Electromagnetism: Worked Examples

University of Oxford

Second Year, Part A2

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Contents

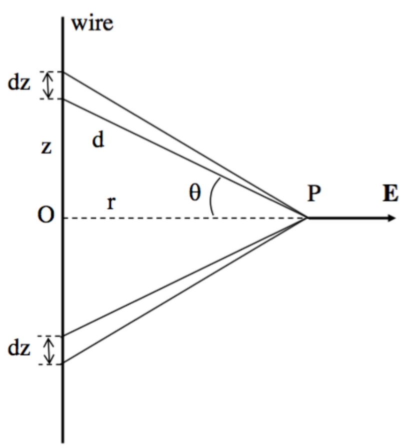
Most of the problems are taken from: David J. Griffiths, *Introduction to Electrodynamics*, 4th edition (Cambridge University Press)

Chapter 1

Potentials

1.1 Potential of an infinite line charge

(a) *A straight wire, assumed infinitely long, has a constant linear charge density λ. Calculate the electric field due to the wire. Use this result to calculate the electric potential.*



Given the symmetry of the system, the electric field at the point P is:

$$E = \int_{\text{wire}} \frac{\lambda dz}{4\pi\epsilon_0 d^2} \cos\theta.$$

Using $d = r/\cos\theta$ and $z = r\tan\theta$, which yields $dz = rd\theta/\cos^2\theta$, we get:

$$E = \frac{\lambda}{4\pi\epsilon_0 r}\int_{-\pi/2}^{\pi/2} \cos\theta d\theta = \frac{\lambda}{2\pi\epsilon_0 r}.$$

Using cylindrical coordinates, $\mathbf{E} = -\boldsymbol{\nabla} V$ yields:

$$V = -\frac{\lambda}{2\pi\epsilon_0}\ln r + C,$$

where C is a constant. We see that the reference point for V cannot be taken at infinity (that is, we cannot have $V = 0$ when $r \to +\infty$).

(b) *Show that equation (1.12) from the lecture notes cannot be used to calculate the potential.*

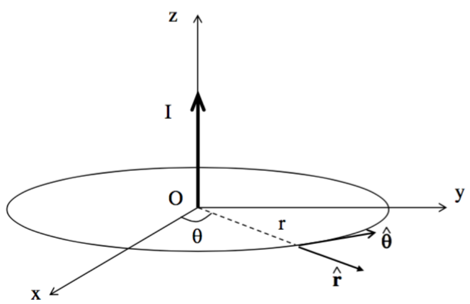
Using equation (1.12) would yield:

$$V = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty} \frac{dz}{\sqrt{r^2+z^2}} = \frac{\lambda}{4\pi\epsilon_0}\left[\text{arcsinh}(x)\right]_{-\infty}^{+\infty},$$

which gives an infinite result. Equation (1.12) is not valid here because it assumes that the reference point for the potential is at infinity, which cannot be satisfied when the wire extends to infinity.

1.2 Vector potential of an infinite wire

(a) *A straight wire, assumed infinitely long, carries a current I. Calculate the magnetic field due to the wire. Use this result to calculate the vector potential.*



Assuming the current is along the z–direction, Ampère's law gives the magnetic field:

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\,\hat{\boldsymbol{\theta}},$$

where $\hat{\boldsymbol{\theta}}$ is the unit vector in the azimuthal direction.

We look for $\mathbf{A}$ such that $\boldsymbol{\nabla}\times\mathbf{A} = \mathbf{B} = B\hat{\boldsymbol{\theta}}$. In cylindrial coordinates, the θ–component of the curl is $\partial A_r/\partial z - \partial A_z/\partial r$. Given the symmetry of the system, $\mathbf{A}$ cannot depend on z, so that we are left with:

$$-\frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi r} \Longrightarrow \mathbf{A} = -\frac{\mu_0 I}{2\pi}\ln r\,\hat{\mathbf{z}}.$$

Note that any vector with zero curl could be added to this solution.

(b) *Show that equation (1.13) (or equation [1.16]) from the lecture notes cannot be used to calculate the vector potential.*

Using the z–component of equation (1.16) would yield:

$$A_z = \frac{\mu_0}{4\pi}\int_{-\infty}^{+\infty}\frac{I dz}{\sqrt{r^2+z^2}} = \frac{\mu_0}{4\pi}\left[\text{arcsinh}(x)\right]_{-\infty}^{+\infty},$$

which gives an infinite result. As in the case of the infinite line charge above, equation (1.16) is not valid here because it assumes that the reference point for the vector potential is at infinity, which cannot be satisfied when the wire extends to infinity.

1.3 Potential of a sphere with a charge density $\sigma = k\cos\theta$

We consider a sphere of radius R and a spherical coordinate system (r, θ, φ) with origin at the centre of the sphere. A charge density $\sigma = k\cos\theta$ is glued over the surface of the sphere. Calculate the potential inside and outside the sphere using separation of variables.

There is no charge inside and outside the sphere so that the potential V satisfies Laplace's equation $\nabla^2 V = 0$ at any $r \neq R$. As the charge density does not depend on φ, V is also independent of φ. The general solution of Laplace's equation in the axisymmetric case is given by equation (1.32) from the lecture notes. We can then write the potential inside and outside the sphere as:

$$\boxed{\begin{aligned} V_{\text{in}}(r,\theta) &= \sum_{l=0}^{\infty}\left(\alpha_l^{\text{in}} r^l + \frac{\beta_l^{\text{in}}}{r^{l+1}}\right) P_l(\cos\theta), \\ V_{\text{out}}(r,\theta) &= \sum_{l=0}^{\infty}\left(\alpha_l^{\text{out}} r^l + \frac{\beta_l^{\text{out}}}{r^{l+1}}\right) P_l(\cos\theta). \end{aligned}} \tag{1.1}$$

Boundary conditions:

$$\begin{aligned} &\text{(i)} \quad V_{\text{out}}(r,\theta) \to 0 \quad \text{as} \quad r \to +\infty, \\ &\text{(ii)} \quad V_{\text{in}}(r,\theta) \ \text{finite inside the sphere}, \\ &\text{(iii)} \quad V_{\text{in}}(R,\theta) = V_{\text{out}}(R,\theta) \ \text{(V continuous)}, \\ &\text{(iv)} \quad \left|\mathbf{E}_{\text{out}}^{\perp}\right| - \left|\mathbf{E}_{\text{in}}^{\perp}\right| = \frac{\sigma}{\epsilon_0} \Rightarrow -\frac{\partial V_{\text{out}}}{\partial r} + \frac{\partial V_{\text{in}}}{\partial r} = \frac{k\cos\theta}{\epsilon_0}. \end{aligned}$$

General method (always works): we use the boundary conditions to calculate all the coefficients.

- Condition (i) implies $\alpha_l^{\text{out}} = 0 \ \ \forall\, l$, condition (ii) implies $\beta_l^{\text{in}} = 0 \ \ \forall\, l$, so that equations (1.1) become:

$$\boxed{\begin{aligned} V_{\text{in}}(r,\theta) &= \sum_{l=0}^{\infty} \alpha_l^{\text{in}} r^l P_l(\cos\theta), \\ V_{\text{out}}(r,\theta) &= \sum_{l=0}^{\infty} \frac{\beta_l^{\text{out}}}{r^{l+1}} P_l(\cos\theta). \end{aligned}} \tag{1.2}$$

- Then condition (iii) implies:

$$\sum_{l=0}^{\infty} \alpha_l^{\text{in}} R^l P_l(\cos\theta) = \sum_{l'=0}^{\infty} \frac{\beta_{l'}^{\text{out}}}{R^{l'+1}} P_{l'}(\cos\theta),$$

where we have renamed the index of summation l' in the right–hand side to make things more clear.

We now multiply each side of this equality by $P_m(\cos\theta)\sin\theta$ and integrate from 0 to π:

$$\sum_{l=0}^{\infty}\alpha_l^{\text{in}}R^l\int_0^{\pi}P_l(\cos\theta)P_m(\cos\theta)\sin\theta d\theta=\sum_{l'=0}^{\infty}\frac{\beta_{l'}^{\text{out}}}{R^{l'+1}}\int_0^{\pi}P_{l'}(\cos\theta)P_m(\cos\theta)\sin\theta d\theta.$$

Using the orthogonality of the Legendre polynomials:

$$\int_0^{\pi}P_l(\cos\theta)P_m(\cos\theta)\sin\theta d\theta=\frac{2}{2l+1}\delta_{l,m},$$

we obtain $\alpha_m^{\text{in}}R^m=\beta_m^{\text{out}}/R^{m+1}\ \ \forall\, m$. Therefore, equations (1.2) become:

$$\boxed{\begin{aligned}V_{\text{in}}(r,\theta)&=\sum_{l=0}^{\infty}\alpha_l^{\text{in}}r^lP_l(\cos\theta),\\ V_{\text{out}}(r,\theta)&=\sum_{l=0}^{\infty}\alpha_l^{\text{in}}\frac{R^{2l+1}}{r^{l+1}}P_l(\cos\theta).\end{aligned}}\tag{1.3}$$

- Then condition (iv) implies:

$$\sum_{l=0}^{\infty}(2l+1)\alpha_l^{\text{in}}R^{l-1}P_l(\cos\theta)=\frac{k\cos\theta}{\epsilon_0}=\frac{k}{\epsilon_0}P_1(\cos\theta),$$

where we have used the fact that $\cos\theta=P_1(\cos\theta)$.
Here again, we multiply each side of this equality by $P_m(\cos\theta)\sin\theta$ and integrate from 0 to π:

$$\sum_{l=0}^{\infty}(2l+1)\alpha_l^{\text{in}}R^{l-1}\int_0^{\pi}P_l(\cos\theta)P_m(\cos\theta)\sin\theta d\theta=\frac{k}{\epsilon_0}\int_0^{\pi}P_1(\cos\theta)P_m(\cos\theta)\sin\theta d\theta$$

The orthogonality of the Legendre polynomials then yields:

$$2\alpha_m^{\text{in}}R^{m-1}=\frac{2k}{3\epsilon_0}\delta_{1,m},$$

that is, $\alpha_m^{\text{in}}=0$ for $m\neq 1$ and $\alpha_1^{\text{in}}=k/(3\epsilon_0)$.
Finally, equations (1.3) then become:

$$\boxed{\begin{aligned}V_{\text{in}}(r,\theta)&=\frac{k}{3\epsilon_0}r\cos\theta,\\ V_{\text{out}}(r,\theta)&=\frac{k}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta.\end{aligned}}\tag{1.4}$$

Shorter method (works in this particular case): we "guess" some of the coefficients and use the boundary conditions to calculate the others.

As the surface charge density is proportional to $\cos\theta$, we expect that the potential itself is propotional to $\cos\theta$. Therefore, we can assume that all the terms in equations (1.1) are 0 except for $l = 1$. We will check *a posteriori* that this assumption is correct. We then write directly:

$$\boxed{\begin{aligned} V_{\rm in}(r,\theta) &= \left(\alpha^{\rm in} r + \frac{\beta^{\rm in}}{r^2}\right)\cos\theta, \\ V_{\rm out}(r,\theta) &= \left(\alpha^{\rm out} r + \frac{\beta^{\rm out}}{r^2}\right)\cos\theta. \end{aligned}} \tag{1.5}$$

As above, the boundary conditions (i) and (ii) yield $\beta^{\rm in} = 0$ and $\alpha^{\rm out} = 0$. Conditions (iii) and (iv) can then be used to calculate $\alpha^{\rm in}$ and $\beta^{\rm out}$.

Finally, we have found a solution that satisfies Laplace's equation and the boundary conditions. As the solution is unique, this is the correct one, and that justifies *a posteriori* the assumption we have made to start with.

1.4 Multipole expansion

A sphere of radius R, centered at the origin, carries charge density:

$$\rho(r,\theta) = k\frac{R}{r^2}(R-2r)\sin\theta,$$

where k is a constant, and r, θ are the usual spherical coordinates. Find the approximate potential for points on the z axis, far from the sphere.

We use equation (1.35) from the lecture notes with O the center of the sphere and M on the z–axis, so that $\gamma = \theta$ and $r = z$.

- Monopole term:

$$V_0(z) = \frac{1}{4\pi\epsilon_0 z}\iiint_{\mathcal{V}} \rho(\mathbf{r})d\tau = \frac{Q}{4\pi\epsilon_0 z},$$

  with $d\tau = r^2 \sin\theta \, dr \, d\theta \, d\varphi$ and Q is the total charge of the sphere. Here:

$$Q = kR\int_{r=0}^{R}(R-2r)\,dr\int_{\varphi=0}^{2\pi}d\varphi\int_{\theta=0}^{\pi}\sin^2\theta d\theta.$$

  The integral over r is 0 so that $\boxed{V_0(z) = 0}$.

- Dipole term:

$$V_1(z) = \frac{1}{4\pi\epsilon_0 z^2}\iiint_{\mathcal{V}} r\cos\theta\rho(\mathbf{r})d\tau.$$

  Here:

$$V_1(z) = \frac{kR}{4\pi\epsilon_0 z^2}\int_{r=0}^{R}(R-2r)\,rdr\int_{\varphi=0}^{2\pi}d\varphi\int_{\theta=0}^{\pi}\sin^2\theta\cos\theta d\theta.$$

  The integral over θ is 0 so that $\boxed{V_1(z) = 0}$.

- Quadrupole term:

$$V_2(z) = \frac{1}{4\pi\epsilon_0 z^3}\iiint_{\mathcal{V}} r^2\frac{1}{2}\left(3\cos^2\theta - 1\right)\rho(\mathbf{r})d\tau.$$

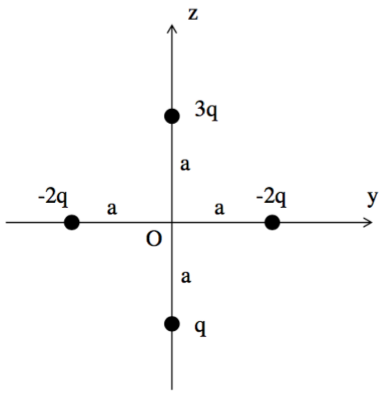
  Here:

$$V_2(z) = \frac{kR}{8\pi\epsilon_0 z^3}\int_{r=0}^{R}(R-2r)\,r^2dr\int_{\varphi=0}^{2\pi}d\varphi\int_{\theta=0}^{\pi}\left(3\cos^2\theta - 1\right)\sin^2\theta d\theta.$$

  The integral over r is $-R^4/6$ and the integral over θ is $-\pi/8$, so that:

$$\boxed{V(z) \simeq V_3(z) = \frac{1}{4\pi\epsilon_0}\frac{k\pi^2R^5}{48z^3}}.$$

1.5 Electric dipole moment



Four charges are placed as shown. Calculate the electric dipole moment of the distribution and find an approximate formula for the potential, valid far from the origin (use spherical coordinates).

Since the total charge is 0, the monopole term is also 0. The dipole term in the expansion of the potential is given by:

$$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2},$$

(equation 1.37 from the lecture notes) where $\mathbf{r}$ is the position vector, $\hat{\mathbf{r}}$ is the unit vector along the radial direction in spherical coordinates and $\mathbf{p}$ is the electric dipole moment. We have:

$$\mathbf{p} = \sum_i q_i \mathbf{r}'_i,$$

where $\mathbf{r}'_i$ denotes the position of the charge q_i. Since the total charge is 0, $\mathbf{p}$ does not depend on the choice of the origin from which we measure the positions. We use O as the origin. Then we have:

$$\mathbf{p} = (3qa - qa)\hat{\mathbf{z}} + (-2qa + 2qa)\hat{\mathbf{y}} = 2qa\hat{\mathbf{z}},$$

where $\hat{\mathbf{z}}$ and $\hat{\mathbf{y}}$ are unit vectors. Therefore:

$$V(\mathbf{r}) \simeq V_1(\mathbf{r}) = \boxed{\frac{1}{4\pi\epsilon_0}\frac{2qa\cos\theta}{r^2}},$$

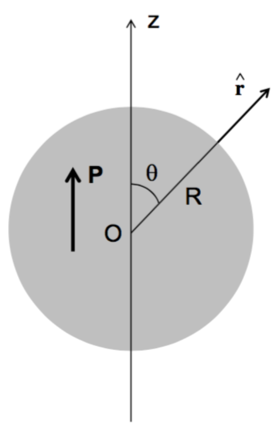
where we have used $\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = \cos\theta$.

Chapter 2

Electric fields in matter

2.1 Field of a uniformly polarized sphere

(a) *Find the electric field produced by a uniformly polarized sphere of radius R. (Hint: use the results of example 1.3)*



Inside the sphere, the volume charge density is $\rho_p = -\nabla \cdot \mathbf{P}$ which is 0 as the polarization is uniform. At the surface of the sphere, there is a surface charge density $\sigma_p = \mathbf{P} \cdot \hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is the unit vector in the radial direction. We choose the z–axis in the direction of $\mathbf{P}$, so that $\sigma_p = P\cos\theta$ in spherical coordinates.

In example 1.3 we have calculated the potential of a sphere with a surface charge density proportional to $\cos\theta$:

$$\boxed{\begin{aligned} V(r,\theta) &= \frac{P}{3\epsilon_0} r\cos\theta, \quad \text{for} \quad r \le R, \\ V(r,\theta) &= \frac{P}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta, \quad \text{for} \quad r \ge R. \end{aligned}} \tag{2.1}$$

The electric field is:

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}},$$

where $\hat{\boldsymbol{\theta}}$ is the unit vector in the azimuthal direction. Therefore:

$$\boxed{\begin{aligned} \mathbf{E} &= -\frac{P}{3\epsilon_0}\hat{\mathbf{z}}, \quad \text{for} \quad r < R, \\ \mathbf{E} &= \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}\hat{\mathbf{r}} + \frac{p\sin\theta}{4\pi\epsilon_0 r^3}\hat{\boldsymbol{\theta}}, \quad \text{for} \quad r > R, \end{aligned}} \tag{2.2}$$

where we have used $r\cos\theta = z$ and we have defined the total dipole moment of the sphere $p = (4/3)\pi R^3 P$.

We see that **the field inside the sphere is uniform**, whereas **the field outside the sphere is that of a perfect dipole** p at the origin.

(b) *Calculate the potential directly using equation 2.7 from the lecture notes.*

Equation 2.7 gives:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\iiint_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}')d\tau' \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}, \tag{2.3}$$

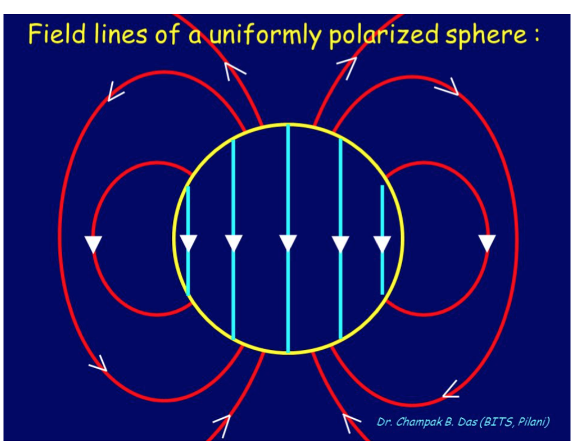
where $\mathcal{V}$ is the volume of the sphere. We have seen in the lectures that this expression is valid for calculating the potential both inside and outside the sphere. Since $\mathbf{P}$ is uniform, it can be taken out of the integral, so that we obtain:

$$V(\mathbf{r}) = \mathbf{P} \cdot \left[\frac{1}{4\pi\epsilon_0}\iiint_{\mathcal{V}} \frac{(\mathbf{r} - \mathbf{r}')d\tau'}{|\mathbf{r} - \mathbf{r}'|^3}\right]. \tag{2.4}$$

The term in brackets is the electric field $\mathbf{E}_1$ of a sphere with a uniform charge density equal to unity. This electric field can be calculated using Gauss's law and is given by:

$$\begin{aligned} \mathbf{E}_1 &= \frac{r}{3\epsilon_0}\hat{\mathbf{r}}, \quad \text{for} \quad r < R, \\ \mathbf{E}_1 &= \frac{R^3}{3\epsilon_0 r^2}\hat{\mathbf{r}}, \quad \text{for} \quad r > R. \end{aligned} \tag{2.5}$$

$V = \mathbf{P} \cdot \mathbf{E}_1$ can then be used to recover the potential.



Note that the parallel component of $\mathbf{E}$ at the surface, $E_\theta(R)$, is continuous, whereas the perpendicular component, $E_r(R)$, is discontinuous:

$$\mathbf{E}^{\perp}_{\text{out}} - \mathbf{E}^{\perp}_{\text{in}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{r}}.$$

2.2 D and E of an insulated straight wire

A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius a. Find the electric displacement and the electric field.

Gauss's law (2.16) from the lecture notes can be used to calculate **D**. Given the symmetry of the problem, and using cylindrical coordinates, **D** depends only on the distance r to the wire and is in the radial direction. We choose the Gaussian surface to be a cylinder of radius r and length L around the wire. Then Gauss's law yields $\mathbf{D} \times 2\pi r L = \lambda L$, that is:

$$\mathbf{D} = \frac{\lambda}{2\pi r}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is the unit vector in the radial direction. This is true for all $r > 0$.

Outside the insulation, $P = 0$ so that:

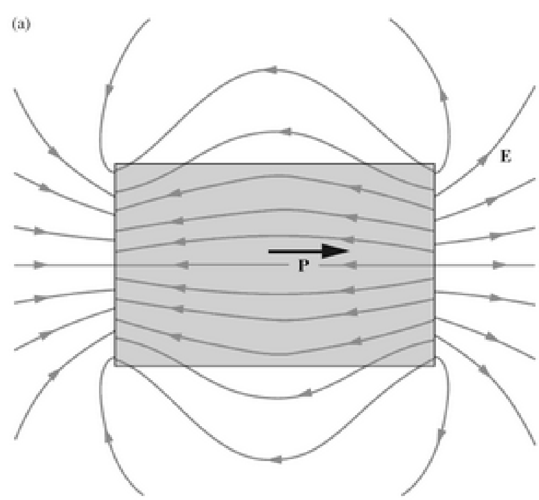
$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{\mathbf{r}}.$$

Inside the insulation, $\mathbf{E} = (\mathbf{D} - \mathbf{P})/\epsilon_0$, which we cannot calculate as we do not know **P**.

2.3 D and E of a cylinder with polarization along the axis

A cylinder of finite size carries a "frozen–in" uniform polarization **P***, parallel to its axis. (Such a cylinder is know as a bar* **electret***). Find the polarization charges, sketch the electric field and the electric displacement (use the boundary conditions).*

Polarization charges: there is no volume charge as **P** is uniform, and the surface charge is $\mathbf{P} \cdot \hat{\mathbf{n}} = +\sigma$ on one end and $-\sigma$ on the other end.



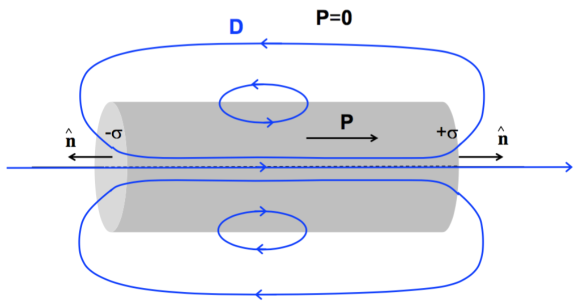
Electric field: The bar is therefore like a physical dipole. Note that electric field lines terminate on charges, whether they are bound or free, with **E** pointing towards the negative charge. (Figure from *Purcell.*)

The perpendicular component of **E** is discontinuous at the surfaces:

$$\mathbf{E}_{\text{out}}^{\perp} - \mathbf{E}_{\text{in}}^{\perp} = \frac{\pm\sigma}{\epsilon_0}\hat{\mathbf{n}},$$

whereas the parallel component of **E** is continuous across the lateral surface.

Electric displacement: $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$, therefore $\mathbf{D} = \mathbf{E}$ outside the cylinder. Since there is no free charge at the surfaces, the perpendicular component of **D** is continuous there, so that **D** inside the cylinder goes from left to right. Note that **D** lines terminate on *free* charges. As there are no free charges here, **D** lines are continuous:



We see that even though there is **no polarization outside** the cylinder, **D** is **non zero** there. This is because the polarization is discontinuous at the surface of the cylinder, and along the lateral surface we have (see equation [2.22] from the lecture notes):

$$\mathbf{D}_{\text{in}}^{\parallel} - \mathbf{D}_{\text{out}}^{\parallel} = \mathbf{P}.$$

The discontinuity of **P** generates a non zero **D** outside.

2.4 Linear dielectric sphere with a charge at the center

A point charge q is embedded at the center of a sphere of linear dielectric material (with susceptibility χ_e and radius R). Find the electric field, the polarization, and the polarization charge densities, ρ_p and σ_p. What is the total polarization charge on the surface? Where is the compensating negative polarization charge located?

Electric displacement: Given the symmetry of the problem, in spherical coordinates, Gauss's law yields:

$$\mathbf{D} = \frac{q}{4\pi r^2}\hat{\mathbf{r}},$$

for all $r > 0$, where $\hat{\mathbf{r}}$ is the unit vector in the radial direction.

Electric field: Inside the sphere, $\mathbf{E} = \mathbf{D}/\epsilon$ with $\epsilon = \epsilon_0(1+\chi_e)$. Therefore:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0(1+\chi_e)r^2}\hat{\mathbf{r}}, \quad \text{for} \quad r < R,$$

and $\mathbf{E} = \mathbf{D}/\epsilon_0$ outside the sphere.

Polarization: $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$ inside the sphere, that is:

$$\mathbf{P} = \frac{\chi_e q}{4\pi(1+\chi_e)r^2}\hat{\mathbf{r}}, \quad \text{for} \quad r < R.$$

Volume polarization charge:

$$\rho_p = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\frac{\chi_e q}{4\pi(1+\chi_e)}\boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right).$$

Using the definition of the δ function in 3 dimensions:

$$\boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^3(\mathbf{r}),$$

we get:

$$\rho_p = -\frac{\chi_e}{1+\chi_e}\, q\, \delta^3(\mathbf{r}).$$

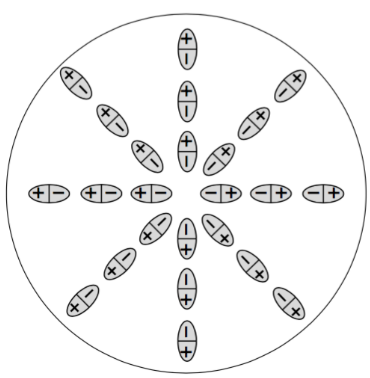
Surface polarization charge:

$$\sigma_p = \mathbf{P}(R)\cdot\hat{\mathbf{r}} = \frac{\chi_e q}{4\pi(1+\chi_e)R^2}.$$

Therefore, the total surface charge is:

$$Q_{\text{surf}} = \sigma_p \times 4\pi R^2 = \frac{\chi_e}{1+\chi_e}\, q.$$

The compensating negative charge is at the center, which is implied by the fact that the δ^3 function is non zero only at $\mathbf{r} = \mathbf{0}$.



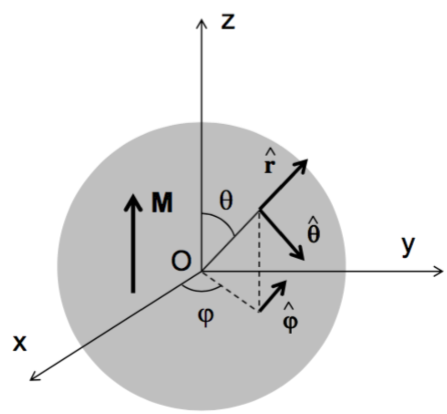
As $\mathbf{P}$ is radial, the dipoles in the sphere are all aligned in the radial direction, leaving a net charge at the center and at the surface only.

Chapter 3

Magnetic fields in matter

3.1 Field of a uniformly magnetized sphere

(a) *Find the magnetic field produced by a uniformly magnetized sphere of radius R. (Hint: use the results of Problem 7 in Problem Set 1)*



Inside the sphere, the volume current density is $\mathbf{J}_m = \boldsymbol{\nabla}\times\mathbf{M}$ which is 0 as the magnetization is uniform. At the surface of the sphere, there is a surface current density $\mathbf{K}_m = \mathbf{M}\times\hat{\mathbf{r}}$. We choose the z–axis in the direction of $\mathbf{M}$, so that $\mathbf{K}_m = M\sin\theta\hat{\boldsymbol{\varphi}}$ in spherical coordinates.

In Problem 7, Problem Set 1, we have calculated the magnetic field due to a spinning sphere, which has a surface current density $\mathbf{K}$ proportional to $\sin\theta$ in the $\hat{\boldsymbol{\varphi}}$–direction. By making the appropriate replacement, we therefore get:

$$\boxed{\begin{aligned}&\mathbf{B} = \frac{2}{3}\mu_0\mathbf{M}, \quad \text{for} \quad r < R,\\ &\mathbf{B} = \frac{\mu_0 M R^3}{3r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right), \quad \text{for} \quad r > R.\end{aligned}} \tag{3.1}$$

We see that **the field inside the sphere is uniform**.

We define the total dipole moment of the sphere $m = (4/3)\pi R^3 M$. Then the field outside the sphere can be written as:

$$\mathbf{B} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right),$$

which is the **field of a perfect dipole** m at the origin.

(b) *Calculate the vector potential* **A** *directly using equation 3.13 from the lecture notes. Derive the magnetic field from* **A**.

Equation 3.13 gives:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\iiint_{\mathcal{V}} \frac{\mathbf{M}(\mathbf{r}')d\tau' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{3.2}$$

where $\mathcal{V}$ is the volume of the sphere. We have seen in the lectures that this expression is valid for calculating the potential both inside and outside the sphere. Since **M** is uniform, it can be taken out of the integral, so that we obtain:

$$\mathbf{A}(\mathbf{r}) = \mathbf{M}\times\left[\frac{\mu_0}{4\pi}\iiint_{\mathcal{V}} \frac{(\mathbf{r}-\mathbf{r}')d\tau'}{|\mathbf{r}-\mathbf{r}'|^3}\right]. \tag{3.3}$$

The term in brackets is the electric field **E** of a sphere with a uniform charge density equal to $\mu_0\epsilon_0$. This electric field can be calculated using Gauss's law and is given by:

$$\begin{aligned} \mathbf{E} &= \frac{\mu_0 r}{3}\hat{\mathbf{r}}, \quad \text{for} \quad r < R, \\ \mathbf{E} &= \frac{\mu_0 R^3}{3r^2}\hat{\mathbf{r}}, \quad \text{for} \quad r > R. \end{aligned} \tag{3.4}$$

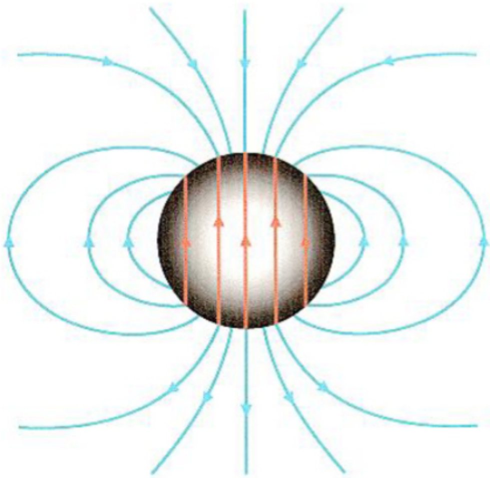
$\mathbf{A} = \mathbf{M}\times\mathbf{E}$ if then equal to:

$$\begin{aligned} \mathbf{A} &= \frac{\mu_0 M r}{3}\sin\theta\,\hat{\boldsymbol{\varphi}}, \quad \text{for} \quad r < R, \\ \mathbf{A} &= \frac{\mu_0 M R^3}{3r^2}\sin\theta\,\hat{\boldsymbol{\varphi}}, \quad \text{for} \quad r > R. \end{aligned} \tag{3.5}$$

The magnetic field is given by:

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ A)\,\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}(rA)\,\hat{\boldsymbol{\theta}},$$

which leads to the same expressions as in (a).



Field lines of a uniformly magnetized sphere. Note that the perpendicular component of $\mathbf{B}$ at the surface, $B_r(R)$, is continuous, whereas the parallel component, $B_\theta(R)$, is discontinuous:

$$\mathbf{B}^{\parallel}_{\text{out}} - \mathbf{B}^{\parallel}_{\text{in}} = \mu_0 K_m \hat{\boldsymbol{\theta}}.$$

3.2 Field of a cylinder with circular magnetization

A long circular cylinder of radius R carries a magnetization $\mathbf{M} = kr^2\hat{\boldsymbol{\varphi}}$, where k is a constant, r the radius in cylindrical coordinates and $\hat{\boldsymbol{\varphi}}$ the unit vector in the azimuthal direction. Find the magnetic field due to $\mathbf{M}$ inside and outside the cylinder. Calculate $\mathbf{H}$ using $\mathbf{B}$ and $\mathbf{M}$ and, independently, using the current distribution.

The volume current density is:

$$\mathbf{J}_m = \boldsymbol{\nabla}\times\mathbf{M} = \frac{1}{r}\frac{d}{dr}(rM)\,\hat{\mathbf{z}} = 3kr\hat{\mathbf{z}}.$$

The surface current density is:

$$\mathbf{K}_m = \mathbf{M}(R)\times\hat{\mathbf{r}} = -kR^2\hat{\mathbf{z}}.$$

Note that the current is flowing up in the volume of the cylinder and returns down along the surface. For a length L of the cylinder, the total current is:

$$\mathbf{I}_{\text{tot}} = -\mathbf{K}_m \times 2\pi RL + L\int_0^R \mathbf{J}_m 2\pi r dr = -2\pi kR^3L\hat{\mathbf{z}} + 6\pi kL\hat{\mathbf{z}}\int_0^R r^2 dr = \mathbf{0},$$

as expected.

Given the symmetry of the system, and using Ampère's law, we obtain:

$$2\pi r\mathbf{B} = \mu_0\hat{\boldsymbol{\varphi}}\int_0^r J_m 2\pi r dr \Longrightarrow \boxed{\mathbf{B} = \mu_0 kr^2\hat{\boldsymbol{\varphi}} = \mu_0\mathbf{M}} \quad \text{for} \quad r < R,$$

and

$$2\pi r\mathbf{B} = \mu_0\hat{\boldsymbol{\varphi}}I_{\text{tot}} \Longrightarrow \boxed{\mathbf{B} = \mathbf{0}} \quad \text{for} \quad r > R.$$

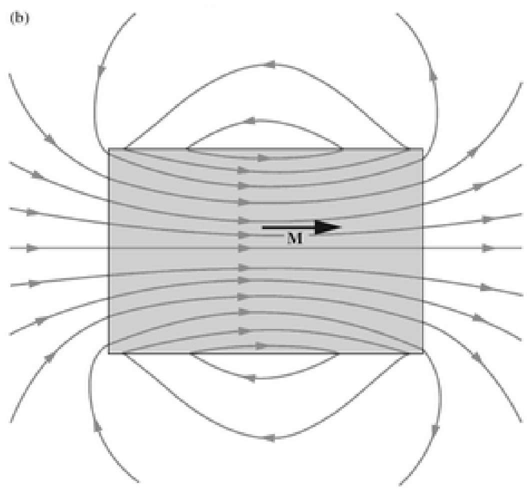
To get $\mathbf{H}$ we use $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$. Then $\boxed{\mathbf{H} = \mathbf{0}}$ everywhere.

This is consistent with Ampère's law, as $\boldsymbol{\nabla}\times\mathbf{H} = \mathbf{J}_f = \mathbf{0}$ and the symmetry of the problem leads to $2\pi r\mathbf{H} = 0\hat{\boldsymbol{\varphi}}$. Note however that, in general, there may be a non zero $\mathbf{H}$ even when there is no free current (see below).

3.3 H and B of a cylinder with magnetization along the axis

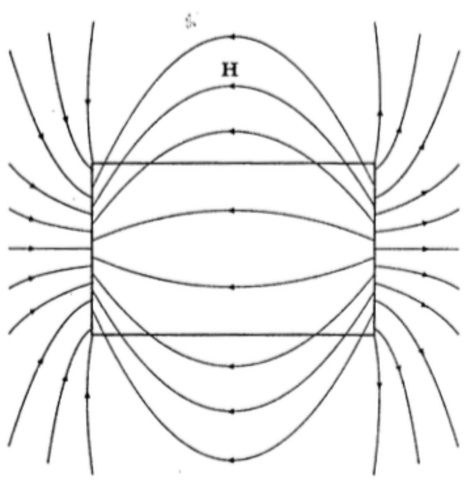
A cylinder of finite size carries a "frozen–in" uniform magnetization **M***, parallel to its axis. (Such a cylinder is know as a bar* **magnet***). Find the magnetization currents, sketch the magnetic field* **B** *and the auxiliary field* **H** *(use the boundary conditions).*

Magnetization currents: Since the magnetization is uniform, the volume current density is $\mathbf{J}_m = \nabla\times\mathbf{M} = \mathbf{0}$. The surface current density is $\mathbf{K}_m = \mathbf{M}\times\hat{\mathbf{n}}$, that is $\boxed{\mathbf{K}_m = M\hat{\boldsymbol{\varphi}}}$ on the lateral surface.



Magnetic field: The bar is like a finite–size solenoid. (Figure from *Purcell.*)
The perpendicular component of **B** is continuous across the surfaces, whereas:

$$\mathbf{B}^{\parallel}_{\text{in}} - \mathbf{B}^{\parallel}_{\text{out}} = \mu_0 K_m \hat{\mathbf{z}}.$$



Auxiliary field: $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$, therefore $\mathbf{H} = \mathbf{B}/\mu_0$ outside the cylinder. Since there is no free current at the surfaces, the parallel component of **H** is continuous there, so that **H** inside the cylinder goes from right to left.

We see that even though there is **no magnetization outside** the cylinder, **H** is **non zero** there. This is because the magnetization is discontinuous at the surface of the cylinder:

$$\mathbf{H}^{\perp}_{\text{out}} - \mathbf{H}^{\perp}_{\text{in}} = \frac{\mathbf{B}^{\perp}_{\text{out}}}{\mu_0} - \frac{\mathbf{B}^{\perp}_{\text{in}}}{\mu_0} + \mathbf{M} = \mathbf{M}.$$

The discontinuity of **M** generates a non zero **H** outside.

3.4 H and B of a linear insulated straight wire

A current I flows along a straight wire or radius a. If the wire is made of linear material with susceptibility χ_m, and the current is distributed uniformly, what is the magnetic field a distance r from the axis? Find all the magnetization currents. What is the net *magnetization current flowing along the wire?*

We define the volume density of free current: $J_f = I/(\pi a^2)$, and we assume that the current I flows up along the z–axis. Given the symmetry of the system, and using cylindrical coordinates, Ampère's law gives:

$$\mathbf{H} \times 2\pi r = \pi r^2 J_f \,\hat{\boldsymbol{\theta}} \Longrightarrow \boxed{\mathbf{H} = \frac{Ir}{2\pi a^2}\,\hat{\boldsymbol{\theta}} \quad \text{for} \quad r < a},$$

and:

$$\mathbf{H} \times 2\pi r = I \,\hat{\boldsymbol{\theta}} \Longrightarrow \boxed{\mathbf{H} = \frac{I}{2\pi r}\,\hat{\boldsymbol{\theta}} \quad \text{for} \quad r > a}.$$

Therefore, $\mathbf{B} = \mu\mathbf{H}$, with $\mu = \mu_0(1 + \chi_m)$ inside the insulated material and $\mu = \mu_0$ outside, yields:

$$\boxed{\begin{aligned} \mathbf{B} &= \frac{\mu_0(1+\chi_m)Ir}{2\pi a^2}\,\hat{\boldsymbol{\theta}} \quad \text{for} \quad r < a, \\ \mathbf{B} &= \frac{\mu_0 I}{2\pi r}\,\hat{\boldsymbol{\theta}} \quad \text{for} \quad r > a. \end{aligned}}$$

Volume density of magnetization current: $\mathbf{J}_m = \boldsymbol{\nabla}\times\mathbf{M} = \boldsymbol{\nabla}\times(\chi_m\mathbf{H}) = \chi_m\boldsymbol{\nabla}\times\mathbf{H}$, that is: $\boxed{\mathbf{J}_m = \chi_m J_f \hat{\mathbf{z}}}$.

Surface density of magnetization current: $\mathbf{K}_m = \mathbf{M}\times\hat{\mathbf{r}} = \chi_m\mathbf{H}(a)\times\hat{\mathbf{r}}$, that is: $\boxed{\mathbf{K}_m = -\chi_m I\hat{\mathbf{z}}/(2\pi a)}$.

Net magnetization current along the wire:

$$\int_0^a \mathbf{J}_m \times 2\pi r dr + \mathbf{K}_m \times 2\pi a = \mathbf{0},$$

as expected.