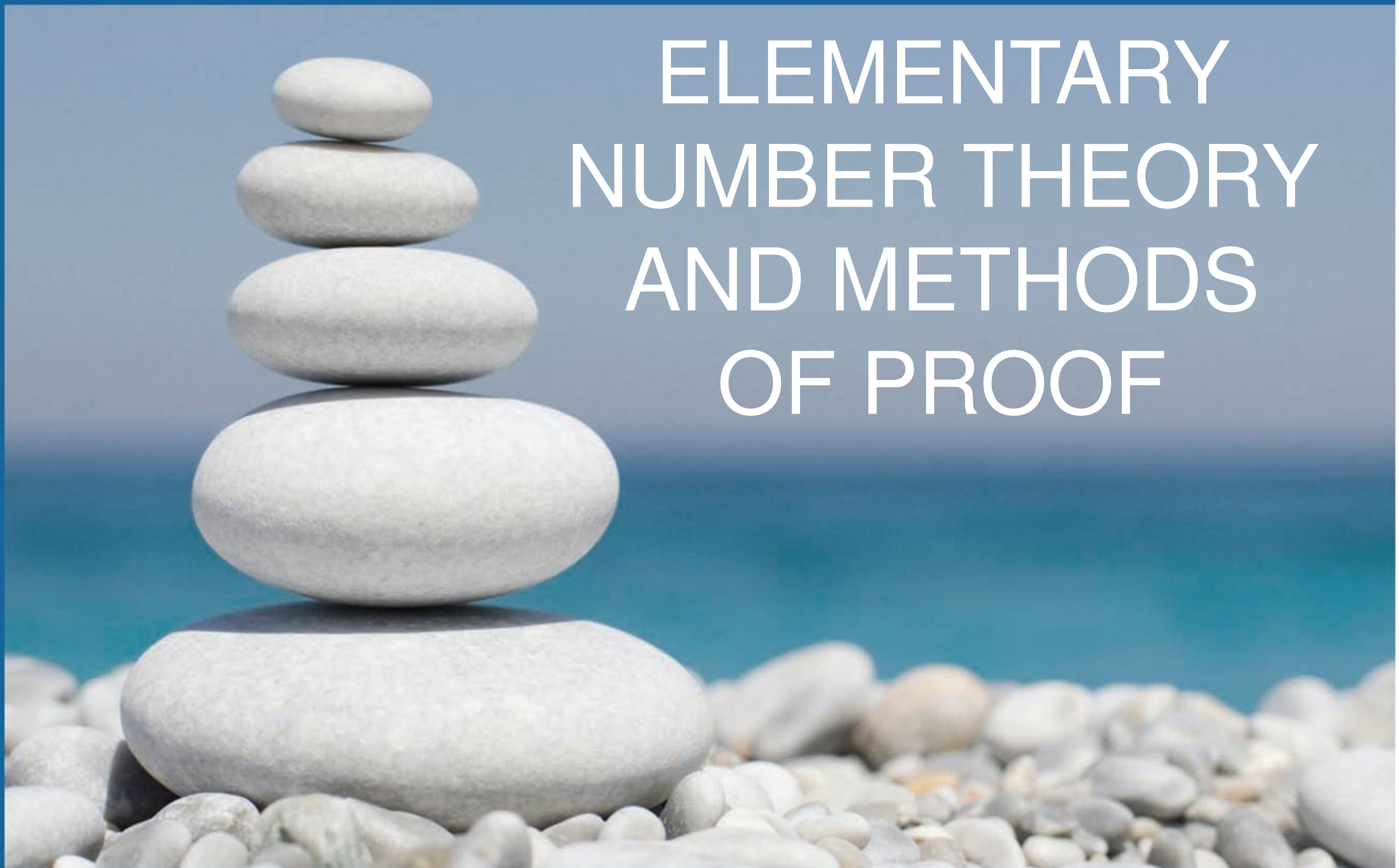
CHAPTER 4

ELEMENTARY NUMBER THEORY AND METHODS OF PROOF



SECTION 4.1

Direct Proof and Counterexample I:Introduction

Direct Proof and Counterexample I: Introduction

Both discovery and proof are integral parts of problem solving. When you think you have discovered that a certain statement is true, try to figure out why it is true.

If you succeed, you will know that your discovery is genuine. Even if you fail, the process of trying will give you insight into the nature of the problem and may lead to the discovery that the statement is false.

Direct Proof and Counterexample I: Introduction

For complex problems, the interplay between discovery and proof is not reserved to the end of the problem-solving process but, rather, is an important part of each step.

Assumptions

- In this text we assume a familiarity with the laws of basic algebra, which are listed in Appendix A.
- We also use the three properties of equality: For all objects A, B, and C, (1) $A = A$, (2) if $A = B$ then $B = A$, and (3) if $A = B$ and $B = C$, then $A = C$.
- In addition, we assume that there is no integer between 0 and 1 and that the set of all integers is closed under addition, subtraction, and multiplication. This means that sums, differences, and products of integers are integers.
- Of course, most quotients of integers are not integers. For example, $3 \div 2$, which equals $3/2$, is not an integer, and $3 \div 0$ is not even a number.

Definitions

Definitions

In order to evaluate the truth or falsity of a statement, you must understand what the statement is about. In other words, you must know the meanings of all terms that occur in the statement. Mathematicians define terms very carefully and precisely and consider it important to learn definitions virtually word for word.

• Definitions

An integer n is **even** if, and only if, n equals twice some integer. An integer n is **odd** if, and only if, n equals twice some integer plus 1.

Symbolically, if n is an integer, then

$$n \text{ is even} \quad \Leftrightarrow \quad \exists \text{ an integer } k \text{ such that } n = 2k.$$

$$n \text{ is odd} \quad \Leftrightarrow \quad \exists \text{ an integer } k \text{ such that } n = 2k + 1.$$

Example 1 – *Even and Odd Integers*

Use the definitions of *even* and *odd* to justify your answers to the following questions.

a. Is 0 even?

b. Is –301 odd?

c. If a and b are integers, is $6a^2b$ even?

d. If a and b are integers, is $10a + 8b + 1$ odd?

e. Is every integer either even or odd?

Solution:

a. Yes, $0 = 2 \cdot 0$.

b. Yes, $-301 = 2(-151) + 1$.

Example 1 – *Solution*

cont'd

c. Yes, $6a^2b = 2(3a^2b)$, and since a and b are integers, so is $3a^2b$ (being a product of integers).

d. Yes, $10a + 8b + 1 = 2(5a + 4b) + 1$, and since a and b are integers, so is $5a + 4b$ (being a sum of products of integers).

e. The answer is yes, although the proof is not obvious.

Definitions

The integer 6, which equals 2 · 3, is a product of two smaller positive integers.

On the other hand, 7 cannot be written as a product of two smaller positive integers; its only positive factors are 1 and 7. A positive integer, such as 7, that cannot be written as a product of two smaller positive integers is called *prime*.

• Definition

An integer n is **prime** if, and only if, $n > 1$ and for all positive integers r and s, if $n = rs$, then either r or s equals n. An integer n is **composite** if, and only if, $n > 1$ and $n = rs$ for some integers r and s with $1 < r < n$ and $1 < s < n$.

In symbols:

n is prime $\Leftrightarrow$ $\forall$ positive integers r and s, if $n = rs$ then either $r = 1$ and $s = n$ or $r = n$ and $s = 1$.

n is composite $\Leftrightarrow$ $\exists$ positive integers r and s such that $n = rs$ and $1 < r < n$ and $1 < s < n$.

Example 2 – *Prime and Composite Numbers*

a. Is 1 prime?

b. Is every integer greater than 1 either prime or composite?

c. Write the first six prime numbers.

d. Write the first six composite numbers.

Solution:

a. No. A prime number is required to be greater than 1.

b. Yes. Let n be any integer that is greater than 1. Consider all pairs of positive integers r and s such that $n = rs$. There exist at least two such pairs, namely $r = n$ and $s = 1$ and $r = 1$ and $s = n$.

Example 2 – *Solution*

cont'd

Moreover, since $n = rs$, all such pairs satisfy the inequalities $1 \leq r \leq n$ and $1 \leq s \leq n$. If n is prime, then the two displayed pairs are the only ways to write n as rs.

Otherwise, there exists a pair of positive integers r and s such that $n = rs$ and neither r nor s equals either 1 or n. Therefore, in this case $1 < r < n$ and $1 < s < n$, and hence n is composite.

c. 2, 3, 5, 7, 11, 13

d. 4, 6, 8, 9, 10, 12

Proving Existential Statements

Proving Existential Statements

We have known that a statement in the form

$$\exists x \in D \text{ such that } Q(x)$$

is true if, and only if,

$$Q(x) \text{ is true for at least one } x \text{ in } D.$$

One way to prove this is to find an x in D that makes $Q(x)$ true.

Another way is to give a set of directions for finding such an x. Both of these methods are called **constructive proofs of existence**.

Example 3 – *Constructive Proofs of Existence*

a. Prove the following: $\exists$ an even integer n that can be written in two ways as a sum of two prime numbers.

b. Suppose that r and s are integers. Prove the following: $\exists$ an integer k such that $22r + 18s = 2k$.

Solution:

a. Let $n = 10$. Then $10 = 5 + 5 = 3 + 7$ and 3, 5, and 7 are all prime numbers.

b. Let $k = 11r + 9s$.

Example 3 – *Solution*

cont'd

Then k is an integer because it is a sum of products of integers; and by substitution, $2k = 2(11r + 9s)$, which equals $22r + 18s$ by the distributive law of algebra.

Proving Existential Statements

A **nonconstructive proof of existence** involves showing either (a) that the existence of a value of x that makes $Q(x)$ true is guaranteed by an axiom or a previously proved theorem or (b) that the assumption that there is no such x leads to a contradiction.

The disadvantage of a nonconstructive proof is that it may give virtually no clue about where or how x may be found.

Disproving Universal Statements by Counterexample

Disproving Universal Statements by Counterexample

To disprove a statement means to show that it is false. Consider the question of disproving a statement of the form

$$\forall x \text{ in } D, \text{ if } P(x) \text{ then } Q(x).$$

Showing that this statement is false is equivalent to showing that its negation is true. The negation of the statement is existential:

$$\exists x \text{ in } D \text{ such that } P(x) \text{ and not } Q(x).$$

Disproving Universal Statements by Counterexample

But to show that an existential statement is true, we generally give an example, and because the example is used to show that the original statement is false, we call it a *counterexample*. Thus the method of disproof by *counterexample* can be written as follows:

Disproof by Counterexample

To disprove a statement of the form "$\forall x \in D$, if $P(x)$ then $Q(x)$," find a value of x in D for which the hypothesis $P(x)$ is true and the conclusion $Q(x)$ is false. Such an x is called a **counterexample.**

Example 4 – *Disproof by Counterexample*

Disprove the following statement by finding a counterexample:

$\forall$ real numbers a and b, if $a^2 = b^2$ then $a = b$.

Solution:

To disprove this statement, you need to find real numbers a and b such that the hypothesis $a^2 = b^2$ is true and the conclusion $a = b$ is false.

The fact that both positive and negative integers have positive squares helps in the search.

Example 4 – *Solution*

cont'd

If you flip through some possibilities in your mind, you will quickly see that 1 and –1 will work (or 2 and –2, or 0.5 and –0.5, and so forth).

Statement: $\forall$ real numbers a and b, if $a^2 = b^2$, then $a = b$.

Counterexample: Let $a = 1$ and $b = -1$. Then $a^2 = 1^2 = 1$ and $b^2 = (-1)^2 = 1$, and so $a^2 = b^2$. But $a \neq b$ since $1 \neq -1$.

Proving Universal Statements

Proving Universal Statements

The vast majority of mathematical statements to be proved are universal. In discussing how to prove such statements, it is helpful to imagine them in a standard form:

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x).$$

When D is finite or when only a finite number of elements satisfy $P(x)$, such a statement can be proved by the method of exhaustion.

Example 5 – *The Method of Exhaustion*

Use the method of exhaustion to prove the following statement:

$\forall n \in \mathbf{Z}$, if n is even and $4 \le n \le 26$, then n can be written as a sum of two prime numbers.

Solution:

$$
\begin{array}{llll}
4 = 2 + 2 & 6 = 3 + 3 & 8 = 3 + 5 & 10 = 5 + 5 \\
12 = 5 + 7 & 14 = 11 + 3 & 16 = 5 + 11 & 18 = 7 + 11 \\
20 = 7 + 13 & 22 = 5 + 17 & 24 = 5 + 19 & 26 = 7 + 19
\end{array}
$$

Proving Universal Statements

The most powerful technique for proving a universal statement is one that works regardless of the size of the domain over which the statement is quantified.

It is called the *method of generalizing from the generic particular*. Here is the idea underlying the method:

Method of Generalizing from the Generic Particular

To show that every element of a set satisfies a certain property, suppose x is a *particular* but *arbitrarily chosen* element of the set, and show that x satisfies the property.

Example 6 – *Generalizing from the Generic Particular*

At some time you may have been shown a “mathematical trick” like the following.

You ask a person to pick any number, add 5, multiply by 4, subtract 6, divide by 2, and subtract twice the original number.

Then you astound the person by announcing that their final result was 7. How does this “trick” work?

Example 6 – *Generalizing from the Generic Particular*
cont'd

Let an empty box • or the symbol x stand for the number the person picks.

Here is what happens when the person follows your directions:

Step	Visual Result	Algebraic Result
Pick a number.	•	x
Add 5.	• |||||	$x + 5$
Multiply by 4.	• ||||| • ||||| • ||||| • |||||	$(x + 5) \cdot 4 = 4x + 20$
Subtract 6.	• || • || • ||||| • |||||	$(4x + 20) - 6 = 4x + 14$
Divide by 2.	• || • |||||	$\dfrac{4x + 14}{2} = 2x + 7$
Subtract twice the original number.	|| |||||	$(2x + 7) - 2x = 7$

Example 6 – *Generalizing from the Generic Particular* cont'd

Thus no matter what number the person starts with, the result will always be 7.

Note that the x in the analysis above is *particular* (because it represents a single quantity), but it is also *arbitrarily chosen* or *generic* (because any number whatsoever can be put in its place).

This illustrates the process of drawing a general conclusion from a particular but generic object.

Proving Universal Statements

When the method of generalizing from the generic particular is applied to a property of the form "If $P(x)$ then $Q(x)$," the result is the method of *direct proof*.

We have known that the only way an if-then statement can be false is for the hypothesis to be true and the conclusion to be false.

Thus, given the statement "If $P(x)$ then $Q(x)$," if you can show that the truth of $P(x)$ compels the truth of $Q(x)$, then you will have proved the statement.

Proving Universal Statements

It follows by the method of generalizing from the generic particular that to show that "$\forall x$, if $P(x)$ then $Q(x)$," is true for *all* elements x in a set D, you suppose x is a particular but arbitrarily chosen element of D that makes $P(x)$ true, and then you show that x makes $Q(x)$ true.

Method of Direct Proof

1. Express the statement to be proved in the form "$\forall x \in D$, if $P(x)$ then $Q(x)$." (This step is often done mentally.)
2. Start the proof by supposing x is a particular but arbitrarily chosen element of D for which the hypothesis $P(x)$ is true. (This step is often abbreviated "Suppose $x \in D$ and $P(x)$.")
3. Show that the conclusion $Q(x)$ is true by using definitions, previously established results, and the rules for logical inference.

Example 7 – *A Direct Proof of a Theorem*

Prove that the sum of any two even integers is even.

Solution:

Whenever you are presented with a statement to be proved, it is a good idea to ask yourself whether you believe it to be true.

In this case you might imagine some pairs of even integers, say 2 + 4, 6 + 10, 12 + 12, 28 + 54, and mentally check that their sums are even.

Example 7 – *Solution*

cont'd

However, since you cannot possibly check all pairs of even numbers, you cannot know for sure that the statement is true in general by checking its truth in these particular instances.

Many properties hold for a large number of examples and yet fail to be true in general.

To prove this statement in general, you need to show that no matter what even integers are given, their sum is even. But given any two even integers, it is possible to represent them as $2r$ and $2s$ for some integers r and s.

Example 7 – *Solution*

cont'd

And by the distributive law of algebra, $2r + 2s = 2(r + s)$, which is even. Thus the statement is true in general.

Suppose the statement to be proved were much more complicated than this. What is the method you could use to derive a proof?

Formal Restatement: $\forall$ integers m and n, if m and n are even then $m + n$ is even.

This statement is universally quantified over an infinite domain. Thus to prove it in general, you need to show that no matter what two integers you might be given, if both of them are even then their sum will also be even.

Example 7 – *Solution*

cont'd

Next ask yourself, "Where am I starting from?" or "What am I supposing?" The answer to such a question gives you the starting point, or first sentence, of the proof.

Starting Point: Suppose m and n are particular but arbitrarily chosen integers that are even.

Or, in abbreviated form:

Suppose m and n are any even integers.

Then ask yourself, "What conclusion do I need to show in order to complete the proof?"

Example 7 – *Solution*

cont'd

To Show: $m + n$ is even.

At this point you need to ask yourself, "How do I get from the starting point to the conclusion?" Since both involve the term *even integer*, you must use the definition of this term—and thus you must know what it means for an integer to be even.

It follows from the definition that since m and n are even, each equals twice some integer.

Example 7 – *Solution*

cont'd

One of the basic laws of logic, called *existential instantiation*, says, in effect, that if you know something exists, you can give it a name.

However, you cannot use the same name to refer to two different things, both of which are currently under discussion.

Existential Instantiation

If the existence of a certain kind of object is assumed or has been deduced then it can be given a name, as long as that name is not currently being used to denote something else.

Example 7 – *Solution*

cont'd

Thus since m equals twice some integer, you can give that integer a name, and since n equals twice some integer, you can also give that integer a name:

$$m = 2r, \text{ for some integer } r \qquad \text{and} \qquad n = 2s, \text{ for some integer } s.$$

Now what you want to show is that $m + n$ is even.
In other words, you want to show that $m + n$ equals $2 \cdot$ (some integer). Having just found alternative representations for m (as $2r$) and n (as $2s$), it seems reasonable to substitute these representations in place of m and n:

$$m + n = 2r + 2s.$$

Example 7 – *Solution*

cont'd

Your goal is to show that $m + n$ is even. By definition of even, this means that $m + n$ can be written in the form

$$2 \cdot (\text{some integer}).$$

This analysis narrows the gap between the starting point and what is to be shown to showing that

$$2r + 2s = 2 \cdot (\text{some integer}).$$

Why is this true? First, because of the distributive law from algebra, which says that

$$2r + 2s = 2(r + s),$$

and, second, because the sum of any two integers is an integer, which implies that $r + s$ is an integer.

Example 7 – *Solution*

cont'd

This discussion is summarized by rewriting the statement as a theorem and giving a formal proof of it. (In mathematics, the word *theorem* refers to a statement that is known to be true because it has been proved.)

Such comments are purely a convenience for the reader and could be omitted entirely. For this reason they are italicized and enclosed in italic square brackets: *[ ]*.

Donald Knuth, one of the pioneers of the science of computing, has compared constructing a computer program from a set of specifications to writing a mathematical proof based on a set of axioms.

Example 7 – *Solution*

cont'd

In keeping with this analogy, the bracketed comments can be thought of as similar to the explanatory documentation provided by a good programmer. Documentation is not necessary for a program to run, but it helps a human reader understand what is going on.

Theorem 4.1.1

The sum of any two even integers is even.

Proof:

Suppose m and n are *[particular but arbitrarily chosen]* even integers. *[We must show that* $m + n$ *is even.]*

Example 7 – *Solution*

cont'd

By definition of even, $m = 2r$ and $n = 2s$ for some integers r and s. Then

$$\begin{aligned} m + n &= 2r + 2s && \text{by substitution} \\ &= 2(r + s) && \text{by factoring out a 2.} \end{aligned}$$

Let $t = r + s$. Note that t is an integer because it is a sum of integers. Hence

$$m + n = 2t \quad \text{where } t \text{ is an integer.}$$

It follows by definition of even that $m + n$ is even.
[This is what we needed to show.]

Directions for Writing Proofs of Universal Statements

Directions for Writing Proofs of Universal Statements

Think of a proof as a way to communicate a convincing argument for the truth of a mathematical statement.

Over the years, the following rules of style have become fairly standard for writing the final versions of proofs:

1. **Copy the statement of the theorem to be proved on your paper.**

2. **Clearly mark the beginning of your proof with the word <u>Proof</u>.**

3. **Make your proof self-contained.**

Directions for Writing Proofs of Universal Statements

This means that you should explain the meaning of each variable used in your proof in the body of the proof. Thus you will begin proofs by introducing the initial variables and stating what kind of objects they are.

At a later point in your proof, you may introduce a new variable to represent a quantity that is known at that point to exist.

4. **Write your proof in complete, gramatically correct sentences.**

This does not mean that you should avoid using symbols and shorthand abbreviations, just that you should incorporate them into sentences.

Directions for Writing Proofs of Universal Statements

5. **Keep your reader informed about the status of each statement in your proof.**

   Your reader should never be in doubt about whether something in your proof has been assumed or established or is still to be deduced. If something is assumed, preface it with a word like *Suppose* or *Assume*.

   If it is still to be shown, preface it with words like, *We must show that* or *In other words*, *we must show that*. This is especially important if you introduce a variable in rephrasing what you need to show.

Directions for Writing Proofs of Universal Statements

6. **Give a reason for each assertion in your proof.**

   Each assertion in a proof should come directly from the hypothesis of the theorem, or follow from the definition of one of the terms in the theorem, or be a result obtained earlier in the proof, or be a mathematical result that has previously been established or is agreed to be assumed.

   Indicate the reason for each step of your proof using phrases such as *by hypothesis*, *by definition of* . . . , and *by theorem* . . . .

Directions for Writing Proofs of Universal Statements

7. **Include the "little words and phrases" that make the logic of your arguments clear.**

   When writing a mathematical argument, especially a proof, indicate how each sentence is related to the previous one.

   Does it follow from the previous sentence or from a combination of the previous sentence and earlier ones? If so, start the sentence by stating the reason why it follows or by writing *Then*, or *Thus*, or *So*, or *Hence*, or *Therefore*, or *Consequently*, or *It follows that*, and include the reason at the end of the sentence.

Directions for Writing Proofs of Universal Statements

If a sentence expresses a new thought or fact that does not follow as an immediate consequence of the preceding statement but is needed for a later part of a proof, introduce it by writing *Observe that*, or *Note that*, or *But*, or *Now*.

Sometimes in a proof it is desirable to define a new variable in terms of previous variables. In such a case, introduce the new variable with the word *Let*.

8. **Display equations and inequalities.**

The convention is to display equations and inequalities on separate lines to increase readability, both for other people and for ourselves so that we can more easily check our work for accuracy.

Variations among Proofs

Variations among Proofs

It is rare that two proofs of a given statement, written by two different people, are identical. Even when the basic mathematical steps are the same, the two people may use different notation or may give differing amounts of explanation for their steps, or may choose different words to link the steps together into paragraph form.

An important question is how detailed to make the explanations for the steps of a proof. This must ultimately be worked out between the writer of a proof and the intended reader, whether they be student and teacher, teacher and student, student and fellow student, or mathematician and colleague.

Common Mistakes

Common Mistakes

The following are some of the most common mistakes people make when writing mathematical proofs.

1. **Arguing from examples.**

   Looking at examples is one of the most helpful practices a problem solver can engage in and is encouraged by all good mathematics teachers.

   However, it is a mistake to think that a general statement can be proved by showing it to be true for some special cases. A property referred to in a universal statement may be true in many instances without being true in general.

Common Mistakes

2. **Using the same letter to mean two different things.**

Some beginning theorem provers give a new variable quantity the same letter name as a previously introduced variable.

3. **Jumping to a conclusion.**

To jump to a conclusion means to allege the truth of something without giving an adequate reason.

4. **Circular reasoning.**

To engage in circular reasoning means to assume what is to be proved; it is a variation of jumping to a conclusion.

Common Mistakes

5. **Confusion between what is known and what is still to be shown.**

   A more subtle way to engage in circular reasoning occurs when the conclusion to be shown is restated using a variable.

6. **Use of *any* rather than *some*.**

   There are a few situations in which the words *any* and *some* can be used interchangeably.

Common Mistakes

7. **Misuse of the word *if*.**

Another common error is not serious in itself, but it reflects imprecise thinking that sometimes leads to problems later in a proof. This error involves using the word *if* when the word *because* is really meant.

Getting Proofs Started

Getting Proofs Started

Believe it or not, once you understand the idea of generalizing from the generic particular and the method of direct proof, you can write the beginnings of proofs even for theorems you do not understand.

The reason is that the starting point and what is to be shown in a proof depend only on the linguistic form of the statement to be proved, not on the content of the statement.

Example 8 – *Identifying the "Starting Point" and the "Conclusion to Be Shown"*

Write the first sentence of a proof (the "starting point") and the last sentence of a proof (the "conclusion to be shown") for the following statement:

Every complete, bipartite graph is connected.

Solution:

It is helpful to rewrite the statement formally using a quantifier and a variable:

Formal Restatement:

$$\forall \overbrace{\text{graphs } G}^{\text{domain}}, \text{ if } \overbrace{G \text{ is complete and bipartite}}^{\text{hypothesis}}, \text{ then } \overbrace{G \text{ is connected}}^{\text{conclusion}}.$$

Example 8 – *Solution*

cont'd

The first sentence, or starting point, of a proof supposes the existence of an object (in this case *G*) in the domain (in this case the set of all graphs) that satisfies the hypothesis of the if-then part of the statement (in this case that *G* is complete and bipartite).

The conclusion to be shown is just the conclusion of the if-then part of the statement (in this case that *G* is connected).

Example 8 – *Solution*

cont'd

Starting Point: Suppose *G* is a *[particular but arbitrarily chosen]* graph such that *G* is complete and bipartite.

Conclusion to Be Shown: *G* is connected.

Thus the proof has the following shape:

Proof:

Suppose *G* is a *[particular but arbitrarily chosen]* graph such that *G* is complete and bipartite.

⋮

Therefore, *G* is connected.

Showing That an Existential Statement Is False

Showing That an Existential Statement Is False

We have known that the negation of an existential statement is universal.

It follows that to prove an existential statement is false, you must prove a universal statement (its negation) is true.

Example 9 – *Disproving an Existential Statement*

Show that the following statement is false:

There is a positive integer n such that $n^2 + 3n + 2$ is prime.

Solution:

Proving that the given statement is false is equivalent to proving its negation is true.

The negation is

For all positive integers n, $n^2 + 3n + 2$ is not prime.

Because the negation is universal, it is proved by generalizing from the generic particular.

Example 9 – *Solution*

cont'd

Claim: The statement "There is a positive integer n such that $n^2 + 3n + 2$ is prime" is false.

Proof:

Suppose n is any *[particular but arbitrarily chosen]* positive integer. *[We will show that $n^2 + 3n + 2$ is not prime.]*

We can factor $n^2 + 3n + 2$ to obtain

$$n^2 + 3n + 2 = (n + 1)(n + 2).$$

We also note that $n + 1$ and $n + 2$ are integers (because they are sums of integers) and that both $n + 1 > 1$ and $n + 2 > 1$ (because $n \geq 1$). Thus $n^2 + 3n + 2$ is a product of two integers each greater than 1, and so $n^2 + 3n + 2$ is not prime.

Conjecture, Proof, and Disproof

Conjecture, Proof, and Disproof

More than 350 years ago, the French mathematician Pierre de Fermat claimed that it is impossible to find positive integers x, y, and z with $x^n + y^n = z^n$ if n is an integer that is at least 3. (For $n = 2$, the equation has many integer solutions, such as $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$.)

Fermat wrote his claim in the margin of a book, along with the comment "I have discovered a truly remarkable PROOF of this theorem which this margin is too small to contain."

Conjecture, Proof, and Disproof

No proof, however, was found among his papers, and over the years some of the greatest mathematical minds tried and failed to discover a proof or a counterexample, for what came to be known as Fermat's last theorem.

One of the oldest problems in mathematics that remains unsolved is the Goldbach conjecture. In Example 5 it was shown that every even integer from 4 to 26 can be represented as a sum of two prime numbers.

More than 250 years ago, Christian Goldbach (1690–1764) conjectured that every even integer greater than 2 can be so represented.

Conjecture, Proof, and Disproof

Explicit computer-aided calculations have shown the conjecture to be true up to at least 10^{18}. But there is a huge chasm between 10^{18} and infinity.

As pointed out by James Gleick of the *New York Times*, many other plausible conjectures in number theory have proved false.

Leonhard Euler (1707–1783), for example, proposed in the eighteenth century that $a^4 + b^4 + c^4 = d^4$ had no nontrivial whole number solutions.

Conjecture, Proof, and Disproof

In other words, no three perfect fourth powers add up to another perfect fourth power. For small numbers, Euler's conjecture looked good.

But in 1987 a Harvard mathematician, Noam Elkies, proved it wrong. One counterexample, found by Roger Frye of Thinking Machines Corporation in a long computer search, is $95{,}800^4 + 217{,}519^4 + 414{,}560^4 = 422{,}481^4$.