2 \rangle, \tag{23}$$

the sum of magnetic field energy, fluid kinetic energy and fluid internal energy. The easily verified identities (for homogeneous boundary conditions)

$$\langle f[g, h] \rangle = \langle g[h, f] \rangle = \langle h[f, g] \rangle \tag{24}$$

can be used to show that H is constant in the ideal ($\eta = 0$) limit. It is therefore a natural choice for the Hamiltonian functional $H(\xi)$.

It is evident that the functional derivatives, H_ξ, are given by

$$H_U = -\phi, \quad H_\psi = -J, \quad H_\chi = \alpha\chi. \tag{25}$$

The minus signs in the first two of these expressions arise from partial integration. Hence the equations of motion can be written as

$$\left.\begin{aligned} \partial U/\partial t &= [H_U, U] + [H_\psi, \psi] - \partial J/\partial z, \\ \partial \chi/\partial t &= [H_U, \chi] + [H_\psi, \psi] - \partial J/\partial z, \\ \partial \psi/\partial t &= [H_U, \psi] + [H_\chi, \psi] - \alpha\, \partial \chi/\partial z. \end{aligned}\right\} \tag{26}$$

We obtain the outer bracket from (26), following the procedure of Morrison & Hazeltine (1984). It is convenient to begin with the two-dimensional (axisymmetric or helically symmetric) case, in which z-derivatives can be presumed to vanish. Then we seek an outer bracket of the generic form.

$$\{F, G\} = \langle \alpha_{ijk}\, \xi_k [F_{\xi_i}, G_{\xi_j}] \rangle, \tag{27}$$

where the α_{ijk} are constants, and a sum over repeated indices is implied. The bracket form of (27) is common for equations that describe two-dimensional continuous media in terms of Eulerian variables. Next, we examine the case $F = \boldsymbol{\xi}$ and $G = H$, and determine the α's by requiring that (18) and (27) reproduce (26). The resulting outer bracket is given by

$$\{F, G\}_2 = \langle U[F_U, G_U] \rangle + \langle \chi[F_\chi, G_\chi] \rangle + \langle \psi([F_\psi, G_U + G_\chi] - [G_\psi, F_U + F_\chi]) \rangle + \langle \chi([F_\chi, G_U] - [G_\chi, F_U]) \rangle, \tag{28}$$

where the subscript on the left-hand side refers to the dimensionality.

This bracket is manifestly linear in F and G, and antisymmetric in the sense of (19). By using the corresponding properties of the inner bracket, one can straightforwardly show that $\{\,,\}$ also satisfies (20) and (21). Verification of the Jacobi identity is similar to previous analysis (Morrison & Hazeltine 1984), so we omit the details. In fact, the quantities $\alpha_{ijk}[\xi_i, \xi_k]$ are the components of a Lie product acting on n-tuples of functions. Here the Lie algebra is an extension of that associated with canonical transformations, which has the product $[\,,]$.

Thus (23) and (28) provide a Hamiltonian formulation of the two-dimensional equations of motion.

The generalization to three dimensions can also be obtained by following previous (Morrison & Hazeltine 1984) procedures. One finds that the outer bracket,

$$\{F, G\} = \{F, G\}_2 + \langle F_U\, \partial G_\psi/\partial z \rangle - \langle G_U\, \partial F_\psi/\partial z \rangle + \langle F_\psi\, \partial G_\chi/\partial z \rangle - \langle G_\psi\, \partial F_\chi/\partial z \rangle + \langle F_\chi\, \partial G_U/\partial z \rangle - \langle G_\chi\, \partial F_U/\partial z \rangle, \tag{29}$$

reproduces the three-dimensional equations of motion, while satisfying (19)–(21).

3.3. *Casimir invariants*

The inclusive system possesses a variety of integral invariants. For example, it can be seen that any integrated linear combination of the basic fields, with $a_i =$ constant,

$$L \equiv \langle a_i\, \xi_i \rangle, \tag{30}$$

is invariant, $\partial L/\partial t = 0$. Another obvious example is the energy, H. One can demonstrate such conservation laws by direct manipulation of the original system; thus energy conservation is verified by integrating the sum of (12) multiplied by ϕ, (13) multiplied by J, and (14) multiplied by $-\alpha\chi$.

Generalized Poisson brackets clarify the study of integral invariants for several reasons. They often simplify the proof that a given integral is constant; thus the constancy of H follows immediately from the Hamiltonian equations (18) and the fact that $\{H, H\} \equiv 0$, i.e. from the antisymmetry of the bracket. Brackets also allow the systematic derivation of constants of motion (Morrison & Hazeltine 1984). Perhaps most importantly, a Poisson bracket formulation permits one to distinguish an especially useful class of integral invariants: the Casimir invariants (Sudarshan & Mukunda 1974; Marsden & Morrison 1984) or 'Casimirs'.

The defining property of a Casimir is easily understood. While any invariant, A, must 'commute' with the Hamiltonian, $\{A, H\} = 0$, the Casimirs are those invariants that commute with all functionals of the dynamical variables

$$\{C, F\} = 0, \quad \text{for all } F(\xi_i). \tag{31}$$

Evidently the invariant H is not a Casimir. On the other hand, since

$$\delta L / \delta \xi_i = a_i = \text{constant}$$

it is easily seen from (29) that L in (30) is a Casimir invariant.

Casimir invariants which (unlike L) depend nonlinearly on the field variables play a crucial role in nonlinear stability investigations, as shown in the following section. Here we display three nonlinear Casimirs for the two-dimensional bracket, (28):

$$C_1 = \langle F(\psi) \rangle, \tag{32}$$

$$C_2 = \langle \chi G(\psi) \rangle, \tag{33}$$

$$C_3 = \langle K(U - \chi) \rangle. \tag{34}$$

Here F, G and K are arbitrary functions of their arguments. Proof that the C_i satisfy (31) is straightforward and omitted.

4. Two-dimensional equilibria and solitary wave equations

4.1. *Generalized potential*

This section is devoted to studying time-independent solutions of the nonlinear field equations, or generalized equilibria. We restrict attention to the two-dimensional system, with helical or axial symmetry. Equation (28) provides the relevant bracket.

The equilibrium field variables, denoted by ξ_e, evidently satisfy

$$\{\xi_e, H(\xi_e)\}_2 = 0. \tag{35}$$

From the form of (28) it is clear that one class of equilibria is described by extrema of H:

$$H_\xi(\xi_e) = 0. \tag{36}$$

But this is far from a general description of equilibrium; in fact, solutions to (36) are usually too restricted to be interesting. For a general description we note that one can add to H, in (35), any combination of the Casimir invariants, C_i, since the C_i generate null equations of motion. Because the C_i in (32)–(34) already involve arbitrary nonlinear functions, it suffices to consider a simple linear combination:

$$I(\xi) \equiv H(\xi) + \sum_i C_i(\xi). \tag{37}$$

The functional I will be called the generalized potential. Equilibria of interest satisfy

$$I_\xi(\xi_e) = 0. \tag{38}$$

RMHD applications of the generalized potential are considered by Hazeltine *et al.* (1984*a*) and Holm *et al.* (1985). The main points are that (i) requiring definiteness in sign of the second variation of $I_{\xi\xi}(\xi_e)$ yields criteria sufficient for linear stability; (ii) in many cases it is possible to show that I is a convex Lyapunov functional of ξ, and thus to investigate nonlinear stability, considering finite departures from equilibrium. Hence, both the linear stability analysis and the nonlinear stability method rely on the generalized potential. Stability of the reduced inclusive system's equilibria will be the subject of future work; here we restrict attention to the equilibrium problem itself.

4.2. *Equilibrium equations*

After substituting (23) and (32)–(34) into (37), one can straightforwardly compute the first variation of I:

$$\delta I = \langle \delta\psi(-J+F_\psi+\chi G_\psi)\rangle + \langle \delta\chi(\alpha\chi+G+K_\chi)\rangle + \langle \delta\phi(-U+\nabla_\perp^2 K_U)\rangle. \tag{39}$$

Thus the equilibrium fields must satisfy

$$J = F'(\psi)+\chi G'(\psi), \tag{40}$$

$$\alpha\chi+G(\psi) = K'(U-\chi), \tag{41}$$

$$\phi = K'(U-\chi), \tag{42}$$

with $U = \nabla_\perp^2 \phi$ and $J = \nabla_\perp^2 \psi$. The primes denote differentiation with respect to argument (so that $K_\chi = -K'$, for example); the e subscript is suppressed.

Note that (42) omits a harmonic term, ϕ_0, where $\nabla_\perp^2 \phi_0 = 0$. Because such zero-vorticity contributions to ϕ would enter (42) alone, their omission is not serious. On the other hand, if the Casimir terms were omitted, using (36) instead of (37), the only resulting equilibria would correspond to $\phi = \phi_0$. In fact the equilibria described by (36) lack both vorticity and parallel current.

It is not hard to verify that (40)–(42) indeed satisfy the time-independent versions of (12)–(14). Thus we have a variational principle for equilibria that in addition to poloidal flux involve poloidal flow and density variation. Equations (40)–(42) taken together can be viewed as a generalization of the large aspect-ratio Grad–Shafranov equation.

4.3. *Conventional limits*

Since (41) and (42) imply

$$\alpha\chi-\phi = -G(\psi), \tag{43}$$

it is clear from (16) that the CHM limit corresponds to $G = 0$. This choice, together with the assumption that the function K' is invertible, quickly yields the conventional description of drift-solitary waves (Meiss & Horton 1983):

$$\nabla_\perp^2\phi-\phi/\alpha = M(\phi), \tag{44}$$

where the function M is defined by

$$M(K'(f)) = f,$$

for any f. Because K is in general unspecified, it is appropriate to consider M to be an arbitrary function of its argument.

The RMHD limit has $\alpha = 0$ and therefore, from (43), $G(\psi) = \phi$. Thus in this case the electrostatic potential is constant on flux surfaces. It follows from the $\alpha = 0$ version of (41) that

$$\chi = U - M(G(\psi)),$$

and (40) becomes

$$\nabla_\perp^2 \psi + G'(\psi)\, \nabla_\perp^2 \phi = F'(\psi). \tag{45}$$

Here we have omitted a term involving MG' because $G'(\psi)$ is already arbitrary. Equation (45) is easily recognized as a generalized, reduced Grad–Shafranov equation; it differs from the conventional (Strauss 1977) ($G = 0$) form in allowing for equilibrium perpendicular flow.

4.4. *Electromagnetic solitary wave equations*

Returning to the general case, we find it instructive to express (43) in terms of the $\mathbf{E} \times \mathbf{B}$ velocity. From (7),

$$\mathbf{v}_E = \mathbf{v}_* - G'\mathbf{B}_p, \tag{46}$$

where $\mathbf{v}_* \equiv \alpha\hat{z} \times \nabla_\perp \chi$ and $\mathbf{B}_p \equiv -\hat{z} \times \nabla_\perp \psi$ measures the poloidal magnetic field. The case of vanishing $\mathbf{v}_*$ corresponds to a stationary, nonlinear Alfvén wave (Morrison & Hazeltine 1984); it has no parallel electric field because (2) implies that $E_\parallel \propto [\psi, \phi]$ in the two-dimensional case. Thus $\mathbf{v}_*$ measures the departure of the fluid velocity from that of an Alfvén wave. In the simplest case, $\chi = \chi(\psi)$ is a flux function, so that any non-Alfvénic flow has $E_\parallel = 0$ and the fluid remains frozen to the magnetic field. More generally, however, $\alpha\chi \neq 0$ allows a non-dissipative slippage between field lines and fluid.

Electromagnetic solitary waves differ from nonlinear Alfvén waves because they allow for finite $\mathbf{v}_*$. They are described by the coupled equations,

$$\nabla_\perp^2 \phi - M(\phi) = G(\psi), \tag{47}$$

$$\nabla_\perp^2 \psi - F(\psi) = G'(\psi)\, \phi, \tag{48}$$

that result from (40) and (41) after χ is eliminated using (43). The arbitrary functions M, G and F differ slightly from those introduced previously.

Containing three arbitrary functions, (47) and (48) of course possess a wealth of solutions. We briefly consider the simplest class, in which

$$G = \gamma\psi, \quad \phi = \kappa\psi, \tag{49}$$

where γ and κ are constants. This choice evidently corresponds to frozen-field-line flow, with $E_\parallel = 0$. However, the non-Alfvénic velocity v_* is evidently proportional to $\gamma - \kappa$ and need not vanish. Substitution yields the consistency condition

$$\kappa F(\psi) - M(\kappa\psi) = \gamma(1-\kappa^2)\, \psi, \tag{50}$$

which is only mildly restrictive. Equation (50) makes (47) and (48) equivalent; in fact they both assume the same form as (44), whose explicit solitary-wave solutions have been examined previously (Meiss & Horton 1983). Thus (49) provides a simple and natural electromagnetic generalization of drift-solitary waves. Of course it is far from the most general solution to (40)–(42).

5. Summary

A previously derived nonlinear system, which includes RMHD and CHM as limiting cases, has been shown to possess a Hamiltonian structure in terms of generalized Poisson brackets. The bracket is given explicitly by (28) in the two-dimensional case and by (39) for three spatial dimensions. These results generalize an earlier Hamiltonian formulation of RMHD (Morrison & Hazeltine 1984) by allowing for parallel electron pressure gradients, or electron adiabaticity. (We remark that Strauss (1976, 1977) considers both low- and high-beta tokamak orderings, while the present work is restricted to low beta.)

Consideration of the bracket structure in the two-dimensional case has allowed us to construct three nonlinear Casimir invariants: exact constants of the motion that Poisson-commute with all functionals. Although related to invariants of RMHD, the Casimirs defined by (32)–(34) are new.

After using the Hamiltonian and Casimirs to construct a generalized potential, we have studied exact nonlinear, stationary solutions to the inclusive system. These two-dimensional equilibria extend previous results of electrostatic theory and RMHD; their general form is expressed in (47) and (48). Special cases of the generalized equilibria include a Grad–Shafranov equation that allows for equilibrium flow, the conventional description of electrostatic drift-solitary waves, and a new solitary wave structure. The latter is similar to the drift-solitary wave, but fully electromagnetic.

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