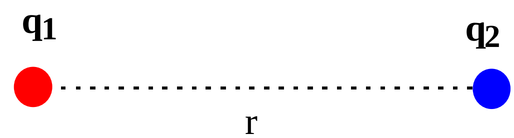
Electrostatic Force and Electric Charge

Electrostatic Force (charges at rest):

- Electrostatic force can be **attractive**
- Electrostatic force can be **repulsive**
- Electrostatic force acts through **empty space**
- Electrostatic force much stronger than **gravity**
- Electrostatic forces are **inverse square law** forces (**proportional to $1/r^2$**)
- Electrostatic force is proportional to the product of the amount of charge on each interacting object



Magnitude of the Electrostatic Force is given by Coulomb's Law:

$$F = K\, q_1 q_2 / r^2 \qquad \text{(Coulomb's Law)}$$

where K depends on the system of units

$$K = 8.99\text{x}10^9\ Nm^2/C^2 \quad \text{(in MKS system)}$$
$$K = 1/(4\pi\varepsilon_0) \quad \text{where} \quad \varepsilon_0 = 8.85\text{x}10^{-12}\ C^2/(Nm^2)$$

Electric Charge:

electron charge = -e	**$e = 1.6\text{x}10^{-19}$ C**
proton charge = e	C = Coulomb

Electric charge is a conserved quantity (*net electric charge is never created or destroyed!*)

Units

MKS System (meters-kilograms-seconds):

also Amperes, Volts, Ohms, Watts

Force:	**F = ma**	**Newtons = kg m / s^2 = 1 N**
Work:	**W = Fd**	**Joule = Nm = kg m^2 / s^2 = 1 J**
Electric Charge:	**Q**	**Coulomb = 1 C**
$F = K q_1q_2/r^2$	**$K = 8.99\times10^9$ Nm^2/C^2 (in MKS system)**	

CGS System (centimeter-grams-seconds):

Force:	**F = ma**	**1 dyne = g cm / s^2**
Work:	**W = Fd**	**1 erg = dyne-cm = g cm^2 / s^2**
Electric Charge:	**Q**	**esu (electrostatic unit)**
$F = q_1q_2/r^2$	**K = 1 (in CGS system)**	

Conversions (MKS - CGS):

Force:	**1 N = 10^5 dynes**
Work:	**1 J = 10^7 ergs**
Electric Charge:	**1 C = 2.99×10^9 esu**

Fine Structure Constant (dimensionless):

$$\alpha = K\, 2\pi e^2/hc$$ **(same in all systems of units)**

h = Plank's Constant c = speed of light in vacuum

Electrostatic Force versus Gravity

Electrostatic Force :

$$F_e = K\, q_1 q_2 / r^2 \qquad \text{(Coulomb's Law)}$$

$$K = 8.99 \times 10^{9}\ \mathrm{Nm^2/C^2} \qquad \text{(in MKS system)}$$

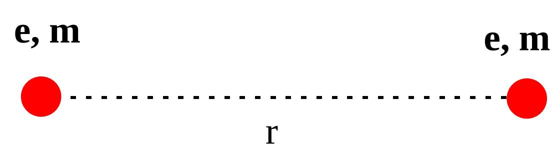
Gravitational Force :

$$F_g = G\, m_1 m_2 / r^2 \qquad \text{(Newton's Law)}$$

$$G = 6.67 \times 10^{-11}\ \mathrm{Nm^2/kg^2} \qquad \text{(in MKS system)}$$

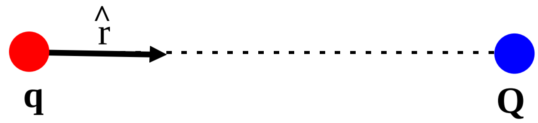
Ratio of forces for two electrons :

$e = 1.6 \times 10^{-19}$ C $\qquad$ $m = 9.11 \times 10^{-31}$ kg



$$F_e / F_g = K e^2 / G m^2 = 4.16 \times 10^{42} \qquad \text{(Huge number !!!)}$$

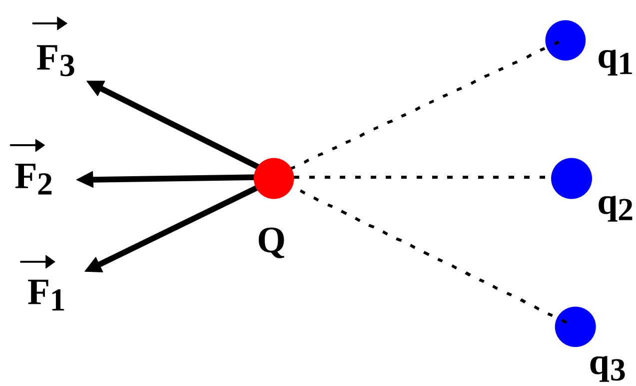
Vector Forces



The Electrostatic Force is a vector:

The force on q due to Q points along the direction r and is given by

$$\vec{F} = \frac{KqQ}{r^2}\hat{r}$$



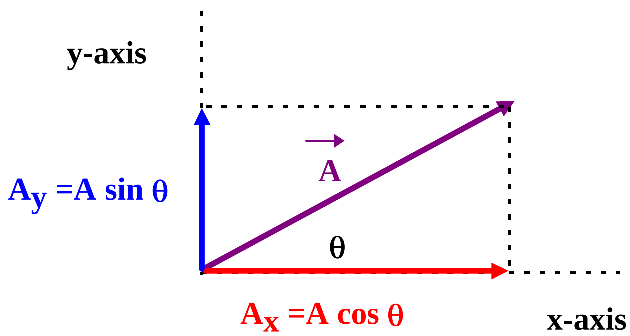
Vector Superposition of Electric Forces:

If several point charges q_1, q_2, q_3, … simultaneously exert electric forces on a charge Q then

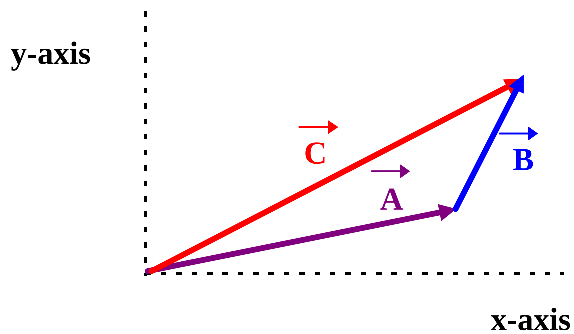
$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots$$

Vectors & Vector Addition

The Components of a vector:



Vector Addition:



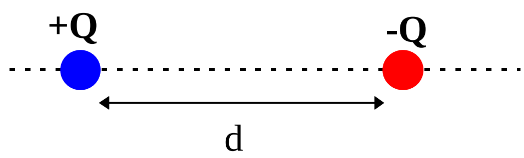
To add vectors you add the components of the vectors as follows:

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$$\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

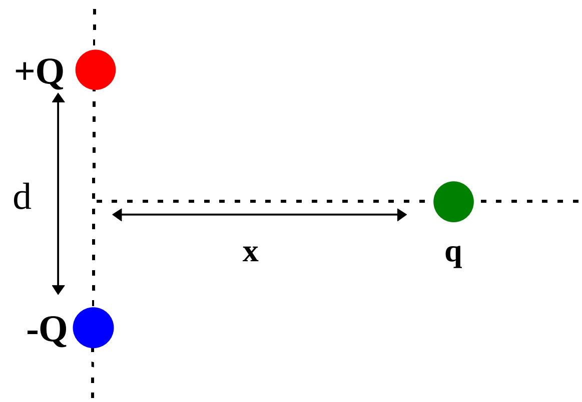
$$\vec{C} = \vec{A} + \vec{B} = (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} + (A_z + B_z)\hat{z}$$

The Electric Dipole



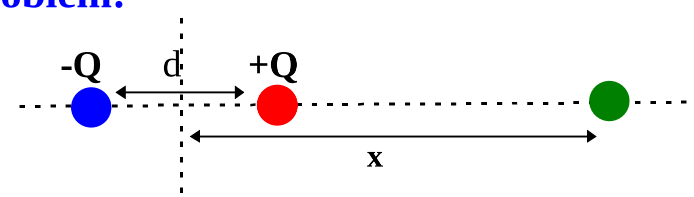
An electric "dipole" is two equal and opposite point charges separated by a distance d. It is an electrically neutral system. The "dipole moment" is defined to be the charge times the separation (dipole moment = Qd).

Example Problem:



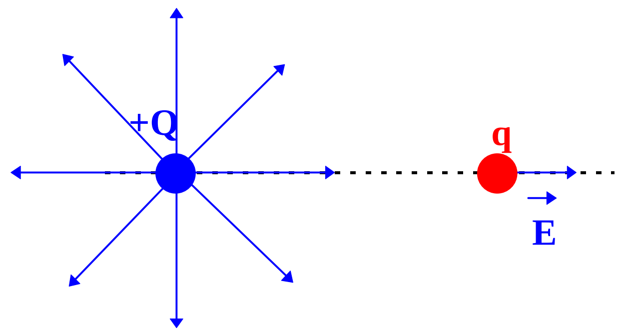
A dipole with charge **Q** and separation **d** is located on the y-axis with its midpoint at the origin. A charge **q** is on the x-axis a distance **x** from the midpoint of the dipole. What is the electric force on **q** due to the dipole and how does this force behave in the limit **x** >>**d** (**dipole approximation**)?

Example Problem:



A dipole with charge **Q** and separation **d** is located on the x-axis with its midpoint at the origin. A charge **q** is on the x-axis a distance **x** from the midpoint of the dipole. What is the electric force on **q** due to the dipole and how does this force behave in the limit **x** >>**d** (**dipole approximation**)?

The Electric Field



The charge Q produces an electric field which in turn produces a force on the charge q. The force on q is expressed as two terms:

$$F = K\,qQ/r^2 = q\,(KQ/r^2) = q\,E$$

The electric field at the point q due to Q is simply the force per unit positive charge at the point q:

$$E = F/q \qquad E = KQ/r^2$$

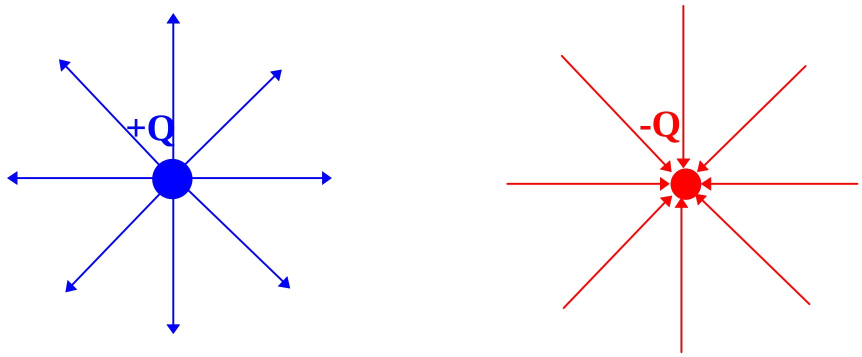
The units of E are Newtons per Coulomb (units = N/C).

The electric field is a physical object which can carry both momentum and energy. It is the mediator (or carrier) of the electric force. The electric field is massless.

The Electric Field is a Vector Field:

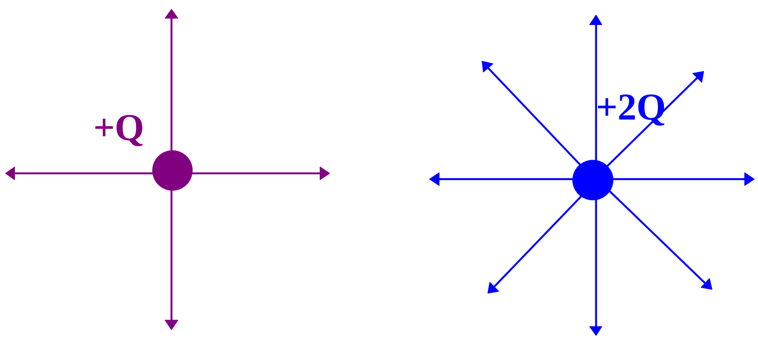
$$\vec{E} = \frac{KQ}{r^2}\hat{r}$$

Electric Field Lines

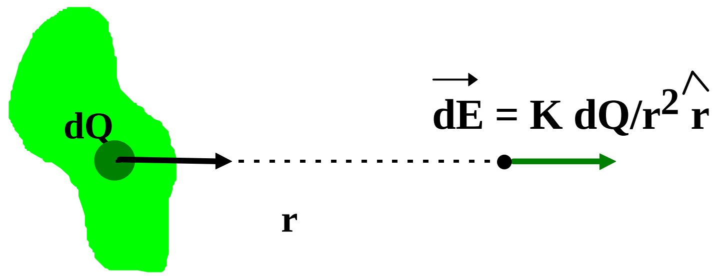


Electric field line diverge from (i.e. start) on positive charge and end on negative charge. The direction of the line is the direction of the electric field.

The number of lines penetrating a unit area that is perpendicular to the line represents the strength of the electric field.



Electric Field due to a Distribution of Charge

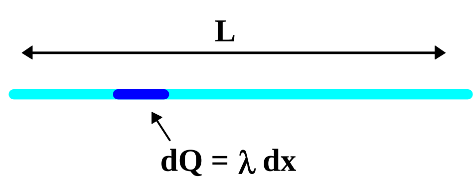


The electric field from a continuous distribution of charge is the superposition (i.e. integral) of all the (infinite) contributions from each infinitesimal dQ as follows:

$$\vec{E} = \int \frac{K}{r^2}\hat{r}\,dQ \quad \text{and} \quad Q = \int dQ$$

Charge Distributions:

- **Linear charge density λ:** **λ(x) = charge/unit length**



For a straight line dQ = λ(x) dx and

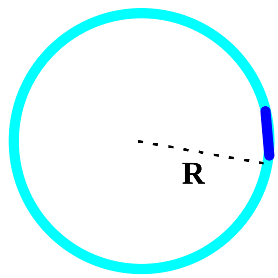
$$Q = \int dQ = \int l(x)dx$$

If λ(x) = λ is constant then dQ = λ dx and Q = λL, where L is the length.

Charge Distributions

Charge Distributions:

- **Linear charge density λ:** **λ(θ) = charge/unit arc length**

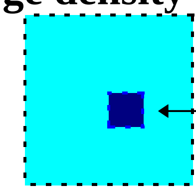


For a circular arc dQ = λ(θ) ds = λ(θ) Rdθ and

$$Q = \int dQ = \int l(q)ds = \int l(q)Rdq$$

If λ(θ) = λ is constant then dQ = λ ds and Q = λs, where s is the arc length.

- **Surface charge density σ:** **σ(x,y) = charge/unit area**

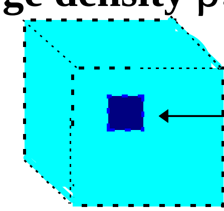


For a surface dQ = σ(x,y) dA and

$$Q = \int dQ = \int s(x,y)dA$$

If σ(x,y) = σ is constant then dQ = σ dA and Q = σA, where A is the area.

- **Volume charge density ρ:** **ρ(x,y,z) = charge/unit volume**



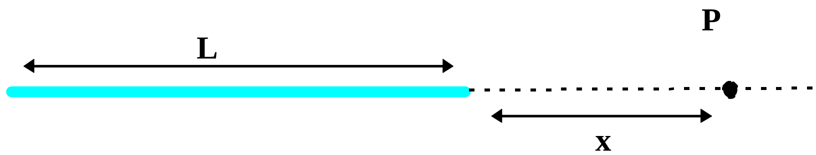
For a surface dQ = ρ(x,y,z) dV and

$$Q = \int dQ = \int r(x,y,z)dV$$

If ρ(x,y,z) = ρ is constant then dQ = ρ dV and Q = ρV, where V is the volume.

Calculating the Electric Field

Example:

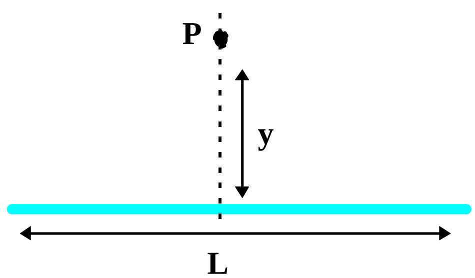


A total amount of charge **Q** is uniformly distributed along a thin **straight rod** of length **L**. What is the electric field a a point **P** on the x-axis a distance **x** from the end of the rod?

Answer: $$\vec{E} = \frac{KQ}{x(x+L)}\hat{x}$$

Example:

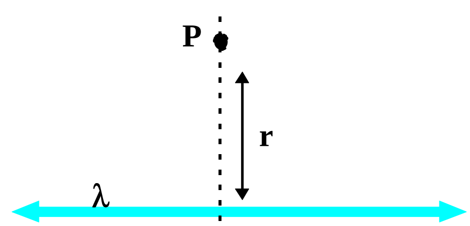
A total amount of charge **Q** is uniformly distributed along a thin **straight rod** of length **L**. What is the electric field a a point **P** on the y-axis a distance **y** from the midpoint of the rod?



Answer: $$\vec{E} = \frac{KQ}{y\sqrt{y^2+(L/2)^2}}\hat{y}$$

Example:

A **infinitely long straight rod** has a uniform charge density λ. What is the electric field a point **P** a perpendicular distance r from the rod?



Answer: $$\vec{E} = \frac{2Kl}{r}\hat{r}$$

Some Useful Math

Approximations:

$$(1+e)^p \underset{e<<1}{\approx} 1+pe$$

$$(1-e)^p \underset{e<<1}{\approx} 1-pe$$

$$e^e \underset{e<<1}{\approx} 1+e$$

$$\tan e \underset{e<<1}{\approx} e \qquad \sin e \underset{e<<1}{\approx} e$$

Indefinite Integrals:

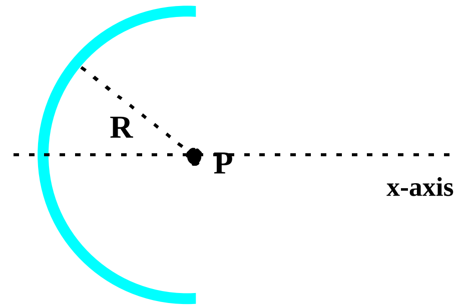
$$\int \frac{a^2}{(x^2+a^2)^{3/2}} dx = \frac{x}{\sqrt{x^2+a^2}}$$

$$\int \frac{x}{(x^2+a^2)^{3/2}} dx = \frac{-1}{\sqrt{x^2+a^2}}$$

Calculating the Electric Field

Example:

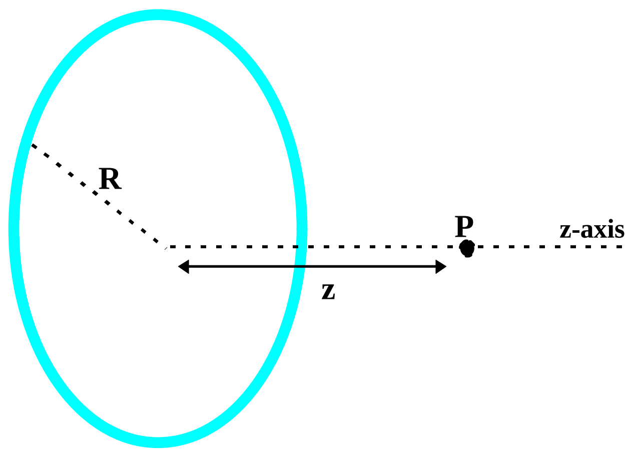
A total amount of charge **Q** is uniformily distributed along a thin **semicircle** of radius **R**. What is the electric field a a point **P** at the center of the circle?



Answer: $$\vec{E} = \frac{2KQ}{pR^2}\hat{x}$$

Example:

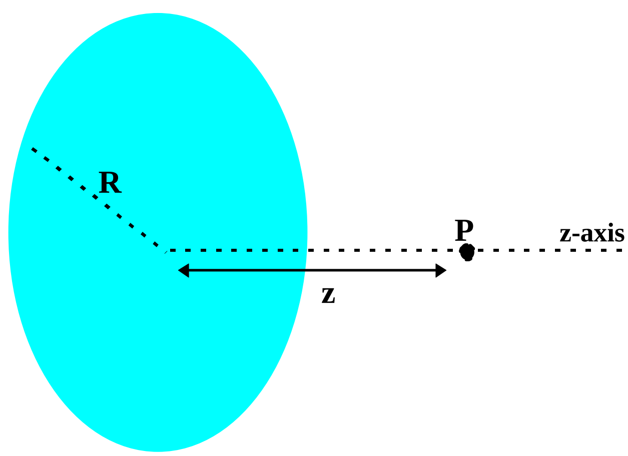
A total amount of charge **Q** is uniformily distributed along a thin **ring** of radius **R**. What is the electric field a point **P** on the z-axis a distance **z** from the center of the ring?



Answer: $$\vec{E} = \frac{KQz}{\left(z^2 + R^2\right)^{3/2}}\hat{z}$$

Example:

A total amount of charge **Q** is uniformily distributed on the surface of a **disk** of radius **R**. What is the electric field a point **P** on the z-axis a distance **z** from the center of the disk?



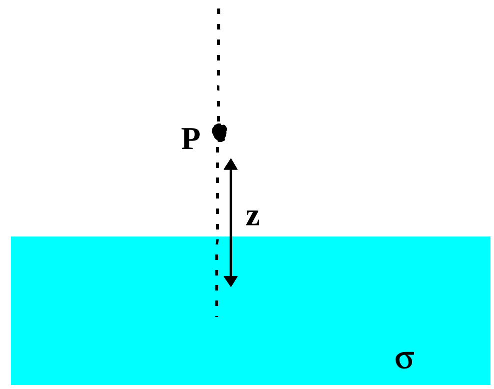
Answer:

$$\vec{E} = \frac{2KQ}{R^2}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right)\hat{z}$$

Calculating the Electric Field

Example:

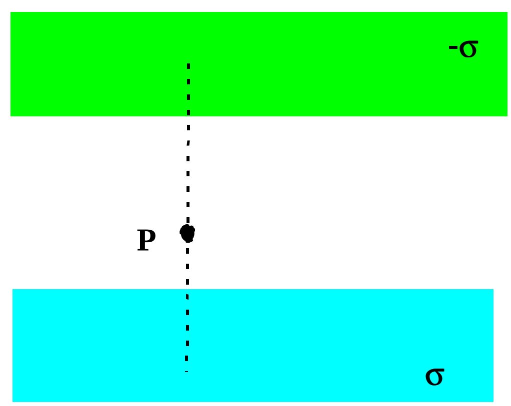
What is the electric field generated by a large (**infinite**) **sheet** carrying a uniform surface charge density of σ coulombs per meter?



Answer: $$\vec{E} = \frac{s}{2e_0}\hat{z}$$

Example:

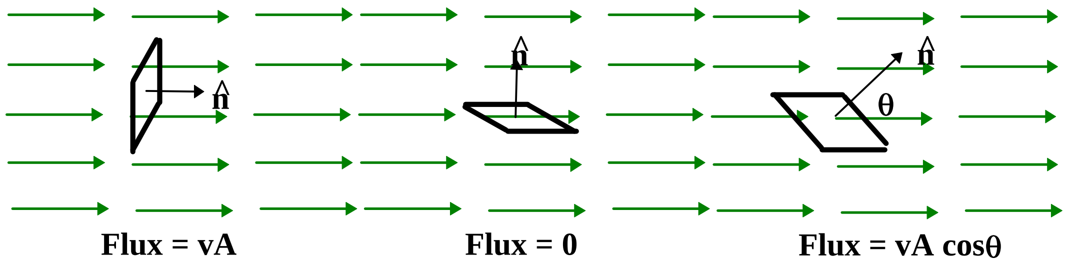
What is the electric field at a point P between **two** large (**infinite**) **sheets** carrying an equal but opposite uniform surface charge density of σ?



Answer: $$\vec{E} = \frac{s}{e_0}\hat{z}$$

Flux of a Vector Field

Fluid Flow:



Consider the fluid with a vector $\vec{v}$ which describes the velocity of the fluid at every point in space and a square with area A = L^2 and normal $\hat{n}$. **The flux is the volume of fluid passing through the square area per unit time.**

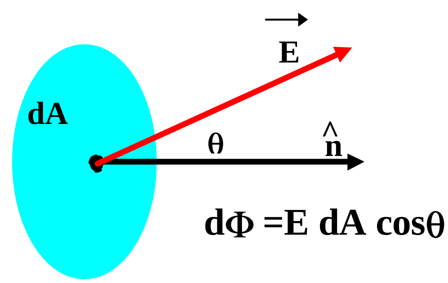
Generalize to the Electric Field:

Electric flux through the infinitesimal area dA is equal to

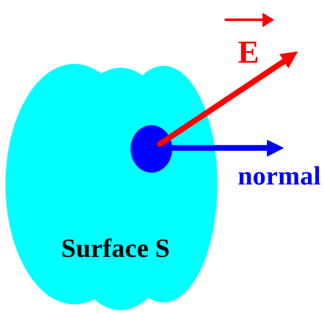
$$d\Phi = \vec{E} \cdot d\vec{A}$$

where

$$d\vec{A} = A\hat{n}$$



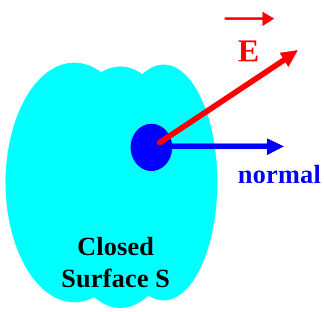
Total Electric Flux through a Closed Surface:



$$\Phi_E = \oint_S \vec{E} \cdot d\vec{A}$$

Electric Flux and Gauss' Law

The electric flux through any closed surface is proportional to the net charge enclosed.



$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{e_0}$$

For the discrete case the total charge enclosed is the sum over all the enclosed charges:

$$Q_{enclosed} = \sum_{i=1}^{N} q_i$$

For the continuous case the total charge enclosed is the integral of the charge density over the volume enclosed by the surface S:

$$Q_{enclosed} = \int r\, dV$$

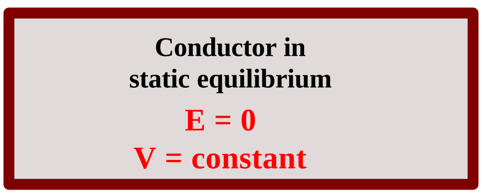
Simple Case: If the electric field is constant over the surface and if it always points in the same direction as the normal to the surface then

$$\Phi_E = \oint_S \vec{E} \cdot d\vec{A} = EA$$

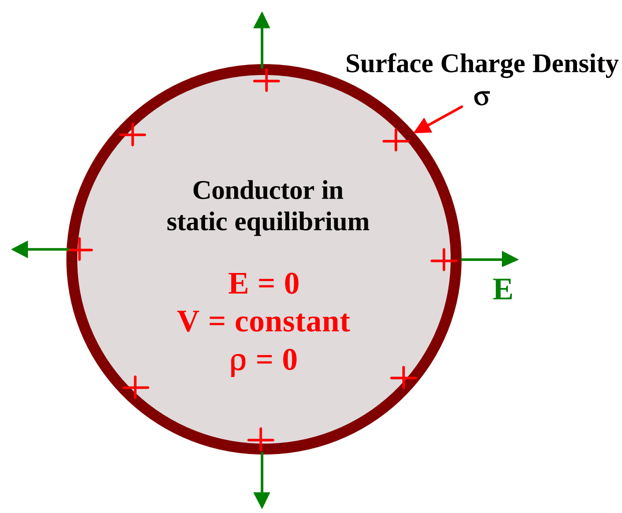
The units for the electric flux are **Nm^2/C**.

Conductors in Static Equilibrium

Conductor: In a conductor some electrons are free to move (**without restraint**) within the volumn of the material (**Examples:** copper, silver, aluminum, gold)



Conductor in Static Equilibrium: When the charge distribution on a conductor reaches **static equilibrium** (**i.e. nothing moving**), the **net electric field withing the conducting material is exactly zero** (**and the electric potential is constant**).



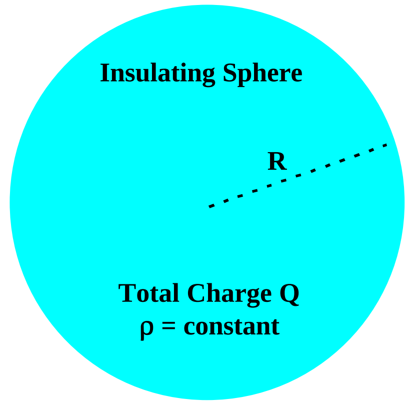
Excess Charge: For a conductor **in static equilibrium** all the (**extra**) electric charge **reside on the surface**. **There is no net electric charge within the volumn of the conductor (i.e. $\rho = 0$).**

Electric Field at the Surface: The electric field at the surface of a conductor **in static equilibrium** is **normal to the surface** and has a magnitude, $\mathbf{E = \sigma/\varepsilon_0}$, where σ is the surface charge density (**i.e. charge per unit area**) and the **net charge** on the conductor is

$$Q = \int_{Surface} s\, dA \,.$$

Gauss' Law Examples

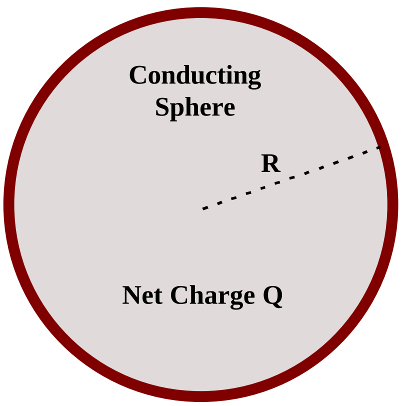
Problem: A solid insulating sphere of radius **R** has charge distributed uniformly throughout its volume. The total charge of the sphere is **Q**. What is the magnitude of the electric field inside and outside the sphere?



Answer:

$$\vec{E}_{out} = \frac{KQ}{r^2}\hat{r}$$

$$\vec{E}_{in} = \frac{KQr}{R^3}\hat{r}$$



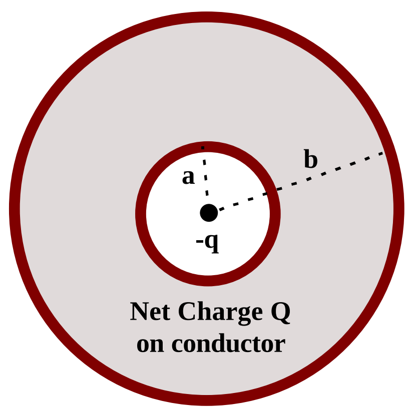
Problem: A solid conducting sphere of radius **R** has a net charge of **Q**. What is the magnitude of the electric field inside and outside the sphere? Where are the charges located?

Answer: Charges are on the surface and

$$\vec{E}_{out} = \frac{KQ}{r^2}\hat{r}$$

$$\vec{E}_{in} = 0$$

Problem: A solid conducting sphere of radius **b** has a spherical hole in it of radius **a** and has a net charge of **Q**. If there is a point charge **-q** located at the center of the hole, what is the magnitude of the electric field inside and outside the conductor? Where are the charges on the conductor located?



Answer: Charges are on the inside and outside surface with Q_{in}=q and Q_{out}=Q-q and

$$\vec{E}_{r>b} = \frac{K(Q-q)}{r^2}\hat{r}$$

$$\vec{E}_{a<r<b} = 0$$

$$E_{r<a} = \frac{-Kq}{r^2}\hat{r}$$

Gravitational Potential Energy

Gravitational Force: $F = G\, m_1 m_2 / r^2$

Gravitational Potential Energy GPE:

$U = GPE = mgh$ **(near surface of the Earth)**

Kinetic Energy: $KE = \frac{1}{2} m v^2$

Total Mechanical Energy: $E = KE + U$

Work Energy Theorem:

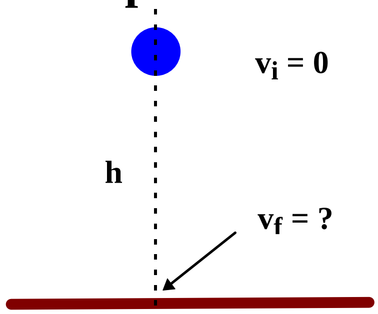
$W = E_B - E_A = (KE_B - KE_A) + (U_B - U_A)$

(work done on the system)

Energy Conservation: $E_A = E_B$

(if no external work done on system)

Example:



A ball is dropped from a height h. What is the speed of the ball when it hits the ground?

Solution: $E_i = KE_i + U_i = mgh$ $E_f = KE_f + U_f = mv_f^2/2$

$$E_i = E_f \Rightarrow v_f = \sqrt{2gh}$$

Electric Potential Energy

Gravitational Force: $F = K q_1 q_2/r^2$

Electric Potential Energy: $EPE = U$ (Units = Joules)

Kinetic Energy: $KE = \frac{1}{2} m v^2$ (Units = Joules)

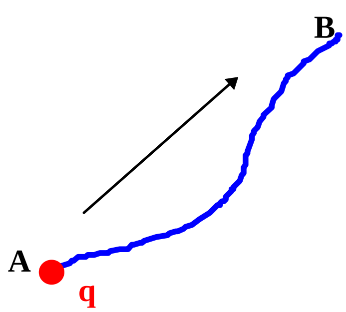
Total Energy: $E = KE + U$ (Units = Joules)

Work Energy Theorem: (work done on the system)

$$W = E_B - E_A = (KE_B - KE_A) + (U_B - U_A)$$

Energy Conservation: $E_A = E_B$ (if no external work done on system)

Electric Potential Difference $\Delta V = \Delta U/q$:



Work done (**against the electric force**) per unit charge in going from **A** to **B** (**without changing the kinetic energy**).

$$\Delta V = W_{AB}/q = \Delta U/q = U_B/q - U_A/q$$

(Units = Volts 1V = 1 J / 1 C)

Electric Potential V = U/q: $U = qV$

Units for the Electric Field (Volts/meter):

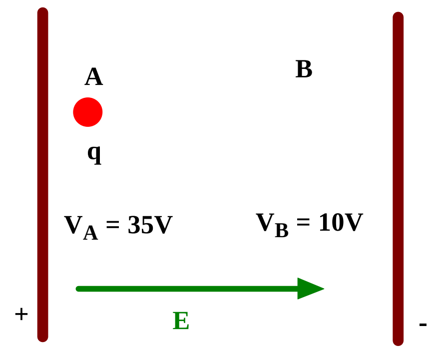
N/C = Nm/(Cm) = J/(Cm) = V/m

Energy Unit (electron-volt): One electron-volt is the amount of kinetic energy gained by an electron when it drops through one Volt potential difference

$$1\ eV = (1.6\text{x}10^{-19}\ C)(1\ V) = 1.6\text{x}10^{-19}\ \text{Joules}$$

1 MeV = 10^6 eV **1 GeV=1,000 MeV** **1 TeV=1,000 GeV**

Accelerating Charged Particles



Example Problem: A particle with mass **M** and charge **q** starts from rest a the point **A**. What is its speed at the point **B** if $\mathbf{V_A=35V}$ and $\mathbf{V_B=10V}$ ($M = 1.8\text{x}10^{-5}$kg, $q = 3\text{x}10^{-5}$C)?

Solution:
The total energy of the particle at **A** and **B** is

$$E_A = KE_A + U_A = 0 + qV_A$$

$$E_B = KE_B + U_B = \frac{1}{2}Mv_B^2 + qV_B \,.$$

Setting $\mathbf{E_A = E_B}$ (**energy conservation**) yields

$$\frac{1}{2}Mv_B^2 = q(V_A - V_B)$$

(Note: the particle gains an amount of kinetic energy equal to its charge, q, time the change in the electric potential.)

Solving for the particle speed gives

$$v_B = \sqrt{\frac{2q(V_A - V_B)}{M}}$$

(Note: positive particles fall from high potential to low potential $V_A > V_B$, while negative particles travel from low potential to high potential, $V_B > V_A$.)

Plugging in the numbers gives

$$v_B = \sqrt{\frac{2(3\times10^{-5}C)(25V)}{1.8\times10^{-5}kg}} = 9.1m/s\,.$$

Potential Energy & Electric Potential

Mechanics (last semester!):

Work done by force F in going from A to B:

$$W_{A \to B}^{byF} = \int_A^B \vec{F} \cdot d\vec{r}$$

Potential Energy Difference ΔU:

$$W_{A \to B}^{againstF} = \Delta U = U_B - U_A = -\int_A^B \vec{F} \cdot d\vec{r}$$

$$\vec{F} = -\vec{\nabla} U = -\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y} - \frac{\partial U}{\partial z}\hat{z}$$

Electrostatics (this semester):

Electrostatic Force: $\vec{F} = q\vec{E}$

Electric Potential Energy Difference ΔU:

(work done against E in moving q from A to B)

$$\Delta U = U_B - U_A = -\int_A^B q\vec{E} \cdot d\vec{r}$$

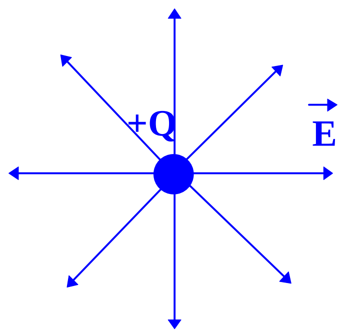
Electric Potential Difference ΔV=ΔU/q:

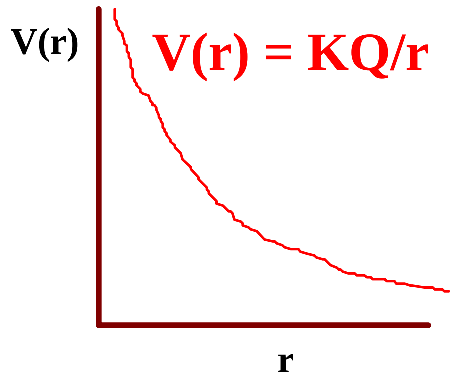
(work done against E per unit charge in going from A to B)

$$\Delta V = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{r}$$

$$\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$$

The Electric Potential of a Point Charge

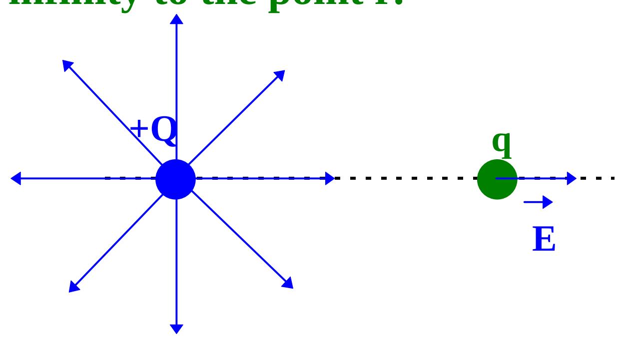




Potential from a point charge:

$$V(r) = \Delta V = V(r) - V(\text{infinity}) = KQ/r$$

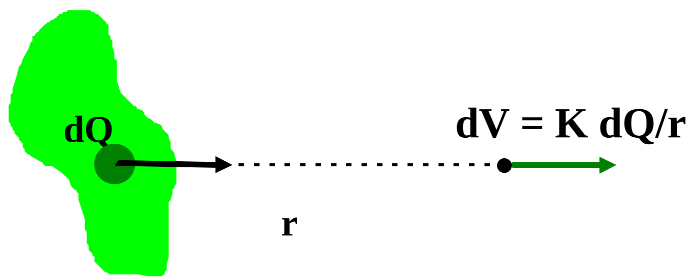
U = qV = work done against the electric force in bringing the charge q from infinity to the point r.



Potential from a system of N point charges:

$$V = \sum_{i=1}^{N} \frac{Kq_i}{r_i}$$

Electric Potential due to a Distribution of Charge



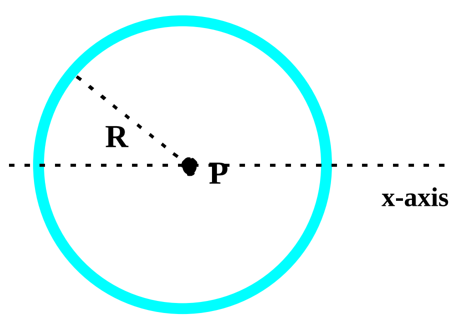
The electric potential from a continuous distribution of charge is the superposition (i.e. integral) of all the (infinite) contributions from each infinitesimal dQ as follows:

$$V = \int \frac{K}{r} dQ \quad \text{and} \quad Q = \int dQ$$

Example:

A total amount of charge Q is uniformly distributed along a thin **circle** of radius R. What is the electric potential at a point P at the center of the circle?

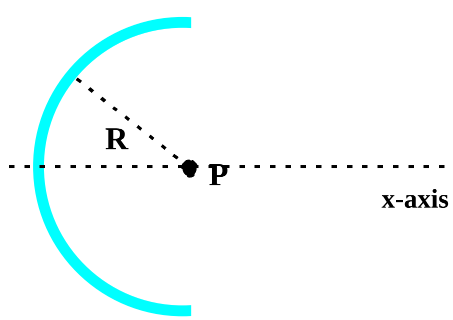
Answer: $V = \frac{KQ}{R}$



Example:

A total amount of charge Q is uniformly distributed along a thin **semicircle** of radius R. What is the electric potential at a point P at the center of the circle?

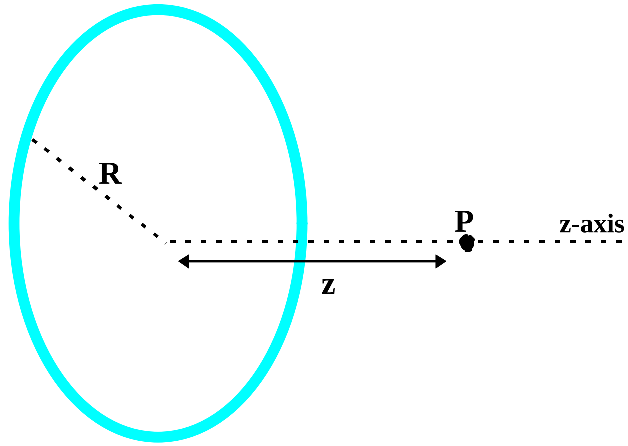
Answer: $V = \frac{KQ}{R}$



Calculating the Electric Potential

Example:

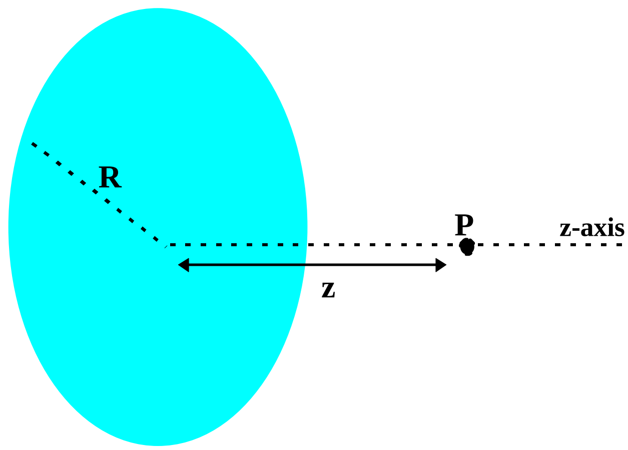
A total amount of charge Q is uniformily distributed along a thin **ring** of radius R. What is the electric potential at a point P on the z-axis a distance z from the center of the ring?



Answer: $V(z) = \dfrac{KQ}{\sqrt{z^2 + R^2}}$

Example:

A total amount of charge Q is uniformily distributed on the surface of a **disk** of radius R. What is the electric potential at a point P on the z-axis a distance z from the center of the disk?



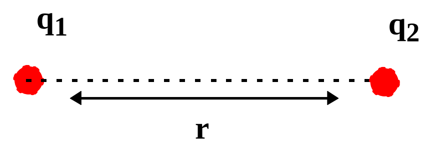
Answer: $V(z) = \dfrac{2KQ}{R^2}\left(\sqrt{z^2 + R^2} - z\right)$

Electric Potential Energy

For a system of point charges:

The potential energy **U** is the **work** required to assemble the final charge configuration starting from an inital condition of infinite separation.

Two Particles:



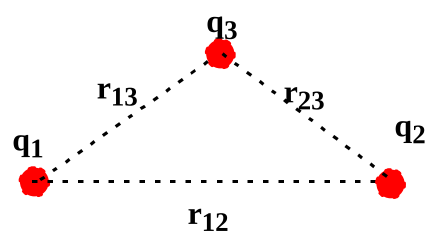
$$U = K\frac{q_1 q_2}{r} = \frac{1}{2}q_1\left(\frac{Kq_2}{r}\right) + \frac{1}{2}q_2\left(\frac{Kq_1}{r}\right)$$

so we see that

$$U = \frac{1}{2}\sum_{i=1}^{2} q_i V_i$$

where $\mathbf{V_i}$ is the electric potential at **i** due to the other charges.

Three Particles:



$$U = K\frac{q_1 q_2}{r_{12}} + K\frac{q_1 q_3}{r_{13}} + K\frac{q_2 q_3}{r_{23}}$$

which is equivalent to

$$U = \frac{1}{2}\sum_{i=1}^{3} q_i V_i$$

where $\mathbf{V_i}$ is the electric potential at **i** due to the other charges.

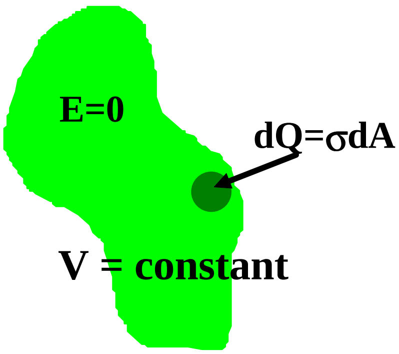
N Particles:

$$U = \frac{1}{2}\sum_{i=1}^{N} q_i V_i$$

Stored Electric Potential Energy

For a conductor with charge Q:

The potential energy **U** is the **work** required to assemble the final charge configuration starting from an inital condition of infinite separation.



For a conductor the total charge Q resides on the surface

$$Q = \int dq = \int s\, dA$$

Also, V is constant on and inside the conductor and

$$dU = \frac{1}{2} dQV = \frac{1}{2} V s\, dA$$

and hence

$$U = \frac{1}{2} \int_{Surface} V dQ = \frac{1}{2} V \int_{Surface} s\, dA = \frac{1}{2} VQ$$

Stored Energy:

$$U_{conductor} = \frac{1}{2} Q V$$

where Q is the charge on the conductor and V is the electric potential of the conductor.

For a System of N Conductors:

$$U = \frac{1}{2} \sum_{i=1}^{N} Q_i V_i$$

where Q_i is the charge on the i-th conductor and V_i is the electric potential of the i-th conductor.

Capacitors & Capacitance

Capacitor:

Any arrangement of **conductors** that is used to store electric charge (**will also store electric potential energy**).

Capacitance: $C=Q/V$ or $C=Q/\Delta V$

Units: 1 farad = 1 F = 1 C/1 V $1\ \mu F=10^{-6}$ F $1\ pF=10^{-9}$ F

Stored Energy:

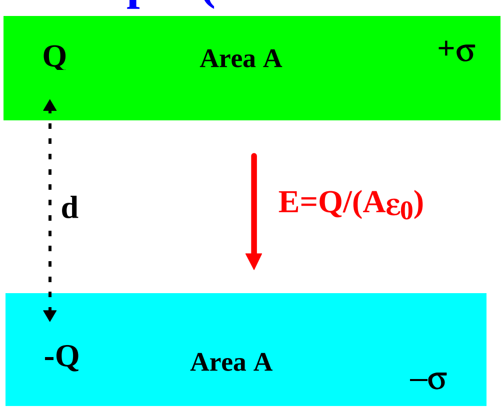
$$U_{conductor} = \frac{1}{2}QV = \frac{Q^2}{2C} = \frac{1}{2}CV^2$$

where **Q** is the charge on the conductor and **V** is the electric potential of the conductor and **C** is the capacitance of the conductor.

Example (Isolated Conducting Sphere):

For an isolated conducting sphere with radius R, $V=KQ/R$ and hence $C=R/K$ and $U=KQ^2/(2R)$.

Example (Parallel Plate Capacitor):

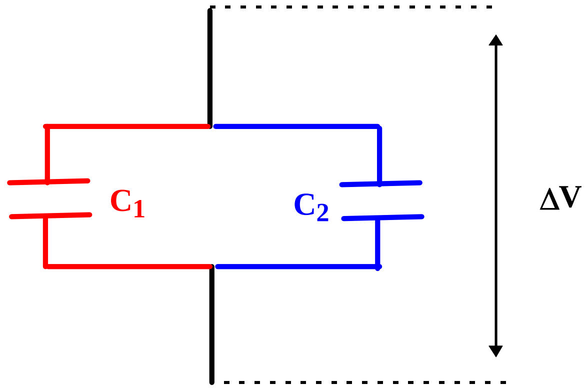


For two parallel conducting plates of area A and separation d we know that $E = \sigma/\varepsilon_0 = Q/(A\varepsilon_0)$ and $\Delta V = Ed = Qd/(A\varepsilon_0)$ so that $C = A\varepsilon_0/d$. The stored energy is $U = Q^2/(2C) = Q^2d/(2A\varepsilon_0)$.

Capacitors in Series & Parallel

Parallel:

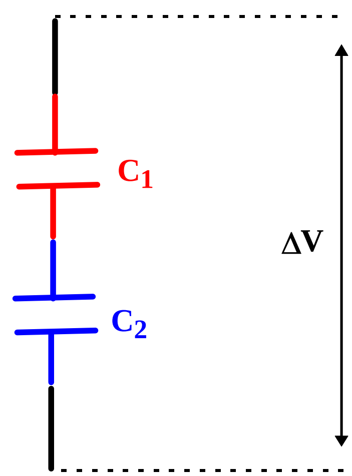
In this case **$\Delta V_1 = \Delta V_2 = \Delta V$** and **$Q = Q_1 + Q_2$**. Hence,
$Q = Q_1 + Q_2 = C_1\Delta V_1 + C_2\Delta V_2 = (C_1+C_2)\Delta V$
so **$C = Q/\Delta V = C_1 + C_2$**, where I used $Q_1 = C_1\Delta V_1$ and $Q_2 = C_2\Delta V_2$.



Capacitors in parallel add.

Series:

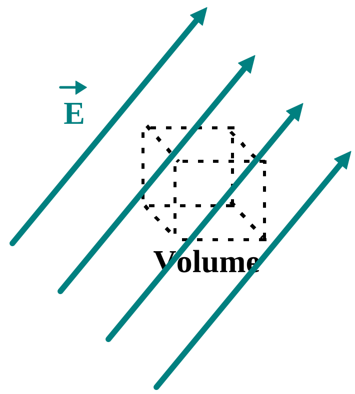
In this case **$\Delta V = \Delta V_1 + \Delta V_2$** and **$Q = Q_1 = Q_2$**. Hence,
$\Delta V = \Delta V_1 + \Delta V_2 = Q_1/C_1 + Q_2/C_2 = (1/C_1 + 1/C_2)Q$
so **$1/C = \Delta V/Q = 1/C_1 + 1/C_2$**, where I used $Q_1 = C_1\Delta V_1$ and $Q_2 = C_2\Delta V_2$.



Capacitors in series add inverses.

Energy Density of the Electric Field

Energy Density u:



Electric field lines contain **energy!** The amount of energy per unit volume is

$$u = e_0 E^2/2,$$

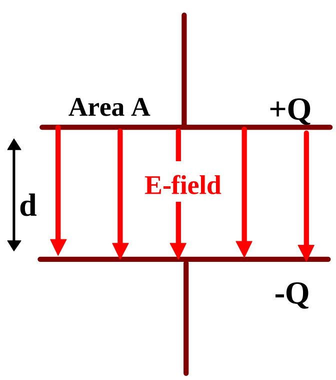
where E is the magnitude of the electric field. The energy density has **units of Joules/m^3**.

Total Stored Energy U:

The total energy strored in the electric field lines in an infinitessimal volume dV is **dU = u dV** and

$$U = \int_{Volume} u\,dV$$

If u is constant throughout the volume, V, then U = u V.



Example: Parallel Plate Capacitor

Think of the work done in bringing in the charges from infinity and placing them on the capacitor as the work necessary to produce the electric field lines and that the energy is **stored in the electric field!**

From before we know that $C = A\varepsilon_0/d$ so that the stored energy in the capacitor is

$$U = Q^2/(2C) = Q^2 d/(2A\varepsilon_0).$$

The energy stored in the electric field is $U = uV = e_0 E^2 V/2$ with $E = \sigma/e_0 = Q/(e_0 A)$ and $V = Ad$, thus

$$U = Q^2 d/(2A\varepsilon_0),$$

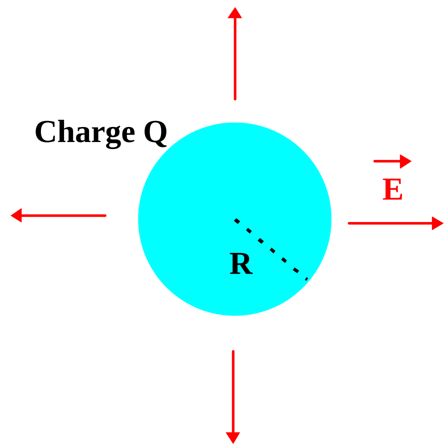
which is the **same as the energy stored in the capacitor!**

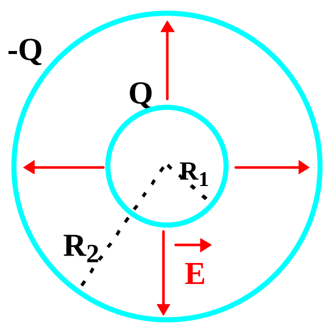
Electric Energy Examples

Example:

How much electric energy is stored by a **solid conducting sphere** of radius R and total charge Q?

Answer: $$U = \frac{KQ^2}{2R}$$





Example:

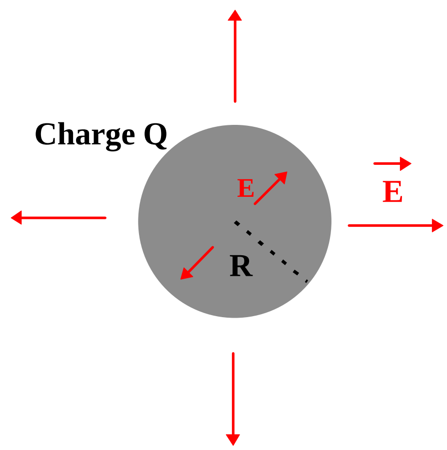
How much electric energy is stored by a two thin spherical conducting shells one of radius R_1 and charge Q and the other of radius R_2 and charge -Q (**spherical capacitor**)?

Answer: $$U = \frac{KQ^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Example:

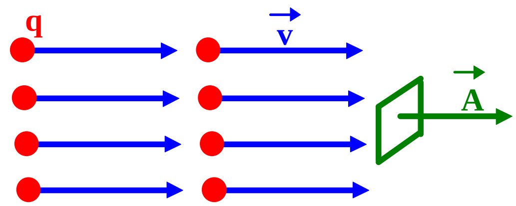
How much electric energy is stored by a **solid insulating sphere** of radius R and total charge Q uniformly distributed throughout its volume?

Answer: $$U = \left(1 + \frac{1}{5}\right)\frac{KQ^2}{2R} = \frac{3}{5}\frac{KQ^2}{R}$$



Charge Transport and Current Density

Consider **n** particles per unit volume all moving with velocity **v** and each carrying a charge **q**.



The number of particles, **ΔN**, passing through the (**directed**) area **A** in a time **Δt** is $\Delta N = n\vec{v} \cdot \vec{A}\Delta t$ and the amount of charge, **ΔQ**, passing through the (**directed**) area **A** in a time **Δt** is

$$\Delta Q = nq\vec{v} \cdot \vec{A}\Delta t \, .$$

The **current**, **I(A)**, is the amount of charge per unit time passing through the (**directed**) area **A**:

$$I(\vec{A}) = \frac{\Delta Q}{\Delta t} = nq\vec{v} \cdot \vec{A} = \vec{J} \cdot \vec{A} \, ,$$

where the **"current density"** is given by $\vec{J} = nq\vec{v}_{drift}$.

The current **I** is measured in **Ampere's** where 1 Amp is equal to one Coulomb per second (**1A = 1C/s**).

For an infinitesimal area (**directed**) area **dA**:

$$dI = \vec{J} \cdot d\vec{A} \quad \text{and} \quad \vec{J} \cdot \hat{n} = \frac{dI}{dA} \, .$$

The **"current density"** is the amount of current per unit area and has units of **A/m^2**. The current passing through the surface S is given by

$$I = \int_S \vec{J} \cdot d\vec{A} \, .$$

The current, I, is the "flux" associated with the vector J.

Electrical Conductivity and Ohms Law

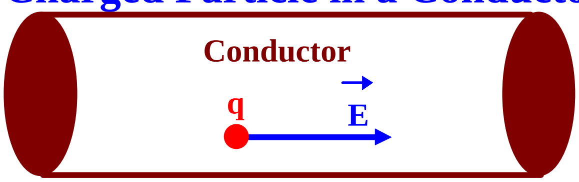
Free Charged Particle:

For a free charged particle in an electric field,

$$\vec{F} = m\vec{a} = q\vec{E} \qquad \text{and thus} \qquad \vec{a} = \frac{q}{m}\vec{E}\,.$$

The **acceleration is proportional to the electric field strength E** and the **velocity of the particle increases with time!**

Charged Particle in a Conductor:



However, for a charged particle in a conductor the **average velocity is proportional to the electric field** strength **E** and since $\vec{J} = nq\vec{v}_{ave}$ we have

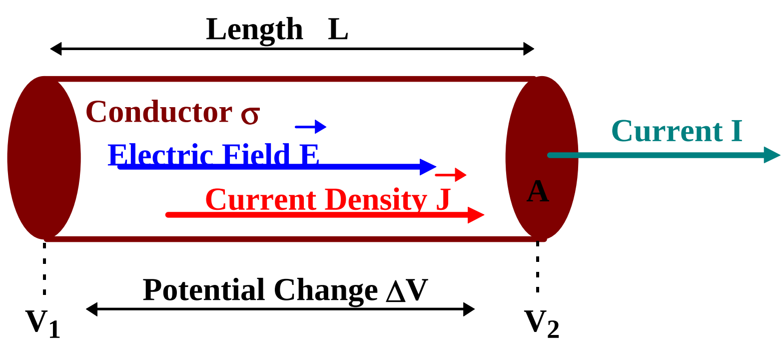
$$\vec{J} = \sigma\vec{E}\,,$$

where σ is the **conductivity** of the material and is a property of the conductor. **The resistivity ρ = 1/σ.**

Ohm's Law:

$$\vec{J} = \sigma\vec{E}$$

$$I = JA = \sigma EA$$



$$\Delta V = EL = \frac{I}{\sigma A}L = \left(\frac{L}{\sigma A}\right)I = RI$$

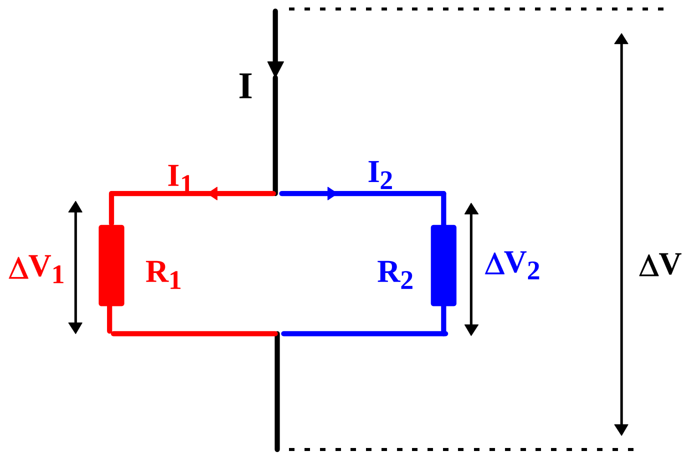
ΔV = IR (Ohm's Law) **R = L/(σA) = ρL/A (Resistance)**

Units for R are Ohms 1Ω = 1V/1A

Resistors in Series & Parallel

Parallel:

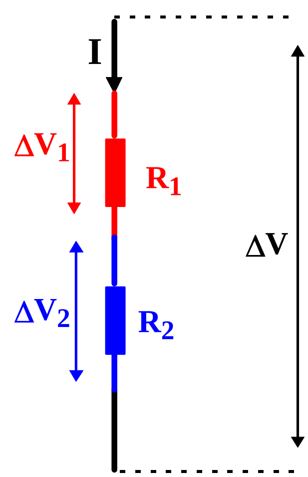
In this case $\Delta V_1 = \Delta V_2 = \Delta V$ and $I = I_1 + I_2$. Hence, $I = I_1 + I_2 = \Delta V_1/R_1 + \Delta V_2/R_2 = (1/R_1 + 1/R_2)\Delta V$ so $1/R = I/\Delta V = 1/R_1 + 1/R_2$, where I used $I_1 = \Delta V_1/R_1$ and $I_2 = \Delta V_2/R_2$. Also, $\Delta V = I_1R_1 = I_2R_2 = IR$ so $I_1 = R_2I/(R_1+R_2)$ and $I_2 = R_1I/(R_1+R_2)$.



Resistors in parallel add inverses.

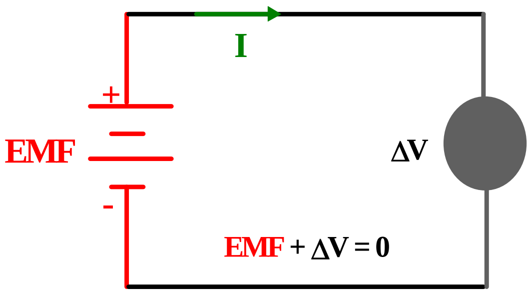
Series:

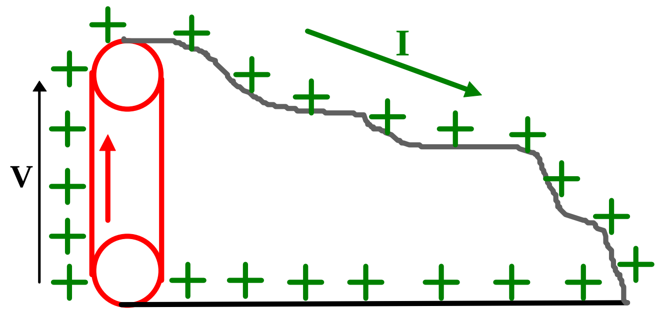
In this case $\Delta V = \Delta V_1 + \Delta V_2$ and $I = I_1 = I_2$. Hence, $\Delta V = \Delta V_1 + \Delta V_2 = I_1R_1 + I_2R_2 = (R_1+R_2)I$ so $R = \Delta V/I = R_1 + R_2$, where I used $\Delta V_1 = I_1R_1$ and $\Delta V_2 = I_2R_2$.



Resistors in series add.

Direct Current (DC) Circuits





Electromotive Force:

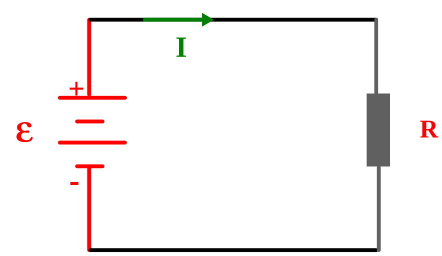
The **electromotive force EMF** of a source of electric potential energy is defined as the amount of **electric energy per Coulomb of positive charge** as the charge passes through the source from low potential to high potental.

EMF = ε = U/q (The units for EMF is Volts)

Single Loop Circuits:

ε - IR = 0 and **I = ε /R**

(Kirchhoff's Rule)



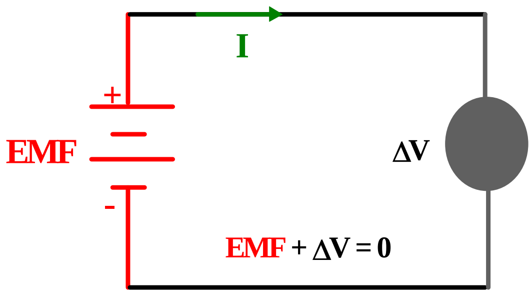
Power Delivered by EMF (P = εI):

$$dW = edq \qquad P = \frac{dW}{dt} = e\frac{dq}{dt} = eI$$

Power Dissipated in Resistor ($P = I^2R$):

$$dU = \Delta V_R dq \qquad P = \frac{dU}{dt} = \Delta V_R \frac{dq}{dt} = \Delta V_R I$$

DC Circuit Rules

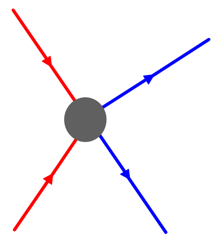


Loop Rule:

The algebraic sum of the **changes in potential** encountered in a complete traversal of any **loop** of a circuit must be **zero**.

$$\sum_{loop} \Delta V_i = 0 \; .$$

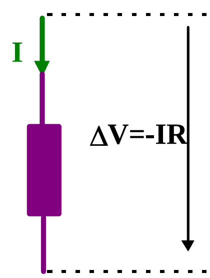
Junction Rule:



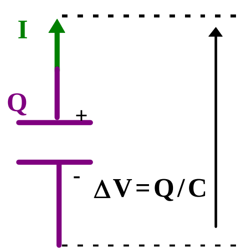
The sum of the currents entering any **junction** must be equal the sum of the currents leaving that junction.

$$\sum_{in} I_i = \sum_{out} I_i$$

Resistor:



If you move across a **resistor** **in the direction of the current flow** then the potential change is **ΔV_R = - IR**.



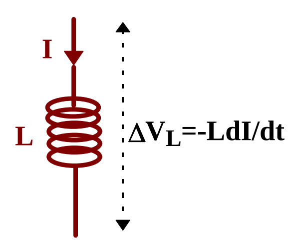
Capacitor:

If you move across a **capacitor** from **minus to plus** then the potential change is

$$\Delta V_C = Q/C,$$

and the current **leaving the capacitor** is **I = -dQ/dt**.

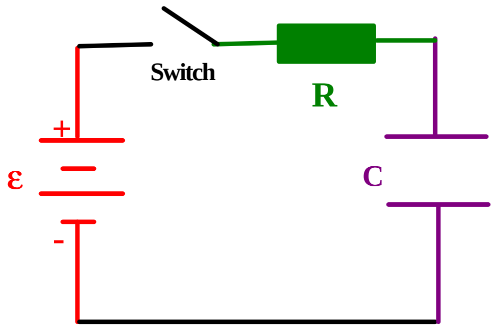
Inductor (Chapter 31):



If you move across an **inductor** **in the direction of the current flow** then the potential change is

$$\Delta V_L = - L\, dI/dt.$$

Charging a Capacitor



After the switch is closed the current is entering the capacitor so that **I = dQ/dt**, where **Q** is the charge on the capacitor and summing all the potential changes in going around the loop gives

$$\varepsilon - IR - \frac{Q}{C} = 0 \ ,$$

where **I(t)** and **Q(t)** are a function of time. If the switch is closed at t=0 then **Q(0)=0** and

$$\varepsilon - R\frac{dQ}{dt} - \frac{Q}{C} = 0 \ ,$$

which can be written in the form

$$\frac{dQ}{dt} = -\frac{1}{\tau}\left(Q - \varepsilon C\right) \ ,$$ where I have define **τ=RC**.

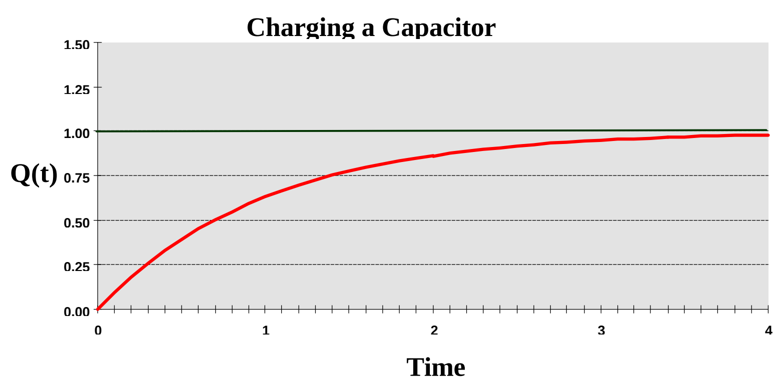
Dividing by (Q-εC) and multipling by dt and integrating gives

$$\int_0^Q \frac{dQ}{\left(Q - \varepsilon C\right)} = -\int_0^t \frac{1}{\tau}dt$$, which implies $$\ln\left(\frac{Q - \varepsilon C}{-\varepsilon C}\right) = -\frac{t}{\tau} \ .$$

Solving for **Q(t)** gives

$$Q(t) = \varepsilon C\left(1 - e^{-t/\tau}\right)$$

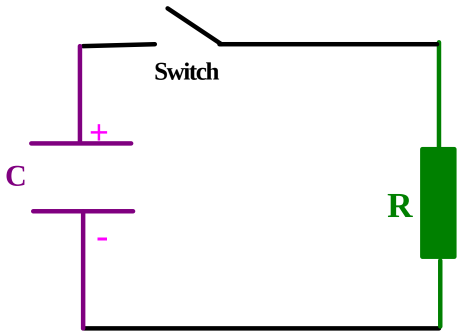
.



The curent is given by **I(t)=dQ/dt** which yields

$$I(t) = \frac{\varepsilon C}{\tau}e^{-t/\tau} = \frac{\varepsilon}{R}e^{-t/\tau}$$. The quantity **τ=RC** is call the **time constant** and has dimensions of time.

Discharging a Capacitor



After the switch is closed the current is leaving the capacitor so that **I = -dQ/dt**, where **Q** is the charge on the capacitor and summing all the potential changes in going around the loop gives

$$\frac{Q}{C} - IR = 0 \ ,$$

where **I(t)** and **Q(t)** are a function of time. If the switch is closed at t=0 then **$Q(0)=Q_0$** and

$$\frac{Q}{C} + R\frac{dQ}{dt} = 0 \ ,$$

which can be written in the form

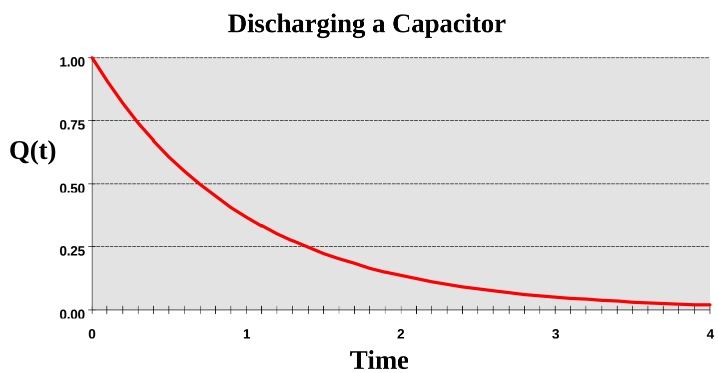
$$\frac{dQ}{dt} = -\frac{1}{t}Q \ ,$$ where I have defined **τ=RC**.

Dividing by Q and multiplying by dt and integrating gives

$$\int_{Q_0}^{Q}\frac{dQ}{Q} = -\int_0^t \frac{1}{t}dt$$, which implies $$\ln\left(\frac{Q}{Q_0}\right) = -\frac{t}{t} \ .$$

Solving for **Q(t)** gives

$$Q(t) = Q_0 e^{-t/t} \ .$$



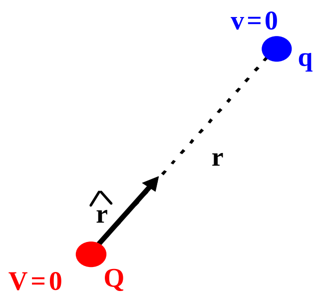
The current is given by **I(t)=-dQ/dt** which yields

$$I(t) = \frac{Q_0}{RC}e^{-t/t} \ .$$

The quantity **τ=RC** is call the **"time constant"** and has dimensions of **time**.

The Electromagnetic Force

The Force Between Two-Charged Particles (at rest):

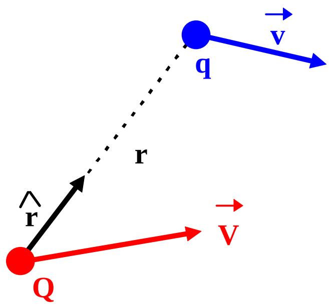


The force between two charged particles **at rest** is the **electrostatic force** and is given by

$$\vec{F}_E = \frac{KQq}{r^2}\hat{r} \quad \textbf{(electrostatic force)},$$

where K = 8.99×10^9 Nm^2/C^2.

The Force Between Two Moving Charged Particles:



The force between two **moving** charged particles is the **electromagnetic force** and is given by

$$\vec{F}_{EM} = \frac{KQq}{r^2}\hat{r} + \frac{KQq}{c^2r^2}\vec{v}\times\vec{V}\times\hat{r}$$

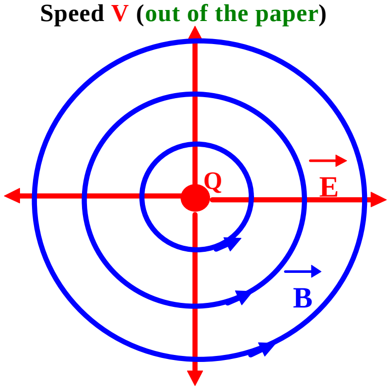
(electromagnetic force)

where K = 8.99×10^9 Nm^2/C^2 and **c = 3×10^8 m/s** (**speed of light in a vacuum**). The first term is the **electric force** and the second (**new**) term is the called the **magnetic force** so that $\vec{F}_{EM} = \vec{F}_E + \vec{F}_B$, with

$$\vec{F}_E = \frac{KQq}{r^2}\hat{r} = q\left(\frac{KQ}{r^2}\right)\hat{r} = q\vec{E}$$

$$\vec{F}_B = \frac{KQq}{c^2r^2}\vec{v}\times\vec{V}\times\hat{r} = q\vec{v}\times\left(\frac{KQ}{c^2r^2}\vec{V}\times\hat{r}\right) = q\vec{v}\times\vec{B}$$

Electric and Magnetic Fields of a Charged Particle Q moving with Speed V (out of the paper)



The **electric** and **magnetic fields** due to the particle **Q** are

$$\vec{E} = \frac{KQ}{r^2}\hat{r}$$

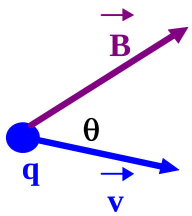
$$\vec{B} = \frac{KQ}{c^2r^2}\vec{V}\times\hat{r}$$

The **electromagnetic force** on **q** is given by

$$\vec{F}_{EM} = q\vec{E} + q\vec{v}\times\vec{B} \quad \textbf{(Lorenz Force)}.$$

The Magnetic Force

The Force on Charged Particle in a Magnetic Field:



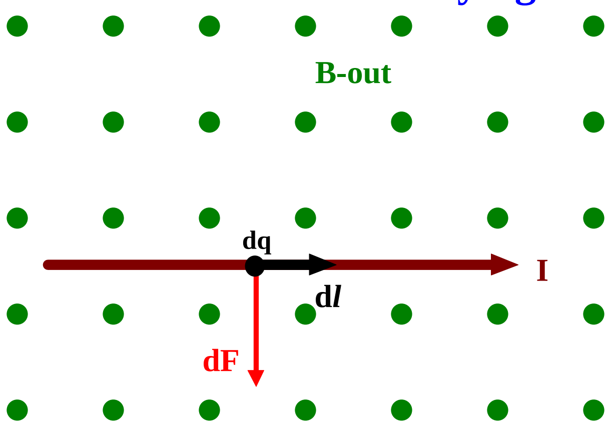
The **magnetic force** an a charged particle **q** in a **magnetic field B** is given by

$$\vec{F}_B = q\vec{v} \times \vec{B} \ .$$

The magnitude of the magnetic force is **F_B = qvB sinθ** and **B = F_B/(qv sinθ)** is the definition of the magnetic field. (**The units for B are Tesla, T, where 1 T = 1 N/(C m/s)**). The **magnetic force** an infinitesimal charged particle **dq** in a **magnetic field B** is given by

$$d\vec{F}_B = dq\vec{v} \times \vec{B} \ .$$

The Force on Wire Carrying a Current in a Magnetic Field:



A current in a wire corresponds to moving charged particles with **I = dq/dt**. The magnetic force on the charge **dq** is

$$d\vec{F}_B = dq\vec{v} \times \vec{B} \ ,$$

and the speed **v=d*l*/dt**. Hence,

$$dq\vec{v} = dq\frac{d\vec{l}}{dt} = Id\vec{l} \ ,$$

and the **magnetic force** on a infinitesimal length **d*l*** of the wire becomes $d\vec{F}_B = Id\vec{l} \times \vec{B}$. The total **magnetic force** on the **wire** is

$$\vec{F}_B = \int d\vec{F}_B = \int Id\vec{l} \times \vec{B} \ ,$$

which for a **straight wire of length L** in a **uniform magnetic field** becomes

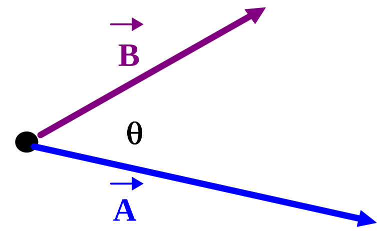
$$\vec{F}_B = I\vec{L} \times \vec{B} \ .$$

Vector Multiplication: Dot & Cross

Two Vectors:

Define two vectors according to

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$
$$\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z} \ .$$



The magnitudes of the vectors is given by

$$\left|\vec{A}\right| = A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$
$$\left|\vec{B}\right| = B = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

Dot Product (Scalar Product):

The dot product, S, is a **scalar** and is given by

$$S = \vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos q = A_x B_x + A_y B_y + A_z B_z$$

Cross Product (Vector Product):

The cross product, $\vec{C}$, is a **vector** and is given by

$$\vec{C} = \vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{x} - (A_xB_z - A_zB_x)\hat{y} + (A_xB_y - A_yB_x)\hat{z}$$

The **magnitude of the cross product** is given by

$$\left|\vec{C}\right| = \vec{A}\times\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\sin q$$

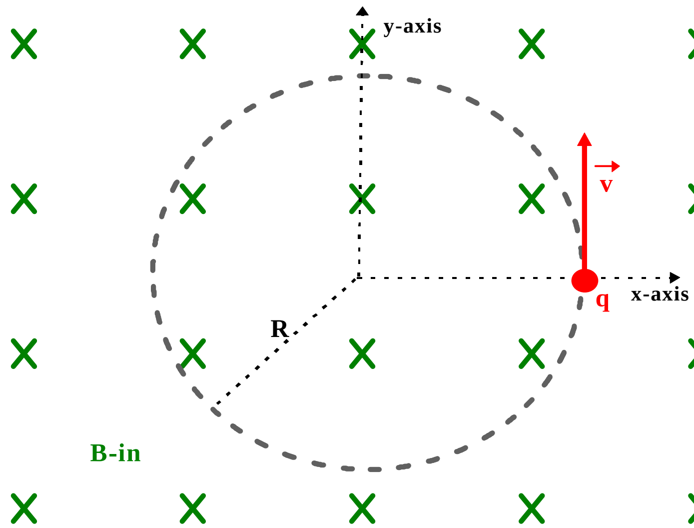
The **direction of the cross product** can be determined from the **"right hand rule"**.

Determinant Method:

The cross product can be constructed by evaluating the following determinant:

$$\vec{C} = \vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Motion of a Charged Particle in a Magnetic Field



Consider a charged particle **q** with **velocity**

$$\vec{v} = v_x\hat{x} + v_y\hat{y}\,,$$

and **kinetic energy**

$$E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v}\cdot\vec{v}\,,$$

in a **uniform magnetic field**

$$\vec{B} = -B\hat{z}\,.$$

The magnetic force on the particle is given by

$$\vec{F}_B = q\vec{v}\times\vec{B}\,.$$

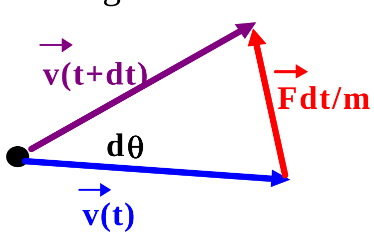
The magnetic force does not change the speed (kinetic energy) of the charged particle. The magnetic force does **no work on the charged particle** since the force is always perpendicular to the path of the particle. There is no change in the particle's kinetic energy and no change in its speed.

Proof: We know that $\vec{F}_B = q\vec{v}\times\vec{B} = m\frac{d\vec{v}}{dt}$ $\quad m\frac{d\vec{v}}{dt} = q\vec{v}\times\vec{B}$. Hence

$$\frac{dE_{kin}}{dt} = \frac{1}{2}m\frac{dv^2}{dt} = \frac{1}{2}m\frac{d(\vec{v}\cdot\vec{v})}{dt} = m\vec{v}\cdot\frac{d\vec{v}}{dt} = q\vec{v}\cdot\vec{v}\times\vec{B} = 0\,,$$

and thus $\mathbf{E_{kin}}$ (and **v**) are constant in time.

The magnetic force can **change the direction a charged particle** but not its speed. The particle undergoes **circular motion** with angular velocity **ω = qB/m**.

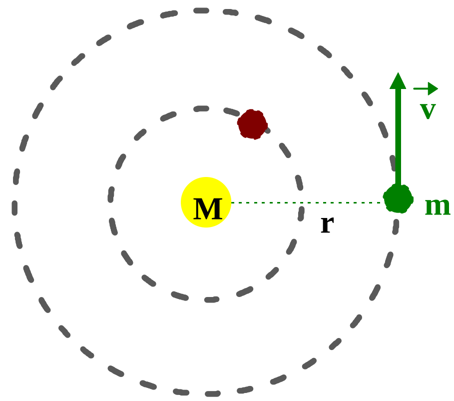


$$v\,dq = \frac{F}{m}dt = \frac{qvB}{m}dt$$

$$w = \frac{dq}{dt} = \frac{qB}{m}$$

Circular Motion: Magnetic vs Gravitational

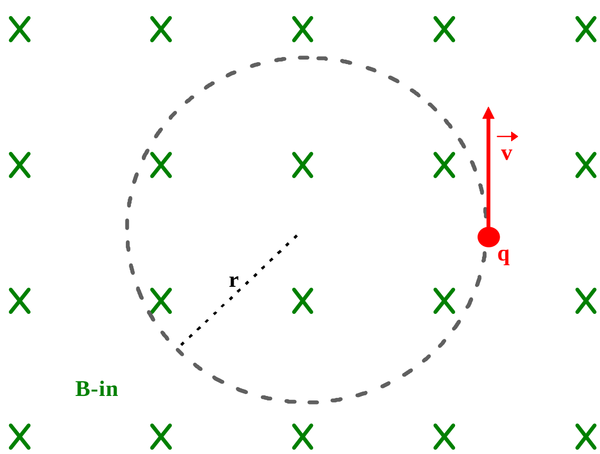
Planetary Motion:



For circular planetary motion the force on the orbiting planet is equal the mass times the **centripetal acceleration**, **$a = v^2/r$**, as follows:

$$F_G = GmM/r^2 = mv^2/r$$

Solving for the radius and speed gives, **$r = GM/v^2$** and **$v = (GM/r)^{1/2}$**. The **period of the rotation** (**time it takes to go around once**) is given by **$T=2\pi r/v=2\pi GM/v^3$** or $T = \frac{2p}{\sqrt{GM}} r^{3/2}$. The **angular velocity**, **$\omega = d\theta/dt$**, and linear velocity **$v = ds/dt$** are related by **$v = r\omega$**, since **$s = r\theta$**. Thus, $w = \sqrt{GM} / r^{3/2}$. The angular velocity an period are related by **$T = 2\pi/\omega$** and the linear frequency **f** and **ω** are related by **$\omega = 2\pi f$** with **$T = 1/f$**. **Planets further from the sum travel slower and thus have a longer period T**.



Magnetism:

For magnetic circular motion the force on the charged particle is equal its mass times the **centripetal acceleration**, **$a = v^2/r$**, as follows:

$$F_B = qvB = mv^2/r.$$

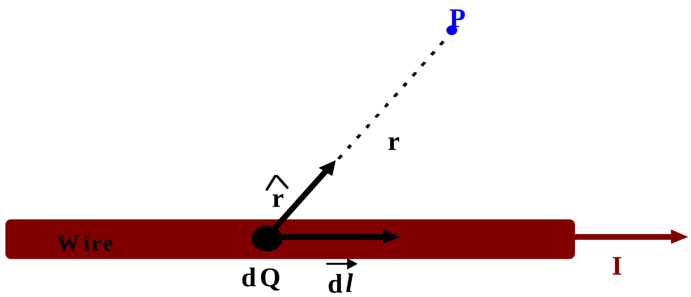
Solving for the radius and speed gives,

$$r = mv/(qB) = p/(qB) ,$$

and **$v = qBr/m$**. The **period of the rotation** is given by **$T = 2\pi r/v = 2\pi m/(qB)$** and is **independent of the radius**! The frequency (**called the cyclotron frequency**) is given by **$f = 1/T= qB/(2\pi m)$** is the **same for all particles with the same charge and mass ($\omega = qB/m$)**.

The Magnetic Field Produced by a Current

The Law of Biot-Savart:



The magnetic field at the point **P** due to a charge **dQ** moving with speed **V** within a **wire carrying a current I** is given by

$$d\vec{B} = \frac{KdQ}{c^2 r^2}\vec{V} \times \hat{r}$$

where K = $8.99\mathrm{x}10^9$ $\mathrm{Nm}^2/\mathrm{C}^2$ and **c = $3\mathrm{x}10^8$ m/s** (**speed of light in a vacuum**).

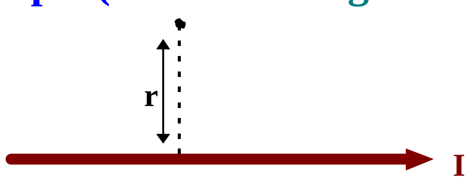
However, we know that **I = dQ/dt** and $\vec{V} = \frac{d\vec{l}}{dt}$ so that $dQ\vec{V} = Id\vec{l}$ and,

$$d\vec{B} = \frac{kI}{r^2}d\vec{l} \times \hat{r}$$ (**Law of Biot-Savart**),

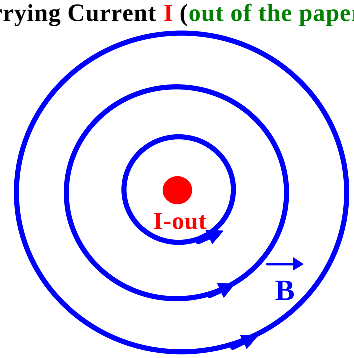
where **k = K/c^2 = 10^{-7} Tm/A**. For historical reasons we define μ_0 as follows:

$$k = \frac{\mu_0}{4\pi} = \frac{K}{c^2},$$ (**μ_0 = 4π x 10^{-7} Tm/A**).

Example (Infinite Straight Wire):



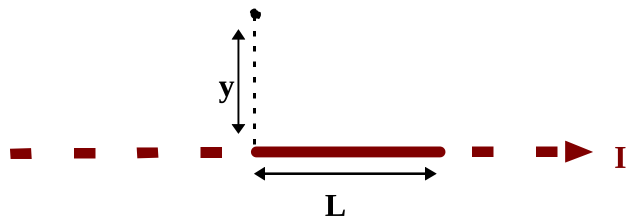
Magnetic Field of an Infinite Wire Carrying Current I (out of the paper)



An infinitely long straight wire carries a steady current I. What is the magnetic field at a distance r from the wire?

Answer: $$B(r) = \frac{2kI}{r}$$

Calculating the Magnetic Field (1)



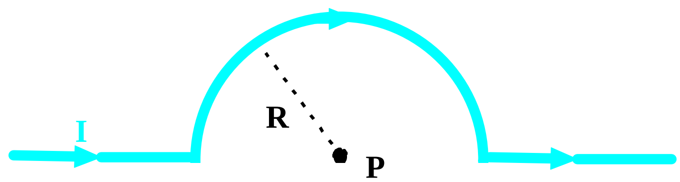
Example (Straight Wire Segment):

An infinitely long straight wire carries a steady current **I**. What is the magnetic field at a distance **y** from the wire due to the segment **0 <x < L**?

Answer: $$B(r) = \frac{kI}{y}\frac{L}{\sqrt{y^2 + L^2}}$$

Example (Semi-Circle):

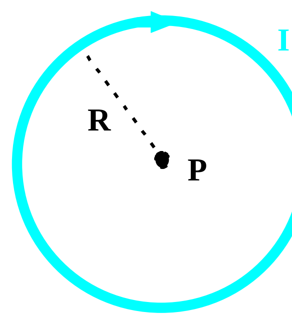
A thin wire carrying a current **I** is bent into a semi-circle of radius **R**. What is the magnitude of magnetic field at the center of the semi-circle?



Answer: $$B = \frac{pkI}{R}$$

Example (Circle):

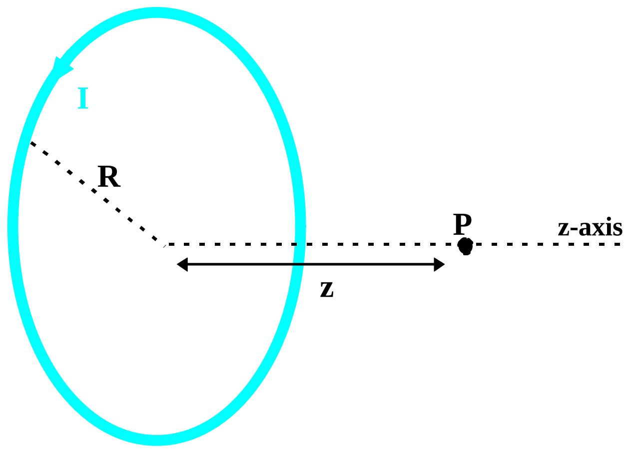
A thin wire carrying a current **I** is forms a circle of radius **R**. What is the magnitude of magnetic field at the center of the semi-circle?



Answer: $$B = \frac{2pkI}{R}$$

Calculating the Magnetic Field (2)

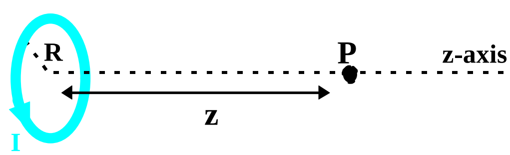
Example (Current Loop):
A thin **ring** of radius **R** carries a current **I**. What is the magnetic field at a point **P** on the z-axis a distance **z** from the center of the ring?



Answer:

$$B_z(z) = \frac{2kIpR^2}{\left(z^2 + R^2\right)^{3/2}}$$

Example (Magnetic Dipole):
A thin **ring** of radius **R** carries a current **I**. What is the magnetic field at a point **P** on the z-axis a distance **z >> R** from the center of the ring?



Answer: $B_z(z) = \frac{2km_B}{z^3}$ $\quad m_B = IpR^2 = IA$

The quantity μ_B is called the **magnetic dipole moment**,

$$\mu_B = NIA,$$

where **N** is the number of loops, **I** is the current and **A** is the area.

Ampere's Law

Gauss' Law for Magnetism:

The **net magnetic flux** emanating from a closed surface **S** is proportional to the amount of **magnetic charge** enclosed by the surface as follows:

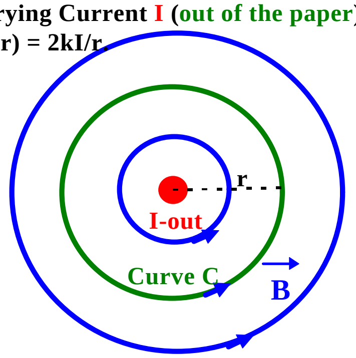
$$\Phi_B = \oint_S \vec{B} \cdot d\vec{A} \propto Q_{enclosed}^{Magnetic} .$$

However, there are **no magnetic charges** (**no magnetic monopoles**) so the **net magnetic flux** emanating from a closed surface **S** is always zero,

$$\Phi_B = \oint_S \vec{B} \cdot d\vec{A} = 0$$ (**Gauss's Law for Magnetism**).

Ampere's Law:

Magnetic Field of an Infinite Wire Carrying Current I (out of the paper) is B(r) = 2kI/r.



The line integral of the magnetic field around a closed loop (**circle**) of radius **r** around a current carrying wire is given by

$$\oint_{Loop} B \cdot d\vec{l} = 2prB(r) = 4pkI = m_0 I .$$

This result is true for **any closed loop** that encloses the current **I**.

The line integral of the magnetic field around any closed path C is equal to μ_0 times the current intercepted by the area spanning the path:

$$\boxed{\oint_C B \cdot d\vec{l} = m_0 I_{enclosed}}$$ **Ampere's Law**

The **current enclosed by the closed curve C** is given by the integral over the **surface S (bounded by the curve C) of the current density J** as follows:

$$I_{enclosed} = \int_S \vec{J} \cdot d\vec{A}$$

Ampere's Law Examples

Example (Infinite Straight Wire with radius R):

An infinitely long straight wire has a circular cross section of radius **R** and carries a uniform current density **J** along the wire. The total current carried by the wire is **I**. What is the magnitude of the magnetic field inside and outside the wire?

Answer:

$$B_{out}(r) = \frac{2kI}{r}$$

$$B_{in}(r) = \frac{2krI}{R^2}.$$

Example (Infinite Solenoid):

An infinitely long thin straight wire carrying current **I** is tightly wound into helical coil of wire (**solenoid**) of radius **R** and infinite length and with **n** turns of wire per unit length. What is the magnitude and direction of the magnetic field inside and outside the solenoid (**assume zero pitch**)?

Answer:

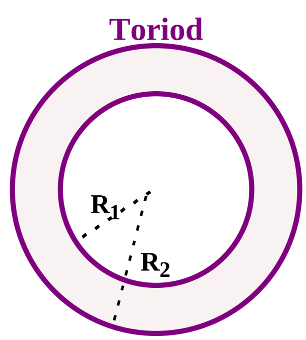
$$B_{out}(r) = 0$$

$$B_{in}(r) = m_0 nI.$$

Example (Toroid):

A solenoid bent into the shape of a doughnut is called a **toriod**. What is the magnitude and direction of the magnetic field inside and outside a toriod of inner radius $\mathbf{R_1}$ and outer radius $\mathbf{R_2}$ and **N** turns of wire carrying a current **I** (**assume zero pitch**)?

Answer:

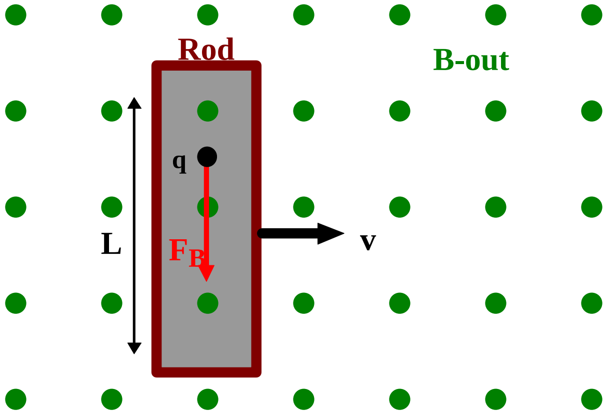


$$B_{out}(r) = 0$$

$$B_{in}(r) = \frac{2kNI}{r}$$

Electromagnetic Induction (1)

Conducting Rod Moving through a Uniform Magnetic Field:



The **magnetic force** on the charge **q** in the rod is

$$\vec{F}_B = q\vec{v} \times \vec{B}\ .$$

The **induced EMF, ε,** is equal to the amount of **work done by the magnetic field in moving a unit charge across the rod**,

$$e = \frac{1}{q}\int \vec{F}_B \cdot d\vec{l} = \int \vec{v} \times \vec{B} \cdot d\vec{l} = vLB\ .$$

In Steady State:

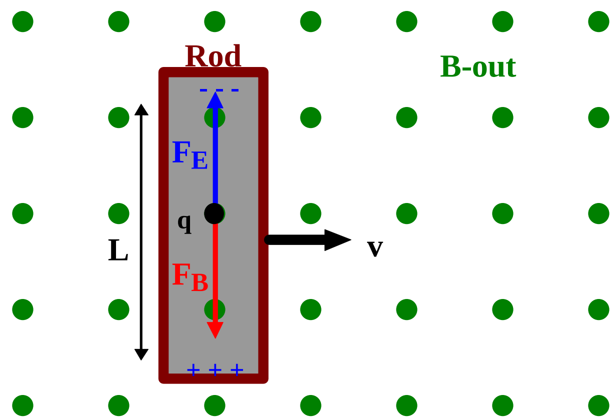
In steady state a charge **q** in the rod experiences **no net force** since,

$$\vec{F}_E + \vec{F}_B = 0\ ,$$

and thus,

$$\vec{E} = -\vec{v} \times \vec{B}\ .$$

The **induced EMF** (**change in electric potential across the rod**) is calculated from the electric field in the usual way,

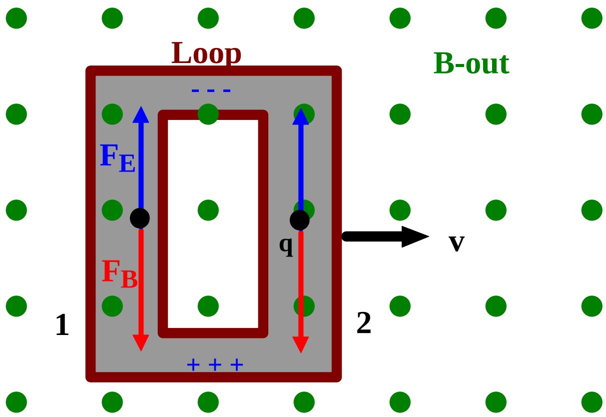


$$e = \int \vec{E} \cdot d\vec{l} = -\int \vec{v} \times \vec{B} \cdot d\vec{l} = vLB\ ,$$

which is the **same as the work done per unit charge by the magnetic field**.

Electromagnetic Induction (2)

Conducting Loop Moving through a Uniform Magnetic Field:



The **magnetic force** on the charge **q** in the loop on **side 1** is,

$$\vec{F}_{B1} = q\vec{v} \times \vec{B}_1 ,$$

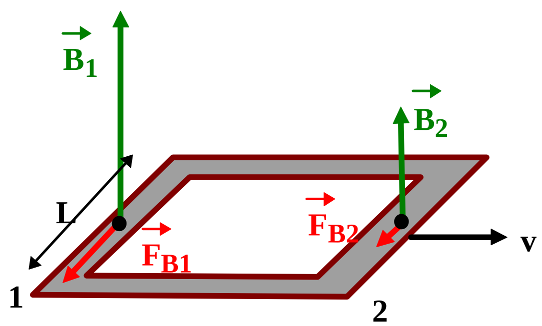
and for a charge **q** on **side 2** to it is,

$$\vec{F}_{B2} = q\vec{v} \times \vec{B}_2 .$$

However, because the magnetic field is uniform, $\vec{B}_1 = \vec{B}_2$, and the induced EMF's on **side 1** and **side 2** are equal, $\varepsilon_1 = \varepsilon_2$, and the **net EMF around the loop** (**counterclockwise**) is zero,

$$e = \frac{1}{q}\int_{Loop} \vec{F}_B \cdot d\vec{l} = e_1 - e_2 = 0 .$$

Conducting Loop Moving through a Non-Uniform Magnetic Field:



If we move a conducting loop through a **non-uniform magnetic field** then induced EMF's on **side 1** and **side 2** are not equal, $\varepsilon_1 = vLB_1$, $\varepsilon_2 = vLB_2$, and the **net EMF around the loop** (**counterclockwise**) is,

$$e = \frac{1}{q}\int_{Loop} \vec{F}_B \cdot d\vec{l} = e_1 - e_2 = vL(B_1 - B_2) .$$

This induced EMF will cause a current to flow around the loop in a counterclockwise direction (if $B_1 > B_2$)!

Faraday's Law of Induction

Magnetic Flux:

The magnetic flux through the surface **S** is defined by,

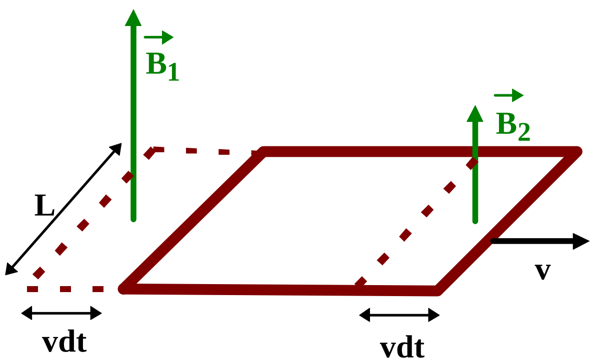
$$\Phi_B = \int_S \vec{B} \cdot d\vec{A} \,.$$

In the simple case where **B is constant and normal to the surface** then

$$\Phi_B = BA.$$

The units for magnetic flux are webbers (1 Wb = 1 Tm^2).

Rate of Change of the Magnetic Flux through Moving Loop:



The **change in magnetic flux**, **dΦ_B**, in a time **dt** through the moving loop is,

$$d\Phi_B = B_2 dA - B_1 dA,$$

with **dA = vdtL** so that

$$\frac{d\Phi_B}{dt} = -vL(B_1 - B_2) = -\varepsilon$$

where **ε** is the induced EMF. Hence,

$$\varepsilon = -\frac{d\Phi_B}{dt}$$ (**Faraday's Law of Induction**).

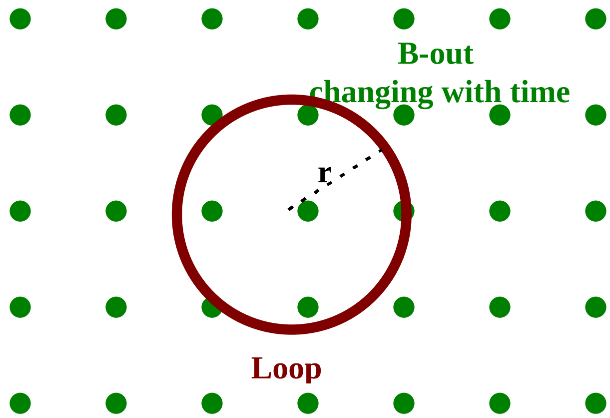
Substituting in the definition of the **induced EMF** and the **magnetic flux** yields,

$$\varepsilon = \oint_{\substack{Closed\\Loop}} \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left(\int_{Surface} \vec{B} \cdot d\vec{A}\right) = -\int_{Surface} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

We see that a changing magnetic field (with time**) can produce an electric field!**

Lenz's Law

Example (Loop of Wire in a Changing Magnetic Field):



A wire loop with a radius, **r**, of 1 meter is placed in a uniform magnetic field. Suppose that the electromagnetic is suddenly switched off and the strength of the **magnetic field decreases at a rate of 20 Tesla per second**. **What is the induced EMF in the loop (in Volts)? If the resistance of the loop, R, is 5 Ohms, what is the induced current in the loop (in Amps)? What is the direction of the induced current? What is the magnitude and direction of the magnetic field produced by the induced current** (**the induced magnetic field**) **at the center of the circle?**

Answers: If I choose my **orientation to be counterclockwise** then $\mathbf{\Phi_B = BA}$ and

$$\boldsymbol{\varepsilon = -d\Phi_B/dt} = -A\,dB/dt = -(\pi r^2)(-20T/s) = \mathbf{62.8\ V}.$$

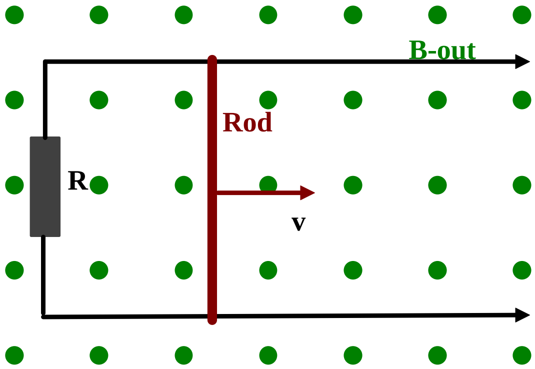
The induced current is $\mathbf{I = \varepsilon/R}$ = (62.8 V)/(5 Ω) = **12.6 A**. Since ε is positive the current is flowing in the direction of my chosen orientation (**counterclockwise**). The induced magnetic field at the center of the circle is given by $\mathbf{B_{ind} = 2\pi kI/r}$ = $(2\pi \times 10^{-7}$ Tm/A)(12.6 A)/(1 m) = **7.9 μT** and points **out of the paper**.

Lenz's Law: **It is a physical fact not a law or not a consequence of sign conventions that an electromagnetic system tends to resist change. Traditionally this is referred to as Lenz's Law:**

Induced EMF's are always in such a direction as to oppose the change that generated them.

Induction Examples

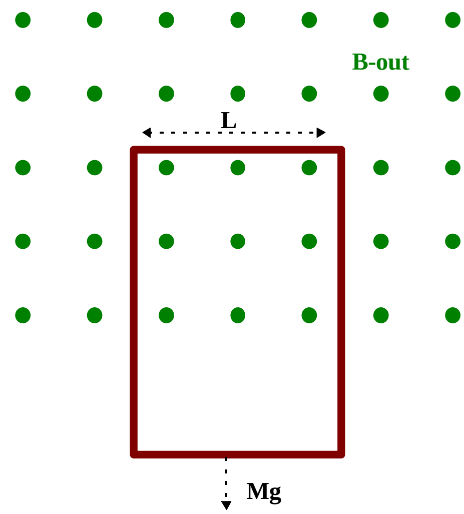
Example (simple generator):



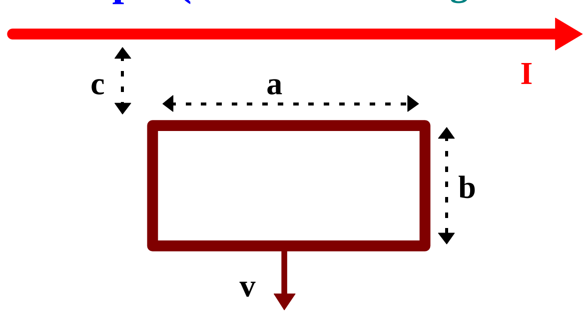
A conducting rod of length **L** is pulled along horizontal, frictionless, conducting rails at a constant speed **v**. A uniform magnetic field (**out of the paper**) fills the region in which the rod moves. The rails and the rod have negligible resistance but are connected by a resistor **R**. **What is the induced EMF in the loop? What is the induced current in the loop? At what rate is thermal energy being generated in the resistor? What force must be applied to the rod by an external agent to keep it in uniform motion? At what rate does this external agent do work on the system?**

Example (terminal velocity):

A long rectangular loop of wire of width **L**, mass **M**, and resistance **R**, falls vertically due to gravity **out of a uniform magnetic field**. Instead of falling with an acceleration, **g**, the loop falls a constant velocity (called **the terminal velocity**). **What is the terminal velocity of the loop?**

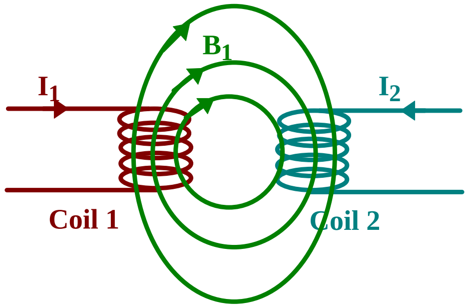


Example (non-uniform magnetic field):



A rectangular loop of wire with length **a**, width **b**, and resistance **R** is moved with velocity **v** away from an infinitely long wire carrying a current **I**. **What is the induced current in the loop when it is a distance c from the wire?**

Mutual & Self Inductance



Mutual Inductance (M):

Consider **two fixed coils** with a varying current I_1 in **coil 1** producing a magnetic field B_1. The **induced EMF** in **coil 2** due to B_1 is proportional to the magnetic flux through **coil 2**, $\Phi_2 = \int_{coil\,2} \vec{B}_1 \cdot d\vec{A}_2 = N_2 \phi_2$,

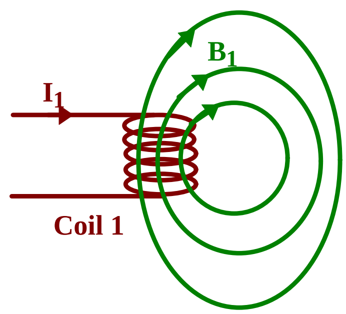
where N_2 is the number of loops in **coil 2** and ϕ_2 is the flux through a **single loop in coil 2**. However, we know that B_1 is proportional to I_1 which means that Φ_2 is proportional to I_1. The **mutual inductance M** is defined to be the **constant of proportionality between** Φ_2 and I_1 and depends on the geometry of the situation,

$M = \frac{\Phi_2}{I_1} = \frac{N_2 \phi_2}{I_1}$ $\quad \Phi_2 = N_2 \phi_2 = M I_1$. The **induced EMF** in **coil 2** due to the varying current in **coil 1** is given by,

$$\varepsilon_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}$$

The units for inductance is a Henry (1 H = Tm^2/A = Vs/A).

Self Inductance (L):



When the current I_1 in **coil 1** is varying there is a **changing magnetic flux** due to B_1 in **coil 1** itself! The **self inductance L** is defined to be the **constant of proportionality between** Φ_1 and I_1 and depends on the geometry of the situation,

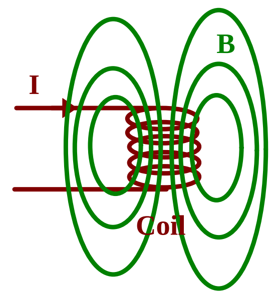
$$L = \frac{\Phi_1}{I_1} = \frac{N_1 \phi_1}{I_1} \quad \Phi_1 = N_1 \phi_1 = L I_1 \,,$$

where N_1 is the number of loops in **coil 1** and ϕ_1 is the flux through a **single loop in coil 1**. The **induced EMF** in **coil 2** due to the varying current in **coil 1** is given by,

$$\varepsilon_1 = -\frac{d\Phi_1}{dt} = -L\frac{dI_1}{dt}$$

Energy Stored in a Magnetic Field

When an **external source of EMF** is connected to an inductor and current begins to flow, the **induced EMF** (called **back EMF**) will oppose the increasing current and the **external EMF must do work** in order to overcome this opposition. This work is **stored in the magnetic field** and can be recovered by removing the external EMF.



Energy Stored in an Inductor L:

The rate at which work is done by the back EMF (power) is

$$P_{back} = \varepsilon I = -LI\frac{dI}{dt},$$

since **ε = -LdI/dt**. The power supplied by the external EMF (rate at which work is done against the back EMF) is

$$P = \frac{dW}{dt} = LI\frac{dI}{dt},$$

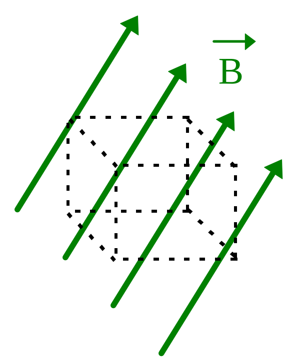
and the **energy stored in the magnetic field of the inductor** is

$$U = \int Pdt = \int_0^t LI\frac{dI}{dt}dt = \int_0^I LIdI = \frac{1}{2}LI^2.$$

Energy Density of the Magnetic Field u:

Magnetic field line contain energy! The amount of energy per unit volume is

$$\boxed{u_B = \frac{1}{2\mu_0}B^2}$$



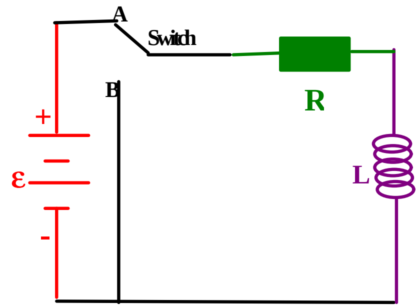
where **B** is the magnitude of the magnetic field. **The magnetic energy density has units of Joules/m^3.** The total amount of energy in an infinitesimal volume dV is **$dU = u_B dV$** and

$$U = \int_{Volume} u_B dV.$$

If **B** is constant through the volume, **V**, then **$U = u_B V$**.

RL Circuits

"Building-Up" Phase:



Connecting the switch to **position A** corresponds to the **"building up" phase of an RL circuit**. Summing all the potential changes in going around the loop gives

$$e - IR - L\frac{dI}{dt} = 0 \ ,$$

where **I(t)** is a function of time. If the switch is closed (**position A**) at t=0 and **I(0)=0** (**assuming the current is zero at t=0**) then

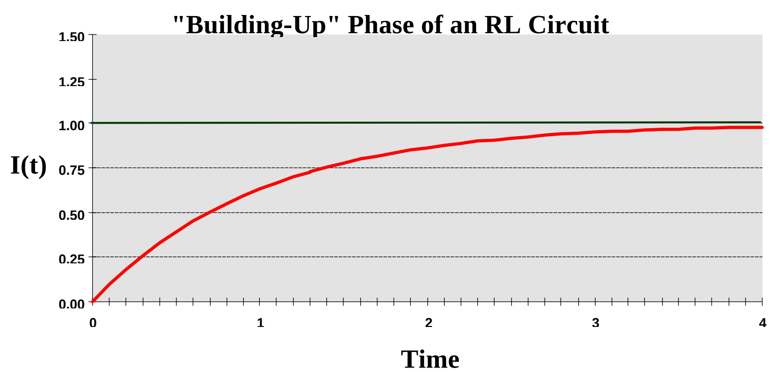
$$\frac{dI}{dt} = -\frac{1}{t}\left(I - \frac{e}{R}\right) \ ,$$ where I have define **τ=L/R**.

Dividing by (I-ε/R) and multiplying by dt and integrating gives

$$\int_0^I \frac{dI}{(I - e/R)} = -\int_0^t \frac{1}{t}dt$$, which implies $$\ln\left(\frac{I - e/R}{-e/R}\right) = -\frac{t}{t} \ .$$

Solving for **I(t)** gives

$$I(t) = \frac{e}{R}\left(1 - e^{-t/t}\right) \ .$$



The potential change across the inductor is given by **ΔV_L(t)=-LdI/dt** which yields

$$\Delta V_L(t) = -ee^{-t/t} \ .$$

The quantity **τ=L/R** is call the **time constant** and has dimensions of time.

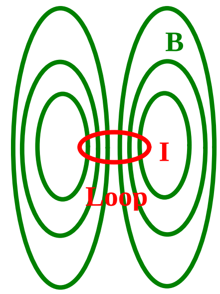
"Collapsing" Phase:

Connecting the switch to **position B** corresponds to the **"collapsing" phase of an RL circuit**. Summing all the potential changes in going around the loop gives $-IR - L\frac{dI}{dt} = 0$, where **I(t)** is a function of time. If the switch is closed (**position B**) at t=0 then **$I(0)=I_0$** and

$$\frac{dI}{dt} = -\frac{1}{t}I \quad \text{and} \quad I(t) = I_0e^{-t/t} \ .$$

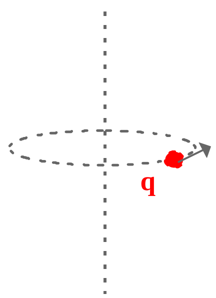
Electrons and Magnetism

Magnetic Dipole:



The magnetic field on the z-axis of a **current loop** with area $\mathbf{A=\pi R^2}$ and current **I** is given by $\mathbf{B_z(z) = 2k\mu/z^3}$, when **z >> R**, where the **magnetic dipole moment** $\mathbf{\mu = IA}$.

Orbital Magnetic Moment:



Consider a single particle with charge **q** and mass **m** undergoing **uniform circular motion** with radius **R** about the z-axis. The period of the orbit is given by **T = 2πR/v**, where **v** is the particles speed. The magnetic moment (**called the orbital magnetic moment**) is

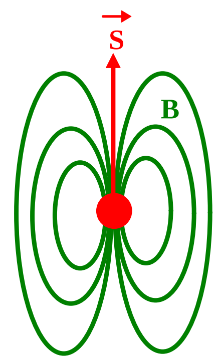
$$m_{orb} = IA = \frac{q}{T} pR^2 = \frac{q}{2} vR ,$$

since **I = q/T**. The orbital magnetic moment can be written in terms of the orbital angular momentum, $\vec{L} = \vec{r} \times \vec{p}$, as follows

$$\boxed{m_{orb} = \frac{q}{2m} L_{orb}},$$

where $\mathbf{L_{orb} = Rmv}$. For an electron,

$$m_{orb} = -\frac{e}{2m_e} L_{orb} .$$



"Spin" Magnetic Moment (Quantum Mechanics):

Certain elementary parrticles (such as electrons) carry **intrensic angular momentum** (called **"spin" angular momentum**) and an **intrensic magnetic moment** (called **"spin" magnetic moment**),

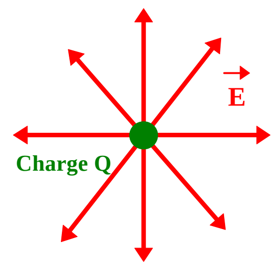
$$m_{spin} = -\frac{e}{2m_e} gS = -\frac{e\hbar}{2m_e} , \quad \text{(electron)}$$

where $S = \hbar/2$ is the spin angular momentum of the electron and **g = 2** is the **gyromagnetic ratio**. ($\hbar = h/2p$ and **h** is **Plank's Constant**.). **Here the units are Bohr Magnitons,** $m_{Bohr} = \frac{e\hbar}{2m_e}$, **with** $\mathbf{\mu_{Bohr} = 9.27 \times 10^{-24}}$ **J/T.**

Maxwell's Equations

I. (Gauss' Law):

$$\Phi_E = \oint_{Surface} \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_{Volume} \rho \, dV$$



Volume Enclosed by Surface

Two Sources of Electric Fields

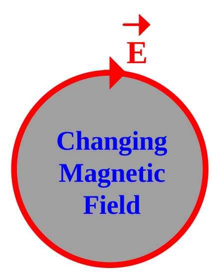
II. (Gauss' Law for Magnetism):

$$\Phi_B = \oint_{Surface} \vec{B} \cdot d\vec{A} = 0$$

No Magnetic Charges!

III. (Faraday's Law of Induction):

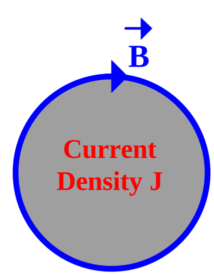
$$\varepsilon = \oint_{Curve} \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -\int_{Surface} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$



Surface Bounded by Curve

IV. (Ampere's Law):

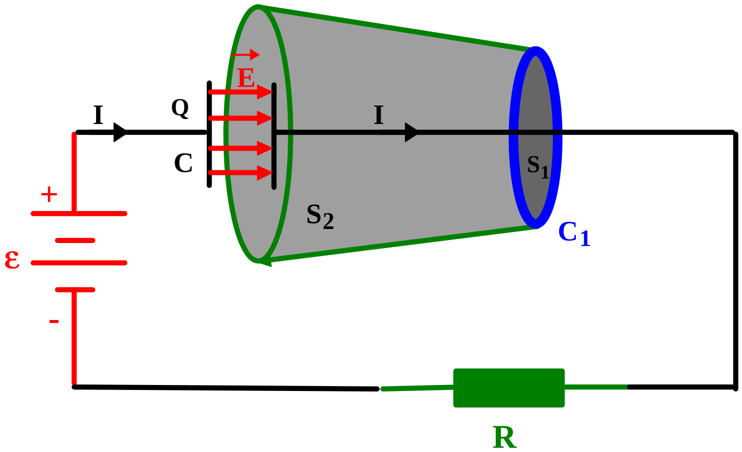
$$\oint_{Curve} \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed} = \mu_0 \int_{Surface} \vec{J} \cdot d\vec{A}$$



Surface Bounded by Curve

One Source of Magnetic Fields

Finding the Missing Term



We are looking for a **new term in Ampere's Law** of the form,

$$\oint_{C1} \vec{B}\cdot d\vec{l} = \mu_0 I + \delta \frac{d\Phi_E}{dt},$$

where δ is an unknown constant and

$$I = \int_S \vec{J}\cdot d\vec{A} \qquad \Phi_E = \int_S \vec{E}\cdot d\vec{A},$$

where **S** is any surface bounded by the curve $\mathbf{C_1}$.

Case I (use surface S_1):

If we use the surface $\mathbf{S_1}$ which is bounded by the curve $\mathbf{C_1}$ then

$$\oint_{C1} \vec{B}\cdot d\vec{l} = \mu_0 I + \delta \frac{d\Phi_E}{dt} = \int_{S1}\left(\mu_0 \vec{J} + \delta \frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{A} = \mu_0 I,$$

since **E = 0 through the surface S_1**.

0

Must be equal, hence $\delta=\mu_0\varepsilon_0$.

Case II (use surface S_2):

If we use the surface $\mathbf{S_2}$ which is bounded by the curve $\mathbf{C_1}$ then

$$\oint_{C1} \vec{B}\cdot d\vec{l} = \mu_0 I + \delta \frac{d\Phi_E}{dt} = \int_{S1}\left(\mu_0 \vec{J} + \delta \frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{A} = \frac{dI}{\varepsilon_0},$$

0

since **J = 0 through the surface S_2** and

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A} \qquad \frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0 A}\frac{dQ}{dt} = \frac{I}{\varepsilon_0 A}.$$

Ampere's Law (complete):

$$\oint_{Curve} \vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \int_{Surface}\left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{A} = \mu_0 (I + I_d),$$

$$I_d = \int_S \vec{J}_d \cdot d\vec{A} \qquad \vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}.$$

"Displacement Current"

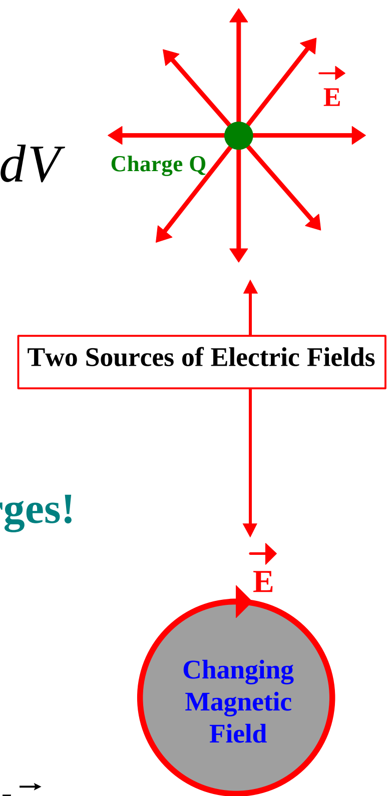
"Displacement Current" Density

Complete Maxwell's Equations

I. (Gauss' Law):

$$\Phi_E = \oint_{Surface} \vec{E}\cdot d\vec{A} = \frac{Q_{enclosed}}{e_0} = \frac{1}{e_0}\int_{Volume} r\,dV$$

Volume Enclosed by Surface



II. (Gauss' Law for Magnetism):

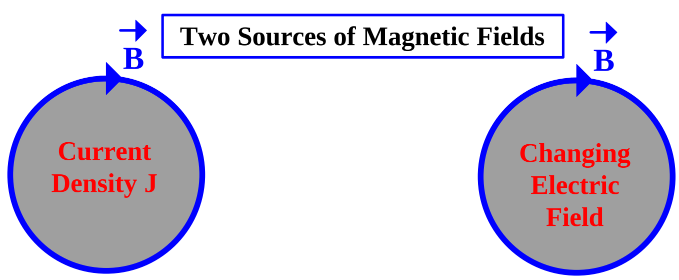
$$\Phi_B = \oint_{Surface} \vec{B}\cdot d\vec{A} = 0$$

No Magnetic Charges!

III. (Faraday's Law of Induction):

$$e = \oint_{Curve} \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -\int_{Surface} \frac{¶\vec{B}}{¶t}\cdot d\vec{A}$$

Surface Bounded by Curve



IV. (Ampere's Law):

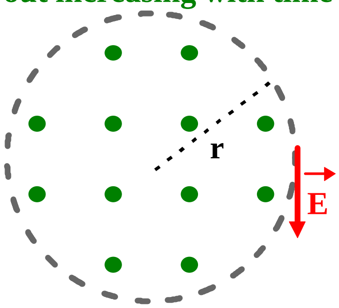
$$\oint_{Curve} \vec{B}\cdot d\vec{l} = m_0 I_{enc} + m_0 e_0 \frac{d\Phi_E}{dt} = m_0 \int_{Surface}\left(\vec{J} + e_0\frac{¶\vec{E}}{¶t}\right)\cdot d\vec{A}$$

Surface Bounded by Curve

Electric & Magnetic Fields that Change with Time

Changing Magnetic Field Produces an Electric Field:

B-out increasing with time



A **uniform magnetic field** is confined to a circular region of radius, **r**, and is **increasing with time**. **What is the direction and magnitude of the induced electric field at the radius r?**

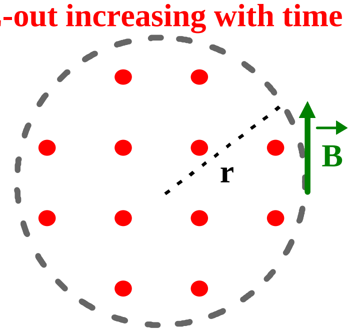
Answer: If I choose my **orientation to be counterclockwise** then **Φ_B = B(t)A** with **A = πr^2**. **Faraday's Law of Induction** tells us that

$$\oint_{Circle} \vec{E}\cdot d\vec{l} = 2\pi r E(r) = -\frac{d\Phi_B}{dt} = -\pi r^2 \frac{dB}{dt},$$

and hence **E(r) = -(r/2) dB/dt**. Since **dB/dt > 0** (**increasing with time**), **E** is negative which means that it points opposite my chosen orientation.

Changing Electric Field Produces a Magnetic Field:

E-out increasing with time



A **uniform electric field** is confined to a circular region of radius, **r**, and is **increasing with time. What is the direction and magnitude of the induced magnetic field at the radius r?**

Answer: If I choose my **orientation to be counterclockwise** then **Φ_E = E(t)A** with **A = πr^2**. **Ampere's Law** (**with J = 0**) tells us that

$$\oint_{Circle} \vec{B}\cdot d\vec{l} = 2\pi r B(r) = \epsilon_0\mu_0 \frac{d\Phi_E}{dt} = \frac{\pi r^2}{c^2}\frac{dE}{dt},$$

and hence **B(r) = (r/2c^2) dE/dt**. Since **dE/dt > 0** (**increasing with time**), **B** is positive which means that it points in the direction of my chosen orientation.

Simple Harmonic Motion

Hooke's Law Spring:

For a **Hooke's Law spring** the restoring force is linearly proportional to the distance from equilibrium, $\mathbf{F_x = -kx}$, where **k** is the spring constant. Since, $\mathbf{F_x = ma_x}$ we have

$$-kx = m\frac{d^2x}{dt^2} \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0, \quad \text{where } x = x(t).$$

General Form of SHM Differential Equation:

The general for of the **simple harmonic motion** (**SHM**) differential equation is

$$\frac{d^2x(t)}{dt^2} + Cx(t) = 0,$$

where **C** is a positive constant (**for the Hooke's Law spring C=k/m**). The most general solution of this **2nd order differential equation** can be written in the following four ways:

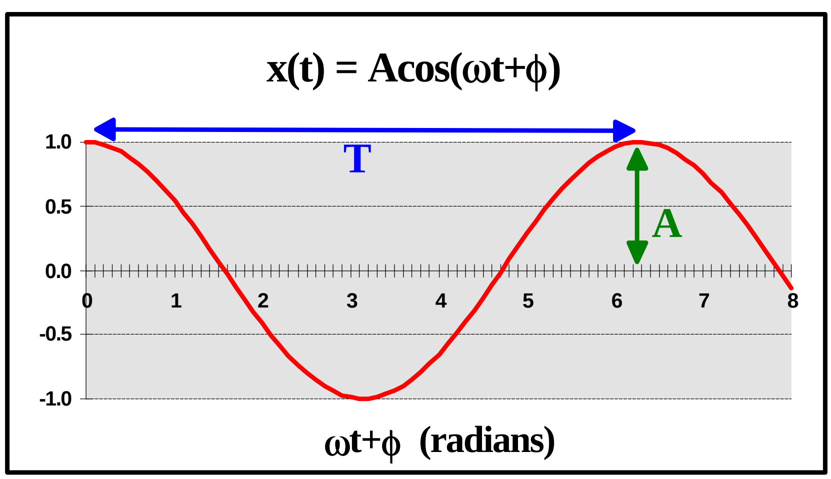
$$x(t) = Ae^{iwt} + Be^{-iwt}$$

$$x(t) = A\cos(wt) + B\sin(wt)$$

$$x(t) = A\sin(wt + f)$$

$$x(t) = A\cos(wt + f)$$

where **A**, **B**, and φ are arbitrary constants and $w = \sqrt{C}$. In the chart, **A is the amplitude** of the oscillations and **T is the period**. The **linear frequency** **f = 1/T** is measured in cycles per second (**1 Hz = 1/sec**). The **angular frequency** **ω = 2πf** and is measured in radians/second. For the Hooke's Law Spring **C = k/m** and thus $w = \sqrt{C} = \sqrt{k/m}$.



SHM Differential Equation

The general for of the **simple harmonic motion** (**SHM**) differential equation is

$$\frac{d^2x(t)}{dt^2} + Cx(t) = 0 \, ,$$

where **C** is a constant. One way to solve this equation is to turn it into an **algebraic equation** by looking for a solution of the form

$$x(t) = Ae^{at} \, .$$

Substituting this into the differential equation yields,

$$a^2 Ae^{at} + CAe^{at} = 0 \quad \text{or} \quad \boxed{a^2 = -C} \, .$$

Case I (C > 0, oscillatory solution):

For positive **C**, $a = \pm i\sqrt{C} = \pm iw$, where $w = \sqrt{C}$. In this case the most general solution of this **2nd order differential equation** can be written in the following four ways:

$$x(t) = Ae^{iwt} + Be^{-iwt}$$

$$x(t) = A\cos(wt) + B\sin(wt)$$

$$x(t) = A\sin(wt + f)$$

$$x(t) = A\cos(wt + f)$$

where **A**, **B**, and ϕ are arbitrary constants (**two arbitrary constants for a 2nd order differential equation).** Remember that $e^{\pm iq} = \cos q \pm i\sin q$ where $i = \sqrt{-1}$.

Case II (C < 0, exponential solution):

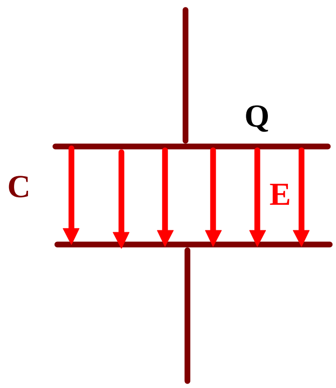
For negative **C**, $a = \pm\sqrt{-C} = \pm g$, where $g = \sqrt{-C}$. In this case, the most general solution of this **2nd order differential equation** can be written as follows:

$$x(t) = Ae^{gt} + Be^{-gt} \, ,$$

where **A** and **B** arbitrary constants.

Capacitors and Inductors

Capacitors Store Electric Potential Energy:

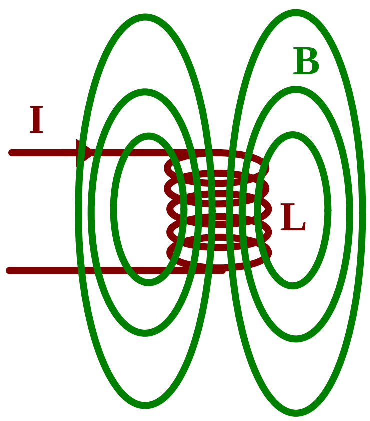


$$U_E = \frac{Q^2}{2C}$$

$$Q = C\Delta V_C \qquad \Delta V_C = Q / C$$

$$u_E = \frac{1}{2} e_0 E^2$$ (**E-field energy density**)

Inductors Store Magnetic Potential Energy:



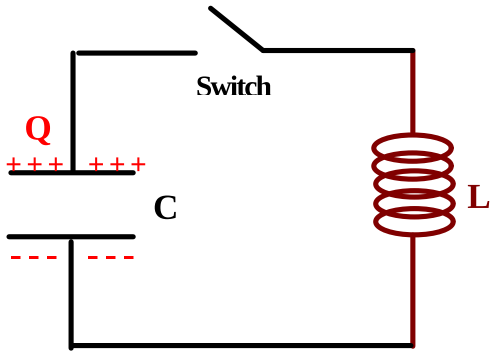
$$U_B = \frac{1}{2} L I^2$$

$$\Phi_B = LI \qquad L = \Phi_B / I$$

$$e_L = -L \frac{dI}{dt}$$

$$u_B = \frac{1}{2 m_0} B^2$$ (**B-field energy density**)

An LC Circuit



At **t = 0** the switch is closed and a capacitor with initial charge $\mathbf{Q_0}$ is connected in series across a inductor (assume there is no resistance). The initial conditions are $\mathbf{Q(0) = Q_0}$ and **I(0) = 0**. Moving around the circuit in the direction of the current flow yields

$$\frac{Q}{C} - L\frac{dI}{dt} = 0 .$$

Since I is flowing out of the capacitor, $I = -dQ/dt$, so that

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0 .$$

This differential equation for **Q(t)** is the **SHM differential equation** we studied earlier with $w = \sqrt{1/LC}$ and solution

$$Q(t) = A\cos wt + B\sin wt .$$

The current is thus,

$$I(t) = -\frac{dQ}{dt} = Aw\sin wt - Bw\cos wt .$$

Applying the initial conditions yields

$$Q(t) = Q_0\cos wt$$

$$I(t) = Q_0 w\sin wt$$

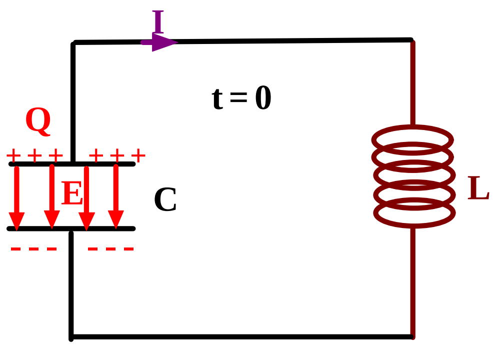
Thus, **Q(t)** and **I(t)** **oscillate with SHM with angular frequency** $w = \sqrt{1/LC}$. The stored energy oscillates between **electric** and **magnetic** according to

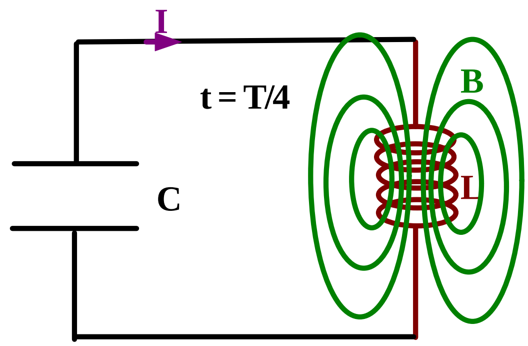
$$U_E(t) = \frac{Q^2(t)}{2C} = \frac{Q_0^2}{2C}\cos^2 wt$$

$$U_B(t) = \frac{1}{2}LI^2(t) = \frac{1}{2}LQ_0^2w^2\sin^2 wt$$

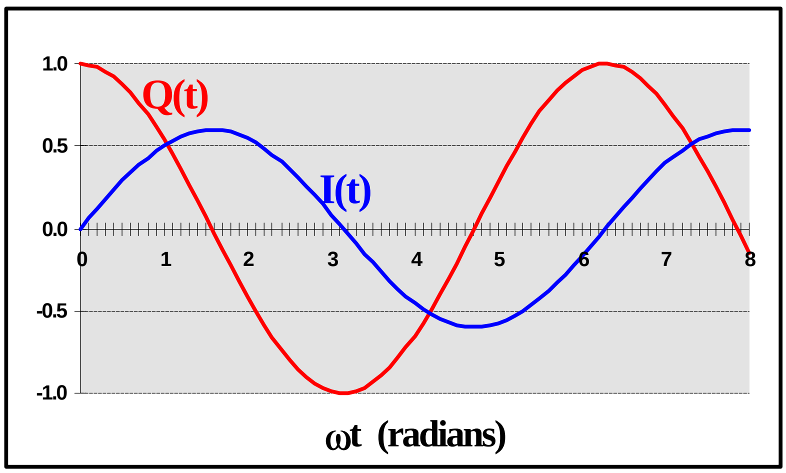
Energy is conserved since $\mathbf{U_{tot}(t) = U_E(t) + U_B(t) = Q_0^2/2C}$ is **constant**.

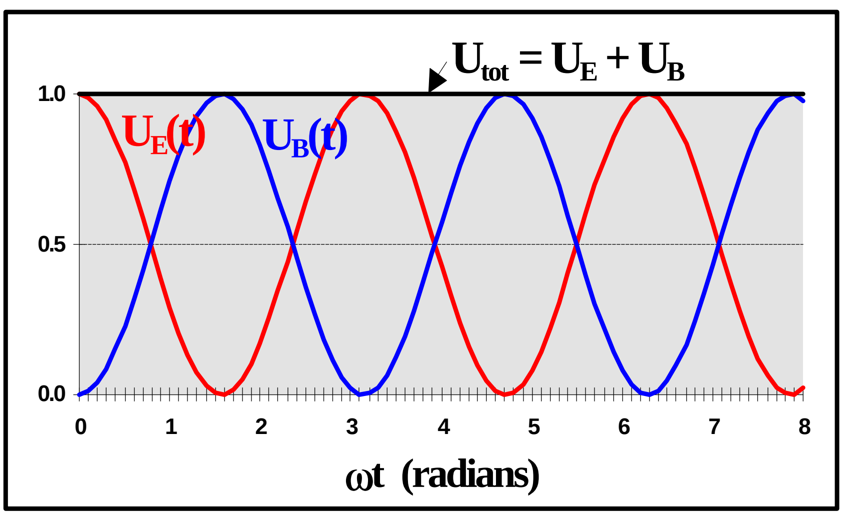
LC Oscillations





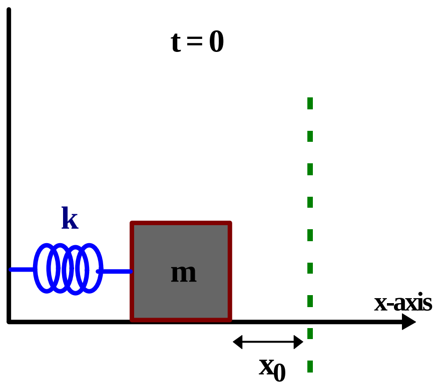
$$Q(t) = Q_0 \cos \omega t$$
$$I(t) = Q_0 \omega \sin \omega t$$

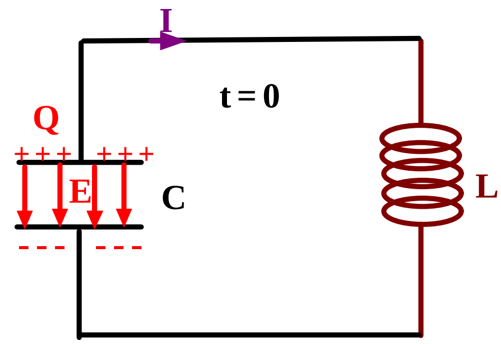




$$U_E(t) = \frac{Q_0^2}{2C} \cos^2 \omega t$$
$$U_B(t) = \frac{Q_0^2}{2C} \sin^2 \omega t$$

Mechanical Analogy





At t = 0:

$$E = \frac{1}{2}kx_0^2$$
$$v = 0$$

At Later t:

$$v = \frac{dx}{dt}$$
$$x(t) = x_0\cos wt$$
$$w = \sqrt{\frac{k}{m}}$$
$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

At t = 0:

$$U = \frac{1}{2C}Q_0^2$$
$$I = 0$$

At Later t:

$$I = -\frac{dQ}{dt}$$
$$Q(t) = Q_0\cos wt$$
$$w = \sqrt{\frac{1}{LC}}$$
$$E = \frac{1}{2}LI^2 + \frac{1}{2C}Q^2$$

Constant

Correspondence:

$$x(t) \leftrightarrow Q(t)$$
$$v(t) \leftrightarrow I(t)$$
$$m \leftrightarrow L$$
$$k \leftrightarrow 1/C$$

Another Differential Equation

Consider the **2nd order** differential equation

$$\frac{d^2x(t)}{dt^2} + D\frac{dx(t)}{dt} + Cx(t) = 0,$$

where **C** and **D** are constants. We solve this equation by turning it into an **algebraic equation** by looking for a solution of the form $x(t) = Ae^{at}$. Substituting this into the differential equation yields,

$$a^2 + Da + C = 0 \quad \text{or} \quad \boxed{a = -\frac{D}{2} \pm \sqrt{\left(\frac{D}{2}\right)^2 - C}}.$$

Case I (C > (D/2)^2, damped oscillations):

For **C > (D/2)^2**, $a = -D/2 \pm i\sqrt{C-(D/2)^2} = -D/2 \pm iw'$, where $w' = \sqrt{C-(D/2)^2}$, and the most general solution has the form:

$$x(t) = e^{-Dt/2}\left(Ae^{iw't} + Be^{-iw't}\right)$$

$$x(t) = e^{-Dt/2}\left(A\cos(w't) + B\sin(w't)\right)$$

$$x(t) = Ae^{-Dt/2}\sin(w't + f)$$

$$x(t) = Ae^{-Dt/2}\cos(w't + f)$$

where **A**, **B**, and ϕ are arbitrary constants.

Case II (C < (D/2)^2, over damped):

For **C < (D/2)^2**, $a = -D/2 \pm \sqrt{(D/2)^2 - C} = -D/2 \pm g$, where $g = \sqrt{(D/2)^2 - C}$. In this case,

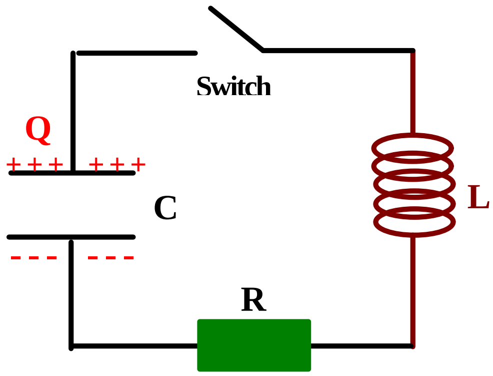
$$x(t) = e^{-Dt/2}\left(Ae^{gt} + Be^{-gt}\right).$$

Case III (C = (D/2)^2, critically damped):

For **C = (D/2)^2**, $a = -D/2$, and

$$x(t) = Ae^{-Dt/2}.$$

An LRC Circuit



At **t = 0** the switch is closed and a capacitor with initial charge $\mathbf{Q_0}$ is connected in series across an inductor and a resistor. The initial conditions are $\mathbf{Q(0) = Q_0}$ and **I(0) = 0**. Moving around the circuit in the direction of the current flow yields

$$\frac{Q}{C} - L\frac{dI}{dt} - IR = 0 .$$

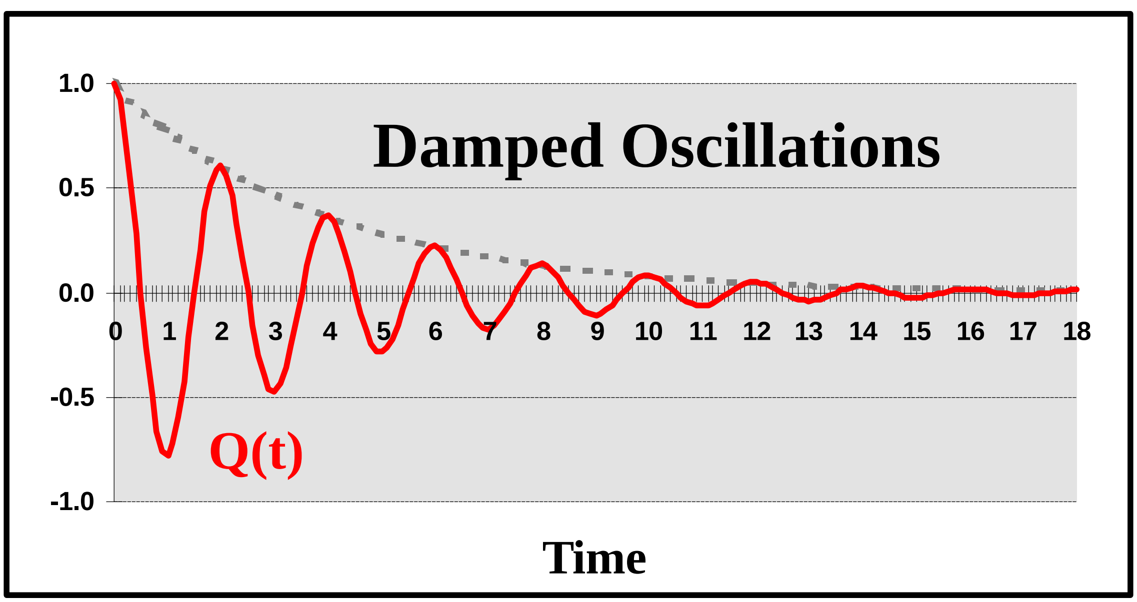
Since I is flowing out of the capacitor, $I = -dQ/dt$, so that

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0 .$$

This differential equation for **Q(t)** is the differential equation we studied earlier. If we take the case where $\mathbf{R^2 < 4L/C}$ (**damped oscillations**) then

$$Q(t) = Q_0 e^{-Rt/2L}\cos w't ,$$

with $w' = \sqrt{w^2 - (R/2L)^2}$ and $w = \sqrt{1/LC}$.

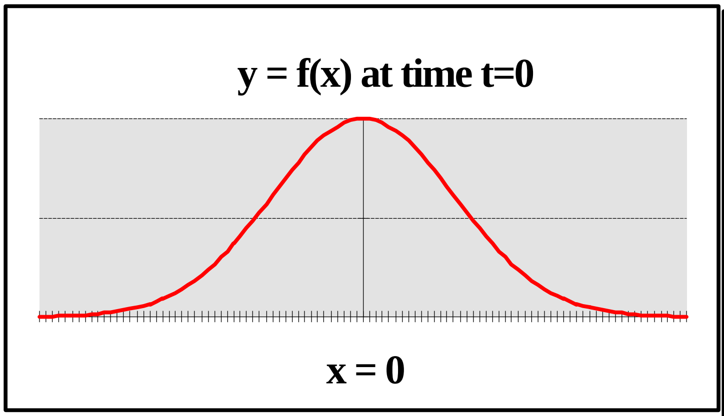


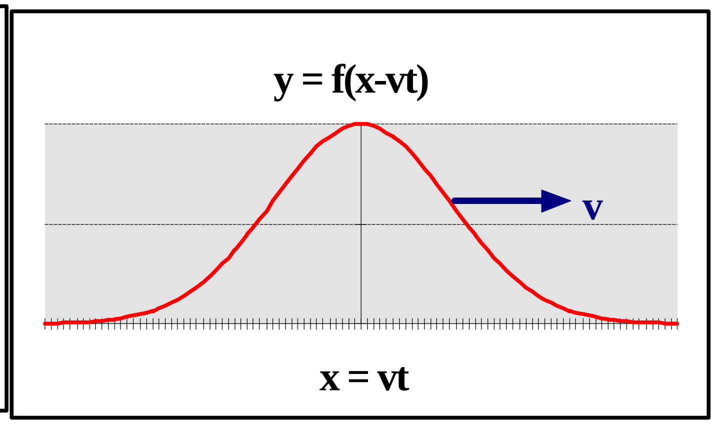
Traveling Waves

A "wave" is a traveling disturbance that transports energy but not matter.

Constructing Traveling Waves:

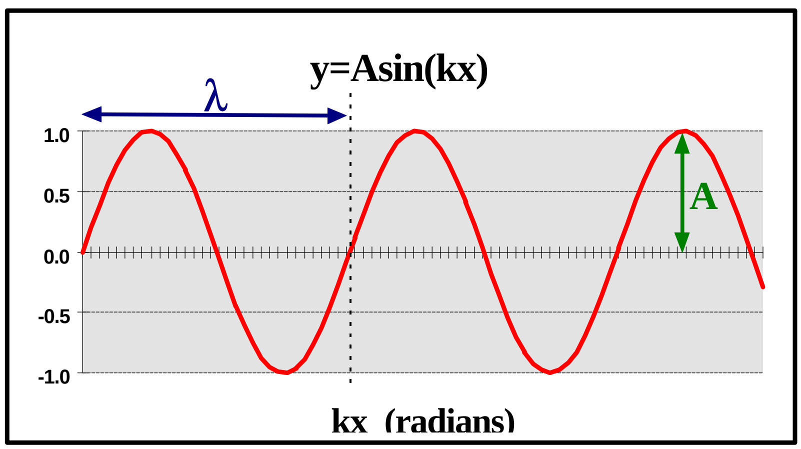
To construct a wave with shape **y = f(x)** at **time t = 0** traveling to the **right with speed v** simply make the replacement $x \rightarrow x - vt$.





Traveling Harmonic Waves:

Harmonic waves have the form **y = A sin(kx)** or **y = Acos(kx)** at time **t = 0**, where **k** is the **"wave number"** (**k = 2π/λ** where **λ** is the **"wave length"**) and **A** is the **"amplitude"**. To construct an harmonic wave traveling to the right with speed **v**, replace **x** by **x-vt** as follows:



y = Asin(k(x-vt) = Asin(kx-ωt) where **ω = kv** (**v = ω/k**). The period of the oscillation, **T = 2π/ω = 1/f**, where **f** is the **linear frequency** (measured in **Hertz** where **1Hz = 1/sec**) and **ω** is the **angular frequency** (**ω = 2πf**). The speed of propagation is given by **v = ω/k = λf** .

y = y(x,t) = Asin(kx-ωt) right moving harmonic wave

y = y(x,t) = Asin(kx+ωt) left moving harmonic wave

The Wave Equation

$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} = 0$$

Whenever analysis of a system results in an equation of the form given above then we know that the system supports traveling waves propagating at speed v.

General Proof:

If **y = y(x,t) = f(x-vt)** then

$$\frac{\partial y}{\partial x} = f' \qquad \frac{\partial^2 y}{\partial x^2} = f''$$

$$\frac{\partial y}{\partial t} = -vf' \qquad \frac{\partial^2 y}{\partial t^2} = v^2 f''$$

and

$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} = f'' - f'' = 0 \, .$$

Proof for Harmonic Wave:

If **y = y(x,t) = Asin(kx-ωt)** then

$$\frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(kx-\omega t) \qquad \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(kx-\omega t)$$

and

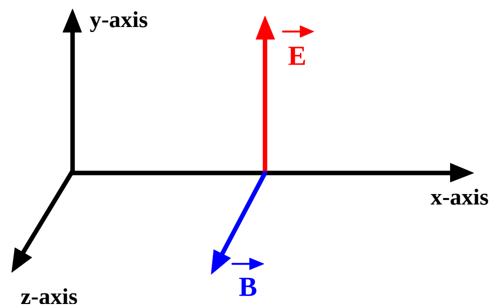
$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} = \left(-k^2 + \frac{\omega^2}{v^2}\right)A\sin(kx-\omega t) = 0 \, ,$$

since **ω = kv**.

Light Propagating in Empty Space

Since there are no charges and no current in empty space, Faraday's Law and Ampere's Law take the form

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} \qquad \oint \vec{B}\cdot d\vec{l} = m_0 e_0 \frac{d\Phi_E}{dt}.$$



Look for a solution of the form

$$\vec{E}(x,t) = E_y(x,t)\hat{y}$$
$$\vec{B}(x,t) = B_z(x,t)\hat{z}$$

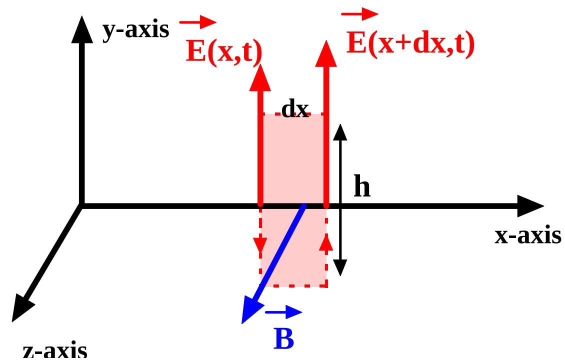
Faraday's Law:

Computing the left and right hand side of **Faraday's Law** using a rectangle (**in the xy-plane**) with width **dx** and **height h** (**counterclockwise**) gives

$$E_y(x+dx,t)h - E_y(x,t)h = -\frac{\P B_z}{\P t}hdx$$

or

$$\boxed{\frac{\P E_y}{\P x} = -\frac{\P B_z}{\P t}}$$



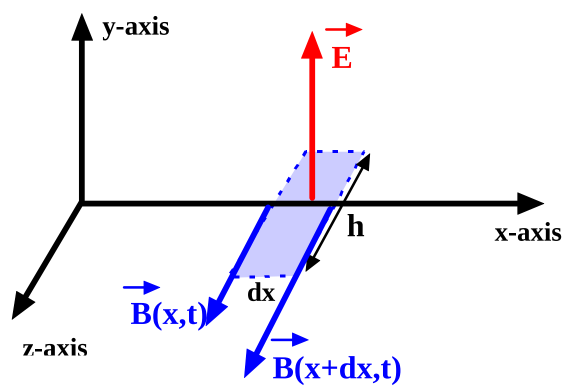
Ampere's Law:

Computing the left and right hand side of **Ampere's Law** using a rectangle (**in the xz-plane**) with width **dx** and **height h** (**counterclockwise**) gives

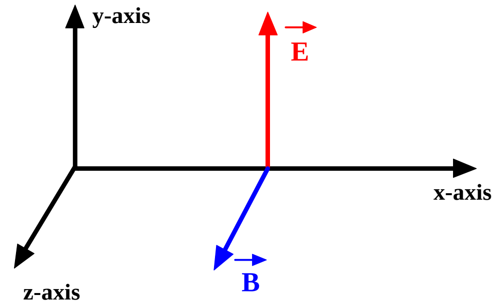
$$B_z(x,t)h - B_z(x+dx,t)h = m_0 e_0 \frac{\P E_y}{\P t}hdx$$

or

$$\boxed{-\frac{\P B_z}{\P x} = m_0 e_0 \frac{\P E_y}{\P t}}$$



Electromagnetic Plane Waves (1)



We have the following two **differential equations** for $\mathbf{E_y(x,t)}$ and $\mathbf{B_z(x,t)}$:

$$\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} \qquad \textbf{(1)}$$

and

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\mu_0\varepsilon_0}\frac{\partial B_z}{\partial x} \qquad \textbf{(2)}$$

Taking the time derivative of **(2)** and using **(1)** gives

$$\frac{\partial^2 E_y}{\partial t^2} = -\frac{1}{\mu_0\varepsilon_0}\frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial x}\right) = -\frac{1}{\mu_0\varepsilon_0}\frac{\partial}{\partial x}\left(\frac{\partial B_z}{\partial t}\right) = \frac{1}{\mu_0\varepsilon_0}\frac{\partial^2 E_y}{\partial x^2}$$

which implies

$$\frac{\partial^2 E_y}{\partial x^2} - \mu_0\varepsilon_0\frac{\partial^2 E_y}{\partial t^2} = 0 .$$

Thus $\mathbf{E_y(x,t)}$ satisfies the wave equation with speed $v = 1/\sqrt{\varepsilon_0\mu_0}$ and has a solution in the form of traveling waves as follows:

$$\mathbf{E_y(x,t) = E_0\sin(kx-\omega t)},$$

where $\mathbf{E_0}$ is the **amplitude of the electric field oscillations** and where the wave has a **unique speed**

$$v = c = \frac{\omega}{k} = \lambda f = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 2.99792\times 10^8 m/s \text{ (\textbf{speed of light})}.$$

From **(1)** we see that

$$\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} = -E_0 k\cos(kx-\omega t),$$

which has a solution given by

$$B_z(x,t) = E_0\frac{k}{\omega}\sin(kx-\omega t) = \frac{E_0}{c}\sin(kx-\omega t),$$

so that

$$\mathbf{B_z(x,t) = B_0\sin(kx-\omega t)},$$

where $\mathbf{B_0 = E_0/c}$ is the **amplitude of the magnetic field oscillations**.

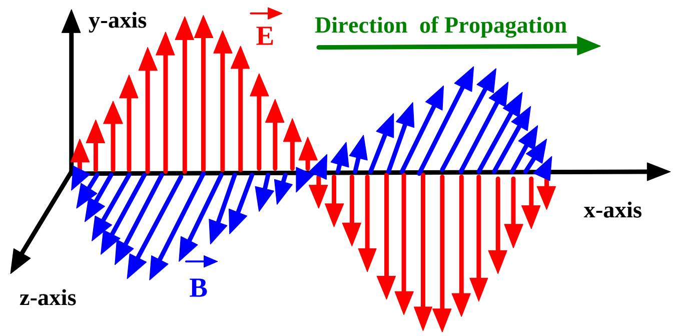
Electromagnetic Plane Waves (2)

The **plane harmonic wave solution** for light with frequency **f** and wavelength λ and speed **c = fλ** is given by

$$\vec{E}(x,t) = E_0 \sin(kx - wt)\hat{y}$$

$$\vec{B}(x,t) = B_0 \sin(kx - wt)\hat{z}$$

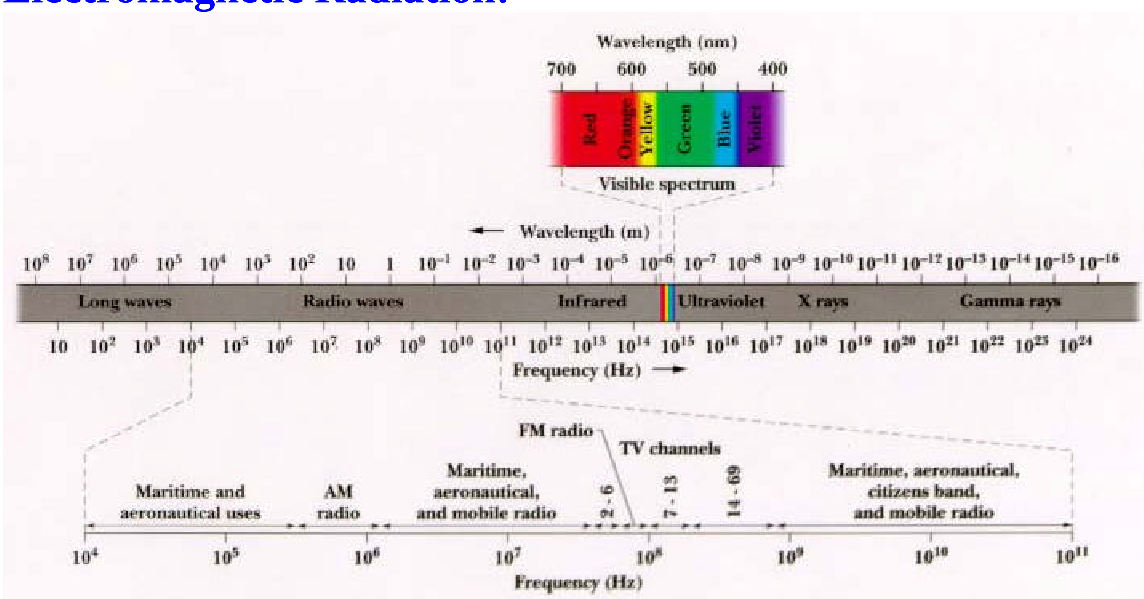
where **k = 2π/λ**, **ω = 2πf**, and **$E_0 = cB_0$**.



Properties of the Electromagnetic Plane Wave:

- **Wave travels at speed c ($c = 1/\sqrt{\mu_0 \epsilon_0}$).**
- **E and B are perpendicular ($\vec{E}\cdot\vec{B} = 0$).**
- **The wave travels in the direction of $\vec{E}\times\vec{B}$.**
- **At any point and time E = cB.**

Electromagnetic Radiation:



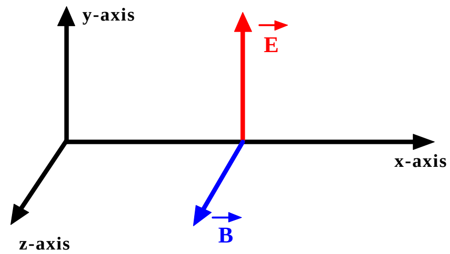
Energy Transport - Poynting Vector

Electric and Magnetic Energy Density:

For an **electromagnetic plane wave**

$E_y(x,t) = E_0\sin(kx\text{-}\omega t)$,

$B_z(x,t) = B_0\sin(kx\text{-}\omega t)$,



where **$B_0 = E_0/c$**. The **electric energy density** is given by

$$u_E = \frac{1}{2}e_0E^2 = \frac{1}{2}e_0E_0^2\sin^2(kx - wt)$$ and the **magnetic energy density** is

$$u_B = \frac{1}{2m_0}B^2 = \frac{1}{2m_0c^2}E^2 = \frac{1}{2}e_0E^2 = u_E\,,$$

where I used **E = cB**. Thus, **for light the electric and magnetic field energy densities are equal** and the **total energy density** is

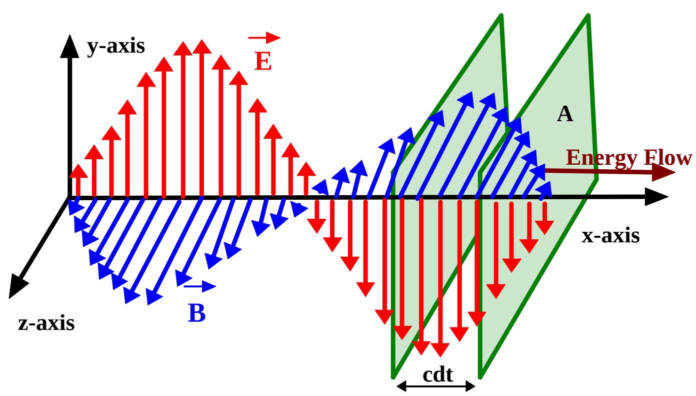
$$u_{tot} = u_E + u_B = e_0E^2 = \frac{1}{m_0}B^2 = e_0E_0^2\sin^2(kx - wt)\,.$$

Poynting Vector ($\vec{S} = \frac{1}{m_0}\vec{E}\times\vec{B}$):

The **direction** of the **Poynting Vector** is the **direction of energy flow** and the **magnitude**

$$S = \frac{1}{m_0}EB = \frac{E^2}{m_0c} = \frac{1}{A}\frac{dU}{dt}$$

is the **energy per unit time per unit area** (units of **Watts/m^2**).



Proof:

$$dU_{tot} = u_{tot}V = e_0E^2Acdt$$ so

$$S = \frac{1}{A}\frac{dU}{dt} = e_0cE^2 = \frac{E^2}{m_0c} = \frac{E_0^2}{m_0c}\sin^2(kx - wt)\,.$$

Intensity of the Radiation (Watts/m^2):

The intensity, **I**, is the **average of S** as follows:

$$I = \bar{S} = \frac{1}{A}\frac{d\bar{U}}{dt} = \frac{E_0^2}{m_0c}\langle\sin^2(kx - wt)\rangle = \frac{E_0^2}{2m_0c}\,.$$

Momentum Transport - Radiation Pressure

Relativistic Energy and Momentum:

$$E^2 = (cp)^2 + (m_0c^2)^2$$

energy momentum rest mass

For light $\mathbf{m_0 = 0}$ and

$$E = cp \quad \text{(for light)}$$

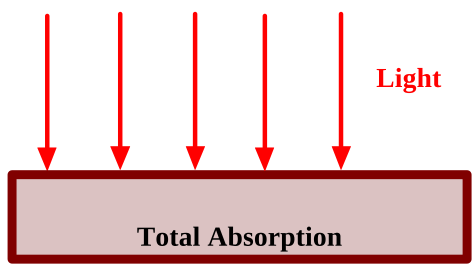
For light the **average momentum per unit time per unit area** is equal to the intensity of the light, **I**, divided by speed of light, **c**, as follows:

$$\frac{1}{A}\frac{d\overline{p}}{dt} = \frac{1}{c}\frac{1}{A}\frac{d\overline{U}}{dt} = \frac{1}{c}I\ .$$

Total Absorption:

$$\overline{F} = \frac{d\overline{p}}{dt} = \frac{1}{c}\frac{d\overline{U}}{dt} = \frac{1}{c}IA$$

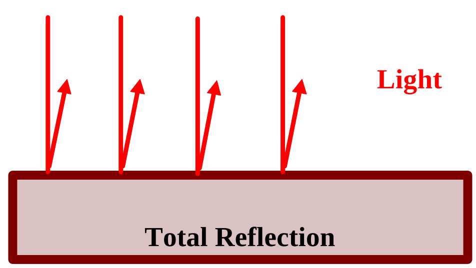
$$P = \frac{\overline{F}}{A} = \frac{1}{c}I \quad \text{(radiation pressure)}$$



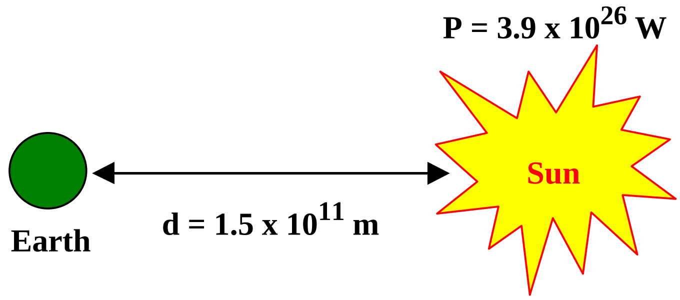
Total Reflection:

$$\overline{F} = \frac{d\overline{p}}{dt} = \frac{2}{c}\frac{d\overline{U}}{dt} = \frac{2}{c}IA\ .$$

$$P = \frac{\overline{F}}{A} = \frac{2}{c}I \quad \text{(radiation pressure)}$$



The Radiation Power of the Sun



Problem:

The **radiation power** of the sun is **$3.9\text{x}10^{26}$ W** and the distance from the Earth to the sun is **$1.5\text{x}10^{11}$ m**.

(a) What is the **intensity** of the electromagnetic radiation from the sun at the surface of the Earth (outside the atmosphere)? (**answer: 1.4 kW/m^2**)

(b) What is the **maximum value of the electric field** in the light coming from the sun? (**answer: 1,020 V/m**)

(c) What is the **maximum energy density of the electric field** in the light coming from the sun? (**answer: $4.6\text{x}10^{-6}$ J/m^3**)

(d) What is the **maximum value of the magnetic field** in the light coming from the sun? (**answer: 3.4 μT**)

(e) What is the **maximum energy density of the magnetic field** in the light coming from the sun? (**answer: $4.6\text{x}10^{-6}$ J/m^3**)

(f) Assuming complete absorption what is the **radiation pressure on the Earth** from the light coming from the sun? (**answer: $4.7\text{x}10^{-6}$ N/m^2**)

(g) Assuming complete absorption what is the **radiation force on the Earth** from the light coming from the sun? The radius of the Earth is about **$6.4\text{x}10^6$ m**. (**answer: $6\text{x}10^8$ N**)

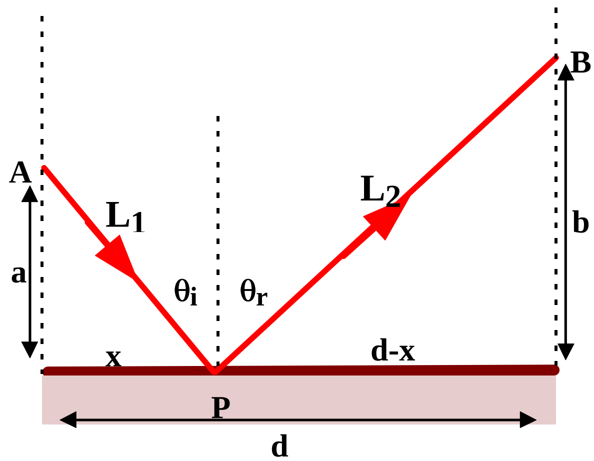
(h) What is the **gravitational force** on the Earth due to the sun. The mass of the Earth and the sun are **$5.98\text{x}10^{24}$ kg** and **$1.99\text{x}10^{30}$ kg**, respectively, and **$G = 6.67\text{x}10^{-11}$ Nm^2/kg^2**. (**answer: $3.5\text{x}10^{22}$ N**)

Geometric Optics

Fermat's Principle:

In traveling from one point to another, light follows the path that requires minimal time compared to the times from the other possible paths.

Theory of Reflection:



Let t_{AB} be the time for light to go from the point **A** to the point **B** reflecting off the point **P**. Thus,

$$t_{AB} = \frac{1}{c}L_1 + \frac{1}{c}L_2\,,$$

where

$$L_1 = \sqrt{x^2 + a^2}$$

$$L_2 = \sqrt{(d-x)^2 + b^2}\,.$$

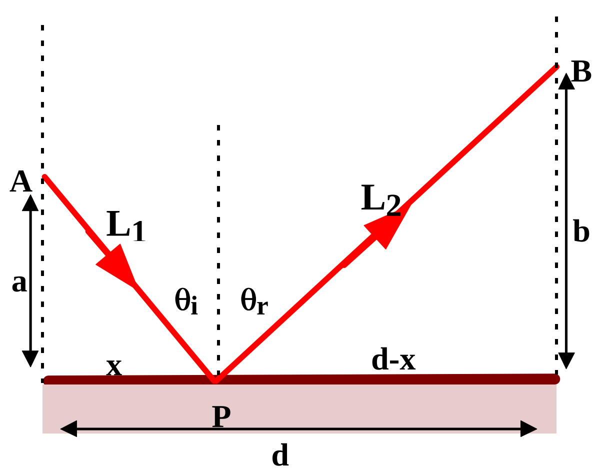
To find the path of **minimal time** we set the derivative of t_{AB} equal to zero as follows:

$$\frac{dt_{AB}}{dx} = \frac{1}{c}\frac{dL_1}{dx} + \frac{1}{c}\frac{dL_2}{dx} = 0\,,$$

which implies

$$\frac{dL_1}{dx} = -\frac{dL_2}{dx}\,,$$

but



$$\frac{dL_1}{dx} = \frac{x}{L_1} = \sin q_i$$

$$\frac{dL_2}{dx} = \frac{-(d-x)}{L_2} = -\sin q_r$$

so that the condition for **minimal time** becomes

$$\boxed{\sin q_i = \sin q_r \qquad q_i = q_r}\,.$$

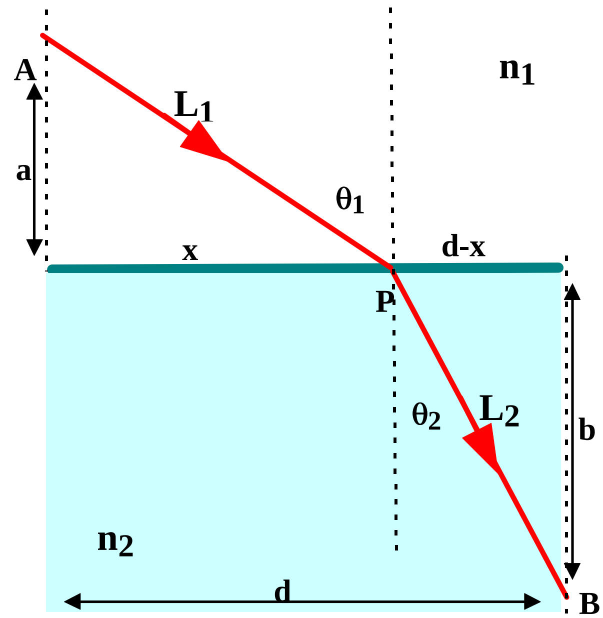
Law of Refraction

Index of Refraction:

Light travels at speed c in a vacuum. It travels at a speed v < c in a medium. The **index for refraction**, **n**, is the ratio of the speed of light in a vacuum to its speed in the medium,

n = c/v,

where n is greater than or equal to one.



Theory of Refraction:

Let t_{AB} be the time for light to go from the point **A** to the point **B** refracting at the point **P**. Thus,

$$t_{AB} = \frac{1}{v_1}L_1 + \frac{1}{v_2}L_2,$$

where

$$L_1 = \sqrt{x^2 + a^2}$$

$$L_2 = \sqrt{(d-x)^2 + b^2}.$$

To find the path of **minimal time** we set the derivative of t_{AB} equal to zero as follows:

$$\frac{dt_{AB}}{dx} = \frac{1}{v_1}\frac{dL_1}{dx} + \frac{1}{v_2}\frac{dL_2}{dx} = 0$$, which implies $\frac{1}{v_1}\frac{dL_1}{dx} = -\frac{1}{v_2}\frac{dL_2}{dx}$, but

$$\frac{dL_1}{dx} = \frac{x}{L_1} = \sin\theta_1$$

$$\frac{dL_2}{dx} = \frac{-(d-x)}{L_2} = -\sin\theta_2$$

so that the condition for **minimal time** becomes

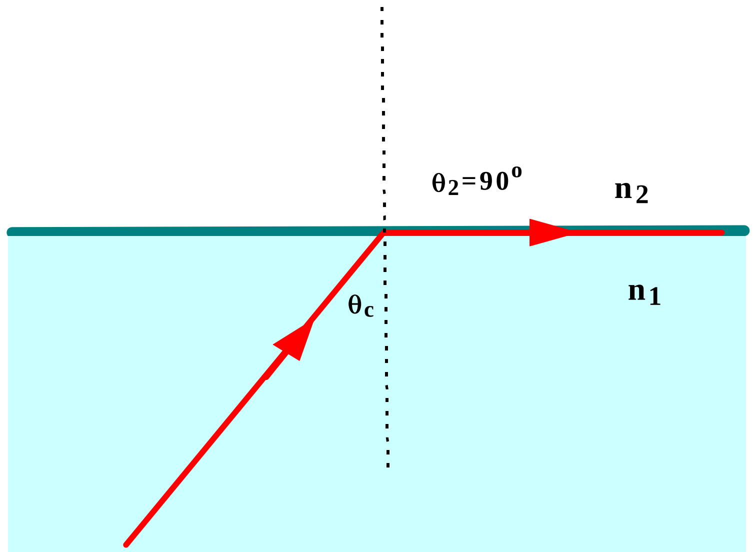
$$\frac{1}{v_1}\sin\theta_1 = \frac{1}{v_2}\sin\theta_2 \qquad n_1\sin\theta_1 = n_2\sin\theta_2$$

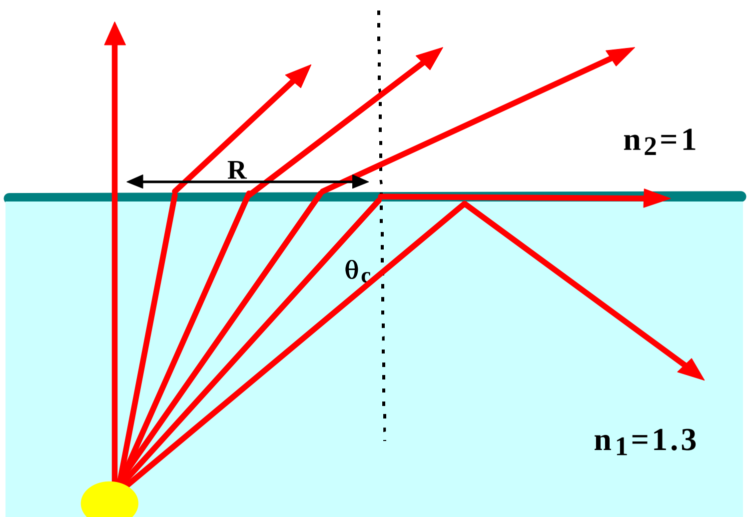
Snell's Law.

Total Internal Reflection

Total internal refection occurs when light travels from medium $\mathbf{n_1}$ to medium $\mathbf{n_2}$ ($\mathbf{n_1} > \mathbf{n_2}$) if θ_1 is greater than or equal to the critical angle, θ_c, where

$$\sin\theta_c = \frac{n_2}{n_1}.$$



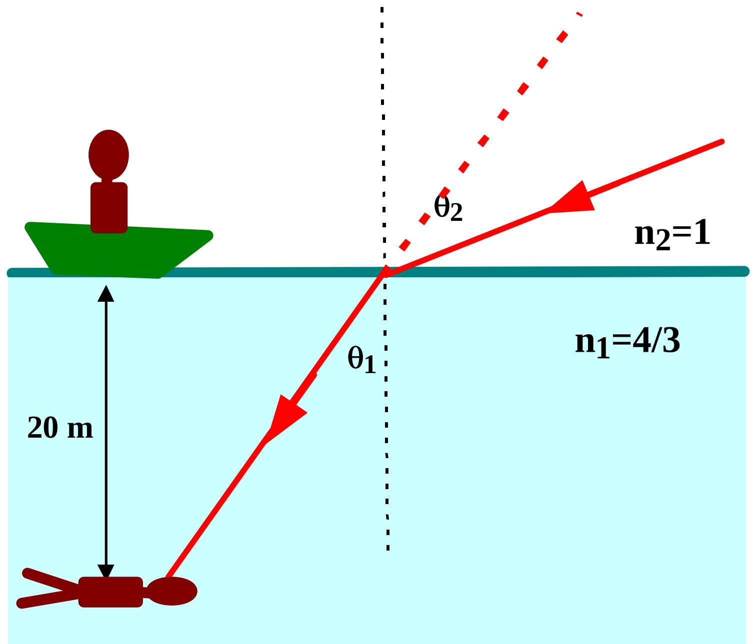


Problem:
A point source of light is located **10 meters** below the surface of a large lake (**n=1.3**). What is the area **(in m²)** of the largest circle on the pool's surface through which light coming directly from the source can emerge? (**answer: 455**)

Refraction Examples

Problem:

A scuba diver **20 meters** beneath the smooth surface of a clear lake looks upward and judges the sun to be **40^o** from directly overhead. At the same time, a fisherman is in a boat directly above the diver.



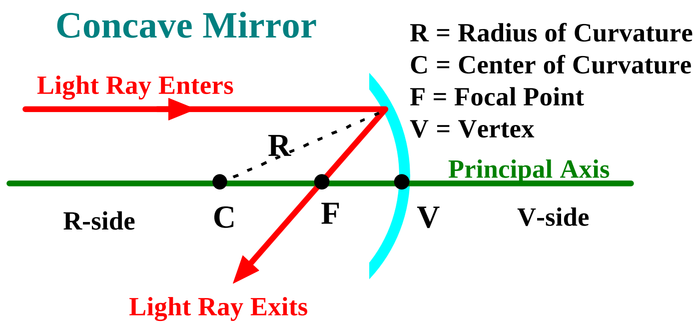
(a) At what angle from the vertical would the fisherman measure the sun? (**answer: 59^o**)

(b) If the fisherman looks downward, at what depth below the surface would he judge the diver to be? (**answer: 15 meters**)

Spherical Mirrors

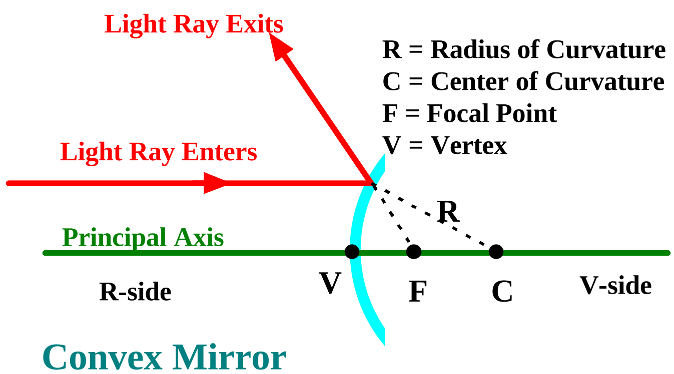
Vertex and Center of Curvature:

The **vertex**, **V**, is the point where the principal axis crosses the mirror and the **center of curvature** is the center of the spherical mirror with radius of curvature **R**.



Real and Virtual Sides:

The **"R"** or **real side** of a spherical mirror is the side of the mirror that the light exits and the other side is the **"V"** or **virtual side**. If the center of curvature lies on the **R-side** then the radius of curvature, **R**, is taken to be **positive** and if the center of curvature lies on the **V-side** then the radius of curvature, **R**, is taken to be **negative**.



Focal Point:

A light ray parallel to the principal axis will pass through the **focal point**, **F**, where **F** lies a distance **f** (**focal length**) from the vertex of the mirror. For spherical mirrors a good approximation is **f = R/2**.

Concave and Convex Mirrors:

A **concave mirror** is one where the center of curvature lies on the **R-side** so the **R** > **0** and **f** > **0** and a **convex mirror** is one where the center of curvature lies on the **V-side** so that **R** < **0** and **f** < **0**.

concave	**f > 0**
convex	**f < 0**

Flat Mirror:

A flat mirror is the limiting case where the radius **R** (and thus the local length **f**) become infinite.

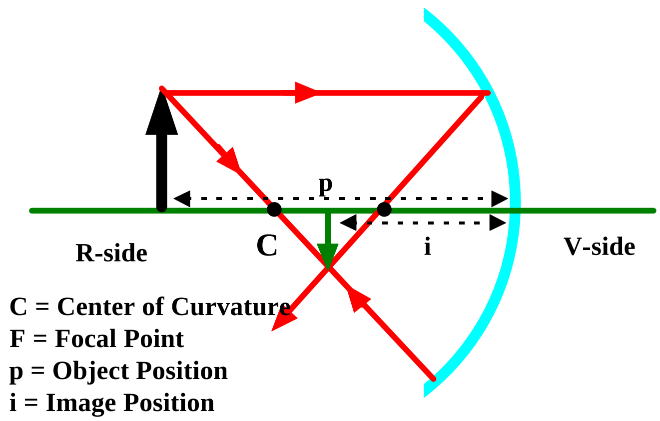
Mirror Equation

Object and Image Position:

For spherical mirrors,

$$\frac{1}{p}+\frac{1}{i}=\frac{1}{f},$$

where **p** is the distance from the vertex to the **object**, **i** is the distance from the vertex to the **image**, and **f** is the **focal length**.



Focal Length:

For spherical mirrors the focal length, **f**, is one-half of the radius of curvature, **R**, as follows:

$$\mathbf{f = R/2}.$$

Magnification:

The **magnification** is

$$m=-\frac{i}{p}, \quad \text{(magnification equation)}$$

where the magnitude of the magnification is the ratio of the height of the image, $\mathbf{h_i}$, to the height of the object, $\mathbf{h_p}$, as follows:

$$|m|=\frac{h_i}{h_p}.$$

Sign Conventions:

Variable	Assigned a Positive Value	Assigned a Negative Value
p (object distance)	always positive	
i (image distance)	if image is on R-side (real image)	if image is on V-side (virtual image)
R (radius of curvature)	if C is on R-side (concave)	if C is on V-side (convex)
f (focal length)	if C is on R-side (concave)	if C is on V-side (convex)
m (magnification)	if the image is not inverted	if the image is inverted

Mirror Examples (1)

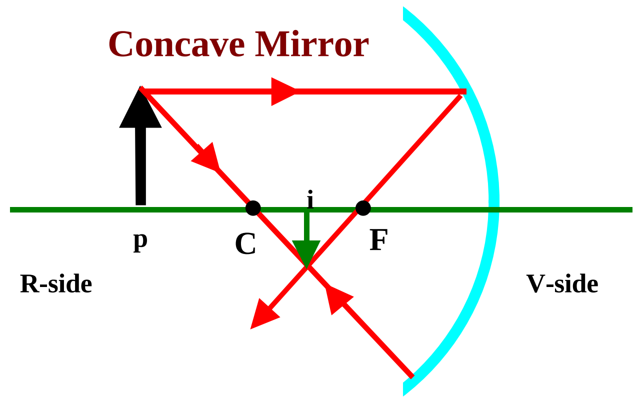
Mirror Equations:

$$f = \frac{R}{2} \qquad \frac{1}{p} + \frac{1}{i} = \frac{1}{f} \qquad m = -\frac{i}{p}$$

Example:

R = 2, p = 3

f = 1, i = 3/2, m = -1/2

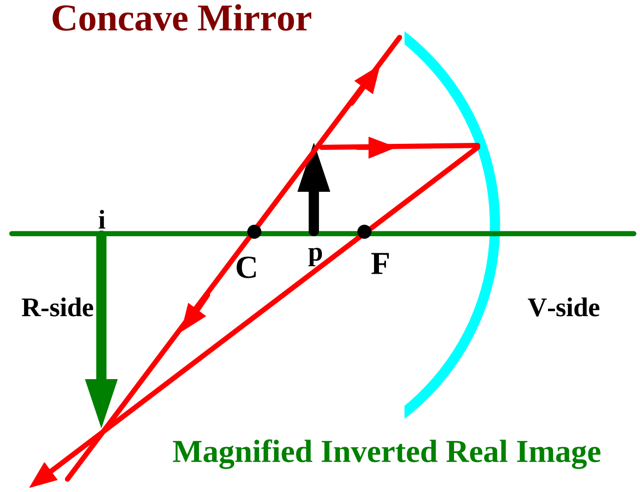


Reduced Inverted Real Image

Example:

R = 2, p = 3/2

f = 1, i = 3, m = -2



Magnified Inverted Real Image

Mirror Examples (2)

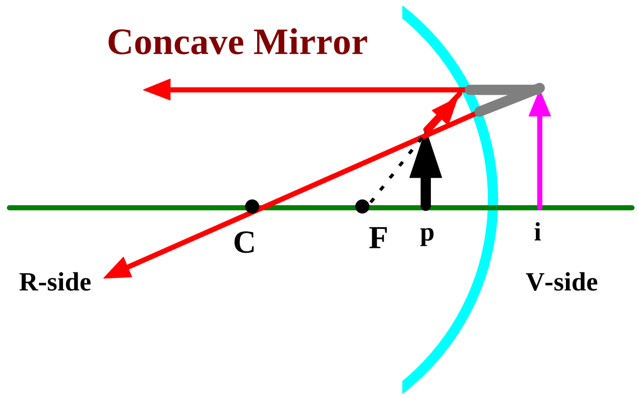
Mirror Equations:

$$f = \frac{R}{2} \qquad \frac{1}{p} + \frac{1}{i} = \frac{1}{f} \qquad m = -\frac{i}{p}$$

Example:

R = 2, p = 1/2

f = 1, i = -1, m = 2

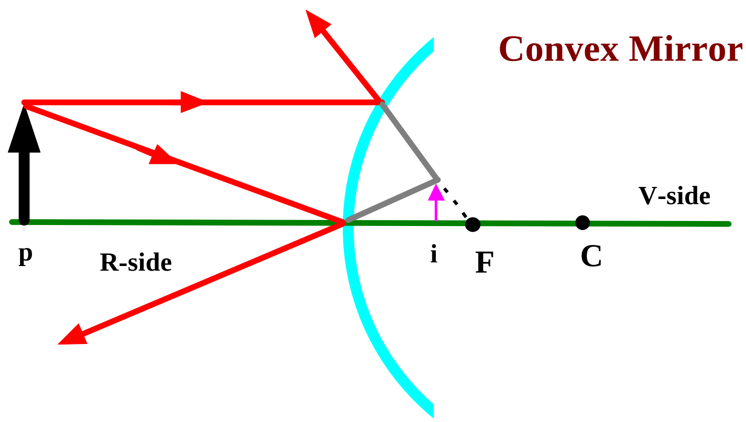


Magnified Non-inverted Virtual Image

Example:

R = -2, p = 3

f = -1, i = -3/4, m = 1/4



Reduced Non-inverted Virtual Image

Thin Lense Formula

Lensemakers Equation:

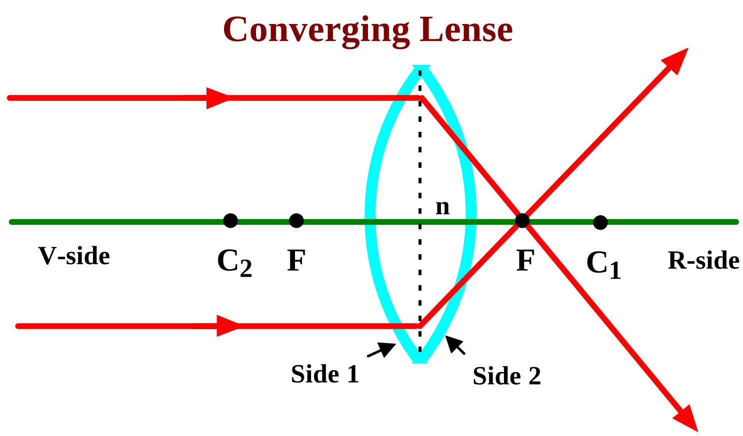
The lensemakers formula is

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right),$$

where **f** is the **focal length**, **n** is the **index of refraction**, $\mathbf{R_1}$ is the radius of curvature of **side 1** (**side that light enters the lense**), and $\mathbf{R_2}$ is the radius of curvature of **side 2** (**side that light exits the lense**).

Lense Equation:

$$\frac{1}{p} + \frac{1}{i} = \frac{1}{f}$$



Magnification:

$$m = -\frac{i}{p}$$

Sign Conventions:

Variable	Assigned a Positive Value	Assigned a Negative Value
p (object distance)	always positive	
i (image distance)	if image is on R-side (real image)	if image is on V-side (virtual image)
R_1 (radius of curvature)	if C_1 is on R-side	if C_1 is on V-side
R_2 (radius of curvature)	if C_2 is on R-side	if C_2 is on V-side
f (focal length)	if f > 0 then converging lense	if f < 0 then diverging lense
m (magnification)	if the image is not inverted	if the image is inverted

Example (converging lense): $R_1 = R \quad R_2 = -R \quad f = \frac{R}{2(n-1)} > 0$

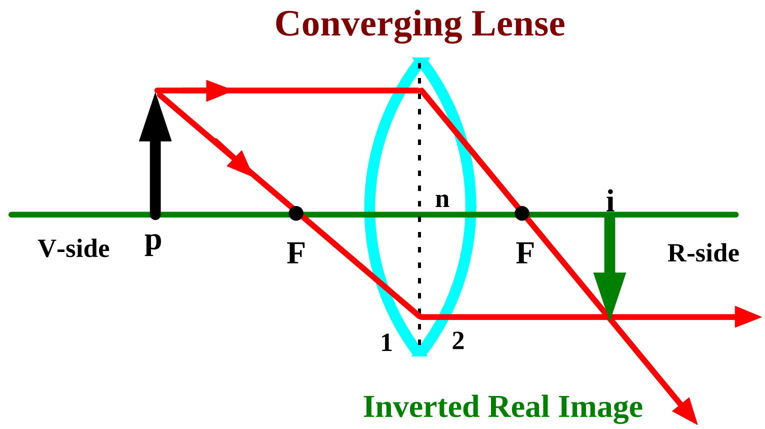
Example (diverging lense): $R_1 = -R \quad R_2 = R \quad f = \frac{-R}{2(n-1)} < 0$

Thin Lenses (Converging)

Example:

f = 1, p = 2

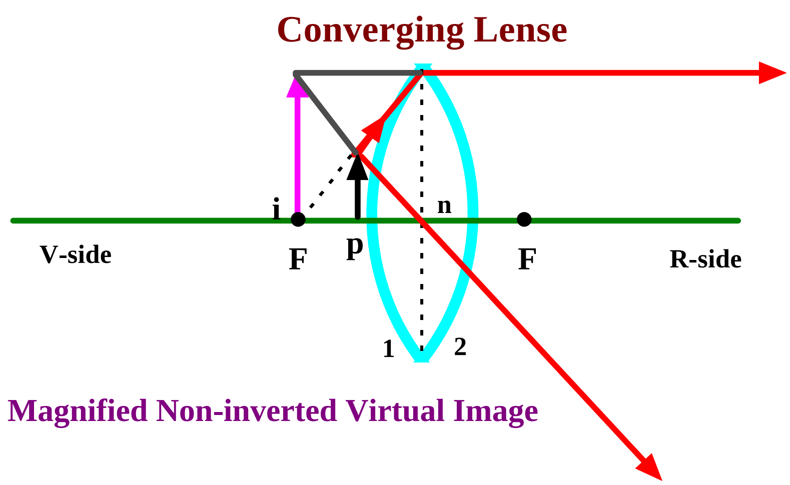
i = 2, m = -1



Example:

f = 1, p = 1/2

i = -1, m = 2

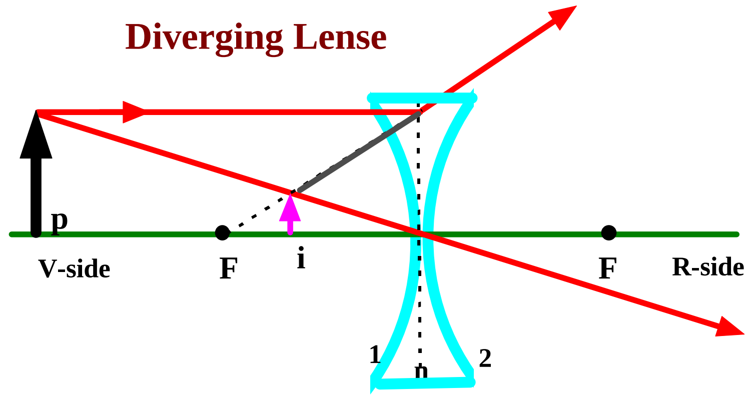


Thin Lenses (Diverging)

Example:

f = -1, p = 2

i = -2/3, m = 1/3

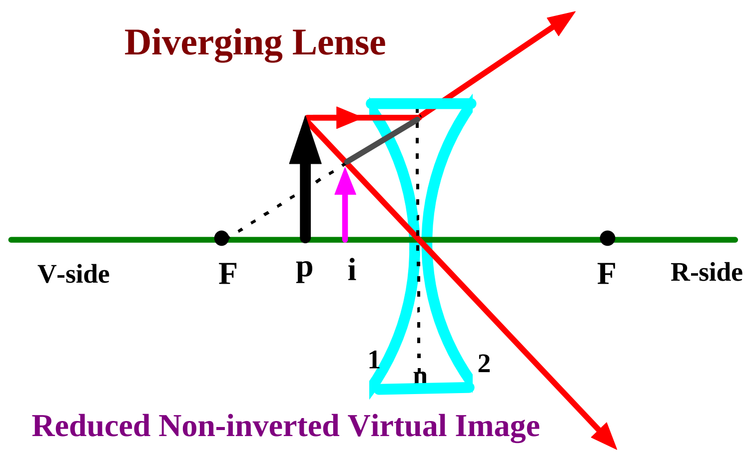


Reduced Non-inverted Virtual Image

Example:

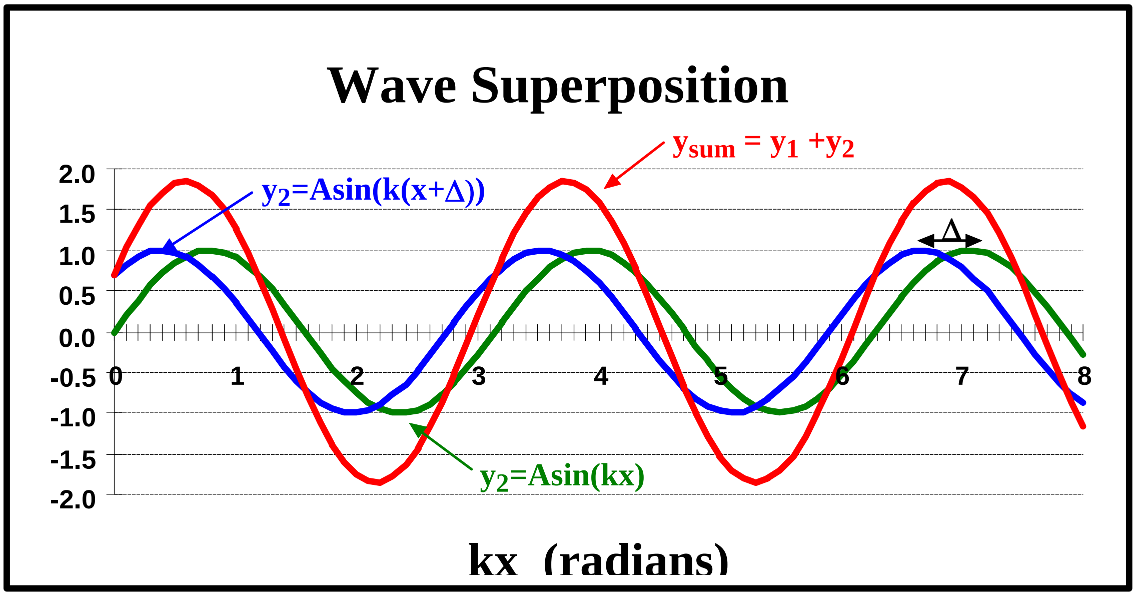
f = -1, p = 1/2

i = -1/3, m = 2/3



Reduced Non-inverted Virtual Image

Interference



Wave Superposition:

Consider the addition (**superposition**) of two waves with the same amplitude and wavelength:

$$y_1 = A\sin(kx)$$

$$y_2 = A\sin(k(x+\Delta))$$

$$y_{sum} = y_1 + y_2$$

The quantity Δ is the **"phase shift"** between the two waves and **k=2π/λ** is the wave number.

Maximal Constructive Interference:

The condition for **maximal constructive** interference is

$$\Delta = m l \qquad m = 0, \pm 1, \pm 2, \ldots \qquad \text{(max constructive)}$$

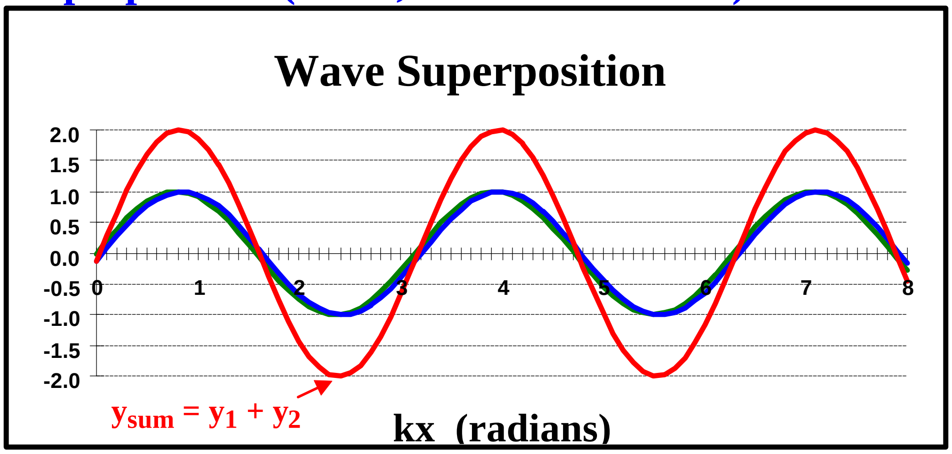
Maximal Destructive Interference:

The condition for **maximal destructive** interference is

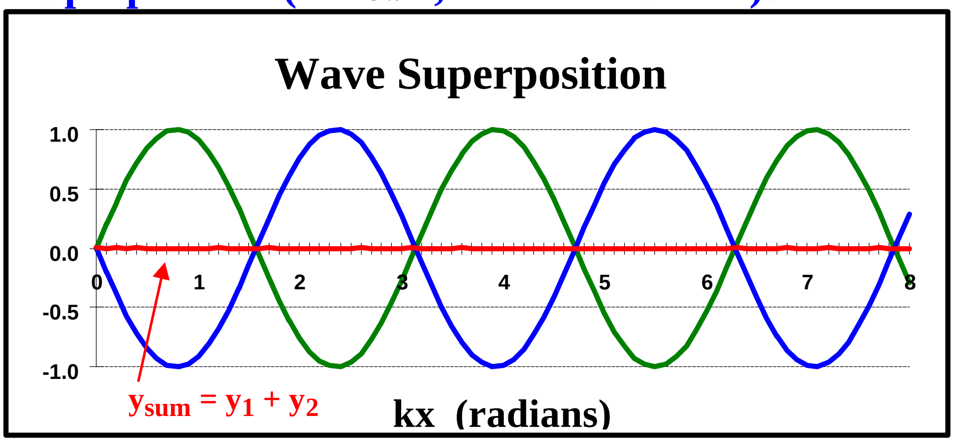
$$\Delta = \left(m + \frac{1}{2}\right) l \qquad m = 0, \pm 1, \pm 2, \ldots \qquad \text{(max destructive)}$$

Interference Examples

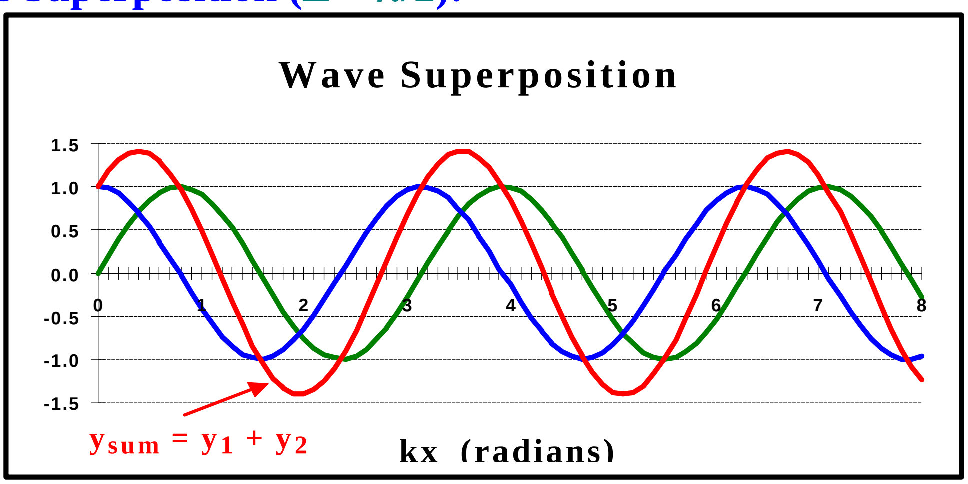
Wave Superposition ($\Delta = \lambda$; max constructive):



Wave Superposition ($\Delta = \lambda/2$; max destructive):

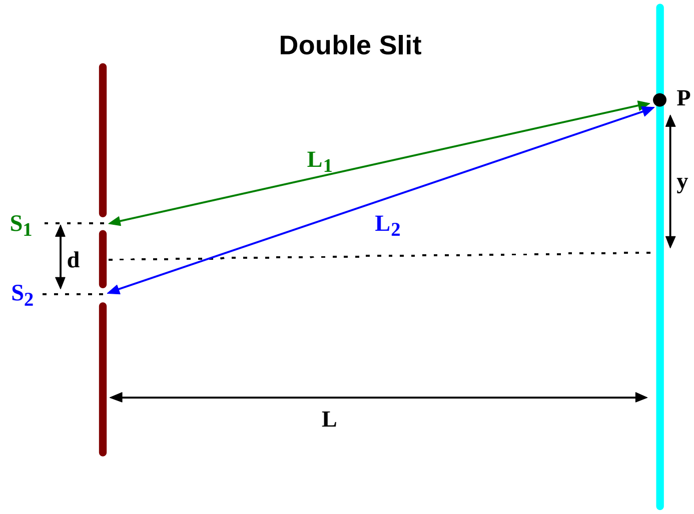


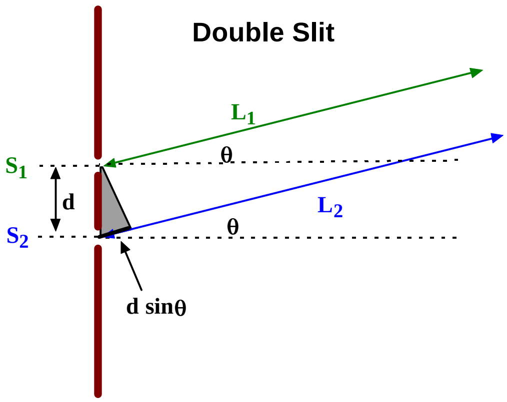
Wave Superposition ($\Delta = \lambda/2$):



Double Slit Interference

The simplest way to produce a phase shift a difference in the path length between the two wave sources, S_1 and S_2 is with a double slit. The point **P** is located on a screen that is a distance **L** away from the slits and the slits are separated by a distance **d**.





If **L** >> **d** then to a good approximation the **path length difference** is,

$$\Delta L = |L_2 - L_1| = d\sin\theta\ ,$$

and thus

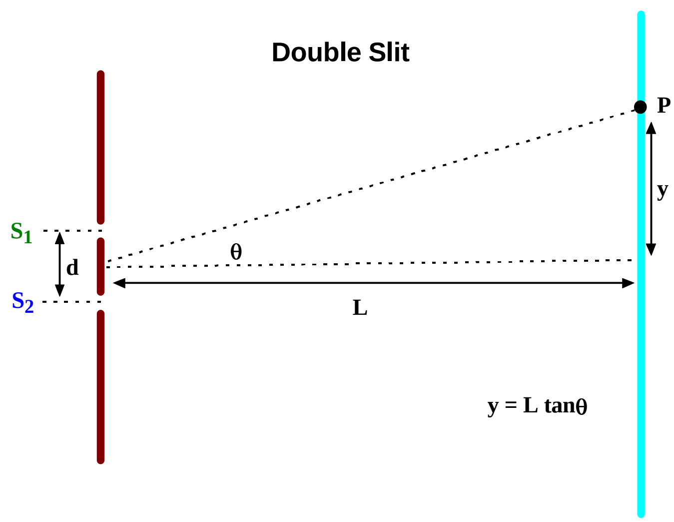
Maximal Constructive Interference:

The condition for **maximal constructive** interference is

$$\sin\theta = m\frac{\lambda}{d} \qquad m = 0,1,2,\ldots$$

(**Bright Fringes - max constructive**)

Order of the Bright Fringe



Maximal Destructive Interference:

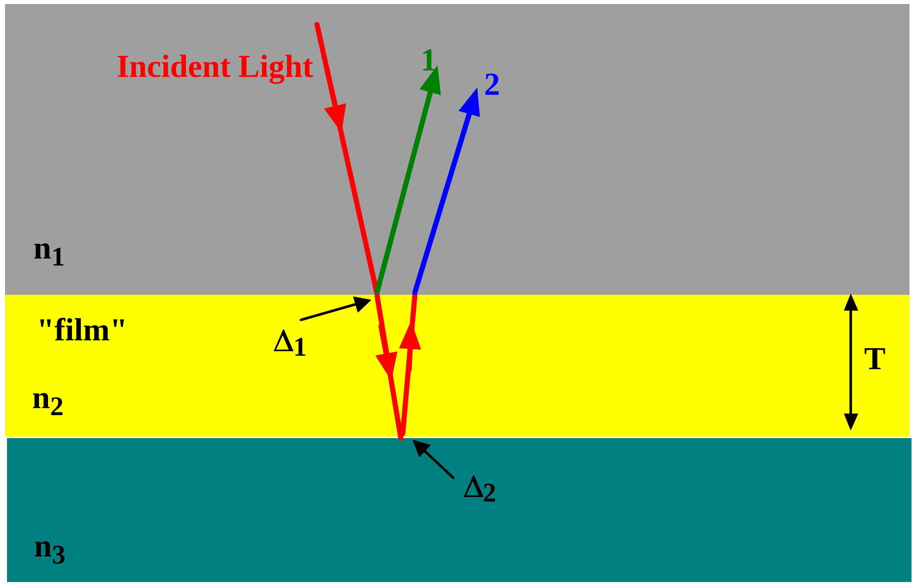
The condition for **maximal destructive** interference is

$$\sin\theta = \left(m + \frac{1}{2}\right)\frac{\lambda}{d} \qquad m = 0,1,2,\ldots$$

(**Dark Fringes - max destructive**)

Thin Film Interference

Thin film interference occurs when a thin layer of material (thickness **T**) with index of refraction $\mathbf{n_2}$ (the **"film"** layer) is sandwiched between two other mediums $\mathbf{n_1}$ and $\mathbf{n_2}$.



The **overall phase shift** between the reflected waves **1** and **2** is given by,

$$\Delta_{overall} = 2T + \Delta_1 + \Delta_2 ,$$

where it is assumed that the incident light ray is nearly perpendicular to the surface and the phase shifts Δ_1 and Δ_2 are given the table.

Phase Shift	Condition	Value
Δ_1	$n_1 > n_2$	0
Δ_1	$n_1 < n_2$	$\lambda_{film}/2$
Δ_2	$n_2 > n_3$	0
Δ_2	$n_2 < n_3$	$\lambda_{film}/2$

Maximal Constructive Interference:

The condition for **maximal constructive** interference is

$$\Delta_{overall} = 2T + \Delta_1 + \Delta_2 = m\lambda_{film} \quad m = 0, \pm 1, \pm 2, \ldots \quad \text{(max constructive)}$$

where $\lambda_{film} = \lambda_0/n_2$, with λ_0 the vacuum wavelength.

Maximal Destructive Interference:

The condition for **maximal destructive** interference is

$$\Delta_{overall} = 2T + \Delta_1 + \Delta_2 = \left(m + \frac{1}{2}\right)\lambda_{film} \quad m = 0, \pm 1, \pm 2, \ldots \text{(max destructive)}$$

Interference Problems

Double Slit Example:

Red light (λ = **664 nm**) is used with slits separated by **d = 1.2x10^{-4} m**. The screen is located a distance from the slits given by **L = 2.75 m**. Find the distance **y** on the screen between the central bright fringe and the **third-order bright fringe**.

Answer: y = 0.0456 m

Thin Film Example:

A thin film of gasoline floats on a puddle of water. Sunlight falls almost perpendicularly on the film and reflects into your eyes. Although the sunlight is white, since it contains all colors, the film has a yellow hue, because destructive interference has occurred eliminating the color of blue (λ_0 **= 469 nm**) from the reflected light. If n_{gas} **= 1.4** and n_{water} **= 1.33**, determine the **minimum thickness of the film**.

Answer: T_{min} = 168 nm

Diffraction Summary

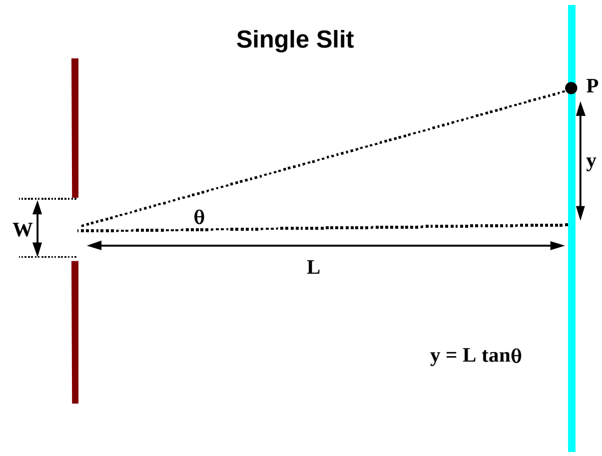
Single Slit-Diffraction:

Angular position of the **dark fringes**:

$$\sin\theta = m\frac{\lambda}{W} \quad m = 1,2,\ldots$$

Width of the Slit

(**Dark Fringes - max destructive**)



Round Hole-Diffraction:

Angular position of the **first dark ring**:

$$\sin\theta = 1.22\frac{\lambda}{D}$$

Diameter of the Hole

(**Dark Ring - max destructive**)

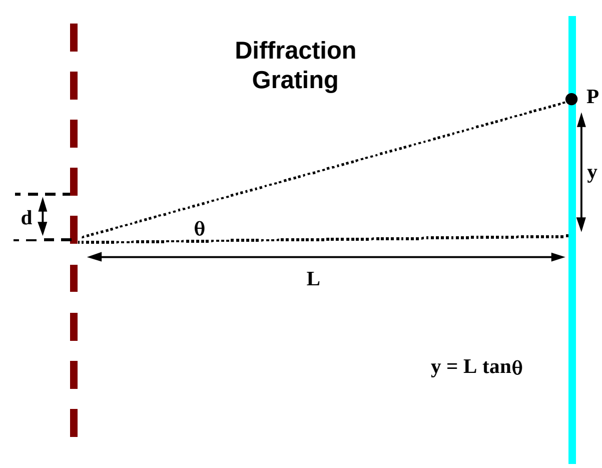
Diffraction Grating:

Angular position of the **bright fringes**:

$$\sin\theta = m\frac{\lambda}{d} \quad m = 0,1,2,\ldots$$

Slit

(**Bright Fringes - max constructive**)



Diffraction Problems

Single Slit Example:

Light passes through a slit and shines on a flat screen that is located **L = 0.4 m** away. The width of the slit is **W = 4x10^{-6} m**. The distance between the middle of the central bright spot and the first dark fringe is **y**. Determine the width **2y** of the central bright spot when the wavelength of light is λ **= 690 nm**.

Answer: 2y = 0.14 m