Elementary Combinatorial Topology

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Motivation

Topological combinatorics uses topological tools to prove combinatorial results. For a better understanding of such a result, purely combinatorial proofs are often desirable. A natural way to get such a proof consists in "combinatorializing" the topological tools. The combinatorial counterparts of classical theorems involving continuous structures belong to *combinatorial topology*, which actually was the name of what later became *algebraic topology*. This quest for a better understanding may have as a first motivation to find constructive or algorithmic proof for results [...]. Another motivation is to find a way to generalize the results, a thing that can not always be done through continuous tools.

According to author's knowledge, there are at least two results in combinatorics that generalize result of topological combinatorics whose only known proofs are purely combinatorial and uses theorems from combinatorial topology. These results belongs to the area of Kneser graphs and hypergraphs. The first one is the complete proof by Chen [] of the equality between the usual chromatic number and the circular chromatic number of Kneser graphs [...]. The other one is the equality between the chromatic number of Kneser hypergraphs and that of quasi-stable Kneser hypergraphs. An interesting thing in Chen's approach is the new theorem in combinatorial topology he needed to prove, which has no counterpart in algebraic topology.

Even if the backbone of this course is combinatorial topology, the applications in combinatorics will play a centrale role, for they ultimately remain the true motivation.

Some results in combinatorics whose proofs are topological

The most classical theorem in topological combinatorics is the Lovasz-Kneser theorem. [...]

THEOREM 1.

$$\chi(KG(n,k)) = n - 2k + 2$$

In 1986, also with a clever use of the Borsuk-Ulam theorem, Alon proved the celebrated splitting necklace theorem. [...] Suppose that the necklace has n beads, each of a certain type i, where $1 \leq i \leq t$. Suppose there is an even number $2a_i$ of beads of type i, $1 \leq i \leq t$, $\sum_{i=1}^{t} 2a_i = n$. A 2-*splitting* of the necklace is a partition of it into 2 parts, each consisting of a finite number of non-overlapping sub-necklaces of beads whose union captures either a_i beads of type i, for every $1 \leq i \leq t$.

THEOREM 2. *Every necklace with an even number $2a_i$ of beads of type i, $1 \leq i \leq t$, has a 2-splitting requiring at most t cuts.*

Applicaton in discrete geometries are numerous. For instance

THEOREM 3. *Any triangulation of a rectangle using triangles of same areas has an even number of triangles.*

In French, this theorem is called "mmre-ppre", which would be something like " ??? ". It is a jeu de mot untranslatable "mme-aire, p paire" which means "same area, even p": if the triangulation has p triangles of same area, p is even.

A graph is *perfect* is, for it and all its induced subgraphs, the maximal size of a clique is equal to the chromatic number. A *kernel* in an oriented graph $D = (V, A)$ is a independent subset K of vertices such that $K \cup N^+(K) = V$. A *clique-acyclic orientation* is an orientation of the edges of a graph in such a way that each clique is acyclic. The following theorem has been conjectured by Berge and Duchet [] and proved by Boros and Gurvich []. The original proof as well as a more recent and simpler one by Aharoni and Holtzmann uses toplogical results ([...]).

THEOREM 4. *A graph is perfect if and only if for every clique-acyclic orientation there is a kernel.*

CHAPTER 1

Basic notions

1. Geometric simplicial complexes

1.1. Simplices and complexes. A *geometric simplex* – or simply *simplex* when there is no ambiguity – is the convex hull of affinely independent points. It is a polytope and has as such facets, faces, edges and vertices. By definition, all faces of a geometric simplex are again geometric simplices. The *dimension* of a geometric simplex σ – denoted $\dim(\sigma)$ – is its number of vertices minus one. The empty set $\emptyset$ is considered as a -1-dimensional simplex.

Geometric simplices in dimension 0, 1, 2, and 3 are explicited in Figure 1. For these values, the simplices have special names: a 0-dimensional simplex is a vertex, a 1-dimensional simplex in an edge, a 2-dimensional simplex is a triangle, and a 3-dimensional simplex is a tetrahedron.

The set of vertices of a simplex σ is denoted $V(\sigma)$.

A geometric simplicial complex Δ is a collection of simplices satsifying the following properties:

- if τ is a face of a simplex $\sigma \in \Delta$, then $\tau \in \Delta$.
- if σ and σ' are two simplices of Δ, then their intersection $\sigma \cap \sigma'$ is a face of both.

Its dimension – denoted $\dim(\Delta)$ – is $\max_{\sigma\in\Delta} \dim\sigma$. Its vertex set is the set of vertices of it simplices: $V(\Delta) = \bigcup_{\sigma\in\mathsf{K}} V(\sigma)$. The set of its edges is denoted $E(\Delta)$ and is the set of its 1-dimensional simplices. The empty set $\emptyset$ is always one of the simplices of a geometric simplicial complex K.

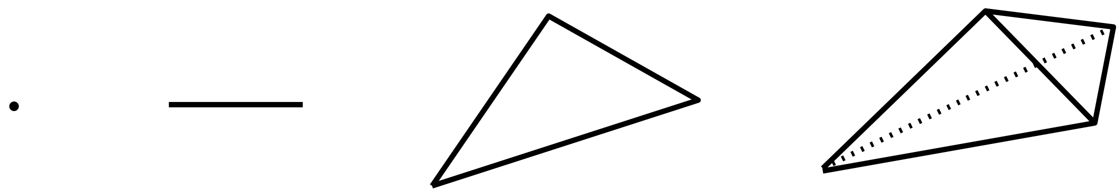
Figure 2 shows a geometric simplex of dimension 2, whereas Figure 3 shows a collection of geometric simplices that do not provide a geometric simplicial complex (the intersection of the two triangles is a face of none of them).

The union $\bigcup_{\sigma\in\Delta} \sigma$ of all simplices of a geometric simplicial complex Δ is called the *polyhedron* of the simplicial complex. It is denoted $||\Delta||$.

Let X be a topological space. A geometric simplicial complex T is a *triangulation* of X if $||\mathsf{T}||$ is homeomorphic to X.

2. Abstract simplicial complexes

2.1. Simplices and complexes. A collection K of subsets of a finite set V is called an *abstract simplcial complex* – or simply simplicial complex if there is no ambiguity – if for all $F \in \mathsf{K}$ and all $F' \subseteq F$ we have $F' \in \mathsf{K}$. Each $F \in \mathsf{K}$ is called an *(abstract) simplex*. Again, the empty set $\emptyset$ is one of these simplices. The elements of V are called the *vertices* of K and is usually denote $V(\mathsf{K})$. The dimension of an abstract simplex F is denoted $\dim(F)$ and is defined



$d = 0$ $d = 1$ $d = 2$ $d = 3$

FIGURE 1. d-dimensional geometric simplices for $d = 0, 1, 2, 3$

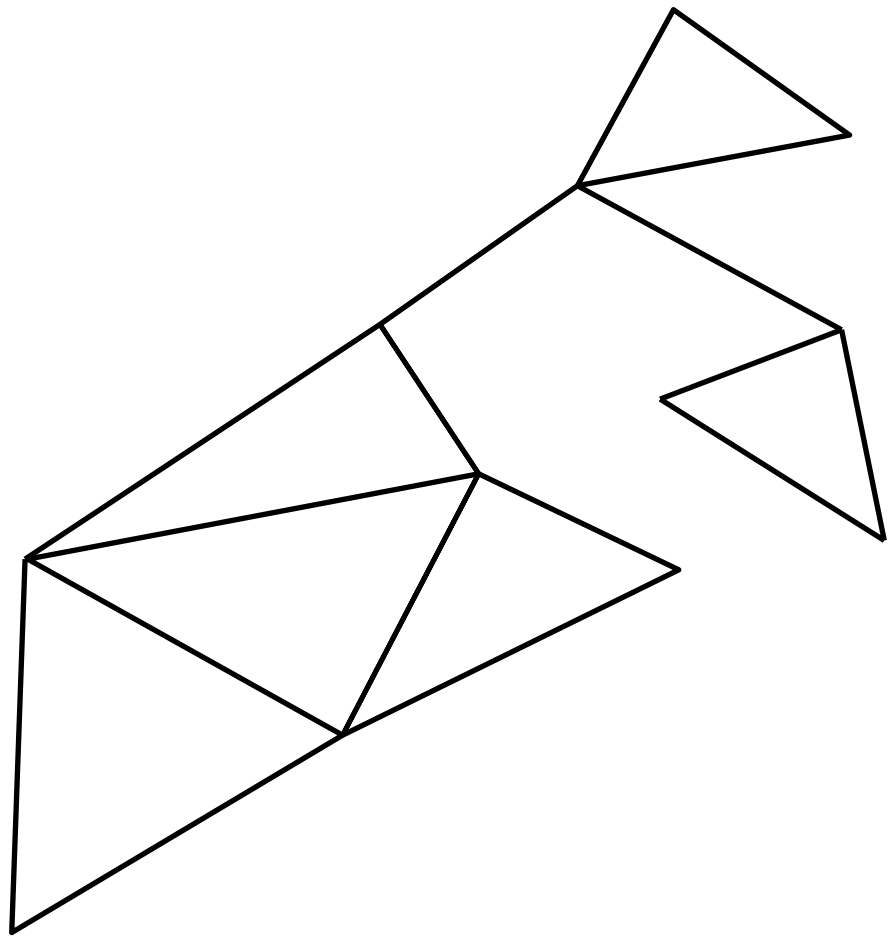


FIGURE 2. A geometric simplicial complex

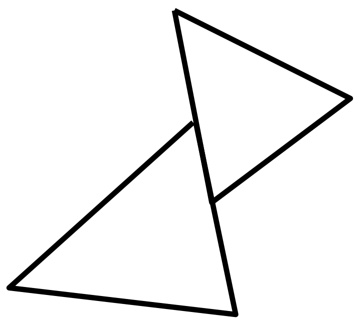


FIGURE 3. Not a geometric simplicial complex

to be its number of vertices minus 1. The dimension of an abstract simplicial complex is the maximal dimension of its simplices.

EXAMPLE 1. Given a graph $G = (V, E)$, the collection of subsets $X \subseteq V$ such that $G[X]$ is a clique is an abstract simplicial complex, called the *clique complex*. The collection of subsets $X \subseteq V$ such that $G[X]$ is independent is also an abstract simplicial complex, called the *independence complex*.

EXAMPLE 2. Given a graph $G = (V, E)$, the collection of subsets $F \subseteq E$ such that F is a forest is an abstract simplicial complex. More generally, the collection of independent sets in a matroid forms an abstract simplicial complex.

Given a geometric simplicial complex Δ, one gets a natural abstract simplicial complex K whose simplices are the $V(\sigma)$ with $\sigma \in \Delta$. We say that Δ *induces* the abstract simplicial complex K.

Conversely, given an abstract simplicial complex K, we can build a geometric simplicial complex Δ such that the $V(\sigma)$'s for $\sigma \in \Delta$ are precisely the simplices of K. Indeed, let $N := |V(\mathsf{K}|$ and let σ^{N-1} be the $(N-1)$-dimensional geometric simplex whose vertices are identified with the vertices of K. Keep from σ^{N-1} the faces whose vertices are the vertex set of a simplex of K. This collection of faces is the geometric simplicial complex we are looking for. It is called the *(geometric) realization* of K.

2.2. Pseudomanifold. An abstract simplicial complex K is a *pseudomanifold* of dimension d if each $(d-1)$-simplex is contained in at least one d-simplex and at most two d-simplices. It is said *without boundary* if each $(d-1)$-simplex is contained in exactly two d-simplices.

3. Simplicial maps

Given two abstract simplicial complexes K and L, a map $\lambda : V(\mathsf{K}) \to V(\mathsf{L})$ is a *simplicial map* is for all $F \in \mathsf{K}$, we have $\lambda(F) \in \mathsf{L}$.

We will sometimes speak of simplicial maps with geometric simplicial complex. It will implicetly mean that the simplcial map are defined on the abstract simplicial complexes induced by the geometric ones.

4. Barycentric subdivision and posets

Let $P = (V, \preceq)$ be a poset. From this poset, we define an abstract simplicial complex $\Delta(P)$, called the *order complex*, whose vertices are the elements of P and whose simplices are the subsets $\{v_0, \ldots, v_k\} \subseteq V$ for some k such that $v_0 \prec v_1 \prec \ldots \prec v_k$.

An abstract simplicial complex can be seen as a poset whose elements are the simplices ordered by the inclusion $\subseteq$. We define the *barycentric subdivision* of K as $\Delta(\mathsf{K} \setminus \{\emptyset\}, \subseteq)$. It is denoted $\mathrm{sd}(\mathsf{K})$.

Given a realization Δ of K, we get easily a realization of $\mathrm{sd}(\mathsf{K})$ as follows: For each $F \in \mathsf{K}$, we denote by v_F the barycenter of the face of Δ having F as vertex set. Then

$$\Delta' := \{\mathrm{conv}(v_{F_0}, \ldots, v_{F_k}) : \{F_0, \ldots, F_k\} \in \mathrm{sd}(\mathsf{K})\}$$

is a realization of $\mathrm{sd}(\mathsf{K})$. The fact that the simplices of $\mathrm{sd}(\mathsf{K})$ are associated to simplices of Δ' is obvious. The only thing that has to be checked is that Δ' is a geometric simplicial complex.

Proposition 1. *Δ' is a geometric simplicial complex.*

Proof. We first prove that two distinct simplices of Δ' cannot simultaneously intersect in their relative interiors.

Suppose for a contradiction that we have $\mathrm{conv}(v_{F_0}, \ldots, v_{F_k}) \cap \mathrm{conv}(v_{F'_0}, \ldots, v_{F'_{k'}})$, with intersection in their relative interiors and $\{F_0, \ldots, F_k\} \neq \{F'_0, \ldots, F'_{k'}\}$. Then there is an x such that

$$x = \sum_{i=0}^{k} \lambda_i v_{F_i} = \sum_{j=0}^{k'} \mu_j v_{F'_j},$$

with $\mu_j, \lambda_i > 0$ and $\sum_{i=0}^{k} \lambda_i = \sum_{j=0}^{k'} \mu_j = 1$. We select such simplices so that $k + k'$ is as small as possible.

Now, note that $v_{F_i} = \sum_{v \in F_i} \frac{1}{|F_i|} v$ and $v_{F'_j} = \sum_{v \in F'_j} \frac{1}{|F'_j|} v$. The point x is in the faces of Δ corresponding to F_k and to $F'_{k'}$. Since Δ is a geometric simplicial complex, x is in the face corresponding to $F_k \cap F'_{k'}$. By identifying for each $v \in V$ its coefficient in $\sum_{i=0}^{k} \lambda_i v_{F_i}$ and in $\sum_{j=0}^{k'} \mu_j v_{F'_j}$, we see that we necessarily have $F_k = F'_{k'}$ (there is uniqueness of the writing of x as a convex combination of the vertices of a face in which it lies). Taking a vertex v in $F_k \setminus F_{k-1}$

shows that $\lambda_k \geq \mu_{k'}$. Taking a vertex in v in $F'_{k'} \setminus F'_{k'-1}$ shows that $\lambda_k \leq \mu_{k'}$. Hence $\lambda_k = \mu_{k'}$. Defining

$$x' = \frac{1}{\sum_{i=0}^{k-1} \lambda_i} \sum_{i=0}^{k-1} \lambda_i v_{F_i} = \frac{1}{\sum_{j=0}^{k'-1} \mu_j} \sum_{j=0}^{k'-1} \mu_j v_{F'_j}$$

contradicts the minimality of $k + k'$. We have proved that two distinct simplices of Δ' cannot simultaneously intersect in their relative interiors.

Now, let us suppose that there are two simplices whose intersection is not contained in a face of one of them. Again, let us write these two simplices $\mathrm{conv}(v_{F_0}, \dots, v_{F_k})$ and $\mathrm{conv}(v_{F'_0}, \dots, v_{F'_{k'}})$, and let x be in their intersection with

$$x = \sum_{i=0}^{k} \lambda_i v_{F_i} = \sum_{j=0}^{k'} \mu_j v_{F'_j}.$$

We keep only F_i's and F_j's with $\mu_j, \lambda_i > 0$ and $\sum_{i=0}^{k} \lambda_i = \sum_{j=0}^{k'} \mu_j = 1$. But according we have just proved, then all F_i and F'_j are then equal.

□

5. Chain complexes and chain maps

5.1. Chain complexes. Let K be an abstract simplicial complex. For each $k = 0, \dots, \dim(\mathsf{K})$, we define the set $C_k(\mathsf{K})$ of all formal sums of k-simplices of K with the relation $\sigma + \sigma = 0$ for any k-simplex σ. The set $C_k(\mathsf{K})$ is therefore a $\mathbb{Z}_2$-vector space with the set of k-simplices of K as a basis. An element c of $C_k(\mathsf{K})$ is called a k-chain and is of the form

$$c = \sum_{\sigma \in \mathsf{K}_k} h_\sigma \sigma,$$

with $h_\sigma \in \mathbb{Z}_2$ for all $\sigma \in \mathsf{K}_k$.

For each $k = 1, \dots, \dim(\mathsf{K})$, we define the *boundary operator* $\partial_k : C_k(\mathsf{K}) \to C_{k-1}(\mathsf{K})$ as follows on any element of the basis and we extend it by linearity.

$$\partial\{v_0, \dots, v_k\} = \sum_{i=0}^{k} \{v_0, \dots, v_k\} \setminus \{v_i\}.$$

We also define $\partial_0 : C_0(\mathsf{K}) \to \{0\}$ (it takes the value 0 on all 0-dimensional simplices.) The collection $\mathcal{C}(\mathsf{K}) = (C_k(\mathsf{K}), \partial_k)_{k=0,\dots,\dim(\mathsf{K})}$ is called the *chain complex* of K.

The following lemma is the fundamental property of the boundary operator. It is the starting point of algebraic topology. Its proof follows from a direct calculation.

Lemma 1. *$\partial_{k-1} \circ \partial_k = 0$ for all $k = 1, \dots, \dim(\mathsf{K})$.*

When there is no risk of confusion, the index is often omitted and the boundary operator is simply denoted ∂.

5.2. Chain maps. Given two simplicial complexes K and L with $\dim(\mathsf{K}) \leq \dim(\mathsf{L})$, a *chain map* $f_\# : \mathcal{C}(\mathsf{K}) \to \mathcal{C}(\mathsf{L})$ is a collection of maps $f_{k\#} : C_k(\mathsf{K}) \to C_k(\mathsf{L})$ commuting with the boundary operator: for all $k \in \{1, \dots, \dim(\mathsf{K})\}$, we have $\partial_k \circ f_{k\#} = f_{(k-1)\#} \circ \partial_k$.

A natural chain map $\lambda_\#$ is associated to each simplicial map $\lambda : \mathsf{K} \to \mathsf{L}$. It is defined as follows for each simplex, and extended by linearity.

$$\lambda_{k\#}(\{v_0, \dots, v_k\}) = \begin{cases} \{\lambda(v_0), \dots, \lambda(v_k)\} & \text{if the } \lambda(v_i) \text{ are pairwise distinct,} \\ 0 & \text{if not.} \end{cases}$$

The fact that $\lambda_\#$ is a chain map follows from a straightfoward calculation and is stated in the following lemma.

Lemma 2. *The map $\lambda_\#$ is a chain map.*

Exercises

5.3. Cyclomatic number. Let $G = (V, E)$ be a graph. A subset $C \subseteq E$ of edges is called a *generalized cycle* if $\deg_C(v) = 0$ for all $v \in V$. We define the sum of two generalized cycles C and C' as their symmetric difference $C \Delta C'$. We denote by $\mathcal{C}(G)$ the set of all generalize cycles with this addition.

1. Prove that $\mathcal{C}$ is a $\mathbb{Z}_2$-vector space.

The *cyclomatic number* of a graph G, denoted $\phi(G)$, is defined as $\dim \mathcal{C}(G)$.

2. Using the boundary operator, prove that

$$|E| - |V| + k(G) = \phi(G),$$

where $k(G)$ is the number of connected components of G.

3. As applications of the above formula, prove that
   (i) if G is a tree, we have $|E| = |V| - 1$,
   (ii) if G is planar and connected, we have $|F| - |E| + |V| = 2$, where F is the number of faces of any plane representation (the infinite face included).

Of course, there are also direct proofs of these facts.

CHAPTER 2

Sperner's lemma, its relatives and applications

1. Statement

Sperner's lemma is usually stated as a lemma. However, we prefer to call it here a theorem [...].

THEOREM 5 (Sperner's lemma). *Let T be a triangulation of a d-dimensional geometric simplex s^d. Let $\lambda : V(T) \to \{0, 1, \ldots, d\}$ be a labelling of the vertices of T such that*

- *each vertex of s^d gets a distinct label (and so we can identify the vertices of s^d with the integers $0, 1, \ldots, d$),*
- *each vertex v of T gets a label in $\lambda(V(s'))$, where s' is the minimal face of s^d containing v.*

Then there is an odd number of simplices $\sigma \in T$ such that $\lambda(\sigma) = \{0, 1, \ldots, d\}$.

Such a simplex σ is said to be *fully-labelled*. We propose first a compact proof, which uses the notions of chain complexes and chain maps defined in Chapter 1. A second proof below is a slightly longer, but has the great advantage to use only elementary tools and to provide an algorithm for finding fully-labelled simplices.

PROOF 1. The proof works by induction and uses the fact that λ is a simplicial map from T to $2^{\{0,1,\ldots,d\}}$, and induces therefore a chain map $\lambda_\#$. If $d = 0$, there is nothing to prove.

Assume now that the theorem is true for $d-1$. Let c be the formal sum of all d-simplices of T and let c'_i be the formal sum of all $(d-1)$-simplices of T on the facet missing label i. We have $\partial c = \sum_{i=1}^d c'_i$. Thus, $\lambda_\#(\partial c) = \sum_{i=1}^d \lambda_\#(c'_i)$. By induction, we have $\lambda_\#(c'_i) = \{0, 1, \ldots, d\} \setminus \{i\}$. We get therefore $\partial \lambda_\#(c) \neq 0$. Since there is only one d-simplex in $2^{\{0,1,\ldots,d\}}$, we get $\lambda_\#(c) = \{0, 1, \ldots, d\}$. □

PROOF 2. This proof works again by induction. If $d = 0$, there is nothing to prove.

Consider the graph $G = (V, E)$ whose vertices are the simplices plus an additional dummy vertex r. Put an edge between two vertices if the corresponding simplices share a common facet labelled $\{0, 1, \ldots, d-1\}$. Put an edge between r and a vertex representing a simplex σ if σ has a facet on the boundary of s^d with labels $\{0, 1, \ldots, d-1\}$.

Now, take a vertex $v \neq r$. We claim that v is of odd degree if and only if the corresponding simplex σ is fully-labelled. Indeed, if σ is fully-labelled, it is of degree 1. If $\lambda(V(\sigma)) = \{0, 1, \ldots, d-1\}$, exactly one of these labels appears twice, giving exactly two facets with labels $\{0, 1, \ldots, d-1\}$. The degree of v is then equal to 2. Finally, if none of these possibilities occurs, the label set of σ does not contain $\{0, \ldots, d-1\}$ and the degree of v is 0.

By induction, r is of odd degree. In a graph, the number of odd degree vertices is even. There is therefore an odd number of vertices $\neq r$ of odd degree, which means that there is an odd number of fully-labelled simplices. □

2. A hypergraph version of Hall's theorem

3. Dissection of a rectangle in triangles of same area

4. Brouwer's theorem

CHAPTER 3

Ky Fan's lemmas

In the previous chapter, we have seen a combinatorial version of Brouwer's theorem. A more general theorem in topology is the Borsuk-Ulam theorem. It will be stated in Section . etc.

1. Ky Fan's lemma

1.1. Combinatorial Stokes Formula.

PROPOSITION 2 (Combinatorial Stokes formula). *Let* M *be a* d*-dimensional pseudomanifold and denote by* $\partial\mathsf{M}$ *its boundary. Let* $\lambda : V(\mathsf{M}) \to \{-1, +1, \ldots, -m, +m\}$ *be a labelling of the vertices such that for any two adjacent vertices* u *and* v*, we have* $\lambda(u) + \lambda(v) \neq 0$*. Denote by* $\alpha(k_0, k_1, \ldots, k_d)$ *the number of* d*-simplices* $F \in \mathsf{M}$ *such that* $\lambda(F) = \{k_0, k_1, \ldots, k_d\}$ *and by* $\beta(k_0, k_1, \ldots, k_{d-1})$ *the number of* $(d-1)$*-simplices* $F' \in \partial\mathsf{M}$ *such that* $\lambda(F') = \{k_0, k_1, \ldots, k_{d-1}\}$*.*
Then

$$\sum_{1\leq j_0<j_1<\ldots<j_d\leq m} \left(\alpha(-j_0, +j_1, \ldots, (-1)^{d-1} j_d\right) + \alpha\left(j_0, -j_1, \ldots, (-1)^d j_d)\right)$$
$$= \sum_{1\leq j_0<j_1<\ldots<j_{d-1}\leq m} \beta\left(j_0, -j_1, \ldots, (-1)^{d-1} j_{d-1}\right) \quad mod\ 2$$

PROOF. Let $u : (x, y) \in (\mathbb{Z}_2)^2 \mapsto (x + y, x + y) \in (\mathbb{Z}_2)^2$ and $\mu_\ell : C_\ell(\mathsf{C}) \to (\mathbb{Z}_2)^2$ defined for an ℓ-dimensional simplex F by

$$\mu_\ell(F) = \begin{cases} (0,1) & \text{if} \\ (1,0) & \text{if} \\ (0,0) & \text{if not.} \end{cases}$$

We have $u \circ \mu_\ell = \mu_{\ell-1} \circ \partial$. There are only three cases. [...] □

1.2. The Ky Fan lemma - general case.

THEOREM 6. *Let* T *be a triangulation of the* d*-dimensional sphere* S^d *such that if* $v \in V(T)$ *then* $-v \in V(T)$*. Let* $\lambda : V(T) \to \{-1, +1, \ldots, -m, +m\}$ *be a labelling of the vertices such that*

- $\lambda(-v) = -\lambda(v)$ *for each* $v \in V(T)$
- $\lambda(u) + \lambda(v) \neq 0$ *for each edge* $uv \in E(T)$*.*

Then there are an odd number of d*-simplices* σ *of* T *such that* $\lambda(V(\sigma))$ *has the form* $\{-j_0, +j_1, \ldots, (-1)^d j_d\}$ *with* $j_0 < j_1 < \ldots < j_d$*. In particular, we have* $m \geq d + 1$*.*

PROOF. We first assume that T refines the triangulation of S^d induced by the d-dimensional cross-polytope.
[...]

□

1.3. The Ky Fan lemma – a purely combinatorial version.

Using the combinatorial interpretation of the barycentric subdivision of the crosspolytope, we get a purely combinatorial version of the Ky Fan lemma. This version will be useful for proving results in combinatorics.

2. Coloring of Kneser graphs

3. Splitting necklaces

Palvogyi + flot alon + conjecture

4. Borsuk-Ulam's theorem

CHAPTER 4

Scarf

kernel

Bibliography