Chapter 4

ELECTROMAGNETISM A

4-2 Fields of a moving charge (Feynman's Equation)

In this Section we'll prove an important equation that Feynman gives in his Lectures without proof. In his own words:

■ When we studied light, we began by writing down equations for the electric and magnetic fields produced by a charge which moves in any arbitrary way. Those equations were [1]

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{e}_{r'}}{r'^2} + \frac{r'}{c}\frac{d}{dt}\left(\frac{\mathbf{e}_{r'}}{r'^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\mathbf{e}_{r'}\right] \tag{21.1}$$

$$c\mathbf{B} = \mathbf{e}_{r'} \times \mathbf{E}$$

If a charge moves in an arbitrary way, the electric field we would find *now* at some point depends only on the position and motion of the charge not now, but at an *earlier* time-at an instant which is earlier by the time it would take light, going at the speed c, to travel the distance r' from the charge to the field point. In other words, if we want the electric field at point (1) at the time t, we must calculate the location $(2')$ of the charge and its motion at the time $(t - r'/c)$, where r' is the distance to the point (1) from the

[1] see [15], *The Feynman Lectures on Physics, Volume II-Mainly Electromagnetism and Matter*, Chapter 21, equation (21.1)

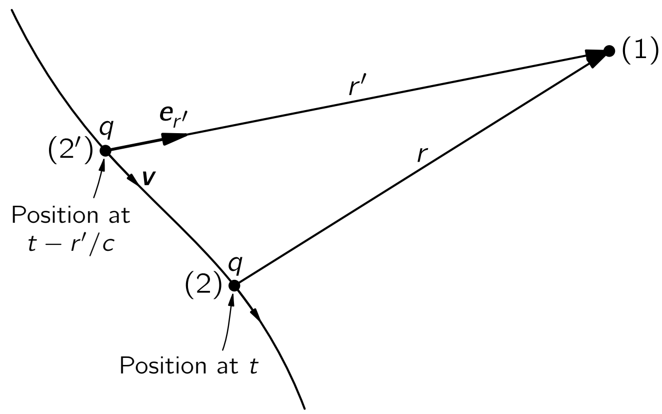

Fig. 21-1. The fields at (1) at the time t depend on the position $(2')$ occupied by the charge q at the time $(t - r'/c)$.

position of the charge $(2')$ at the time $(t - r'/c)$. The prime is to remind you that r' is the so-called retarded distance from the point $(2')$ to the point (1), and not the actual distance between point (2), the position of the charge at the time t, and the field point (1)(see Fig. 21-1) ■

This Section is split in Subsections. The main job is done in the first Subsection, while in the Subsections that follow proofs or explanations are given in detail for the calculation jumps in the first one, in order to have an uninterrupted continuity in the main job.

4-2.1 The scalar $\phi(\mathbf{x}, t)$ and vector $\mathbf{A}(\mathbf{x}, t)$ potentials

Since

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{4-2.1}$$

and

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{4-2.2}$$

we start with the retarded potentials, scalar and vector :

$$\phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_o} \iiint \frac{\rho\left(\mathbf{x}', t - \dfrac{\|\mathbf{x}' - \mathbf{x}\|}{c}\right)}{\|\mathbf{x}' - \mathbf{x}\|} d^3\mathbf{x}' , \quad \text{scalar potential} \tag{4-2.3}$$

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_o}{4\pi} \iiint \frac{\mathbf{j}\left(\mathbf{x}', t - \dfrac{\|\mathbf{x}' - \mathbf{x}\|}{c}\right)}{\|\mathbf{x}' - \mathbf{x}\|} d^3\mathbf{x}' , \quad \text{vector potential} \tag{4-2.4}$$

By these two equations we'll find the potentials at field point $\mathbf{x} = (x_1, x_2, x_3)$ and time t, taking into account the contributions of charges and their currents from all points $\mathbf{x}' = (x_1', x_2', x_3')$ at the retarded time

$$t' = t - \frac{\|\mathbf{x}' - \mathbf{x}\|}{c} \tag{4-2.5}$$

since a time period $\Delta t = \|\mathbf{x}' - \mathbf{x}\|/c$ is needed for this contribution to travel with the speed of light c from $\mathbf{x}'$ to $\mathbf{x}$.

Note that the retarded time t' is a function of $\mathbf{x}, \mathbf{x}', t$

$$\begin{aligned} t' = t'(\mathbf{x}, \mathbf{x}', t) &= t - \frac{\|\mathbf{x}' - \mathbf{x}\|}{c} \\ &= t - \frac{\sqrt{(x_1' - x_1)^2 + (x_2' - x_2)^2 + (x_3' - x_3)^2}}{c} \\ &= t - \frac{[(x_\sigma' - x_\sigma)(x_\sigma' - x_\sigma)]^{\frac{1}{2}}}{c}, \quad \text{Einstein's convention on } \sigma \end{aligned} \tag{4-2.6}$$

Let a point charge q moving with position vector $\boldsymbol{\xi}(t)$, as in Fig.4.1. [2] We suppose that

$$\left\| \frac{d\boldsymbol{\xi}(t)}{dt} \right\| < c \tag{4-2.7}$$

The volume charge density would be expressed via Dirac $\delta-$function [3]

$$\rho(\mathbf{x}, t) = q \cdot \delta^3\big(\mathbf{x} - \boldsymbol{\xi}(t)\big) \tag{4-2.8}$$

as well as the charge current density

$$\mathbf{j}(\mathbf{x}, t) = q \cdot \delta^3\big(\mathbf{x} - \boldsymbol{\xi}(t)\big) \cdot \frac{d\boldsymbol{\xi}(t)}{dt} = q \cdot \delta^3\big(\mathbf{x} - \boldsymbol{\xi}(t)\big) \cdot \mathbf{v}(t) \tag{4-2.9}$$

where

$$\mathbf{v}(t) = \big(v_1(t), v_2(t), v_3(t)\big) = \left(\frac{d\xi_1(t)}{dt}, \frac{d\xi_2(t)}{dt}, \frac{d\xi_3(t)}{dt}\right) = \frac{d\boldsymbol{\xi}(t)}{dt} \tag{4-2.10}$$

the velocity of the charge. The potentials have the following expressions

$$\phi(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_o} \iiint \frac{\delta^3\left(\mathbf{x}' - \boldsymbol{\xi}\left(t - \frac{\|\mathbf{x}' - \mathbf{x}\|}{c}\right)\right)}{\|\mathbf{x}' - \mathbf{x}\|} d^3\mathbf{x}' \tag{4-2.11}$$

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_o q}{4\pi} \iiint \frac{\delta^3\left(\mathbf{x}' - \boldsymbol{\xi}\left(t - \frac{\|\mathbf{x}' - \mathbf{x}\|}{c}\right)\right) \cdot \mathbf{v}\left(t - \frac{\|\mathbf{x}' - \mathbf{x}\|}{c}\right)}{\|\mathbf{x}' - \mathbf{x}\|} d^3\mathbf{x}' \tag{4-2.12}$$

[2] see 3D version of Figure 4.1 in Chapter I, Figure I.7

[3] if $\mathbf{r} = (x, y, z)$ the 3-dimensional $\delta-$function $\delta^3(\mathbf{x})$ is the product of the three 1-dimensional $\delta-$functions

$$\delta^3(\mathbf{x}) = \delta(x)\,\delta(y)\,\delta(z)$$

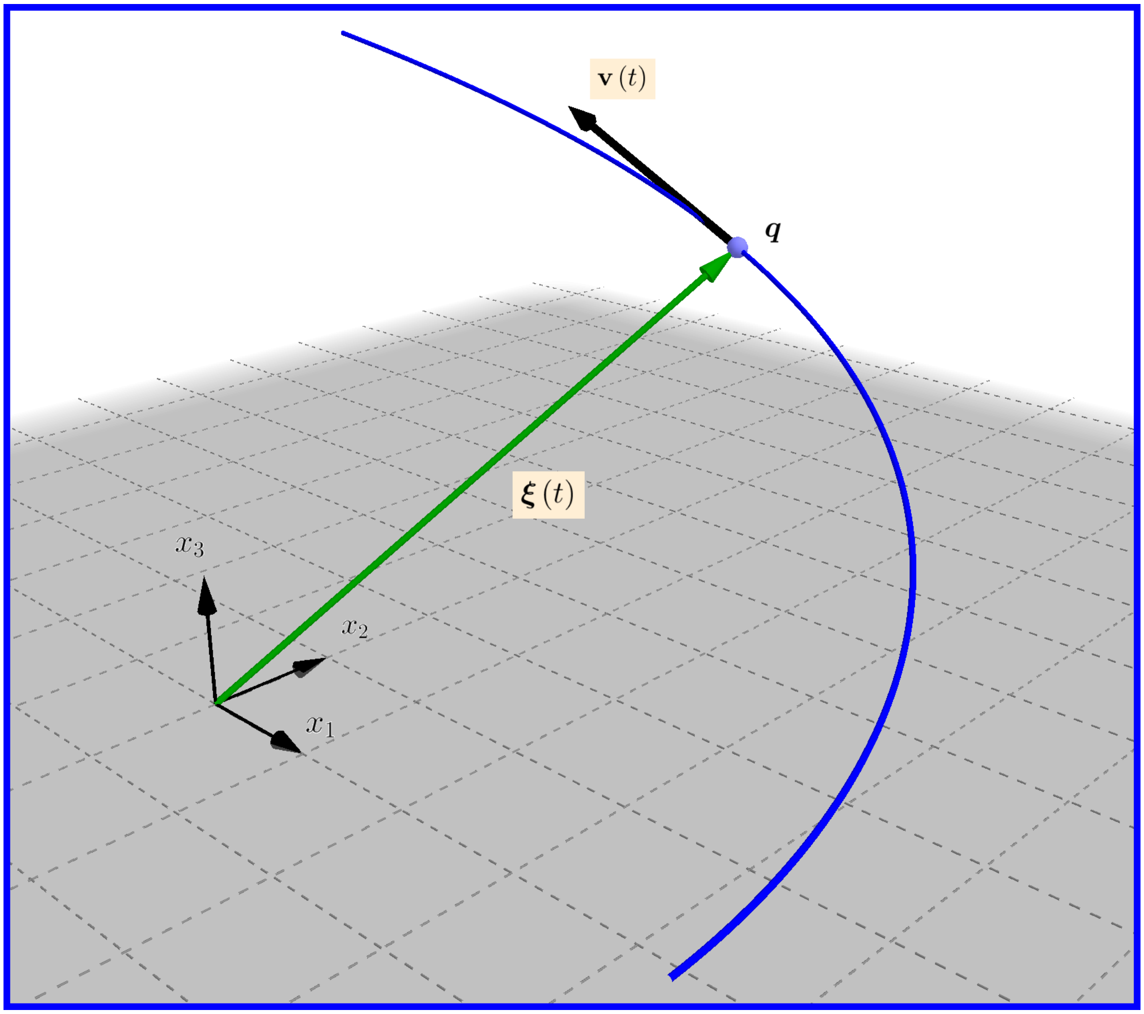

Figure 4.1: Charge q moving in any arbitrary way $\boldsymbol{\xi}(t)$.

As explained in Subsection 4-2.3, we proceed to the following variable change from $\mathbf{x}'$ to $\mathbf{u}$

$$\mathbf{u} = \mathbf{x}' - \boldsymbol{\xi}\left(t - \frac{\|\mathbf{x}' - \mathbf{x}\|}{c}\right) = \mathbf{F}(\mathbf{x}') \tag{4-2.13}$$

as in equation (4-2.63) in Subsection 4-2.3.

In Subsection 4-2.4 we prove that the function $\mathbf{F}(\mathbf{x}')$ is invertible and so

$$d^3\mathbf{x}' = \frac{\partial(x_1', x_2', x_3')}{\partial(u_1, u_2, u_3)} d^3\mathbf{u} = \left[\frac{\partial(u_1, u_2, u_3)}{\partial(x_1', x_2', x_3')}\right]^{-1} d^3\mathbf{u} \tag{4-2.14}$$

as proved in Subsection 4-2.3, see equation (4-2.91). We'll use the relation containing the Jacobian $\dfrac{\partial(u_1, u_2, u_3)}{\partial(x_1', x_2', x_3')}$ for convenience as we'll shall see in the following calculations.

Now, in equations (4-2.11) and (4-2.12) we make the following substitutions

$$\mathbf{F}\left(\mathbf{x}'\right)=\mathbf{x}'-\boldsymbol{\xi}\left(t-\frac{\|\mathbf{x}'-\mathbf{x}\|}{c}\right)\longrightarrow\mathbf{u},\quad \mathbf{x}'\longrightarrow\mathbf{F}^{-1}\left(\mathbf{u}\right),\quad d^3\mathbf{x}'\longrightarrow\left[\frac{\partial\left(u_1,u_2,u_3\right)}{\partial\left(x_1',x_2',x_3'\right)}\right]^{-1}d^3\mathbf{u} \tag{4-2.15}$$

as in equation (4-2.79) in Subsection 4-2.3.

$$\phi\left(\mathbf{x},t\right)=\frac{q}{4\pi\epsilon_o}\iiint\frac{\delta^3\left(\mathbf{u}\right)}{\|\mathbf{F}^{-1}\left(\mathbf{u}\right)-\mathbf{x}\|\cdot\dfrac{\partial\left(u_1,u_2,u_3\right)}{\partial\left(x_1',x_2',x_3'\right)}}d^3\mathbf{u} \tag{4-2.16}$$

$$\mathbf{A}\left(\mathbf{x},t\right)=\frac{\mu_o q}{4\pi}\iiint\frac{\delta^3\left(\mathbf{u}\right)\cdot\mathbf{v}\left(t-\dfrac{\|\mathbf{F}^{-1}\left(\mathbf{u}\right)-\mathbf{x}\|}{c}\right)}{\|\mathbf{F}^{-1}\left(\mathbf{u}\right)-\mathbf{x}\|\cdot\dfrac{\partial\left(u_1,u_2,u_3\right)}{\partial\left(x_1',x_2',x_3'\right)}}d^3\mathbf{u} \tag{4-2.17}$$

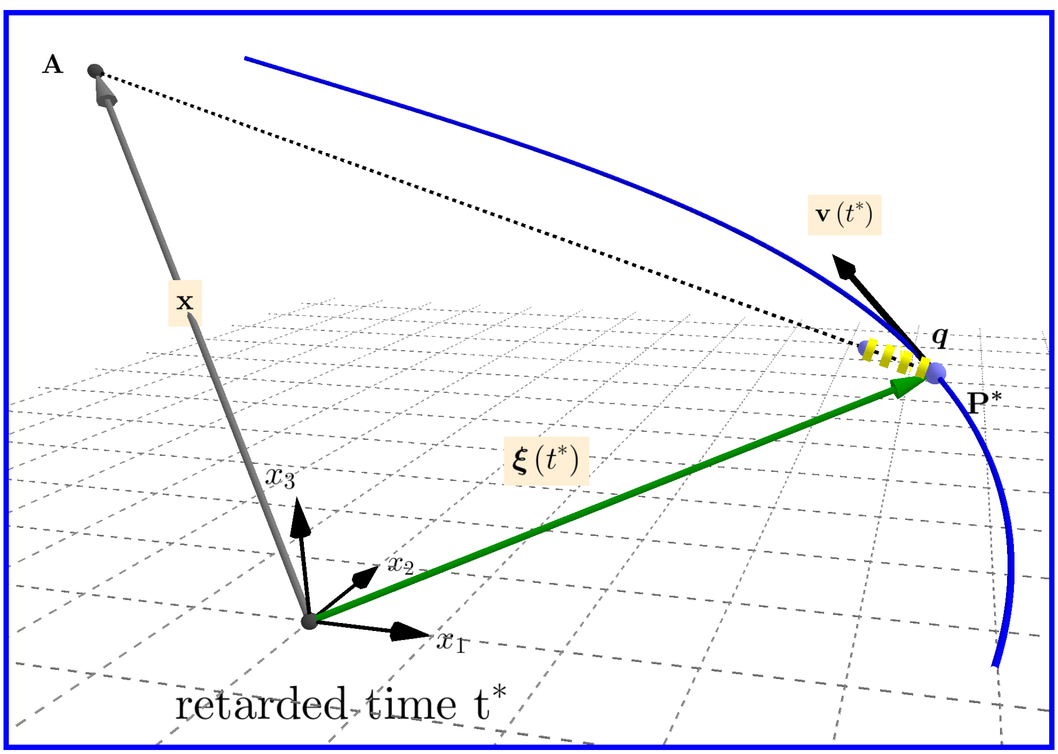

Figure 4.2: Charge q at point P* emits a light beam at time t^* towards field point A.

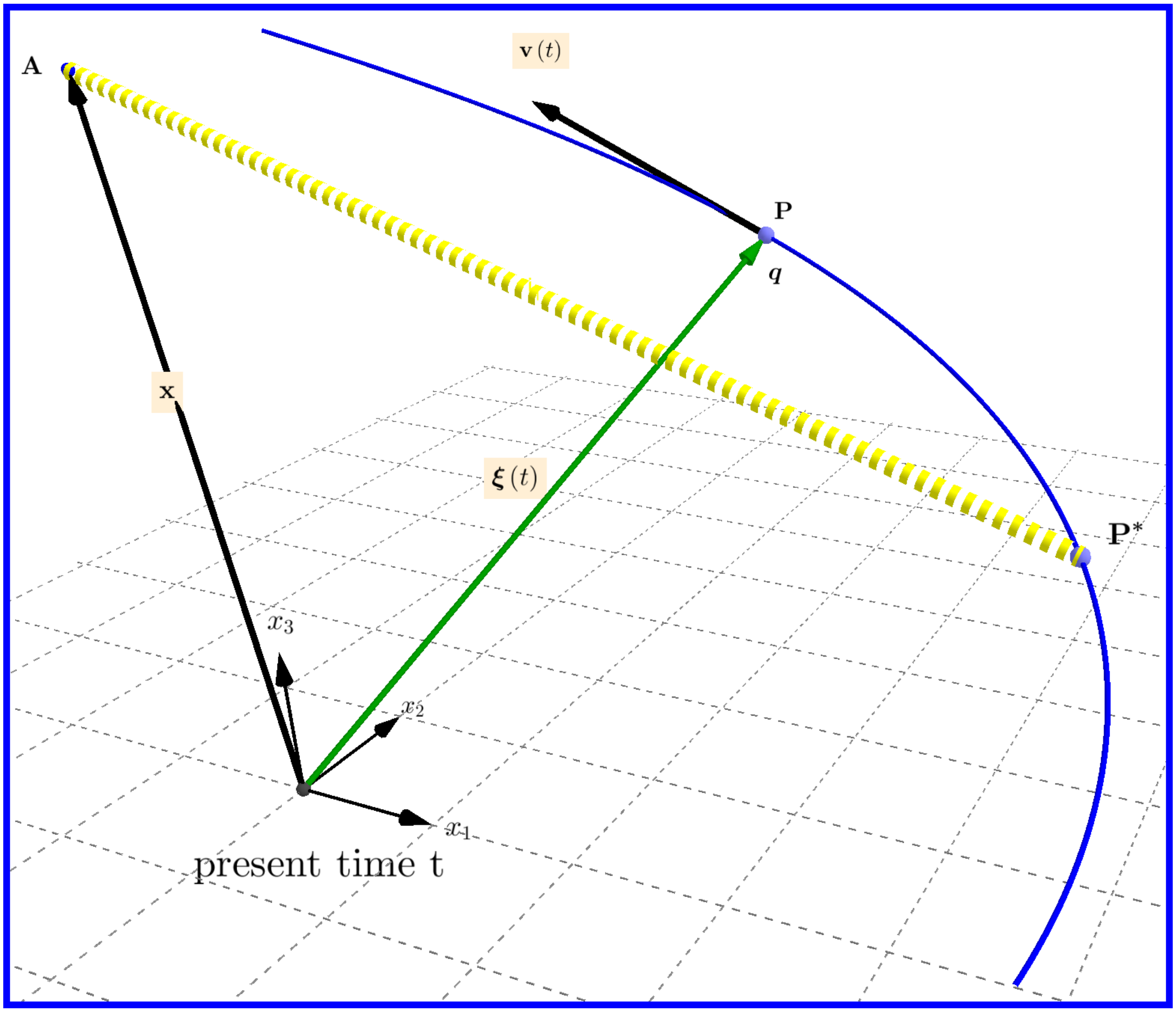

Figure 4.3: Charge q is at point P at time t when its emitted (from P* at time t^*) light beam arrives at field point A.

So

$$\phi(\mathbf{x},t) = \frac{q}{4\pi\epsilon_o}\frac{1}{\left\|\mathbf{F}^{-1}(\mathbf{0}) - \mathbf{x}\right\| \cdot \left[\dfrac{\partial(u_1,u_2,u_3)}{\partial(x_1',x_2',x_3')}\right]_{\mathbf{u}=\mathbf{0}}} \tag{4-2.18}$$

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_o q}{4\pi}\frac{\mathbf{v}\left(t - \dfrac{\left\|\mathbf{F}^{-1}(\mathbf{0}) - \mathbf{x}\right\|}{c}\right)}{\left\|\mathbf{F}^{-1}(\mathbf{0}) - \mathbf{x}\right\| \cdot \left[\dfrac{\partial(u_1,u_2,u_3)}{\partial(x_1',x_2',x_3')}\right]_{\mathbf{u}=\mathbf{0}}} \tag{4-2.19}$$

Now, in order to examine in detail what is behind these two equations we must find what are the quantities $\mathbf{F}^{-1}(\mathbf{0})$, $\left[\dfrac{\partial(u_1,u_2,u_3)}{\partial(x_1',x_2',x_3')}\right]_{\mathbf{u}=\mathbf{0}}$ and if there exists a physical interpretation.

For $\mathbf{F}^{-1}(\mathbf{0})$ we have to say that putting $\mathbf{u}=\mathbf{0}$ in equation (4-2.13) this quantity is a solution with respect to $\mathbf{x}'$ of the equation

$$\mathbf{0}=\mathbf{x}'-\boldsymbol{\xi}\left(t-\frac{\|\mathbf{x}'-\mathbf{x}\|}{c}\right) \tag{4-2.20}$$

So let $\mathbf{x}^*$ a solution of above equation, that is :

$$\mathbf{x}^* \quad \overset{\text{def}}{:} \quad \mathbf{x}^*-\boldsymbol{\xi}\left(t-\frac{\|\mathbf{x}^*-\mathbf{x}\|}{c}\right)=\mathbf{0} \tag{4-2.21}$$

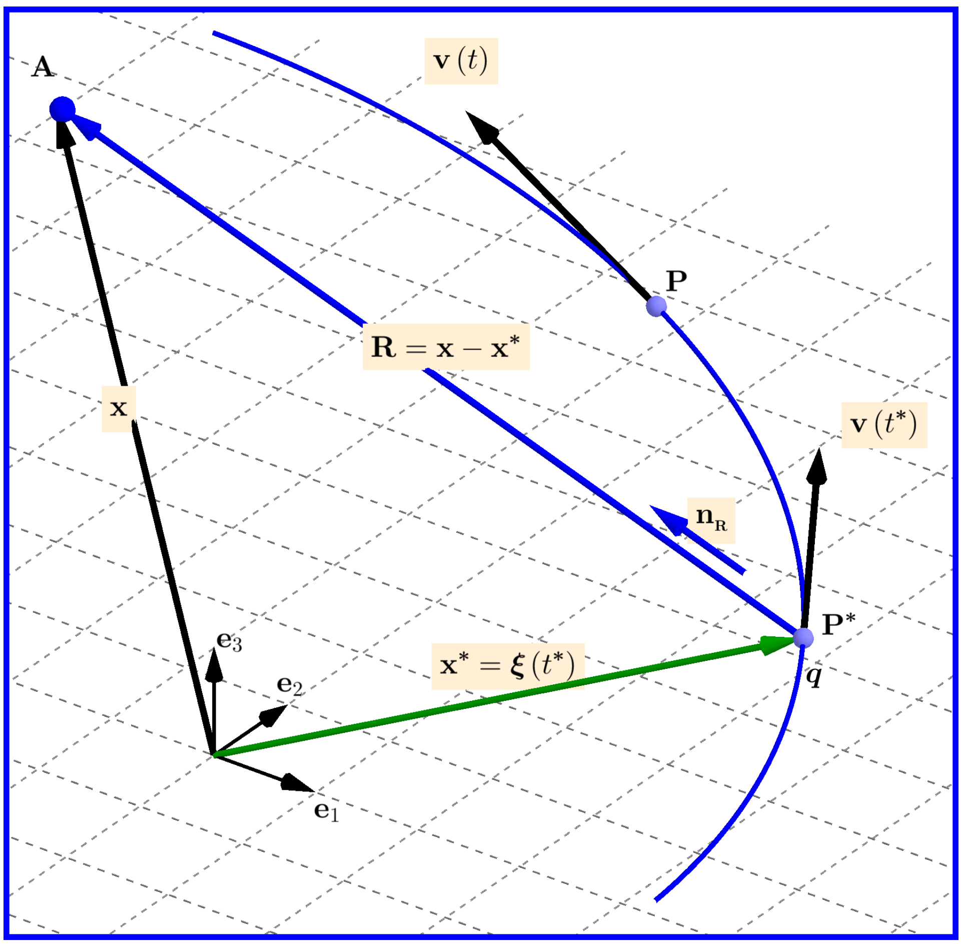

Figure 4.4: The field point A as seen by charge q from the retarded point P^* and time t^* of the later.

In Subsection 4-2.4 we prove not only that there exists a solution $\mathbf{x}^*$ but moreover that this solution is unique. The proof is based on the assumption that the speed of the charged particle is always less than that of light c, see equation (4-2.7). The physical interpretation for $\mathbf{x}^*$ runs as follows :

1. The vector $\mathbf{x}^*$ is the position vector of the charge on its trajectory at a retarded point P^* at a retarded time

$$t^* \overset{\text{def}}{\equiv} t-\frac{\|\mathbf{x}^*-\mathbf{x}\|}{c} \tag{4-2.22}$$

and

$$\mathbf{x}^* = \boldsymbol{\xi}(t^*) = \mathbf{F}^{-1}(\mathbf{0}) \tag{4-2.23}$$

2. If the charge emits a light beam from the retarded point P^* (position vector $\mathbf{x}^*$) and the retarded time t^* towards the field point A (position vector $\mathbf{x}$), see Figure 4.2, [4] then after a time period $\Delta t = t - t^* = \|\mathbf{x}^* - \mathbf{x}\|/c$ the light beam will arrive at point A and the charge q at its present point P at the present time t , see Figure 4.3. [5]

For given equation of motion $\boldsymbol{\xi}(t)$ with $\left\|\dfrac{d\boldsymbol{\xi}(t)}{dt}\right\| < c$ for any t , the retarded position $\mathbf{x}^*$ and retarded time t^* are functions of $\mathbf{x}$ and t .

Now, for the Jacobian the following expression is proved in Subsection 4-2.6, see equation (4-2.119) and definitions (4-2.120) to (4-2.122) repeated here

$$\begin{aligned}\left[\frac{\partial(u_1,u_2,u_3)}{\partial(x_1',x_2',x_3')}\right]_{\mathbf{u}=\mathbf{0}} &= \left[\frac{\partial(u_1,u_2,u_3)}{\partial(x_1',x_2',x_3')}\right]_{\mathbf{x}'=\mathbf{x}^*} = 1 - \frac{\mathbf{v}(t^*)\circ\mathbf{n_R}}{c} \\ &= 1 - \frac{\mathbf{v}\left(t - \dfrac{\|\mathbf{x}-\mathbf{x}^*\|}{c}\right)\circ\left(\dfrac{\mathbf{x}-\mathbf{x}^*}{\|\mathbf{x}-\mathbf{x}^*\|}\right)}{c} \\ &= 1 - \frac{\mathbf{v}\left(t-\dfrac{R}{c}\right)\circ\left(\dfrac{\mathbf{R}}{R}\right)}{c} \overset{\text{def}}{\equiv} \varkappa \end{aligned} \tag{repeat4-2.119}$$

where

$$\mathbf{R} = \mathbf{x} - \mathbf{x}^* = \mathbf{x} - \boldsymbol{\xi}(t^*) \tag{repeat4-2.120}$$

$$R = \|\mathbf{R}\| = \|\mathbf{x} - \mathbf{x}^*\| \tag{repeat4-2.121}$$

$$\mathbf{n_R} = \frac{\mathbf{R}}{\|\mathbf{R}\|} = \frac{\mathbf{x}-\mathbf{x}^*}{\|\mathbf{x}-\mathbf{x}^*\|} \tag{repeat4-2.122}$$

all shown in Figure 4.4. [6]

Using expressions (4-2.23) and (4-2.119) for $\mathbf{F}^{-1}(\mathbf{0})$ and $\left[\dfrac{\partial(u_1,u_2,u_3)}{\partial(x_1',x_2',x_3')}\right]_{\mathbf{u}=\mathbf{0}}$ respectively, equations (4-2.18) and (4-2.19) for the potentials yield

$$\boxed{\phi(\mathbf{x},t) = \frac{q}{4\pi\epsilon_o}\frac{1}{\|\mathbf{x}-\mathbf{x}^*\|\cdot\left[1 - \mathbf{v}\left(t-\dfrac{\|\mathbf{x}-\mathbf{x}^*\|}{c}\right)\circ\left(\dfrac{\mathbf{x}-\mathbf{x}^*}{c\,\|\mathbf{x}-\mathbf{x}^*\|}\right)\right]}} \tag{4-2.24}$$

[4] see 3D version of Figure 4.2 in Appendix I, Figure I.8
[5] see 3D version of Figure 4.3 in Appendix I, Figure I.9
[6] see 3D version of Figure 4.4 in Appendix I, Figure I.10

$$\mathbf{A}\left(\mathbf{x},t\right)=\frac{\mu_{o}q}{4\pi}\frac{\mathbf{v}\left(t-\dfrac{\|\mathbf{x}-\mathbf{x}^{*}\|}{c}\right)}{\|\mathbf{x}-\mathbf{x}^{*}\|\cdot\left[1-\mathbf{v}\left(t-\dfrac{\|\mathbf{x}-\mathbf{x}^{*}\|}{c}\right)\circ\left(\dfrac{\mathbf{x}-\mathbf{x}^{*}}{c\,\|\mathbf{x}-\mathbf{x}^{*}\|}\right)\right]} \tag{4-2.25}$$

If we have in mind that the retarded position is an implicit function of $\mathbf{x},t$, that is $\mathbf{x}^{*}=\mathbf{x}^{*}\left(\mathbf{x},t\right)$, then we can find the field components $\mathbf{E},\mathbf{B}$ from equations (4-2.1),(4-2.2) by differentiations with respect to the components of $\mathbf{x}$ and to time t .

$$\phi\left(\mathbf{x},t\right)=\frac{q}{4\pi\epsilon_{o}}\frac{1}{\left[1-\dfrac{\mathbf{v}\left(t^{*}\right)}{c}\circ\mathbf{n}_{\mathbf{R}}\right]\cdot\|\mathbf{x}-\mathbf{x}^{*}\|}=\frac{q}{4\pi\epsilon_{o}}\frac{1}{\varkappa\cdot R} \tag{4-2.26}$$

$$\mathbf{A}\left(\mathbf{x},t\right)=\frac{\mu_{o}q}{4\pi}\frac{\mathbf{v}\left(t^{*}\right)}{\left[1-\dfrac{\mathbf{v}\left(t^{*}\right)}{c}\circ\mathbf{n}_{\mathbf{R}}\right]\cdot\|\mathbf{x}-\mathbf{x}^{*}\|}=\frac{\mu_{o}q}{4\pi}\frac{\mathbf{v}\left(t^{*}\right)}{\varkappa\cdot R}=\frac{\mathbf{v}\left(t^{*}\right)}{c^{2}}\phi\left(\mathbf{x},t\right) \tag{4-2.27}$$

Note that in equation (4-2.26) the scalar potential $\phi\left(\mathbf{x},t\right)$ seems to be the electrostatic one, not caused by the charge q but by a charge $q/\varkappa$ that is greater than, less than or equal to q depending upon the relation of $\varkappa$ to 1 : $\varkappa<1$, $\varkappa>1$, $\varkappa=1$ respectively. That is if the charge is coming closer, is running away or nothing of these two respectively.

In above two equations $\mathbf{x}^{*}$, t^{*} , $\mathbf{R}$, R , $\mathbf{n}_{\mathbf{R}}$ and $\varkappa$ are all implicit functions of $\mathbf{x},t$:

$$\mathbf{x}^{*}\left(\mathbf{x},t\right),\quad t^{*}\left(\mathbf{x},t\right),\quad \mathbf{R}\left(\mathbf{x},t\right),\quad R\left(\mathbf{x},t\right),\quad \mathbf{n}_{\mathbf{R}}\left(\mathbf{x},t\right),\quad \varkappa\left(\mathbf{x},t\right) \tag{4-2.28}$$

We must have in mind this dependence when we differentiate with respect to t and the components of $\mathbf{x}$.

4-2.2 The electric $\mathbf{E}\left(\mathbf{x},t\right)$ and magnetic $\mathbf{B}\left(\mathbf{x},t\right)$ fields vectors

We'll use the expressions (4-2.26),(4-2.27) for the potentials to find the electric and magnetic field vectors by equations (4-2.1),(4-2.2)

$$\mathbf{E}=-\boldsymbol{\nabla}\phi-\frac{\partial\mathbf{A}}{\partial t} \tag{repeat4-2.1}$$

and

$$\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A} \tag{repeat4-2.2}$$

Note that the Feynman's Lectures equation (21.1), see at the beginning of this Section 4-2, is expressed by a unit vector $\mathbf{e}_{r'}$ and a scalar $r'=\|\mathbf{r}'\|$, as in Figure (21.1) of the Lectures, shown also at the beginning of the aforementioned Section. Comparing this Figure with Figure 4.4 we see that there exists the following correspondence

$$\left(\mathbf{r}'\right)\longrightarrow\left(\mathbf{R}\right),\quad\left(r'=\|\mathbf{r}'\|\right)\longrightarrow\left(R=\|\mathbf{R}\|\right),\quad\left(\mathbf{e}_{r'}=\frac{\mathbf{r}'}{r'}\right)\longrightarrow\left(\mathbf{n}_{\mathbf{R}}=\frac{\mathbf{R}}{R}\right) \tag{4-2.29}$$

and Feynman's equation with hither symbols is

$$\mathbf{E}=\frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{n_R}}{R^2}+\frac{R}{c}\frac{d}{dt}\left(\frac{\mathbf{n_R}}{R^2}\right)+\frac{1}{c^2}\frac{d^2}{dt^2}\mathbf{n_R}\right] \tag{4-2.30}$$

To reach above equation from (4-2.1) by differentiations of the potentials equations (4-2.26),(4-2.27) it would be useful, convenient and overall necessary to express any quantity appeared in these last ones as function of $R, \mathbf{n_R}$ and their partial derivatives with respect to t and the components of $\mathbf{x}$.. There are two quantities we must handle : $\varkappa$ and $\mathbf{v}(t^*)$. But from equation (4-2.119)

$$\varkappa=1-\frac{\mathbf{v}(t^*)}{c}\circ\mathbf{n_R} \tag{4-2.31}$$

So we start with the remaining $\mathbf{v}(t^*)$. Since $\boldsymbol{\xi}(t^*)=\mathbf{x}^*=\mathbf{x}-\mathbf{R}$, see (4-2.120),

$$\frac{\partial\boldsymbol{\xi}(t^*)}{\partial t}=-\frac{\partial\mathbf{R}}{\partial t}\Rightarrow\underbrace{\frac{d\boldsymbol{\xi}(t^*)}{dt^*}}_{\mathbf{v}(t^*)}\frac{\partial t^*}{\partial t}=-\frac{\partial(R\mathbf{n_R})}{\partial t}=-\left(\frac{\partial R}{\partial t}\mathbf{n_R}+R\frac{\partial\mathbf{n_R}}{\partial t}\right)$$

that is

$$\mathbf{v}(t^*)\frac{\partial t^*}{\partial t}=-\frac{\partial\mathbf{R}}{\partial t}=-\left(\frac{\partial R}{\partial t}\mathbf{n_R}+R\frac{\partial\mathbf{n_R}}{\partial t}\right) \tag{4-2.32}$$

From equation (4-2.22)

$$t^*\stackrel{\text{def}}{\equiv}t-\frac{\|\mathbf{x}^*-\mathbf{x}\|}{c}=t-\frac{R}{c} \tag{repeat4-2.22}$$

so

$$\frac{\partial t^*}{\partial t}=1-\frac{\partial R}{c\partial t} \tag{4-2.33}$$

and (4-2.32) yields

$$\mathbf{v}(t^*)=-\frac{\dfrac{\partial\mathbf{R}}{\partial t}}{\left(1-\dfrac{\partial R}{c\partial t}\right)}=-\left[\left\{\frac{\dfrac{\partial R}{\partial t}}{\left(1-\dfrac{\partial R}{c\partial t}\right)}\right\}\mathbf{n_R}+\left\{\frac{R}{\left(1-\dfrac{\partial R}{c\partial t}\right)}\right\}\frac{\partial\mathbf{n_R}}{\partial t}\right] \tag{4-2.34}$$

From above expression and equation (4-2.31)

$$\varkappa=\frac{1}{\left(1-\dfrac{\partial R}{c\partial t}\right)} \tag{4-2.35}$$

since :

$$\mathbf{n_R}\circ\frac{\partial\mathbf{n_R}}{\partial t}=\frac{1}{2}\frac{\partial(\mathbf{n_R}\circ\mathbf{n_R})}{\partial t}=\frac{1}{2}\frac{\partial\|\mathbf{n_R}\|^2}{\partial t}=0 \tag{4-2.36}$$

Replacing expressions (4-2.34),(4-2.35) in equations (4-2.26),(4-2.27) yields the following ones for the potentials as functions of $\mathbf{R}, R, \mathbf{n_R}$ and their partial derivatives with respect to t

$$\phi(\mathbf{x},t) = \frac{q}{4\pi\epsilon_o}\frac{\left(1-\dfrac{\partial R}{c\partial t}\right)}{R} \tag{4-2.37}$$

$$\mathbf{A}(\mathbf{x},t) = -\frac{q}{4\pi\epsilon_o c^2}\frac{1}{R}\frac{\partial \mathbf{R}}{\partial t} = -\frac{q}{4\pi\epsilon_o c^2}\left(\frac{1}{R}\frac{\partial R}{\partial t}\mathbf{n_R}+\frac{\partial \mathbf{n_R}}{\partial t}\right) \tag{4-2.38}$$

We'll try now to find $\boldsymbol{\nabla}\phi$ by differentiation of (4-2.37)

$$-\frac{4\pi\epsilon_o}{q}\boldsymbol{\nabla}\phi(\mathbf{x},t) = -\boldsymbol{\nabla}\left[\frac{\left(1-\dfrac{\partial R}{c\partial t}\right)}{R}\right] = -\left[\left(1-\frac{\partial R}{c\partial t}\right)\boldsymbol{\nabla}\left(\frac{1}{R}\right)+\frac{1}{R}\boldsymbol{\nabla}\left(1-\frac{\partial R}{c\partial t}\right)\right]$$

so

$$-\frac{4\pi\epsilon_o}{q}\boldsymbol{\nabla}\phi(\mathbf{x},t) = \frac{\left(1-\dfrac{\partial R}{c\partial t}\right)}{R^2}\boldsymbol{\nabla}R+\frac{1}{cR}\frac{\partial(\boldsymbol{\nabla}R)}{\partial t} \tag{4-2.39}$$

It's necessary now to handle $\boldsymbol{\nabla}R$

$$\boldsymbol{\nabla}R = \left(\frac{\partial R}{\partial x_1},\frac{\partial R}{\partial x_2},\frac{\partial R}{\partial x_3}\right) \tag{4-2.40}$$

$$\frac{\partial R}{\partial x_j} = \frac{1}{2R}\frac{\partial R^2}{\partial x_j} = \frac{1}{2R}\frac{\partial(\mathbf{R}\circ\mathbf{R})}{\partial x_j} = \frac{\mathbf{R}}{R}\circ\frac{\partial \mathbf{R}}{\partial x_j} = \mathbf{n_R}\circ\frac{\partial \mathbf{R}}{\partial x_j} \tag{4-2.41}$$

Since $\mathbf{R}=\mathbf{x}-\boldsymbol{\xi}(t^*)$, see (4-2.120),

$$\frac{\partial \mathbf{R}}{\partial x_j} = \frac{\partial[\mathbf{x}-\boldsymbol{\xi}(t^*)]}{\partial x_j} = \frac{\partial \mathbf{x}}{\partial x_j}-\frac{\partial \boldsymbol{\xi}(t^*)}{\partial x_j} = \mathbf{e}_j-\frac{d\boldsymbol{\xi}(t^*)}{dt^*}\frac{\partial t^*}{\partial x_j} = \mathbf{e}_j-\mathbf{v}(t^*)\frac{\partial t^*}{\partial x_j} \tag{4-2.42}$$

where

$$\mathbf{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \mathbf{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \tag{4-2.43}$$

the basic vectors of the orthonormal system of coordinates $Ox_1x_2x_3$.

From equation (4-2.22)

$$t^* \overset{\text{def}}{\equiv} t-\frac{\|\mathbf{x}^*-\mathbf{x}\|}{c} = t-\frac{R}{c} \tag{repeat4-2.22}$$

that is

$$\frac{\partial t^*}{\partial x_j} = -\frac{\partial R}{c\partial x_j}$$

so

$$\boldsymbol{\nabla}t^* = -\frac{\boldsymbol{\nabla}R}{c} \tag{4-2.44}$$

and (4-2.42) yields

$$\frac{\partial \mathbf{R}}{\partial x_j} = \mathbf{e}_j + \frac{\mathbf{v}\left(t^*\right)}{c}\frac{\partial R}{\partial x_j} \tag{4-2.45}$$

But from (4-2.41)

$$\frac{\partial R}{\partial x_j} = \mathbf{n}_{\mathbf{R}} \circ \frac{\partial \mathbf{R}}{\partial x_j} = \mathbf{n}_{\mathbf{R}} \circ \left[\mathbf{e}_j + \frac{\mathbf{v}\left(t^*\right)}{c}\frac{\partial R}{\partial x_j}\right] = \left(\mathbf{n}_{\mathbf{R}} \circ \mathbf{e}_j\right) + \left[\frac{\mathbf{v}\left(t^*\right)}{c} \circ \mathbf{n}_{\mathbf{R}}\right]\frac{\partial R}{\partial x_j}$$

so

$$\frac{\partial R}{\partial x_j} = \frac{\left(\mathbf{n}_{\mathbf{R}} \circ \mathbf{e}_j\right)}{\left[1 - \frac{\mathbf{v}\left(t^*\right)}{c} \circ \mathbf{n}_{\mathbf{R}}\right]} = \frac{\left(\mathbf{n}_{\mathbf{R}}\right)_j}{\varkappa} = \left(1 - \frac{\partial R}{c\partial t}\right)\left(\mathbf{n}_{\mathbf{R}}\right)_j$$

or

$$\boldsymbol{\nabla} R = \frac{\mathbf{R}}{\varkappa R} = \frac{\mathbf{n}_{\mathbf{R}}}{\varkappa} = \left(1 - \frac{\partial R}{c\partial t}\right)\mathbf{n}_{\mathbf{R}} \tag{4-2.46}$$

Inserting this in equation (4-2.39)

$$-\frac{4\pi\epsilon_o}{q}\boldsymbol{\nabla}\phi\left(\mathbf{x},t\right) = \frac{\left(1 - \frac{\partial R}{c\partial t}\right)^2}{R^2}\mathbf{n}_{\mathbf{R}} + \frac{1}{cR}\frac{\partial}{\partial t}\left[\left(1 - \frac{\partial R}{c\partial t}\right)\mathbf{n}_{\mathbf{R}}\right]$$

that is

$$-\boldsymbol{\nabla}\phi\left(\mathbf{x},t\right) = \frac{q}{4\pi\epsilon_o}\left[\left\{\frac{1}{R^2} - \frac{2}{cR^2}\frac{\partial R}{\partial t} + \frac{1}{c^2R^2}\left(\frac{\partial R}{\partial t}\right)^2 - \frac{1}{c^2R}\frac{\partial^2 R}{\partial t^2}\right\}\mathbf{n}_{\mathbf{R}} + \left(\frac{1}{cR} - \frac{1}{c^2R}\frac{\partial R}{\partial t}\right)\frac{\partial \mathbf{n}_{\mathbf{R}}}{\partial t}\right] \tag{4-2.47}$$

From equation (4-2.38)

$$-\frac{4\pi\epsilon_o c^2}{q}\frac{\partial \mathbf{A}\left(\mathbf{x},t\right)}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{R}\frac{\partial R}{\partial t}\mathbf{n}_{\mathbf{R}} + \frac{\partial \mathbf{n}_{\mathbf{R}}}{\partial t}\right) = \left\{\frac{\partial}{\partial t}\left(\frac{1}{R}\frac{\partial R}{\partial t}\right)\right\}\mathbf{n}_{\mathbf{R}} + \left(\frac{1}{R}\frac{\partial R}{\partial t}\right)\frac{\partial \mathbf{n}_{\mathbf{R}}}{\partial t} + \frac{\partial^2 \mathbf{n}_{\mathbf{R}}}{\partial t^2}$$

so

$$-\frac{\partial \mathbf{A}\left(\mathbf{x},t\right)}{\partial t} = \frac{q}{4\pi\epsilon_o}\left[\left\{\frac{1}{c^2R}\frac{\partial^2 R}{\partial t^2} - \frac{1}{c^2R^2}\left(\frac{\partial R}{\partial t}\right)^2\right\}\mathbf{n}_{\mathbf{R}} + \left(\frac{1}{c^2R}\frac{\partial R}{\partial t}\right)\frac{\partial \mathbf{n}_{\mathbf{R}}}{\partial t} + \frac{\partial^2 \mathbf{n}_{\mathbf{R}}}{c^2\partial t^2}\right] \tag{4-2.48}$$

Adding equations (4-2.47),(4-2.48) yields

$$\begin{aligned}
\mathbf{E}\left(\mathbf{x},t\right) &= -\boldsymbol{\nabla}\phi\left(\mathbf{x},t\right) - \frac{\partial \mathbf{A}\left(\mathbf{x},t\right)}{\partial t} \\
&= \frac{q}{4\pi\epsilon_o}\left[\left\{\frac{1}{R^2} - \frac{2}{cR^2}\frac{\partial R}{\partial t} + \frac{1}{c^2R^2}\left(\frac{\partial R}{\partial t}\right)^2 - \frac{1}{c^2R}\frac{\partial^2 R}{\partial t^2}\right\}\mathbf{n}_{\mathbf{R}} + \left(\frac{1}{cR} - \frac{1}{c^2R}\frac{\partial R}{\partial t}\right)\frac{\partial \mathbf{n}_{\mathbf{R}}}{\partial t}\right] \\
&\quad + \frac{q}{4\pi\epsilon_o}\left[\left\{\frac{1}{c^2R}\frac{\partial^2 R}{\partial t^2} - \frac{1}{c^2R^2}\left(\frac{\partial R}{\partial t}\right)^2\right\}\mathbf{n}_{\mathbf{R}} + \left(\frac{1}{c^2R}\frac{\partial R}{\partial t}\right)\frac{\partial \mathbf{n}_{\mathbf{R}}}{\partial t} + \frac{\partial^2 \mathbf{n}_{\mathbf{R}}}{c^2\partial t^2}\right] \\
&= \frac{q}{4\pi\epsilon_o}\left[\left\{\frac{1}{R^2} - \frac{2}{cR^2}\frac{\partial R}{\partial t}\right\}\mathbf{n}_{\mathbf{R}} + \left(\frac{1}{cR} - \frac{1}{c^2R}\frac{\partial R}{\partial t}\right)\frac{\partial \mathbf{n}_{\mathbf{R}}}{\partial t}\right]
\end{aligned}$$

so

$$\mathbf{E}(\mathbf{x},t)=\frac{q}{4\pi\epsilon_o}\left[\frac{\mathbf{n_R}}{R^2}+\frac{R}{c}\underbrace{\left(-\frac{2}{R^3}\frac{\partial R}{\partial t}\mathbf{n_R}+\frac{1}{R^2}\frac{\partial \mathbf{n_R}}{\partial t}\right)}_{\frac{\partial}{\partial t}\left(\frac{\mathbf{n_R}}{R^2}\right)}+\frac{\partial^2\mathbf{n_R}}{c^2\partial t^2}\right]$$

and finally

$$\mathbf{E}(\mathbf{x},t)=\frac{q}{4\pi\epsilon_o}\left[\frac{\mathbf{n_R}}{R^2}+\frac{R}{c}\frac{\partial}{\partial t}\left(\frac{\mathbf{n_R}}{R^2}\right)+\frac{\partial^2\mathbf{n_R}}{c^2\partial t^2}\right] \tag{4-2.49}$$

For given field point, that is position vector $\mathbf{x}$

$$\frac{\partial}{\partial t}\equiv\frac{d}{dt} \tag{4-2.50}$$

and then equation (4-2.49) yields the Feynman Lectures one

$$\boxed{\mathbf{E}=\frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{n_R}}{R^2}+\frac{R}{c}\frac{d}{dt}\left(\frac{\mathbf{n_R}}{R^2}\right)+\frac{1}{c^2}\frac{d^2}{dt^2}\mathbf{n_R}\right]} \tag{4-2.51}$$

....

For the magnetic flux density vector $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$, equation (4-2.38) with the expression (4-2.46) for $\boldsymbol{\nabla}R$ yields [7] For the magnetic flux density vector $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$, equation (4-2.38) with the expression (4-2.46) for $\boldsymbol{\nabla}R$ yields [8]

$$\begin{aligned}-\frac{4\pi\epsilon_o c^2}{q}\left[\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x},t)\right]&=\boldsymbol{\nabla}\times\left(\frac{1}{R}\frac{\partial\mathbf{R}}{\partial t}\right)=\boldsymbol{\nabla}\left(\frac{1}{R}\right)\times\frac{\partial\mathbf{R}}{\partial t}+\frac{1}{R}\left(\boldsymbol{\nabla}\times\frac{\partial\mathbf{R}}{\partial t}\right)\\&=\left(-\frac{1}{R^2}\boldsymbol{\nabla}R\right)\times\frac{\partial\mathbf{R}}{\partial t}+\frac{1}{R}\frac{\partial\left(\boldsymbol{\nabla}\times\mathbf{R}\right)}{\partial t}\\&=\left[\left\{-\frac{1}{R^2}\left(1-\frac{\partial R}{c\partial t}\right)\mathbf{n_R}\right\}\times\left\{\frac{\partial R}{\partial t}\mathbf{n_R}+R\frac{\partial\mathbf{n_R}}{\partial t}\right\}\right]+\frac{1}{R}\frac{\partial\left(\boldsymbol{\nabla}\times\mathbf{R}\right)}{\partial t}\end{aligned}$$

so

$$\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x},t)=\frac{q}{4\pi\epsilon_o c^2}\frac{1}{R}\left[\left(1-\frac{\partial R}{c\partial t}\right)\left(\mathbf{n_R}\times\frac{\partial\mathbf{n_R}}{\partial t}\right)-\frac{\partial\left(\boldsymbol{\nabla}\times\mathbf{R}\right)}{\partial t}\right] \tag{4-2.52}$$

So, it remains to express $\boldsymbol{\nabla}\times\mathbf{R}$ as function of $R,\mathbf{n_R}$ and their derivatives. Starting from the definition

$$\boldsymbol{\nabla}\times\mathbf{R}=\begin{bmatrix}\mathbf{e}_1&\mathbf{e}_2&\mathbf{e}_3\\ \dfrac{\partial}{\partial x_1}&\dfrac{\partial}{\partial x_2}&\dfrac{\partial}{\partial x_3}\\ R_1&R_2&R_3\end{bmatrix}=\begin{bmatrix}\dfrac{\partial R_3}{\partial x_2}-\dfrac{\partial R_2}{\partial x_3}\\ \dfrac{\partial R_1}{\partial x_3}-\dfrac{\partial R_3}{\partial x_1}\\ \dfrac{\partial R_2}{\partial x_1}-\dfrac{\partial R_1}{\partial x_2}\end{bmatrix} \tag{4-2.53}$$

[7] we make use of the identity

$$\boldsymbol{\nabla}\times(\psi\mathbf{a})=\boldsymbol{\nabla}\psi\times\mathbf{a}+\psi\boldsymbol{\nabla}\times\mathbf{a} \tag{repeatA-2.9}$$

see equation (A-2.9) in Appendix A

[8] we make use of the identity

$$\boldsymbol{\nabla}\times(\psi\mathbf{a})=\boldsymbol{\nabla}\psi\times\mathbf{a}+\psi\boldsymbol{\nabla}\times\mathbf{a} \tag{repeatA-2.9}$$

see equation (A-2.9) in Appendix A

But we have already the expressions of $\partial R_i/\partial x_j$ in equation (4-2.45) written componentwise as

$$\frac{\partial R_i}{\partial x_j} = \delta_{ij} + \frac{\upsilon_i\left(t^*\right)}{c}\frac{\partial R}{\partial x_j} \tag{4-2.54}$$

Above equation could be written as a so-called Jacobian matrix, which in our case is also the so-called directional derivative of $\mathbf{R}$ with respect to $\mathbf{x}$

$$\frac{D\mathbf{R}}{D\mathbf{x}} \overset{\text{def}}{\equiv} \left\{\frac{\partial R_i}{\partial x_j}\right\} = \begin{bmatrix} \partial R_1/\partial x_1 & \partial R_1/\partial x_2 & \partial R_1/\partial x_3 \\ \partial R_2/\partial x_1 & \partial R_2/\partial x_2 & \partial R_2/\partial x_3 \\ \partial R_3/\partial x_1 & \partial R_3/\partial x_2 & \partial R_3/\partial x_3 \end{bmatrix}$$
$$= \begin{bmatrix} 1+\dfrac{\upsilon_1\left(t^*\right)}{c}\dfrac{\partial R}{\partial x_1} & \dfrac{\upsilon_1\left(t^*\right)}{c}\dfrac{\partial R}{\partial x_2} & \dfrac{\upsilon_1\left(t^*\right)}{c}\dfrac{\partial R}{\partial x_3} \\ \dfrac{\upsilon_2\left(t^*\right)}{c}\dfrac{\partial R}{\partial x_1} & 1+\dfrac{\upsilon_2\left(t^*\right)}{c}\dfrac{\partial R}{\partial x_2} & \dfrac{\upsilon_2\left(t^*\right)}{c}\dfrac{\partial R}{\partial x_3} \\ \dfrac{\upsilon_3\left(t^*\right)}{c}\dfrac{\partial R}{\partial x_1} & \dfrac{\upsilon_3\left(t^*\right)}{c}\dfrac{\partial R}{\partial x_2} & 1+\dfrac{\upsilon_3\left(t^*\right)}{c}\dfrac{\partial R}{\partial x_3} \end{bmatrix} \tag{4-2.55}$$

With the help of above equation (4-2.53) reads

$$\boldsymbol{\nabla}\times\mathbf{R} = \begin{bmatrix} \dfrac{\partial R_3}{\partial x_2} - \dfrac{\partial R_2}{\partial x_3} \\ \dfrac{\partial R_1}{\partial x_3} - \dfrac{\partial R_3}{\partial x_1} \\ \dfrac{\partial R_2}{\partial x_1} - \dfrac{\partial R_1}{\partial x_2} \end{bmatrix} = \frac{1}{c}\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial R}{\partial x_1} & \dfrac{\partial R}{\partial x_2} & \dfrac{\partial R}{\partial x_3} \\ \upsilon_1\left(t^*\right) & \upsilon_2\left(t^*\right) & \upsilon_3\left(t^*\right) \end{bmatrix}$$

so

$$\boldsymbol{\nabla}\times\mathbf{R} = \frac{1}{c}\left[\boldsymbol{\nabla}R\times\mathbf{v}\left(t^*\right)\right] \tag{4-2.56}$$

Inserting the expressions (4-2.46),(4-2.34) of $\boldsymbol{\nabla}R, \mathbf{v}\left(t^*\right)$ respectively, repeated here for convenience

$$\boldsymbol{\nabla}R = \frac{\mathbf{R}}{\varkappa R} = \frac{\mathbf{n}_\mathbf{R}}{\varkappa} = \left(1-\frac{\partial R}{c\partial t}\right)\mathbf{n}_\mathbf{R} \tag{repeat4-2.46}$$

$$\mathbf{v}\left(t^*\right) = -\frac{\dfrac{\partial\mathbf{R}}{\partial t}}{\left(1-\dfrac{\partial R}{c\partial t}\right)} = -\left[\left\{\frac{\dfrac{\partial R}{\partial t}}{\left(1-\dfrac{\partial R}{c\partial t}\right)}\right\}\mathbf{n}_\mathbf{R} + \left\{\frac{R}{\left(1-\dfrac{\partial R}{c\partial t}\right)}\right\}\frac{\partial\mathbf{n}_\mathbf{R}}{\partial t}\right] \tag{repeat4-2.34}$$

we have

$$\boldsymbol{\nabla}\times\mathbf{R} = -\frac{R}{c}\left(\mathbf{n}_\mathbf{R}\times\frac{\partial\mathbf{n}_\mathbf{R}}{\partial t}\right) \tag{4-2.57}$$

so

$$\frac{\partial\left(\boldsymbol{\nabla}\times\mathbf{R}\right)}{\partial t}=-\frac{1}{c}\left\{\left(\frac{\partial R}{\partial t}\right)\left(\mathbf{n}_{\mathbf{R}}\times\frac{\partial\mathbf{n}_{\mathbf{R}}}{\partial t}\right)+R\left(\mathbf{n}_{\mathbf{R}}\times\frac{\partial^{2}\mathbf{n}_{\mathbf{R}}}{\partial t^{2}}\right)\right\}\tag{4-2.58}$$

Inserting this expression in (4-2.53) yields

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{A}\left(\mathbf{x},t\right)&=\frac{q}{4\pi\epsilon_{o}c^{2}}\frac{1}{R}\left[\left(1-\frac{\partial R}{c\partial t}\right)\left(\mathbf{n}_{\mathbf{R}}\times\frac{\partial\mathbf{n}_{\mathbf{R}}}{\partial t}\right)+\frac{1}{c}\left\{\left(\frac{\partial R}{\partial t}\right)\left(\mathbf{n}_{\mathbf{R}}\times\frac{\partial\mathbf{n}_{\mathbf{R}}}{\partial t}\right)+R\left(\mathbf{n}_{\mathbf{R}}\times\frac{\partial^{2}\mathbf{n}_{\mathbf{R}}}{\partial t^{2}}\right)\right\}\right]\\&=\frac{q}{4\pi\epsilon_{o}c}\left[\mathbf{n}_{\mathbf{R}}\times\left(\frac{1}{cR}\frac{\partial\mathbf{n}_{\mathbf{R}}}{\partial t}+\frac{1}{c^{2}}\frac{\partial^{2}\mathbf{n}_{\mathbf{R}}}{\partial t^{2}}\right)\right]\end{aligned}$$

so

$$\mathbf{B}\left(\mathbf{x},t\right)=\boldsymbol{\nabla}\times\mathbf{A}\left(\mathbf{x},t\right)=\frac{q}{4\pi\epsilon_{o}c}\left[\mathbf{n}_{\mathbf{R}}\times\left(\frac{1}{cR}\frac{\partial\mathbf{n}_{\mathbf{R}}}{\partial t}+\frac{1}{c^{2}}\frac{\partial^{2}\mathbf{n}_{\mathbf{R}}}{\partial t^{2}}\right)\right]\tag{4-2.59}$$

Above equation with the help of (4-2.49) is expressed as

$$\mathbf{B}\left(\mathbf{x},t\right)=\frac{1}{c}\mathbf{n}_{\mathbf{R}}\times\underbrace{\left[\frac{q}{4\pi\epsilon_{o}}\left(\frac{1}{cR}\frac{\partial\mathbf{n}_{\mathbf{R}}}{\partial t}+\frac{1}{c^{2}}\frac{\partial^{2}\mathbf{n}_{\mathbf{R}}}{\partial t^{2}}\right)\right]}_{\mathbf{E}(\mathbf{x},t)-\frac{q}{4\pi\epsilon_{o}}\left(\frac{1}{R^{2}}-\frac{2}{cR^{2}}\right)\mathbf{n}_{\mathbf{R}}}$$

and finally the 2nd equation of Feynman Lectures

$$c\,\mathbf{B}\left(\mathbf{x},t\right)=\mathbf{n}_{\mathbf{R}}\times\mathbf{E}\left(\mathbf{x},t\right)\tag{4-2.60}$$

4-2.3 Integrals with Dirac $\delta-$function

The integrals in (4-2.11) and (4-2.12) are of the form

$$\phi\left(\mathbf{x},t\right)=\frac{q}{4\pi\epsilon_{o}}\iiint\delta^{3}\Big(\mathbf{F}\left(\mathbf{x}'\right)\Big)\mathrm{H}\left(\mathbf{x}'\right)\mathrm{d}^{3}\mathbf{x}'\tag{4-2.61}$$

$$\mathbf{A}\left(\mathbf{x},t\right)=\frac{\mu_{o}q}{4\pi}\iiint\delta^{3}\Big(\mathbf{F}\left(\mathbf{x}'\right)\Big)\mathbf{G}\left(\mathbf{x}'\right)d^{3}\mathbf{x}'\tag{4-2.62}$$

where $\mathbf{F}\left(\mathbf{x}'\right),\mathbf{G}\left(\mathbf{x}'\right)$ vector functions and $\mathrm{H}\left(\mathbf{x}'\right)$ scalar function of the vector variable $\mathbf{x}'$

$$\mathbf{F}\left(\mathbf{x}'\right)=\mathbf{x}'-\boldsymbol{\xi}\left(t-\frac{\|\mathbf{x}'-\mathbf{x}\|}{c}\right)\tag{4-2.63}$$

$$\mathbf{G}\left(\mathbf{x}'\right)=\frac{\mathbf{v}\left(t-\dfrac{\|\mathbf{x}'-\mathbf{x}\|}{c}\right)}{\|\mathbf{x}'-\mathbf{x}\|}\tag{4-2.64}$$

$$\mathrm{H}\left(\mathbf{x}'\right)=\frac{1}{\|\mathbf{x}'-\mathbf{x}\|}\tag{4-2.65}$$

We can handle easily integrals where the vector variable of integration, let $\mathbf{u}$, is the argument of the $\delta-$function, for example

$$\iiint \delta^3(\mathbf{u})\,\mathrm{L}(\mathbf{u})\,\mathrm{d}^3\mathbf{u} = \mathrm{L}(\mathbf{0}) \tag{4-2.66}$$

$$\iiint \delta^3(\mathbf{u})\,\mathbf{M}(\mathbf{u})\,d^3\mathbf{u} = \mathbf{M}(\mathbf{0}) \tag{4-2.67}$$

But to handle integrals of the form (4-2.61) and (4-2.62)

$$\iiint \delta^3\Big(\mathbf{F}(\mathbf{x}')\Big)\,\mathrm{H}(\mathbf{x}')\,\mathrm{d}^3\mathbf{x}' \tag{4-2.68}$$

$$\iiint \delta^3\Big(\mathbf{F}(\mathbf{x}')\Big)\,\mathbf{G}(\mathbf{x}')\,d^3\mathbf{x}' \tag{4-2.69}$$

where $\mathbf{F}(\mathbf{x}') \neq \mathbf{x}'$, that is the argument of the $\delta-$function is not the variable of integration, we must proceed to a change of the vector variable from $\mathbf{x}'$ to $\mathbf{u}$

$$\mathbf{u} = \mathbf{F}(\mathbf{x}') \tag{4-2.70}$$

and check with care if we can convert without complications these integrals to expressions like (4-2.66)and (4-2.67).

Indeed, if the vector function $\mathbf{F}$ in (4-2.70) is invertible then

$$\mathbf{x}' = \mathbf{F}^{-1}(\mathbf{u}) \tag{4-2.71}$$

It remains one step : to find the relation between the infinitesimal volumes $d^3\mathbf{x}' = dx'_1dx'_2dx'_3$ and $d^3\mathbf{u} = du_1du_2du_3$. We have the following linear transformation between infinitesimals

$$dx'_1 = \frac{\partial x'_1}{\partial u_1}du_1 + \frac{\partial x'_1}{\partial u_2}du_2 + \frac{\partial x'_1}{\partial u_3}du_3 \tag{4-2.72}$$

$$dx'_2 = \frac{\partial x'_2}{\partial u_1}du_1 + \frac{\partial x'_2}{\partial u_2}du_2 + \frac{\partial x'_2}{\partial u_3}du_3 \tag{4-2.73}$$

$$dx'_3 = \frac{\partial x'_3}{\partial u_1}du_1 + \frac{\partial x'_3}{\partial u_2}du_2 + \frac{\partial x'_3}{\partial u_3}du_3 \tag{4-2.74}$$

or

$$d\mathbf{x}' = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x'_1}{\partial u_1} & \dfrac{\partial x'_1}{\partial u_2} & \dfrac{\partial x'_1}{\partial u_3} \\ \dfrac{\partial x'_2}{\partial u_1} & \dfrac{\partial x'_2}{\partial u_2} & \dfrac{\partial x'_2}{\partial u_3} \\ \dfrac{\partial x'_3}{\partial u_1} & \dfrac{\partial x'_3}{\partial u_2} & \dfrac{\partial x'_3}{\partial u_3} \end{bmatrix} \begin{bmatrix} du_1 \\ du_2 \\ du_3 \end{bmatrix} = \mathbf{J}\left(\mathbf{F}^{-1}\right)d\mathbf{u} \tag{4-2.75}$$

where

$$\mathbf{J}\left(\mathbf{F}^{-1}\right) \overset{\text{def}}{\equiv} \begin{bmatrix} \dfrac{\partial x'_1}{\partial u_1} & \dfrac{\partial x'_1}{\partial u_2} & \dfrac{\partial x'_1}{\partial u_3} \\ \dfrac{\partial x'_2}{\partial u_1} & \dfrac{\partial x'_2}{\partial u_2} & \dfrac{\partial x'_2}{\partial u_3} \\ \dfrac{\partial x'_3}{\partial u_1} & \dfrac{\partial x'_3}{\partial u_2} & \dfrac{\partial x'_3}{\partial u_3} \end{bmatrix} \tag{4-2.76}$$

the so-called Jacobian matrix of the vector function $\mathbf{F}^{-1}$, a matrix function of $\mathbf{u}$. We know that for an invertible linear transformation the ratio of the transformed to the initial volume is equal to the determinant of the respective matrix. So

$$d^3\mathbf{x}' = \frac{\partial\left(x_1', x_2', x_3'\right)}{\partial\left(u_1, u_2, u_3\right)} d^3\mathbf{u} \tag{4-2.77}$$

where

$$\frac{\partial\left(x_1', x_2', x_3'\right)}{\partial\left(u_1, u_2, u_3\right)} \overset{\text{def}}{\equiv} \det\left[\mathbf{J}\left(\mathbf{F}^{-1}\right)\right] = \begin{vmatrix} \dfrac{\partial x_1'}{\partial u_1} & \dfrac{\partial x_1'}{\partial u_2} & \dfrac{\partial x_1'}{\partial u_3} \\ \dfrac{\partial x_2'}{\partial u_1} & \dfrac{\partial x_2'}{\partial u_2} & \dfrac{\partial x_2'}{\partial u_3} \\ \dfrac{\partial x_3'}{\partial u_1} & \dfrac{\partial x_3'}{\partial u_2} & \dfrac{\partial x_3'}{\partial u_3} \end{vmatrix} \tag{4-2.78}$$

the so-called Jacobian of the vector function $\mathbf{F}^{-1}$, the determinant of the Jacobi matrix $\mathbf{J}\left(\mathbf{F}^{-1}\right)$. The Jacobian is a scalar function of $\mathbf{u}$.

Now, if in equations (4-2.68)and (4-2.69) we make the following substitutions

$$\mathbf{F}\left(\mathbf{x}'\right) \longrightarrow \mathbf{u}\,, \quad \mathbf{x}' \longrightarrow \mathbf{F}^{-1}\left(\mathbf{u}\right)\,, \quad d^3\mathbf{x}' \longrightarrow \frac{\partial\left(x_1', x_2', x_3'\right)}{\partial\left(u_1, u_2, u_3\right)} d^3\mathbf{u} \tag{4-2.79}$$

according to equations (4-2.70),(4-2.71) and (4-2.77) respectively, then these integrals are converted to the form of (4-2.66) and (4-2.67), that is

$$\begin{aligned} \iiint \delta^3\Big(\mathbf{F}\left(\mathbf{x}'\right)\Big)\mathrm{H}\left(\mathbf{x}'\right)\mathrm{d}^3\mathbf{x}' &= \iiint \delta^3\left(\mathbf{u}\right)\mathrm{H}\left(\mathbf{F}^{-1}\left(\mathbf{u}\right)\right)\frac{\partial\left(\mathrm{x}_1', \mathrm{x}_2', \mathrm{x}_3'\right)}{\partial\left(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3\right)}\mathrm{d}^3\mathbf{u} \\ &= \mathrm{H}\left(\mathbf{F}^{-1}\left(\mathbf{0}\right)\right)\left[\frac{\partial\left(\mathrm{x}_1', \mathrm{x}_2', \mathrm{x}_3'\right)}{\partial\left(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3\right)}\right]_{\mathbf{u}=\mathbf{0}} \end{aligned} \tag{4-2.80}$$

$$\begin{aligned} \iiint \delta^3\Big(\mathbf{F}\left(\mathbf{x}'\right)\Big)\mathbf{G}\left(\mathbf{x}'\right)d^3\mathbf{x}' &= \iiint \delta^3\left(\mathbf{u}\right)\mathbf{G}\left(\mathbf{F}^{-1}\left(\mathbf{u}\right)\right)\frac{\partial\left(x_1', x_2', x_3'\right)}{\partial\left(u_1, u_2, u_3\right)}d^3\mathbf{u} \\ &= \mathbf{G}\left(\mathbf{F}^{-1}\left(\mathbf{0}\right)\right)\left[\frac{\partial\left(x_1', x_2', x_3'\right)}{\partial\left(u_1, u_2, u_3\right)}\right]_{\mathbf{u}=\mathbf{0}} \end{aligned} \tag{4-2.81}$$

Note that starting from equation (4-2.71) we found equations (4-2.72) to (4-2.78) concerning the vector function $\mathbf{F}^{-1}$. With similar steps we can start from (4-2.70) and find the respective equations for the vector function $\mathbf{F}$. Indeed

$$du_1 = \frac{\partial u_1}{\partial x_1'}dx_1' + \frac{\partial u_1}{\partial x_2'}dx_2' + \frac{\partial u_1}{\partial x_3'}dx_3' \tag{4-2.82}$$

$$du_2 = \frac{\partial u_2}{\partial x_1'}dx_1' + \frac{\partial u_2}{\partial x_2'}dx_2' + \frac{\partial u_2}{\partial x_3'}dx_3' \tag{4-2.83}$$

$$du_3 = \frac{\partial u_3}{\partial x_1'}dx_1' + \frac{\partial u_3}{\partial x_2'}dx_2' + \frac{\partial u_3}{\partial x_3'}dx_3' \tag{4-2.84}$$

or

$$d\mathbf{u}=\begin{bmatrix}du_1\\ du_2\\ du_3\end{bmatrix}=\begin{bmatrix}\frac{\partial u_1}{\partial x_1'} & \frac{\partial u_1}{\partial x_2'} & \frac{\partial u_1}{\partial x_3'}\\ \frac{\partial u_2}{\partial x_1'} & \frac{\partial u_2}{\partial x_2'} & \frac{\partial u_2}{\partial x_3'}\\ \frac{\partial u_3}{\partial x_1'} & \frac{\partial u_3}{\partial x_2'} & \frac{\partial u_3}{\partial x_3'}\end{bmatrix}\begin{bmatrix}dx_1\\ dx_2\\ dx_3\end{bmatrix}=\mathbf{J}\left(\mathbf{F}\right)d\mathbf{x}' \tag{4-2.85}$$

where

$$\mathbf{J}\left(\mathbf{F}\right)\overset{\text{def}}{\equiv}\begin{bmatrix}\frac{\partial u_1}{\partial x_1'} & \frac{\partial u_1}{\partial x_2'} & \frac{\partial u_1}{\partial x_3'}\\ \frac{\partial u_2}{\partial x_1'} & \frac{\partial u_2}{\partial x_2'} & \frac{\partial u_2}{\partial x_3'}\\ \frac{\partial u_3}{\partial x_1'} & \frac{\partial u_3}{\partial x_2'} & \frac{\partial u_3}{\partial x_3'}\end{bmatrix} \tag{4-2.86}$$

the Jacobian matrix of the vector function $\mathbf{F}$, a matrix function of $\mathbf{x}'$. So

$$d^3\mathbf{u}=\frac{\partial\left(u_1,u_2,u_3\right)}{\partial\left(x_1',x_2',x_3'\right)}d^3\mathbf{x}' \tag{4-2.87}$$

where

$$\frac{\partial\left(u_1,u_2,u_3\right)}{\partial\left(x_1',x_2',x_3'\right)}\overset{\text{def}}{\equiv}\det\left[\mathbf{J}\left(\mathbf{F}\right)\right]=\begin{vmatrix}\frac{\partial u_1}{\partial x_1'} & \frac{\partial u_1}{\partial x_2'} & \frac{\partial u_1}{\partial x_3'}\\ \frac{\partial u_2}{\partial x_1'} & \frac{\partial u_2}{\partial x_2'} & \frac{\partial u_2}{\partial x_3'}\\ \frac{\partial u_3}{\partial x_1'} & \frac{\partial u_3}{\partial x_2'} & \frac{\partial u_3}{\partial x_3'}\end{vmatrix} \tag{4-2.88}$$

the Jacobian of the vector function $\mathbf{F}$, the determinant of the Jacobi matrix $\mathbf{J}\left(\mathbf{F}\right)$. The Jacobian is a scalar function of $\mathbf{x}'$. From equations (4-2.76) and (4-2.86) we have

$$\begin{aligned}\mathbf{J}\left(\mathbf{F}^{-1}\right)\cdot\mathbf{J}\left(\mathbf{F}\right)&=\begin{bmatrix}\frac{\partial x_1'}{\partial u_1} & \frac{\partial x_1'}{\partial u_2} & \frac{\partial x_1'}{\partial u_3}\\ \frac{\partial x_2'}{\partial u_1} & \frac{\partial x_2'}{\partial u_2} & \frac{\partial x_2'}{\partial u_3}\\ \frac{\partial x_3'}{\partial u_1} & \frac{\partial x_3'}{\partial u_2} & \frac{\partial x_3'}{\partial u_3}\end{bmatrix}\begin{bmatrix}\frac{\partial u_1}{\partial x_1'} & \frac{\partial u_1}{\partial x_2'} & \frac{\partial u_1}{\partial x_3'}\\ \frac{\partial u_2}{\partial x_1'} & \frac{\partial u_2}{\partial x_2'} & \frac{\partial u_2}{\partial x_3'}\\ \frac{\partial u_3}{\partial x_1'} & \frac{\partial u_3}{\partial x_2'} & \frac{\partial u_3}{\partial x_3'}\end{bmatrix}\\ &=\begin{bmatrix}\frac{\partial x_1'}{\partial x_1'} & 0 & 0\\ 0 & \frac{\partial x_2'}{\partial x_2'} & 0\\ 0 & 0 & \frac{\partial x_2'}{\partial x_2'}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\mathbf{I}\end{aligned}$$

so

$$\mathbf{J}\left(\mathbf{F}^{-1}\right)=\left[\mathbf{J}\left(\mathbf{F}\right)\right]^{-1} \tag{4-2.89}$$

This means that for the Jacobian determinants we have

$$\frac{\partial\left(x_1',x_2',x_3'\right)}{\partial\left(u_1,u_2,u_3\right)}=\left[\frac{\partial\left(u_1,u_2,u_3\right)}{\partial\left(x_1',x_2',x_3'\right)}\right]^{-1} \tag{4-2.90}$$

and equation (4-2.77) is completed to

$$d^3\mathbf{x}' = \frac{\partial\left(x_1', x_2', x_3'\right)}{\partial\left(u_1, u_2, u_3\right)} d^3\mathbf{u} = \left[\frac{\partial\left(u_1, u_2, u_3\right)}{\partial\left(x_1', x_2', x_3'\right)}\right]^{-1} d^3\mathbf{u} \tag{4-2.91}$$

4-2.4 Properties of function : $\mathbf{F}\left(\mathbf{x}'\right) = \mathbf{x}' - \boldsymbol{\xi}\left(t - \dfrac{\|\mathbf{x}' - \mathbf{x}\|}{c}\right)$

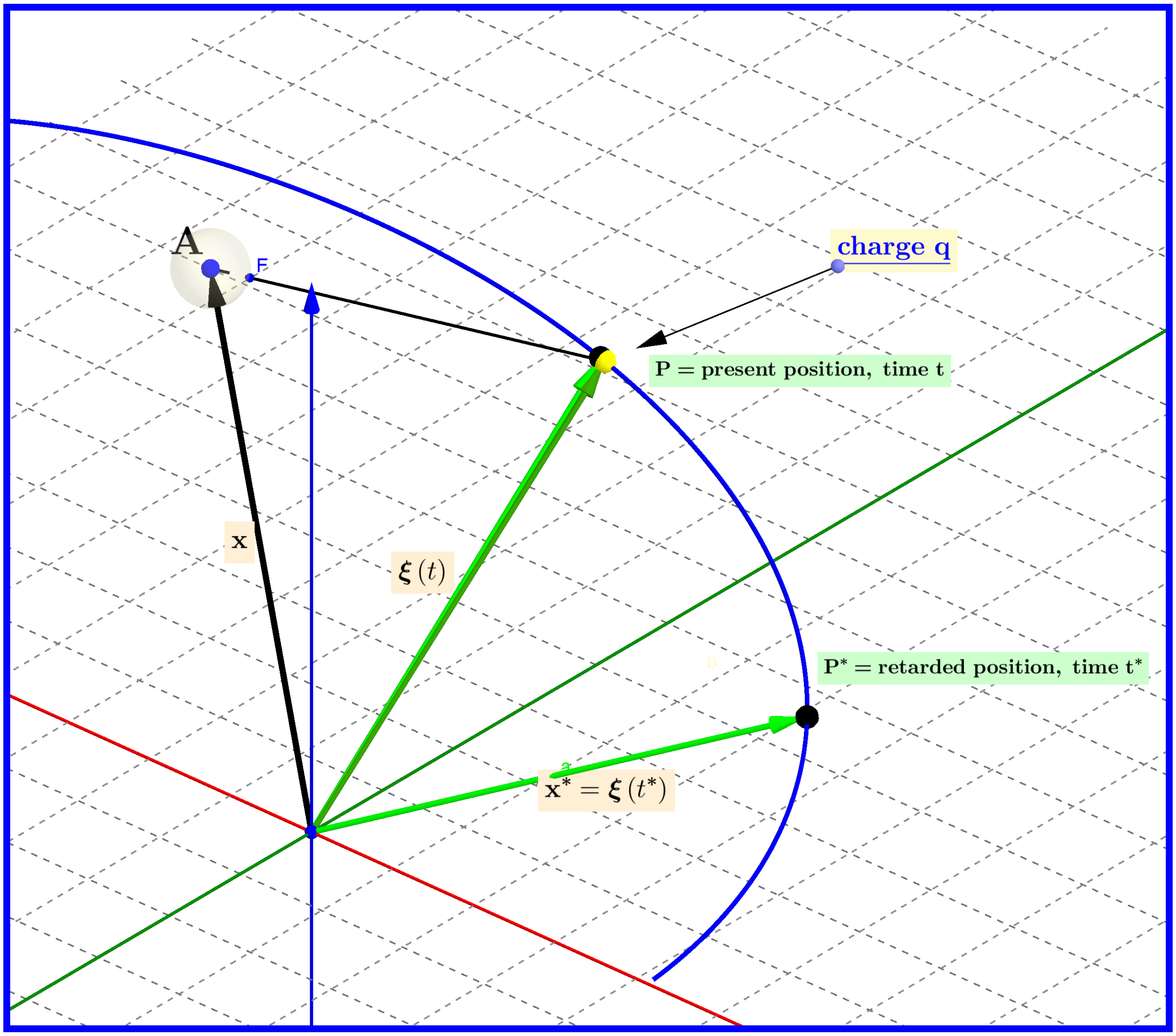

Figure 4.5: When the charge q is at its present position, point P time t, a spherical light wave is emitted from field point A to the past and the video of the motion of the charge is played from t backwards in time.

We see that to handle the integrals in equations (4-2.11),(4-2.12) in Subsection 4-2.1, we proceeded to the following variable change from $\mathbf{x}'$ to $\mathbf{u}$, equation (4-2.13), repeated below

$$\mathbf{u} = \mathbf{x}' - \boldsymbol{\xi}\left(t - \frac{\|\mathbf{x}' - \mathbf{x}\|}{c}\right) = \mathbf{F}\left(\mathbf{x}'\right) \tag{repeat4-2.13}$$

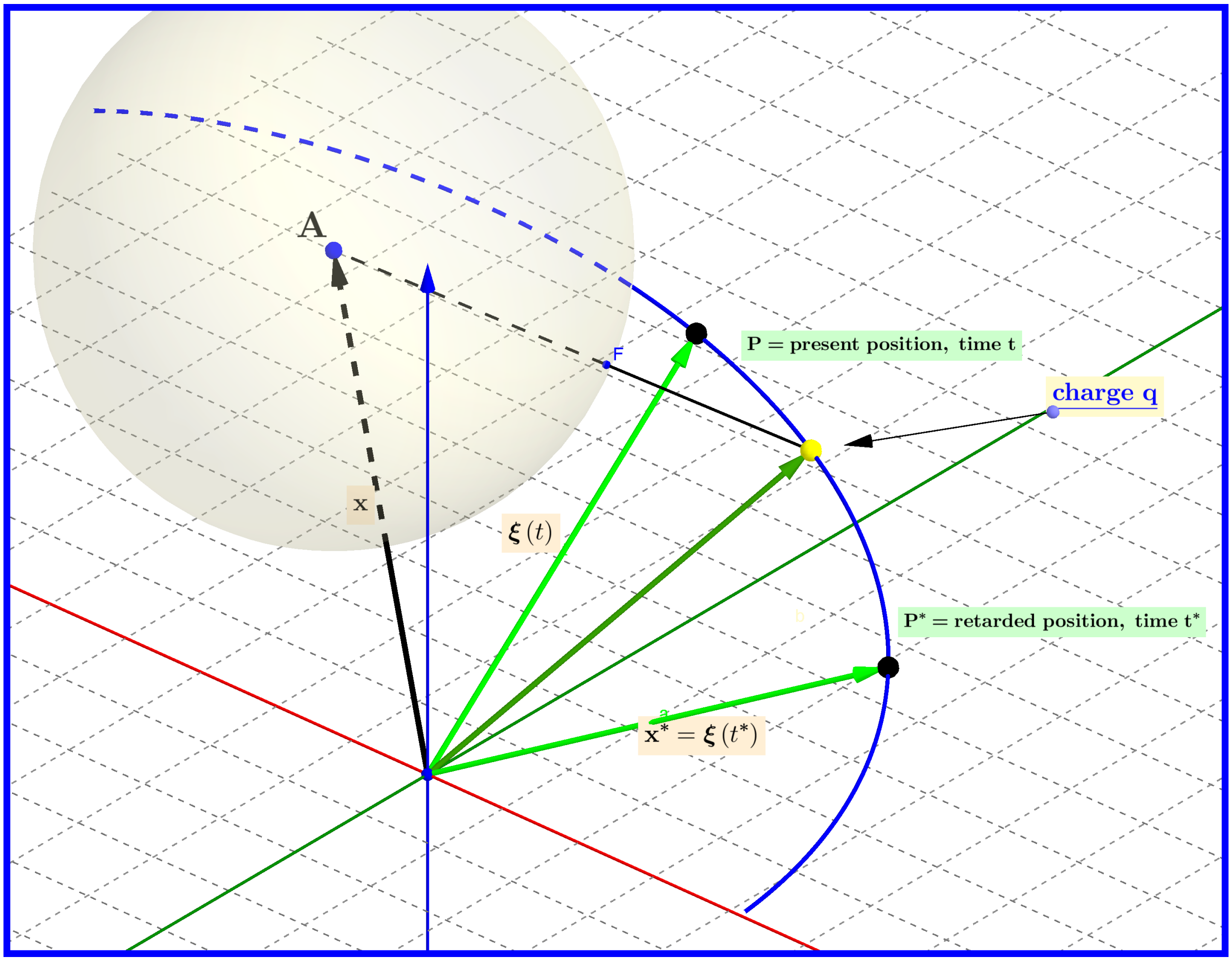

Figure 4.6: The spherical light wave front emitted to the past, see Figure 4.5, is moving with radial speed c greater than the radial speed of the charge, so coming closer and closer to arrest on its trajectory the backwards in time moving charge q.

as also in equation (4-2.63) in Subsection 4-2.3.

This procedure would have sense if the vector function $\mathbf{F}(\mathbf{x}')$ of the vector variable $\mathbf{x}'$ is invertible, which means that for every vector $\mathbf{u}$ not only there exists a vector $\mathbf{x}'$ satisfying above equation but also that this vector is unique. This inverse existence ensures a non-zero Jacobian, see Subsection 4-2.6.

Now, it is proved below that the existence and uniqueness of the solution of equation (4-2.13) with respect to $\mathbf{x}'$ for any $\mathbf{u}$ is equivalent to the existence and uniqueness of the solution of equation (4-2.13) with respect to $\mathbf{x}'$ for $\mathbf{u} = \mathbf{0}$ that is of equation (4-2.20) repeated here

$$\mathbf{0} = \mathbf{x}' - \boldsymbol{\xi}\left(t - \frac{\|\mathbf{x}' - \mathbf{x}\|}{c}\right) \qquad \text{(repeat4-2.20)}$$

So, let suppose that above equation has a solution $\mathbf{x}^*(\mathbf{x}, t)$ (for any $\mathbf{x}$ and t) and this solution is unique, see also equation (4-2.21) repeated here

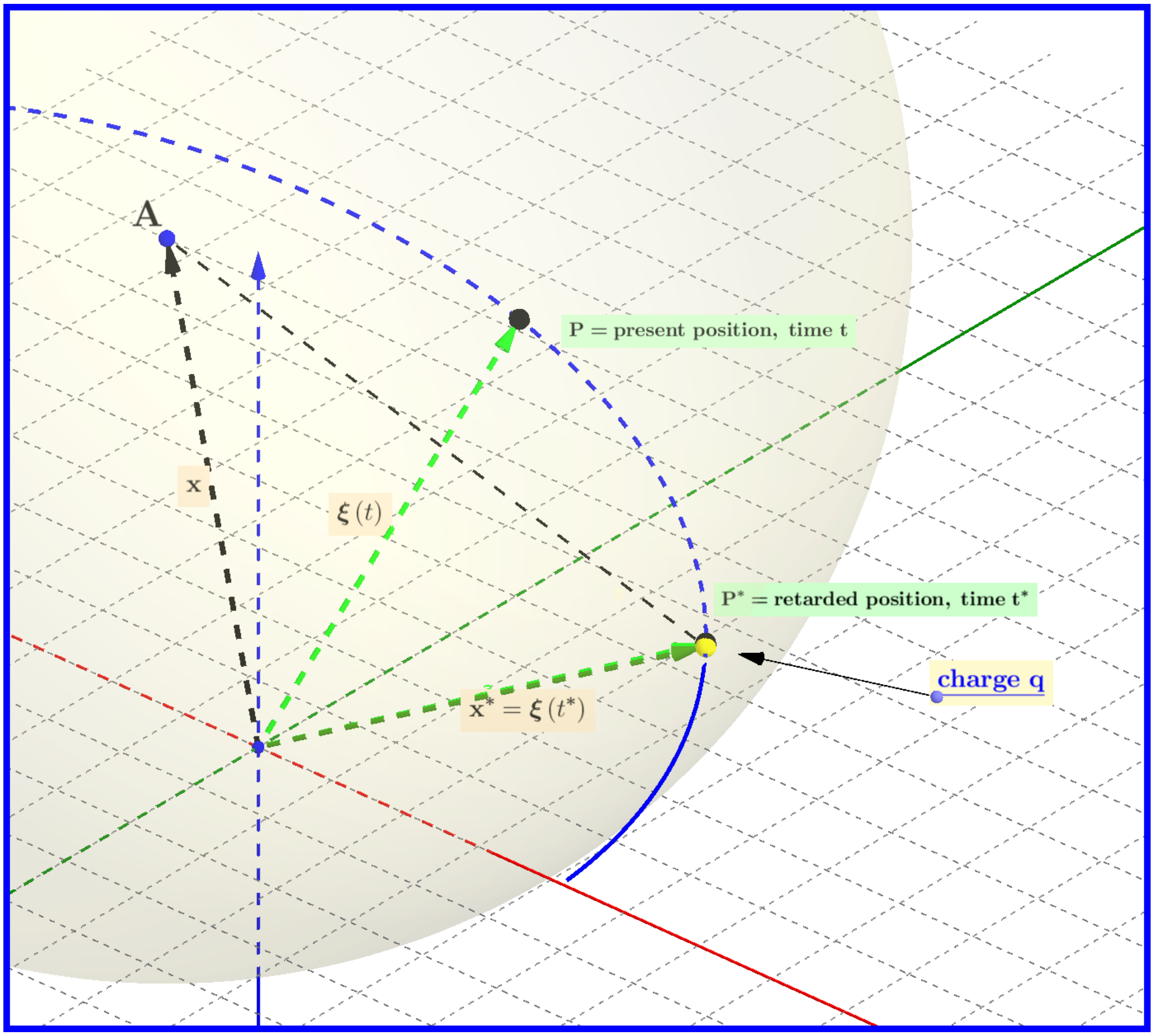

Figure 4.7: The spherical light wave front emitted to the past, see Figures 4.5 and 4.6 , is arresting at the retarded time t^* on the retarded point P* on its trajectory the backwards in time moving charge q.

$$\mathbf{x}^* \quad \overset{\text{def}}{:} \quad \mathbf{x}^* - \boldsymbol{\xi}\left(t - \frac{\|\mathbf{x}^* - \mathbf{x}\|}{c}\right) = \mathbf{0} \qquad \text{(repeat4-2.21)}$$

Then equation (4-2.13) is written as

$$\overbrace{(\mathbf{x}' - \mathbf{u})}^{\mathbf{y}'} - \boldsymbol{\xi}\left(t - \frac{\|\overbrace{(\mathbf{x}' - \mathbf{u})}^{\mathbf{y}'} - \overbrace{(\mathbf{x} - \mathbf{u})}^{\mathbf{y}}\|}{c}\right) = \mathbf{0} \tag{4-2.92}$$

or

$$\mathbf{y}' - \boldsymbol{\xi}\left(t - \frac{\|\mathbf{y}' - \mathbf{y}\|}{c}\right) = \mathbf{0} \tag{4-2.93}$$

with

$$\mathbf{y}' = \mathbf{x}' - \mathbf{u}\,, \qquad \mathbf{y} = \mathbf{x} - \mathbf{u} \tag{4-2.94}$$

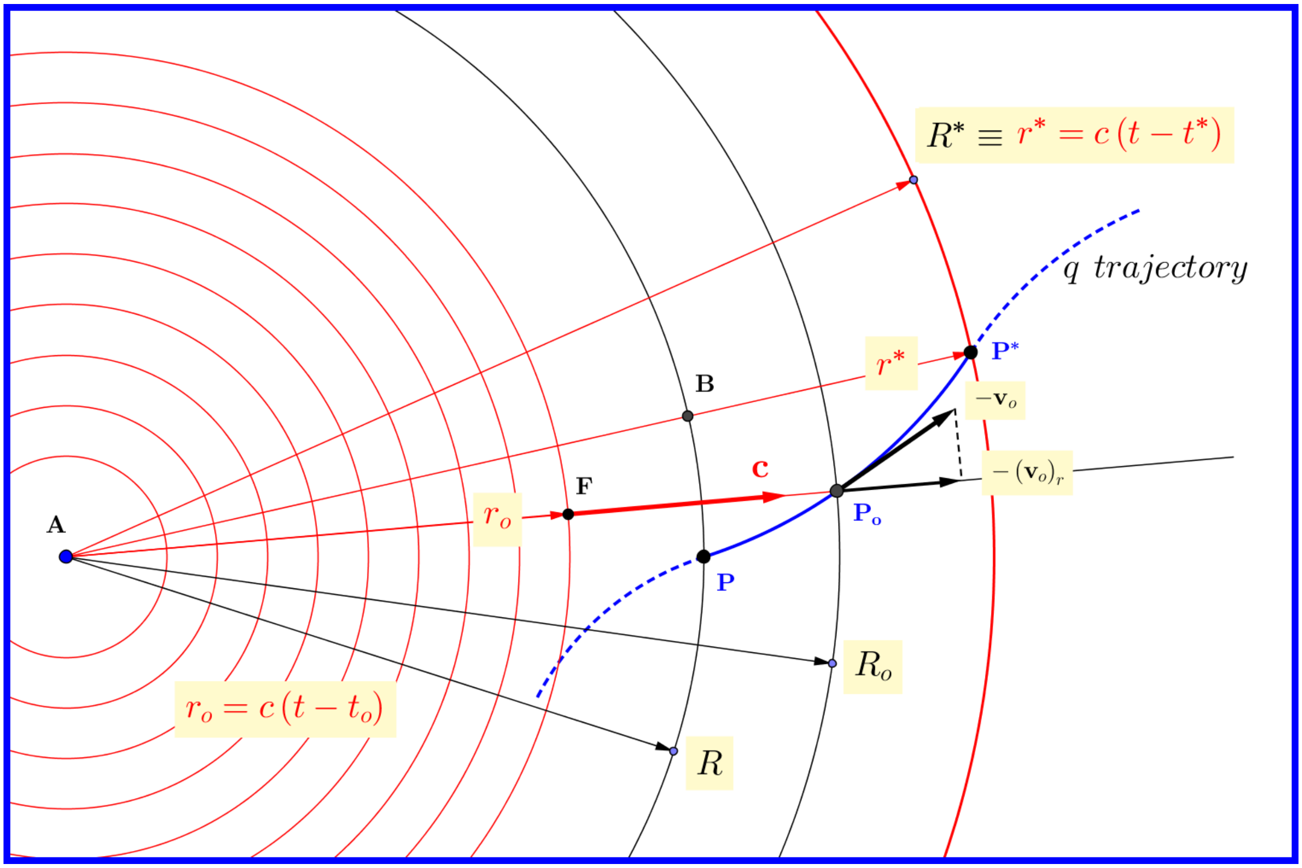

Figure 4.8: Proving the existence of a retarded position and time

with existing and unique solution

$$\mathbf{y}' = \mathbf{x}^*(\mathbf{y}, t) = \mathbf{x}^*(\mathbf{x} - \mathbf{u}, t) \tag{4-2.95}$$

so

$$\mathbf{x}' = \mathbf{u} + \mathbf{y}' = \mathbf{u} + \mathbf{x}^*(\mathbf{x} - \mathbf{u}, t) = \mathbf{F}^{-1}(\mathbf{u}) \tag{4-2.96}$$

Let proceed now to prove the existence and uniqueness of the solution $\mathbf{x}^*(\mathbf{x}, t)$ (for any $\mathbf{x}$ and t) of equation (4-2.21).

We'll travel backwards in time as shown in Figure 4.5: [9] when the charge is at present point P on present time t, a spherical light wave is emitted from field point A (position vector $\mathbf{x}$) to the past. The charge q starts moving backwards in time from its present position P. In Figure 4.6 [10] events are shown at a moment, say t_o $(t^* < t_o < t)$, as the system is moving backwards in time. In Figure 4.7 [11] the light wave, "running" with speed c, is arresting on the retarded time t^* at the retarded point P* on its trajectory the backwards in time moving charge q. That this would happen at least once is shown in Figure 4.8: the charge, being at point $\mathrm{P_o}$ on time t_o has radial speed $\|(\mathbf{v}_o)_r\|$ less than c since

[9] see 3D version of Figure 4.5 in Appendix I, Figure I.11
[10] see 3D version of Figure 4.6 in Appendix I, Figure I.12
[11] see 3D version of Figure 4.7 in Appendix I, Figure I.13

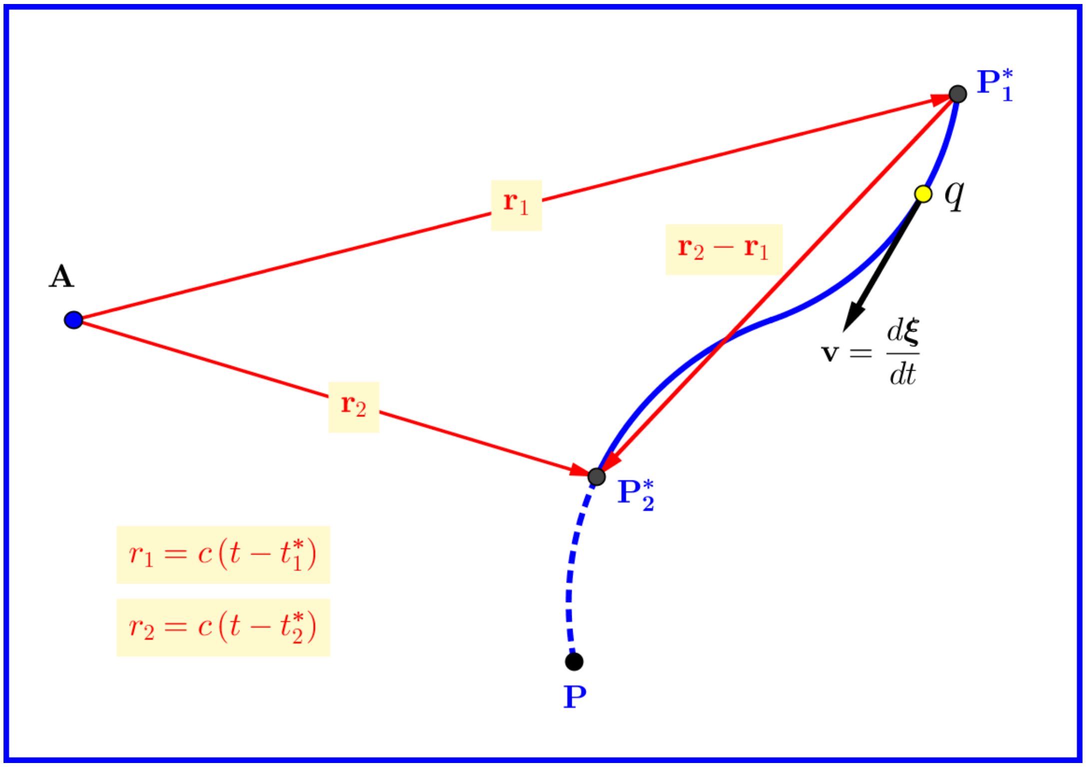

Figure 4.9: Proving the uniqueness of the retarded position and time

$$\|\,(\mathbf{v}_o)_r\,\| \le \|\,(\mathbf{v}_o)\,\| = \left\|\frac{d\boldsymbol{\xi}\,(t_o)}{dt}\right\| < c \tag{4-2.97}$$

according to the assumption that the instantaneous speed of the charge q never exceeds that of light speed c, equation (4-2.7). All will be more clear if we watch the events on the radius $\mathrm{AP_o}$ joining the charge q to the field point A. The spherical wave front on this radius, point F, is moving with speed c always greater than the radial speed $\|\,(\mathbf{v}_o)_r\,\|$ of the charge. So, we have proved the existence of at least one retarded point. Of course, we assume that the charge exists deep in the past (it's not created near the present time, for example).

Now, we'll proceed on the same foot to the proof of uniqueness : suppose that to the present position P of the charge at time t (position vector $\boldsymbol{\xi}\,(t)$) and to the field point A (position vector $\mathbf{x}$) there correspond two different retarded time moments t_1^*, t_2^* with different in general [12] retarded positions $\mathrm{P}_1^*, \mathrm{P}_2^*$ respectively, as in Figure 4.9. Then the distance Δs travelled by the charge on its trajectory in the time interval $[t_1^*, t_2^*]$ is

$$\Delta s = \left|\int_{t_1^*}^{t_2^*}\left\|\frac{d\boldsymbol{\xi}\,(t)}{dt}\right\|dt\right| \tag{4-2.98}$$

This is the length of the generally curved trajectory of the charge between points $\mathrm{P}_1^*, \mathrm{P}_2^*$, greater or equal to the length of the straight segment $\mathrm{P}_1^*\mathrm{P}_2^*$, so

[12] in general, since there exists the special case of the charge describing a closed loop, that is $\mathrm{P}_2^* \equiv \mathrm{P}_1^*$.

$$\Delta s = \left| \int_{t_1^*}^{t_2^*} \left\| \frac{d\boldsymbol{\xi}(t)}{dt} \right\| dt \right| \geq \|\mathbf{r}_2 - \mathbf{r}_1\| \geq |\|\mathbf{r}_2\| - \|\mathbf{r}_1\|| = |r_2 - r_1| = c\,|t_2^* - t_1^*| \tag{4-2.99}$$

that is

$$\frac{\Delta s}{|t_2^* - t_1^*|} = \frac{1}{|t_2^* - t_1^*|} \left| \int_{t_1^*}^{t_2^*} \left\| \frac{d\boldsymbol{\xi}(t)}{dt} \right\| dt \right| \geq c \tag{4-2.100}$$

which means that ”the mean value of the charge speed in the referred time interval is greater or equal to that of light c”, in contradiction to the hypothesis of equation (4-2.7)

$$\left\| \frac{d\boldsymbol{\xi}(t)}{dt} \right\| < c \tag{repeat4-2.7}$$

This completes the proof about the uniqueness of the retarded time and position.

4-2.5 The Jacobian $\dfrac{\partial(u_1, u_2, u_3)}{\partial(x_1', x_2', x_3')}$ of $\mathbf{u} = \mathbf{F}(\mathbf{x}') = \mathbf{x}' - \boldsymbol{\xi}\left(t - \dfrac{\|\mathbf{x}' - \mathbf{x}\|}{c}\right)$

In this Subsection we'll find a general expression for the Jacobian of the vector function $\mathbf{F}(\mathbf{x}')$ of the vector variable $\mathbf{x}'$ that is defined in equation (4-2.13) and is repeated here for convenience

$$\mathbf{u} = \mathbf{x}' - \boldsymbol{\xi}\left(t - \frac{\|\mathbf{x}' - \mathbf{x}\|}{c}\right) = \mathbf{x}' - \boldsymbol{\xi}(t') = \mathbf{F}(\mathbf{x}') \tag{4-2.101}$$

where

$$t' = t - \frac{\|\mathbf{x}' - \mathbf{x}\|}{c} = t - \frac{\sqrt{(x_1' - x_1)^2 + (x_2' - x_2)^2 + (x_3' - x_3)^2}}{c} \tag{4-2.102}$$

as defined in equation (4-2.5) repeated also here.

Note that this function represents a transformation or better a variable change from $\mathbf{x}'$ to $\mathbf{u}$.

The Jacobian of a vector function is a determinant as defined in equation (4-2.88) written also as

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1', x_2', x_3')} = \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1'} & \dfrac{\partial u_1}{\partial x_2'} & \dfrac{\partial u_1}{\partial x_3'} \\ \dfrac{\partial u_2}{\partial x_1'} & \dfrac{\partial u_2}{\partial x_2'} & \dfrac{\partial u_2}{\partial x_3'} \\ \dfrac{\partial u_3}{\partial x_1'} & \dfrac{\partial u_3}{\partial x_2'} & \dfrac{\partial u_3}{\partial x_3'} \end{vmatrix} = \nabla' u_1 \circ (\nabla' u_2 \times \nabla' u_2) \tag{4-2.103}$$

where

$$\nabla' \stackrel{\text{def}}{\equiv} \left(\frac{\partial}{\partial x_1'}, \frac{\partial}{\partial x_2'}, \frac{\partial}{\partial x_3'}\right) \tag{4-2.104}$$

Although we'll find a general expression for $\dfrac{\partial u_j}{\partial x'_k}$ it would be better to write down equation (4-2.101) in components, in order to understand what is going on with a large number of differentiations :

$$u_1 = x'_1 - \xi_1(t') = x'_1 - \xi_1\left(t - \frac{\sqrt{(x'_1 - x_1)^2 + (x'_2 - x_2)^2 + (x'_3 - x_3)^2}}{c}\right) \tag{4-2.105}$$

$$u_2 = x'_2 - \xi_2(t') = x'_2 - \xi_2\left(t - \frac{\sqrt{(x'_1 - x_1)^2 + (x'_2 - x_2)^2 + (x'_3 - x_3)^2}}{c}\right) \tag{4-2.106}$$

$$u_3 = x'_3 - \xi_3(t') = x'_3 - \xi_3\left(t - \frac{\sqrt{(x'_1 - x_1)^2 + (x'_2 - x_2)^2 + (x'_3 - x_3)^2}}{c}\right) \tag{4-2.107}$$

or in one stroke

$$u_j = x'_j - \xi_j(t') = x'_j - \xi_j\left(t - \frac{\sqrt{(x'_1 - x_1)^2 + (x'_2 - x_2)^2 + (x'_3 - x_3)^2}}{c}\right) \tag{4-2.108}$$

From this last equation

$$\frac{\partial u_j}{\partial x'_k} = \frac{\partial x_j}{\partial x'_k} - \frac{\partial \xi_j(t')}{\partial x'_k} = \delta_{jk} - \frac{d\xi_j(t')}{dt'}\frac{\partial t'}{\partial x'_k} \tag{4-2.109}$$

From equation (4-2.102)

$$\frac{\partial t'}{\partial x'_k} = -\frac{(x'_k - x_k)}{c\sqrt{(x'_1 - x_1)^2 + (x'_2 - x_2)^2 + (x'_3 - x_3)^2}} = \frac{(x_k - x'_k)}{c\left\|\mathbf{x} - \mathbf{x}'\right\|} = \frac{n_k}{c} \tag{4-2.110}$$

where $\mathbf{n} = (n_1, n_2, n_3)$ the unit vector

$$\mathbf{n} \overset{\text{def}}{\equiv} \frac{\mathbf{x} - \mathbf{x}'}{\left\|\mathbf{x} - \mathbf{x}'\right\|} \tag{4-2.111}$$

so equation (4-2.109) yields

$$\frac{\partial u_j}{\partial x'_k} = \delta_{jk} - \frac{\upsilon_j(t')}{c} n_k \tag{4-2.112}$$

where $\upsilon_j(t)$ the j-component of the charge q velocity vector $\mathbf{v}(t)$, see equation (4-2.10).

Now, let the basic vectors of the orthonormal system of coordinates $O'x'_1x'_2x'_3$ be

$$\mathbf{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \mathbf{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \tag{4-2.113}$$

Equation (4-2.112), under the Einstein's convention for the summation with respect to repeated indices, yields

$$\nabla' u_j = \frac{\partial u_j}{\partial x'_k}\mathbf{e}_k = \delta_{jk}\mathbf{e}_k - \frac{\upsilon_j(t')}{c}n_k\mathbf{e}_k \tag{4-2.114}$$

that is

$$\nabla' u_j = \mathbf{e}_j - \frac{\upsilon_j(t')}{c}\mathbf{n} \tag{4-2.115}$$

After this detailed analysis on differentiations and since now we have a feeling what's going on, we note that equation (4-2.115) could be extracted in one stroke applying the operator ∇' to equation (4-2.108)

$$\nabla' u_j = \underbrace{\nabla' x'_j}_{\mathbf{e}_j} - \nabla'\xi_j(t') = \mathbf{e}_j - \frac{d\xi_j(t')}{dt'}\nabla' t' = \mathbf{e}_j - \frac{\upsilon_j(t')}{c}\mathbf{n} \tag{4-2.116}$$

since from equation (4-2.108)

$$\nabla' t' = \frac{\mathbf{n}}{c} = \frac{\mathbf{x}-\mathbf{x}'}{c\,\|\mathbf{x}-\mathbf{x}'\|} \tag{4-2.117}$$

Returning now to our Jacobian, equation (4-2.103) we have

$$\begin{aligned}
\frac{\partial(u_1,u_2,u_3)}{\partial(x'_1,x'_2,x'_3)} &= \begin{vmatrix} \dfrac{\partial u_1}{\partial x'_1} & \dfrac{\partial u_1}{\partial x'_2} & \dfrac{\partial u_1}{\partial x'_3} \\ \dfrac{\partial u_2}{\partial x'_1} & \dfrac{\partial u_2}{\partial x'_2} & \dfrac{\partial u_2}{\partial x'_3} \\ \dfrac{\partial u_3}{\partial x'_1} & \dfrac{\partial u_3}{\partial x'_2} & \dfrac{\partial u_3}{\partial x'_3} \end{vmatrix} = \nabla' u_1 \circ (\nabla' u_2 \times \nabla' u_2) \\
&= \left(\mathbf{e}_1 - \frac{\upsilon_1(t')}{c}\mathbf{n}\right) \circ \left[\left(\mathbf{e}_2 - \frac{\upsilon_2(t')}{c}\mathbf{n}\right) \times \left(\mathbf{e}_3 - \frac{\upsilon_3(t')}{c}\mathbf{n}\right)\right] \\
&= \left(\mathbf{e}_1 - \frac{\upsilon_1(t')}{c}\mathbf{n}\right) \circ \left[\underbrace{(\mathbf{e}_2 \times \mathbf{e}_3)}_{\mathbf{e}_1} - \frac{\upsilon_2(t')}{c}(\mathbf{n}\times\mathbf{e}_3) - \frac{\upsilon_3(t')}{c}(\mathbf{e}_2\times\mathbf{n})\right] \\
&= (\mathbf{e}_1\circ\mathbf{e}_1) - \frac{\upsilon_1(t')}{c}(\mathbf{e}_1\circ\mathbf{n}) - \frac{\upsilon_2(t')}{c}\underbrace{[\mathbf{e}_1\circ(\mathbf{n}\times\mathbf{e}_3)]}_{(\mathbf{e}_2\circ\mathbf{n})} - \frac{\upsilon_3(t')}{c}\underbrace{[\mathbf{e}_1\circ(\mathbf{e}_2\times\mathbf{n})]}_{(\mathbf{e}_3\circ\mathbf{n})}
\end{aligned}$$

so

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x'_1,x'_2,x'_3)} = 1 - \left(\frac{\upsilon_1(t')\,\mathbf{e}_1 + \upsilon_2(t')\,\mathbf{e}_2 + \upsilon_3(t')\,\mathbf{e}_3}{c}\right)\circ\mathbf{n}$$

and finally

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x'_1,x'_2,x'_3)} = 1 - \frac{\mathbf{v}(t')\circ\mathbf{n}}{c} = 1 - \frac{\mathbf{v}\left(t - \dfrac{\|\mathbf{x}-\mathbf{x}'\|}{c}\right)\circ\left(\dfrac{\mathbf{x}-\mathbf{x}'}{\|\mathbf{x}-\mathbf{x}'\|}\right)}{c} \tag{4-2.118}$$

In the expressions (4-2.18) and (4-2.19) of the scalar and vector potentials respectively, the value of this Jacobian is needed at $\mathbf{u} = \mathbf{0}$ or equivalently at $\mathbf{x}' = \mathbf{x}^*\,(= \boldsymbol{\xi}(t^*))$

$$\left[\frac{\partial\left(u_1,u_2,u_3\right)}{\partial\left(x_1',x_2',x_3'\right)}\right]_{\mathbf{u}=\mathbf{0}} = \left[\frac{\partial\left(u_1,u_2,u_3\right)}{\partial\left(x_1',x_2',x_3'\right)}\right]_{\mathbf{x}'=\mathbf{x}^*} = 1-\frac{\mathbf{v}\left(t^*\right)\circ\mathbf{n}_{\mathbf{R}}}{c}$$
$$= 1-\frac{\mathbf{v}\left(t-\dfrac{\|\mathbf{x}-\mathbf{x}^*\|}{c}\right)\circ\left(\dfrac{\mathbf{x}-\mathbf{x}^*}{\|\mathbf{x}-\mathbf{x}^*\|}\right)}{c}$$
$$= 1-\frac{\mathbf{v}\left(t-\dfrac{R}{c}\right)\circ\left(\dfrac{\mathbf{R}}{R}\right)}{c} \stackrel{\text{def}}{\equiv} \varkappa \tag{4-2.119}$$

where

$$\mathbf{R}=\mathbf{x}-\mathbf{x}^*=\mathbf{x}-\boldsymbol{\xi}\left(t^*\right) \tag{4-2.120}$$

$$R=\|\mathbf{R}\|=\|\mathbf{x}-\mathbf{x}^*\| \tag{4-2.121}$$

$$\mathbf{n}_{\mathbf{R}}=\frac{\mathbf{R}}{\|\mathbf{R}\|}=\frac{\mathbf{x}-\mathbf{x}^*}{\|\mathbf{x}-\mathbf{x}^*\|} \tag{4-2.122}$$

Note that $\mathbf{R}$ is the position vector of the field point as seen from the retarded position of the charge, see Figure 4.4 in Subsection 4-2.1

Appendix I

3D FIGURES

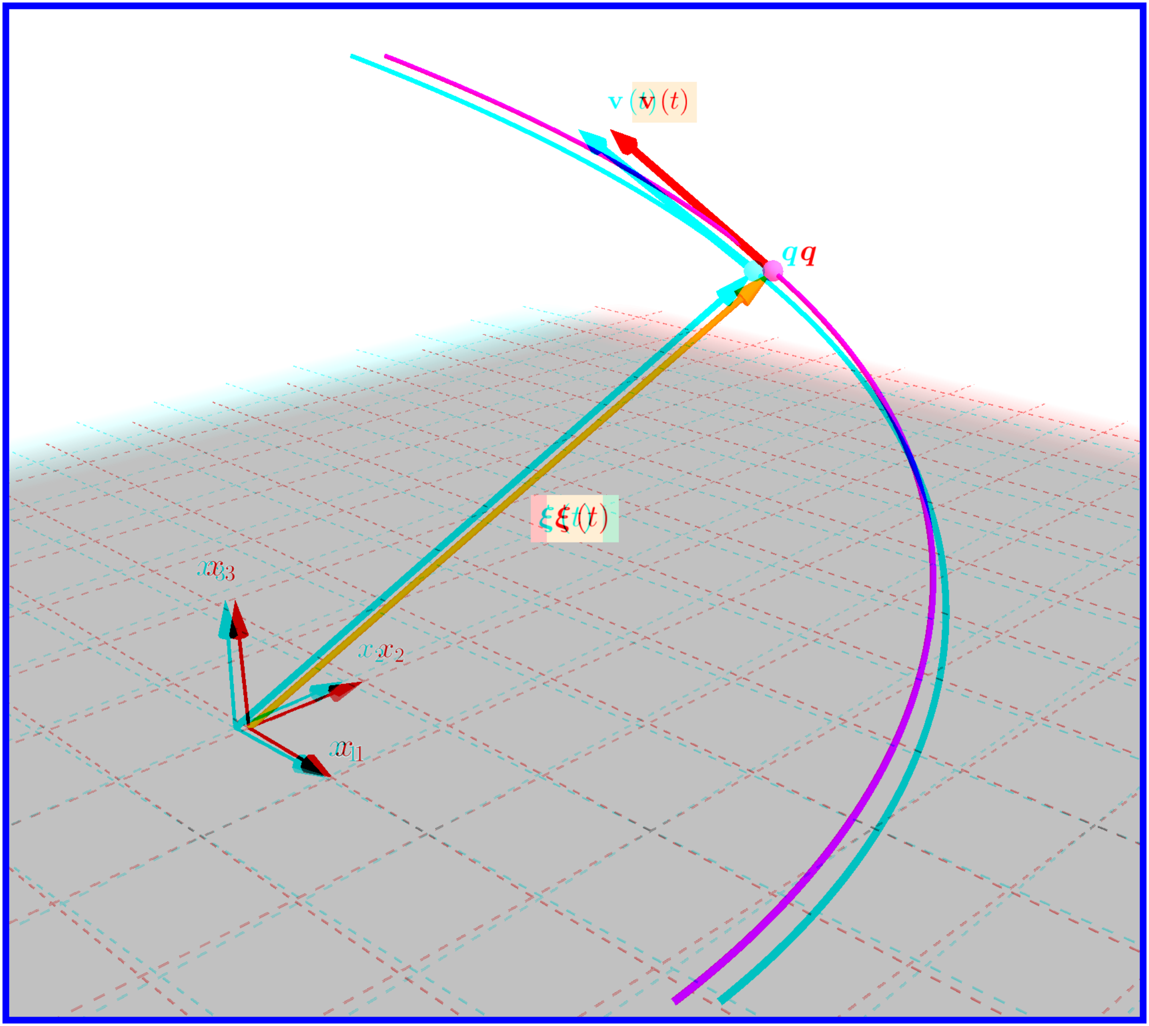

Figure I.7: Charge q moving in any arbitrary way $\boldsymbol{\xi}\,(t)$.

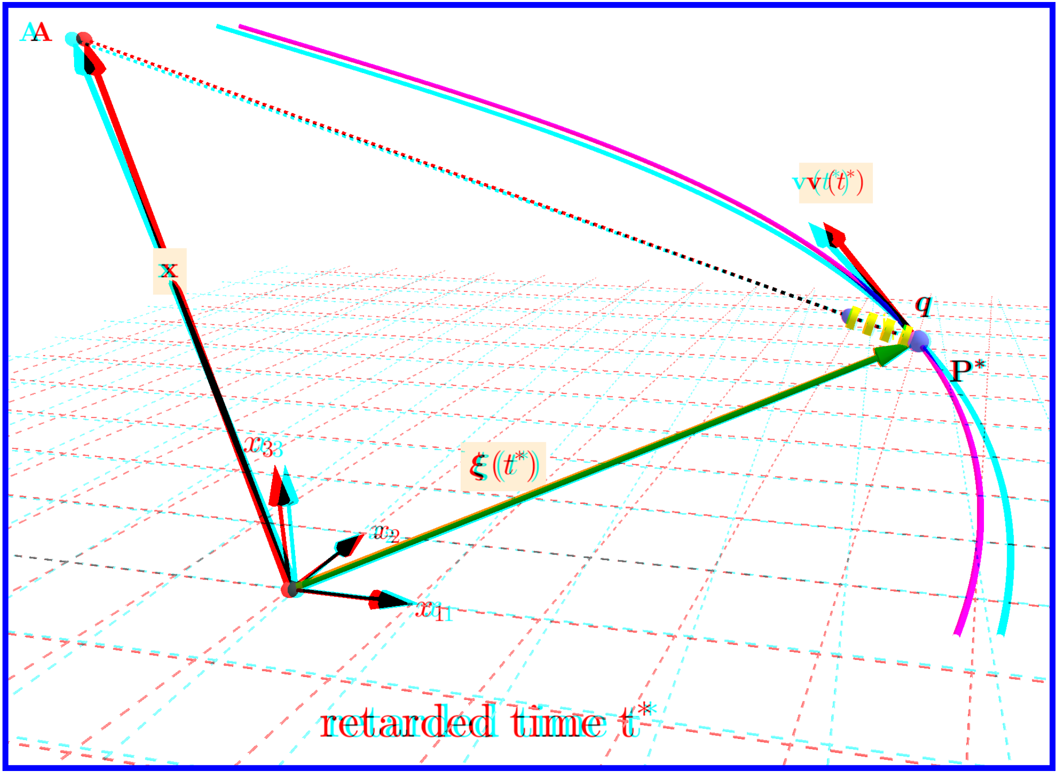

Figure I.8: Charge q at point P^* emits a light beam at time t^* towards field point A.

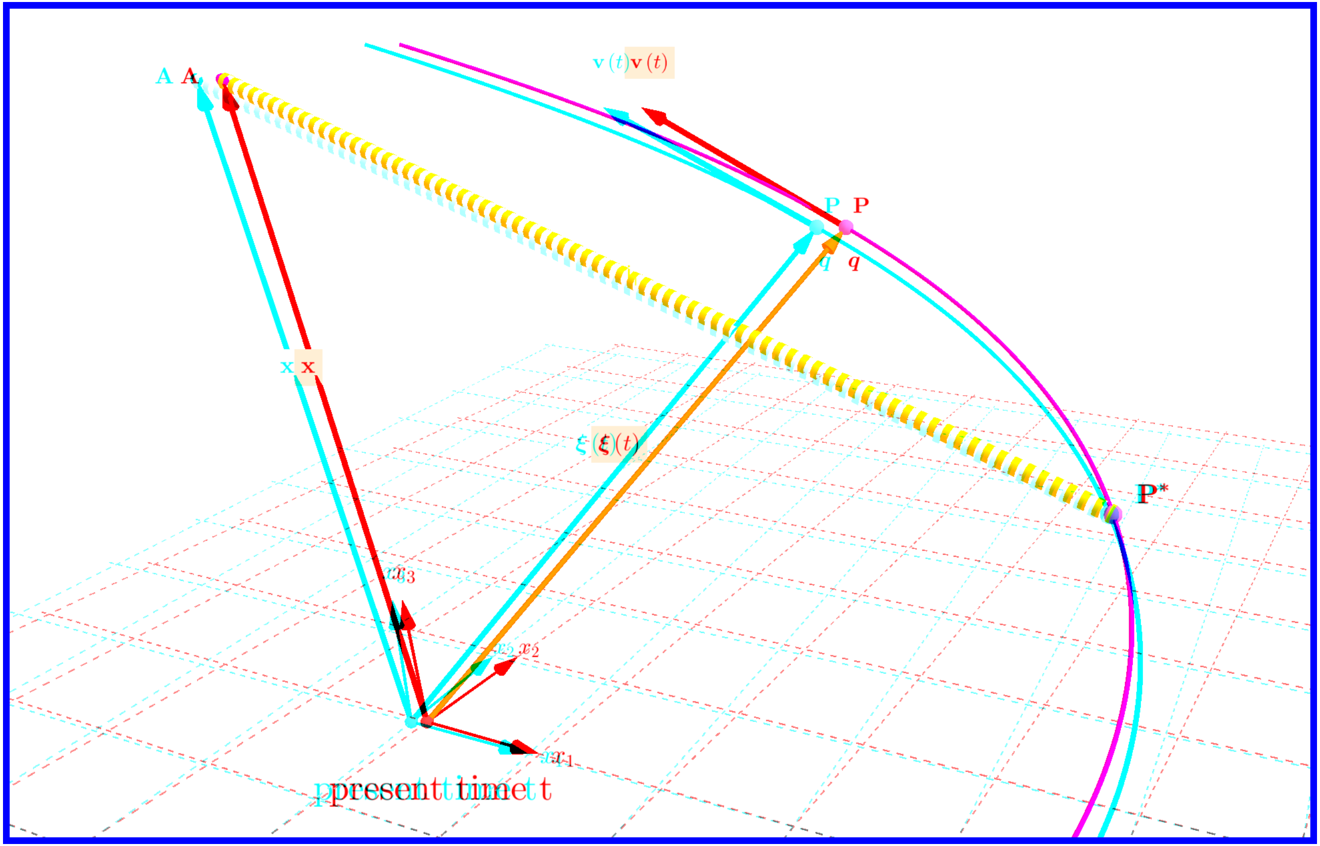

Figure I.9: Charge q is at point P at time t when its emitted (from P* at time t^*) light beam arrives at field point A.

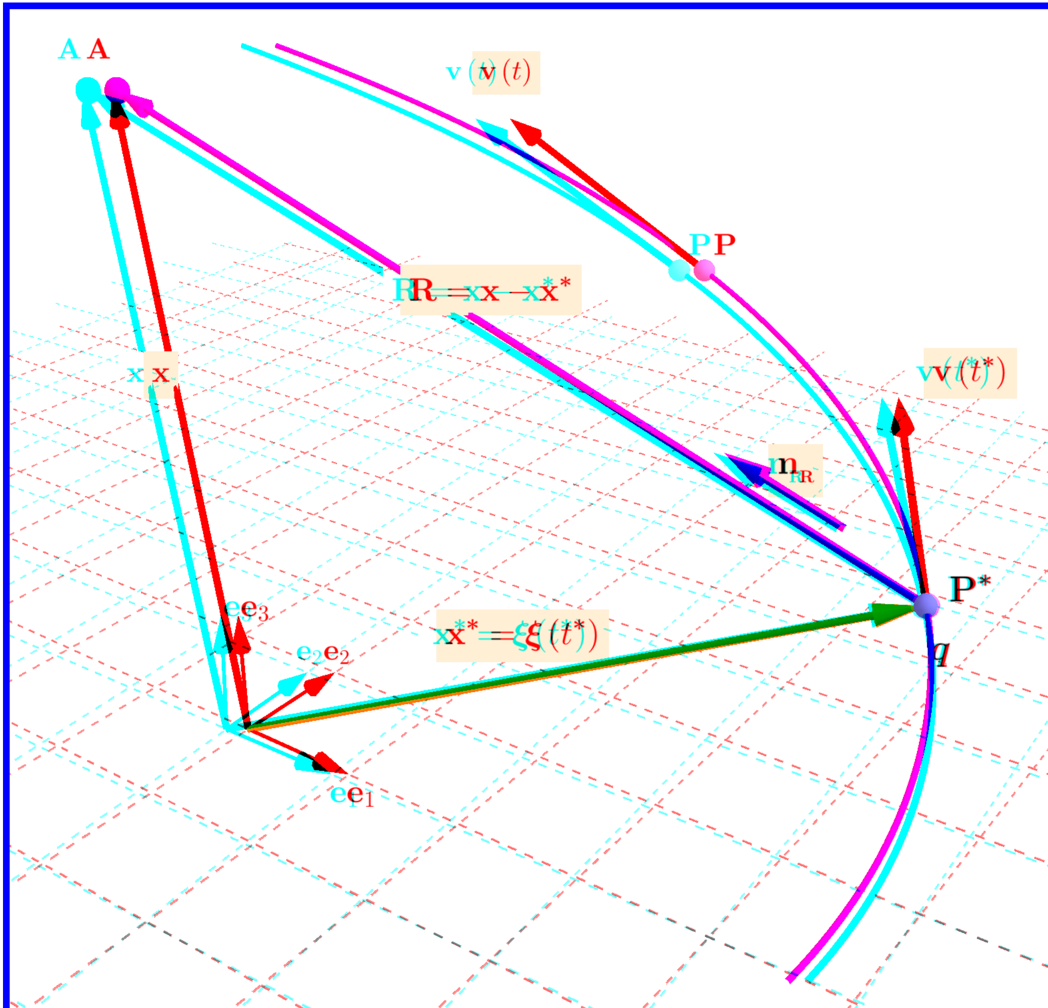

Figure I.10: The field point A as seen by charge q from the retarded point P^* and time t^* of the later.

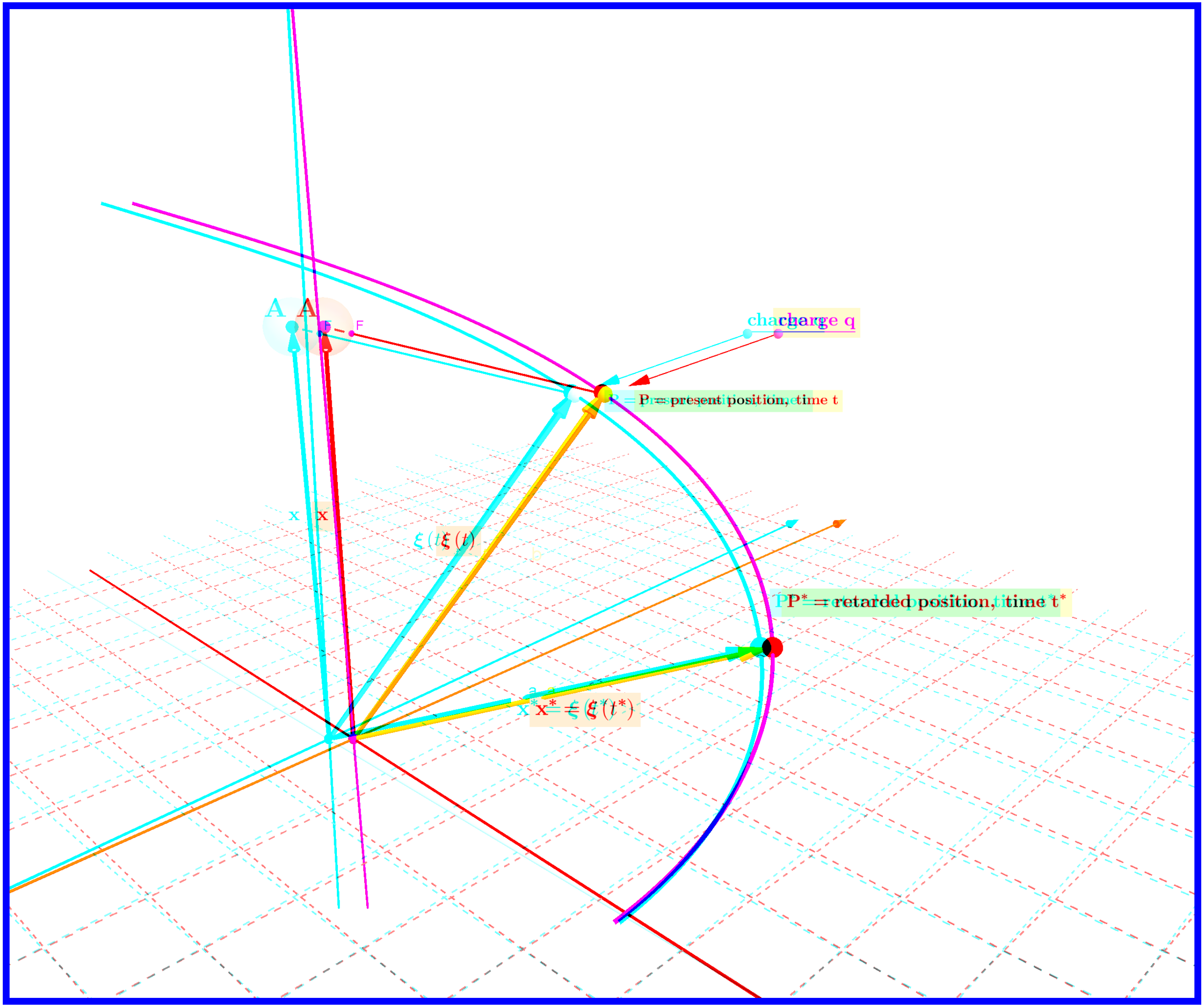

Figure I.11: When the charge q is at its present position, point P time t, a spherical light wave is emitted from field point A to the past and the video of the motion of the charge is played from t backwards in time.

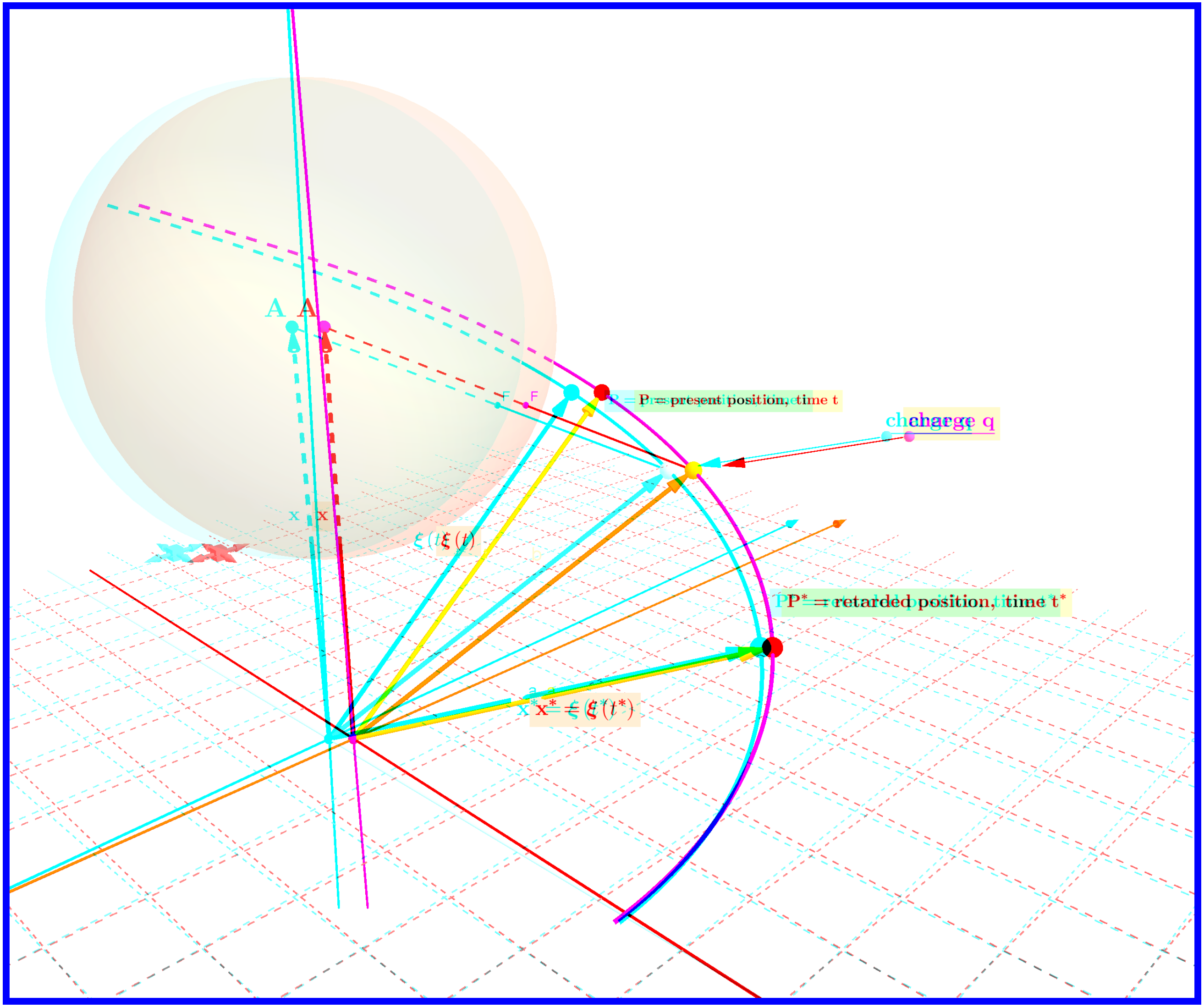

Figure I.12: The spherical light wave front emitted to the past, see Figure I.11, is moving with radial speed c greater than the radial speed of the charge, so coming closer and closer to arrest on its trajectory the backwards in time moving charge q .

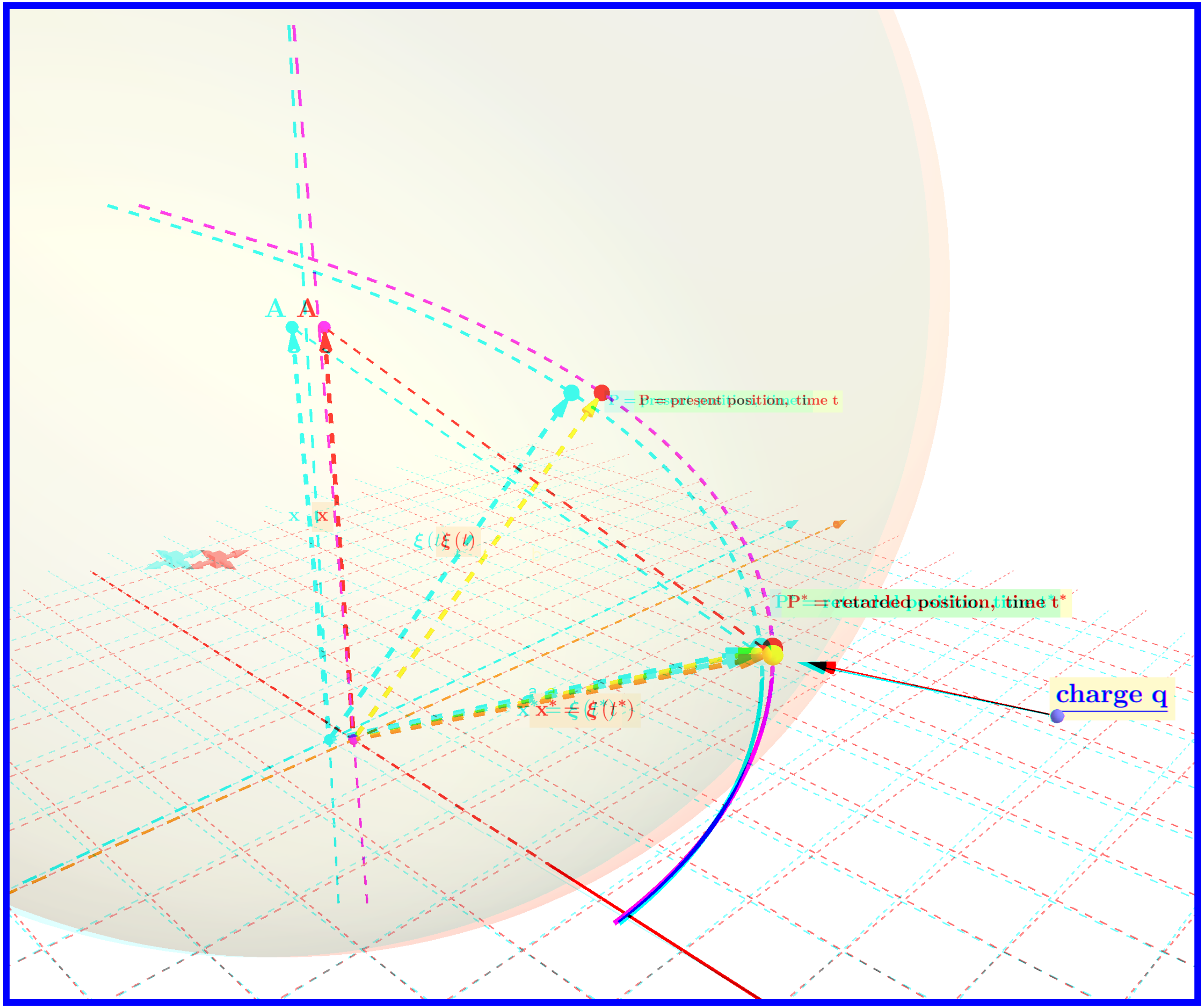

Figure I.13: The spherical light wave front emitted to the past, see Figures I.11 and I.12 , is arresting at the retarded time t^* on the retarded point P^* on its trajectory the backwards in time moving charge q .