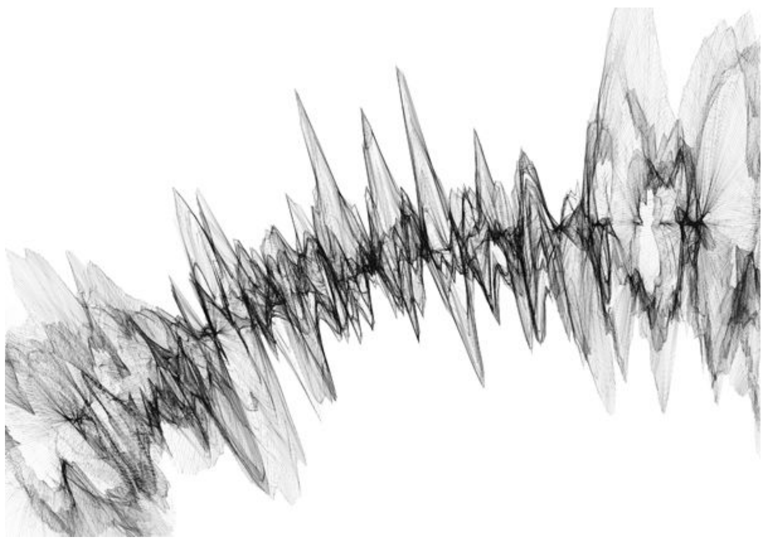
Elementary Real Analysis

MAT 2125

Winter 2017



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Preface

These are notes for the course *Elementary Real Analysis* (MAT 2125) at the University of Ottawa. We begin with an examination of the real numbers. We review the completeness properties of the reals, and then discuss the concepts of supremum, infimum, lim sup, and lim inf. We also explore the topology of Euclidean space $\mathbb{R}^n$, including the concepts of open sets, closed sets, and compact sets. In particular, we will see the Heine–Borel Theorem, which characterizes compact subsets of $\mathbb{R}^n$.

The second part of the course will focus on real-valued functions. We will discuss uniform continuity, the Riemann integral, and the fundamental theorem of calculus. We then focus on sequences and series of functions and uniform convergence. We conclude the course with a brief treatment of the subject of Fourier series.

This is a proof-based course. While proofs of some minor statements will be left as exercises, we will prove almost all the major results in class. The majority of the course work (assignments, quizzes, and exams) will consist of proving mathematical statements, as opposed to computation.

We will assume that the reader is familiar with basic notions of sets, simple proof techniques (proof by contraposition, by contradiction, induction, etc.) and basic logic (quantifiers, implications, etc.). Students are *strongly* encouraged to read through [TBB, Appendix: Background] as a refresher. The notes [Sava] for the course MAT 1362: Mathematical Reasoning & Proofs can also serve as a useful reference for background material.

For the most part, we will follow the notation of [TBB]. The one exception is that we will always use the symbol $\subseteq$ to denote set inclusion. We will avoid the use of the symbol $\subset$, which can cause confusion since it is sometimes used to denote inclusion (i.e. $\subseteq$) and sometimes used to denote proper inclusion (i.e. $\subsetneq$).

Acknowledgements: Portions of these notes follow the handwritten notes of Barry Jessup, while other portions follow the open access textbooks [TBB, Leb], which are the official references for the course.

Alistair Savage Ottawa, 2017.

Course website: `http://alistairsavage.ca/mat2125`

Chapter 1

The real numbers

In this first chapter, we introduce one of the main ingredients of this course: the real numbers. We will assume the existence of the real numbers and that they satisfy certain key axioms. Using these axioms, we will prove some other important properties of the real numbers that will be used throughout the course. A good reference for the material in this chapter is [TBB, Ch. 1].

1.1 Algebraic structure: the field axioms

We will assume there is a set, denoted $\mathbb{R}$, with binary operations $+$ (*addition*) and $\cdot$ (*multiplication*) satisfying the following axioms.

A1 For all $a, b \in \mathbb{R}$, we have $a + b = b + a$. (*commutativity of addition*)

A2 For all $a, b, c \in \mathbb{R}$, we have $(a + b) + c = a + (b + c)$. (*associativity of addition*)

A3 There is a unique element $0 \in \mathbb{R}$ such that, for all $a \in \mathbb{R}$, $a + 0 = 0 + a = a$. The element 0 is called the *additive identity*.

A4 For any $a \in \mathbb{R}$, there exists an element $-a \in \mathbb{R}$ such that $a + (-a) = 0$. The elements $-a$ is called the *additive inverse* of a.

M1 For all $a, b \in \mathbb{R}$, we have $a \cdot b = b \cdot a$. (*commutativity of addition*)

M2 For all $a, b, c \in \mathbb{R}$, we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. (*associativity of multiplication*)

M3 There is a unique element $1 \in \mathbb{R}$ such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in \mathbb{R}$. The element 1 is called the *multiplicative identity*.

M4 For any $a \in \mathbb{R}$, $a \neq 0$, there exists an element $a^{-1} \in \mathbb{R}$ such that $aa^{-1} = 1$. The element a^{-1} is called the *multiplicative inverse* of a.

AM1 For any $a, b, c \in \mathbb{R}$, we have $(a + b) \cdot c = a \cdot c + b \cdot c$. (*distributivity*)

The elements of $\mathbb{R}$ are called *real numbers*. We will often denote multiplication by *juxtaposition*. That is, we will write ab instead of $a \cdot b$. Note that the axiom labels indicate the operations involved: A1–A4 involve addition, M1–M4 involve multiplication, and AM1 involves both. These axioms are called the *field axioms*. Any set with two binary operations satisfying these axioms is called a *field*. Thus, $\mathbb{R}$ is a field.

Remark 1.1. In A3, we do not actually need to *assume* that the additive identity 0 is unique. Indeed, if 0 and $0'$ are two real numbers with the given property, then

$$0 = 0 + 0' = 0'.$$

Similarly, one does not need to assume that the multiplicative identity is unique. Furthermore, one can show that additive and multiplicative inverses are unique. See Exercise 1.1.1.

We can now define subtraction and division in $\mathbb{R}$. For $a, b \in \mathbb{R}$, we define

$$a - b := a + (-b).$$

If we also have $b \neq 0$, then we define

$$\frac{a}{b} = ab^{-1}.$$

Since addition and multiplication are associative (axioms A2 and M2), we can omit parentheses when adding or multiplying more than two real numbers. For instance, we can write expression such as

$$a + b + c + d \qquad \text{and} \qquad wxyz$$

and there is no ambiguity since any way of adding in parentheses yields the same result.

Exercises.

For the following exercises, use *only* axioms A1–AM1.

1.1.1. (a) Show that one does not need the uniqueness assumption in M3. That is, show that if $1, 1' \in \mathbb{R}$ satisfy the conditions $a \cdot 1 = a$ and $a \cdot 1' = a$ for all $a \in \mathbb{R}$, then $1 = 1'$.

(b) Show that additive inverses are unique. That is, show that if $a, b, c \in \mathbb{R}$ satisfy $a + b = 0$ and $a + c = 0$, then $b = c$.

(c) Show that multiplicative inverses are unique. That is, show that if $a, b, c \in \mathbb{R}$ satisfy $ab = 1$ and $ac = 1$, then $b = c$.

1.1.2 ([TBB, Ex. 1.3.5]). Using only the field axioms, show that

$$(x + 1)^2 = x^2 + 2x + 1$$

for all $x \in \mathbb{R}$. Would this identity be true in any field (i.e. in any set with two binary operations $+$ and $\cdot$ satisfying the field axioms)?

1.1.3 ([TBB, Ex. 1.3.6]). Define operations of addition and multiplication on $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ as follows:

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Show that $\mathbb{Z}_5$ satisfies all the field axioms.

1.1.4 ([TBB, Ex. 1.3.6]). Define operations of addition and multiplication on $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ as follows:

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Which of the field axioms does $\mathbb{Z}_6$ fail to satisfy?

1.2 Order structure

In addition to their algebraic structure (coming from addition and multiplication), the real numbers also have an ordering. In particular, we assume that the following axioms hold. Here $a < b$ is a *statement*, which is either true or false.

O1 For any $a, b \in \mathbb{R}$, exactly one of the following statements is true:

- $a = b$
- $a < b$
- $b < a$.

O2 For $a, b, c \in \mathbb{R}$ such that $a < b$ and $b < c$, we have $a < c$. (We say that $<$ is *transitive*.)

O3 For $a, b \in \mathbb{R}$ such that $a < b$, we have $a + c < b + c$ for all $c \in \mathbb{R}$.

O4 For $a, b \in \mathbb{R}$ such that $a < b$, we have $ac < bc$ for all $c \in \mathbb{R}$ satisfying $c > 0$.

We will also write $a > b$ to indicate that $b < a$. We write $a \leq b$ if $a < b$ *or* $a = b$ and we write $a \geq b$ if $a > b$ *or* $a = b$.

Together with the field axioms A1–AM1, axioms O1–O4 imply that $\mathbb{R}$ is an *ordered field*.

Lemma 1.2. *If $a, b \in \mathbb{R}$ with $a < b$, then there exists $c \in \mathbb{R}$ such that $a < c < b$.*

Proof. We have

$$a < b \implies 2a < a + b < 2b \implies a < \frac{a+b}{2} < b. \qquad \square$$

Exercises.

For the following exercises, use only axioms O1–O4 (and algebraic properties of $\mathbb{R}$).

1.2.1 ([TBB, Ex. 1.4.1]). Prove that if $a < b$ and $c < d$, then $ad + bc < ac + bd$.

1.2.2. Suppose $a, b, c \in \mathbb{R}$ such that $a < b$ and $c < 0$. Show that $ac > bc$.

1.2.3. Prove that if $a \leq b$ and $b \leq a$, then $a = b$.

1.3 Bounds and the completeness axiom

Definition 1.3 (Upper bound, bounded above)**.** Suppose E is a set of real numbers. A real number M is an *upper bound* for E if $x \leq M$ for all $x \in E$. If E has an upper bound, then we say E is *bounded above*. Otherwise, we say it is *unbounded above*.

Definition 1.4 (Lower bound, bounded below)**.** Suppose E is a set of real numbers. A real number m is an *lower bound* for E if $m \leq x$ for all $x \in E$. If E has a lower bound, then we say E is *bounded below*. Otherwise, we say it is *unbounded below*.

A set that is *both* bounded above and bounded below is said to be *bounded*.

Definition 1.5 (Maximum/largest element)**.** Suppose E is a set of real numbers. If $M \in E$ satisfies $x \leq M$ for all $x \in E$, then M is the *maximum* or *largest element* of the set E and we write $M = \max E$.

Definition 1.6 (Minimum/smallest element)**.** Suppose E is a set of real numbers. If $m \in E$ satisfies $m \leq x$ for all $x \in E$, then m is the *minimum* or *smallest element* of the set E and we write $m = \min E$.

Example 1.7. The empty set $\varnothing$ is bounded above and below by all real numbers, but it has no maximum or minimum.

Example 1.8. Consider the interval

$$[-1, 3] = \{x : -1 \leq x \leq 3\}.$$

We have $\max[-1, 3] = 3$ and $\min[-1, 3] = -1$. Any $M \in \mathbb{R}$, $M \geq 3$, is an upper bound for $[-1, 3]$ and any $m \in \mathbb{R}$, $m \leq -1$, is a lower bound for $[-1, 3]$. So we see that upper and lower bounds are far from unique. This interval is bounded.

Example 1.9. Consider the interval

$$(-1,3) = \{x : -1 < x < 3\}.$$

This subset of $\mathbb{R}$ has no maximum and no minimum. However, any real number greater than or equal to 3 is an upper bound. Any real number less than or equal to -1 is a lower bound. In particular, this interval is bounded.

Example 1.10. The set

$$[3,\infty) = \{x : 3 \leq x\}$$

has minimum 3, but no maximum and no upper bound. It is bounded below (by any real number less than or equal to 3), but not bounded above, hence not bounded.

As noted in Example 1.9, the set $(-1,3)$ has no maximum, and infinitely many upper bounds. However, it does have a unique *smallest* upper bound, namely 3.

Definition 1.11 (Supremum/least upper bound)**.** Suppose E is a nonempty set of real numbers. If M is an upper bound of E such that $M \leq M'$ for all upper bounds M' of E, we say that M is the *supremum* or *least upper bound* of E, and we write $M = \sup E$.

Definition 1.12 (Infimum/greatest lower bound)**.** Suppose E is a nonempty set of real numbers. If m is a lower bound of E such that $m' \leq m$ for all lower bounds m' of E, we say that m is the *infimum* or *greatest lower bound* of E, and we write $m = \inf E$.

In addition to the above two definitions, we also adopt the following conventions:

- We write $\inf \varnothing = \infty$ and $\sup \varnothing = -\infty$.
- If E is nonempty and unbounded above, we write $\sup E = \infty$.
- If E is nonempty unbounded below, we write $\inf E = -\infty$.

Note that ∞ and $-\infty$ are *not* real numbers. The above are just notational conventions. When we write $\sup E = \infty$, this means, in particular, that E has no supremum.

Example 1.13. If $a < b$, then the interval (a,b) has no minimum and no maximum. However,

$$\inf(a,b) = a \qquad \text{and} \qquad \sup(a,b) = b.$$

Let us prove that $\sup(a,b) = b$. By definition, we have $x < b$ for all $x \in (a,b)$. Thus b is an upper bound for (a,b). It remains to show that it is the *least* upper bound. Suppose $y < b$. We want to show that y is *not* an upper bound for (a,b). Since $a < b$ and $y < b$, we have $\max\{a,y\} < b$. By Lemma 1.2, we can find a real number x such that

$$\max\{a,y\} < x < b.$$

But then $x \in (a,b)$ and $y < x$. So y is not an upper bound for (a,b), completing the proof. The proof that $\inf(a,b) = a$ is similar (see Exercise 1.3.4).

We are now ready to state our final axiom for the real numbers.

Axiom 1.14 (Completeness axiom)**.** Every nonempty set of real numbers that is bounded above has a least upper bound. In other words, if $E \subseteq \mathbb{R}$ is nonempty and bounded above, then $\sup E$ exists (and is a real number).

We can summarize our axioms for $\mathbb{R}$ by stating that $\mathbb{R}$ is a *complete ordered field*. In fact, it is possible to show that, in some sense, $\mathbb{R}$ is the *only* complete ordered field.

Corollary 1.15. *Every nonempty set of real numbers that is bounded below has a greatest lower bound. In other words, if $E \subseteq \mathbb{R}$ is nonempty and bounded below, then* $\inf E$ *exists (and is a real number).*

Proof. The proof of this corollary is left as an exercise (Exercise 1.3.5). □

Exercises.

1.3.1 ([TBB, Ex. 1.6.1]). Show that a set $E \subseteq \mathbb{R}$ is bounded if and only if there exists an $r \in \mathbb{R}$ such that $|x| < r$ for all $x \in E$. (Since Section 1.6 for a discussion of absolute value.)

1.3.2. Prove that maxima and minima are unique if they exist. In other words, show that if m_1 and m_2 are minima for a set E, then $m_1 = m_2$. Similarly, show that if M_1 and M_2 are maxima for a set E, then $M_1 = M_2$.

1.3.3. Prove that suprema and infima are unique if they exist.

1.3.4. Suppose $a < b$. Prove that $\inf(a, b) = a$.

1.3.5. Prove Corollary 1.15. *Hint*: Suppose $E \subseteq \mathbb{R}$ is nonempty and bounded below. Consider the set $\{-x : x \in E\}$.

1.3.6 ([TBB, Ex. 1.6.2]). For each of the following sets, find the supremum, infimum, maximum, and minimum, if they exist.

(a) $\mathbb{N}$

(b) $\mathbb{Z}$

(c) $\mathbb{Q}$

(d) $\mathbb{R}$

(e) $\{-3, 2, 5, 7\}$

(f) $\{x \in \mathbb{R} : x^2 < 2\}$

(g) $\{x \in \mathbb{R} : x^2 - x - 2 < 0\}$

(h) $\{1/n : n \in \mathbb{N}\}$

1.3.7. Suppose $E \subseteq \mathbb{R}$.

(a) Show that if $\max E$ exists, then so does $\sup E$, and $\sup E = \max E$.

(b) Show that if $\sup E$ exists and $\sup E \in E$, then $\max E$ exists and $\max E = \sup E$.

In other words, maxima are precisely suprema that are contained in the set.

1.3.8. Suppose A and B are nonempty subsets of $\mathbb{R}$ that are bounded above. Show that $A \cup B$ has a least upper bound, and that

$$\sup(A \cup B) = \max\{\sup A, \sup B\}.$$

1.3.9. Suppose A is a nonempty set of real numbers and $r > 0$. Let $B = \{ra : a \in A\}$. Show that $\sup A$ exists if and only $\sup B$ exists. Furthermore, if they exist, show that $\sup B = r \sup A$.

1.3.10 ([TBB, Ex. 1.6.17]). Suppose $A \subseteq \mathbb{R}$. Show that x is the supremum of A if and only if $a \le x$ for all $a \in A$ and for every $\varepsilon > 0$ there is an element $a' \in A$ such that $x - \varepsilon < a'$. Similarly, show that y is the infimum of A if and only if $y \le a$ for all $a \in A$ and for every $\varepsilon > 0$ there is an element $a' \in A$ such that $a' < y + \varepsilon$.

1.3.11. Suppose A and B are nonempty sets of real numbers, and that $\sup A$ and $\sup B$ exist. Define

$$C = \{a + b : a \in A,\ b \in B\}.$$

Show that $\sup C$ exists and that $\sup C = \sup A + \sup B$.

1.3.12. Suppose $A \subseteq \mathbb{R}$ and that $\inf A$ and $\sup A$ both exist. Prove the following:

(a) $\sup\{a - b : a, b \in A\} = \sup A - \inf A$.

(b) $\inf\{a - b : a, b \in A\} = \inf A - \sup A$.

(See Exercise 1.6.7 for a continuation of this exercise.)

1.3.13. Suppose $A \subseteq \mathbb{Z}$ is bounded above. Show that $\max A$ exists.

1.4 Natural numbers and induction

The set

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

of *natural numbers* is a subset of $\mathbb{R}$. We will now prove a property of the natural numbers that seems "obvious", but whose proof in fact relies on the completeness axiom for the real numbers.

Theorem 1.16 (Archimedean property of $\mathbb{R}$). *The set of natural numbers $\mathbb{N}$ has no upper bound.*

Proof. We prove this result by contradiction. Assume $\mathbb{N}$ has an upper bound. Then, by the completeness axiom, it has a least upper bound. Let $x = \sup \mathbb{N}$. Then

$$n \le x \quad \forall n \in \mathbb{N}$$

since x is an upper bound for $\mathbb{N}$. In addition, since x is the *least* upper bound for $\mathbb{N}$, the real number $x - 1$ is *not* an upper bound for $\mathbb{N}$. Thus, there exists some $m \in \mathbb{N}$ such that $m > x - 1$. But then $m + 1 \in \mathbb{N}$ and $m + 1 > x$. This contradicts the fact that $x = \sup \mathbb{N}$. □

Corollary 1.17. *(a) For any $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > x$.*

(b) Given any $x, y \in \mathbb{R}$ with $x, y > 0$, there exists an $n \in \mathbb{N}$ such that $nx > y$.

(c) Give any $x \in \mathbb{R}$ with $x > 0$, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

Proof. The proof of this corollary is left as an exercise (Exercise 1.4.1). □

In fact, one can show that the archimedean property follows from each of the statements in Corollary 1.17. Therefore, these statements are actually *equivalent* to the archimedean property.

Theorem 1.18 (Well-ordering property). *Every nonempty subset of $\mathbb{N}$ has a smallest element.*

Proof. Suppose $S \subseteq \mathbb{N}$ and $S \neq \varnothing$. Since S is bounded below (say, by 0), $\alpha = \inf S$ exists by Corollary 1.15. If $\alpha \in S$, then α is a smallest element of S and we are done.

It remains to consider the case $\alpha \notin S$. Since α is the *greatest* lower bound, $\alpha + 1$ is not a lower bound for S. Therefore, there exists $x \in S$ such that $x < \alpha + 1$. Since α is a lower bound for S, we have $\alpha \le x$. We cannot have $\alpha = x$, since we assumed $\alpha \notin S$. Thus we have

$$\alpha < x < \alpha + 1.$$

Since $\alpha < x$, the element x is not a lower bound of S. Thus, there exists another $y \in S$ with

$$\alpha < y < x < \alpha + 1.$$

But then $x, y \in S \subseteq \mathbb{N}$ and $0 < x - y < 1$, which is impossible. This contradiction completes the proof. □

Theorem 1.19 (Principle of induction). *Suppose $S \subseteq \mathbb{N}$ has the following properties:*

- *$1 \in S$,*
- *if $n \in S$, then $n + 1 \in S$.*

Then $S = \mathbb{N}$.

Proof. Let $E = \mathbb{N} \setminus S$. It suffices to show that $E = \varnothing$, and we show this by contradiction. Suppose $E \neq \varnothing$. By Theorem 1.18, there is a smallest element α of E. By hypothesis, $1 \in S$, so $1 \notin E$ and hence $\alpha \neq 1$. Thus, $\alpha - 1 \in \mathbb{N}$. Since α was the smallest element of E, we have $\alpha - 1 \notin E$. Hence $\alpha - 1 \in S$. By hypothesis, it follows that $\alpha = (\alpha - 1) + 1 \in S$. But this contradicts the fact that $\alpha \in E$. This contradiction completes the proof. ☐

Theorem 1.19 is the basis of a powerful method of proof: *proof by induction*. See Exercise 1.4.7 for an example. Students who have not seen proof by induction in previous courses should read [TBB, §A.8].

Exercises.

1.4.1. Prove Corollary 1.17.

1.4.2. Show that the archimedean property follows from Corollary 1.17(c). In other words, assume Corollary 1.17(c) is true and prove the archimedean property using this assumption.

1.4.3 ([TBB, Ex. 1.7.4]). Let $x \in \mathbb{R}$. Show that there is a unique $m \in \mathbb{Z}$ such that

$$m \leq x < m + 1.$$

This m is often denoted $\lfloor x \rfloor$ and is called the *integer part* of x. The function $x \mapsto \lfloor x \rfloor$ is called the *floor function*.

1.4.4. Suppose $a, b \in \mathbb{R}$ and $b - a \geq 1$. Prove that there exists an integer n such that $a < n \leq b$.

1.4.5. Find the infimum and supremum of the set

$$\{3, 5\} \cup \left\{ \frac{1}{2x} : x \in \mathbb{R},\ x \geq 1 \right\}$$

or show that they do not exist. Remember to justify your answer.

1.4.6. Find the infimum and supremum of the set

$$\left\{ 3 + \frac{1}{n} : n \in \mathbb{N} \right\}$$

or show that they do not exist.

1.4.7. Prove that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1} \quad \text{for all } n \in \mathbb{N}.$$

1.4.8. Show that every finite nonempty set has a maximum. *Hint*: Use induction on the size of the set.

1.5 Rational numbers

The set

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z},\ b \neq 0 \right\}$$

of *rational numbers* is a subset of $\mathbb{R}$. The rational numbers form an ordered field. That is, $\mathbb{Q}$ satisfies axioms A1–AM1 and O1–O4. However, it does not satisfy the completeness axiom (Axiom 1.14), as we will see below. However, the rationals are *dense* in the reals, in a sense that we now make precise.

Definition 1.20 (Dense set)**.** A set $E \subseteq \mathbb{R}$ is said to be *dense* in $\mathbb{R}$ if every interval (a, b), $a < b$, contains a point of E.

Theorem 1.21. *The set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$.*

Proof. Suppose $a < b$. We wish to find a rational number in the interval (a, b). By the archimedean property (Theorem 1.16), there is a natural number n such that

$$n > \frac{1}{b-a}.$$

Thus $nb > na + 1$. By Exercise 1.4.3, we can choose $m \in \mathbb{Z}$ such that

$$m \leq na + 1 < m + 1.$$

It follows that

$$m - 1 \leq na < nb - 1 < nb,$$

and so

$$a < \frac{m}{n} \leq a + \frac{1}{n} < b.$$

So we have found a rational number $\frac{m}{n}$ in the interval (a, b), as desired. □

Theorem 1.21 states that between any two distinct real numbers is a rational number. Compare this to Lemma 1.2.

A real number that is not rational is said to be *irrational*. It can be shown that $\sqrt{2}$ is irrational (Exercise 1.5.1). Consider the set

$$\left\{ x \in \mathbb{Q} : 0 \leq x^2 < 2 \right\}.$$

This set is bounded above by a rational number (say, 3), but has no supremum in $\mathbb{Q}$. This shows that the rational numbers are not complete (i.e. they do not satisfy the completeness axiom). Of course, the above set has a supremum in $\mathbb{R}$, namely $\sqrt{2}$.

Exercises.

1.5.1. Prove that $\sqrt{2}$ is irrational. *Hint*: Assume $\sqrt{2} = \frac{m}{n}$, where $m, n \in \mathbb{Z}$ have no common divisors (other than ± 1) and show that this leads to a contradiction.

1.6 Absolute value: the metric structure on $\mathbb{R}$

The real numbers have a useful notion of distance, also called a *metric*, given by absolute value. We discuss here some of the important properties of this metric.

Definition 1.22 (Absolute value)**.** For $x \in \mathbb{R}$, we define the *absolute value* of x to be

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Note that it follows from the definition that $|x| = \max\{x, -x\}$.

Theorem 1.23 (Properties of absolute value)**.** *For all* $x, y \in \mathbb{R}$*, we have*

(a) $|-x| = |x|$,

(b) $-|x| \leq x \leq |x|$,

(c) $|xy| = |x|\,|y|$,

(d) $|x + y| \leq |x| + |y|$, *(triangle inequality)*

(e) $|x| - |y| \leq |x - y|$ *and* $|y| - |x| \leq |x - y|$.

Proof. The proof of this theorem is left as an exercise (Exercise 1.6.1). □

Definition 1.24 (Distance)**.** The *distance* between two real numbers x and y is defined to be

$$d(x, y) = |x - y|.$$

Proposition 1.25 (Properties of distance)**.** *For all* $x, y, z \in \mathbb{R}$*, we have*

(a) $d(x, y) \geq 0$, *(all distances are nonnegative)*

(b) $d(x, y) = 0$ *if and only if* $x = y$, *(distinct points are a positive distance apart)*

(c) $d(x, y) = d(y, x)$, *(distance is symmetric)*

(d) $d(x, y) \leq d(x, z) + d(z, y)$. *(triangle inequality)*

Exercises.

1.6.1. Prove Theorem 1.23. *Hint*: Consider cases based on the definition of the absolute value.

1.6.2 ([TBB, Ex. 1.10.5]). Show that $\big||x| - |y|\big| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

1.6.3 ([TBB, Ex. 1.10.3]). Suppose $x, a, \varepsilon \in \mathbb{R}$ and $\varepsilon > 0$. Show that the inequalities

$$|x - a| < \varepsilon \quad \text{and} \quad a - \varepsilon < x < a + \varepsilon$$

are equivalent.

1.6.4. Suppose $x \in \mathbb{R}$ and $|x - 2| < 2$. Prove that $|x + 1| > 1$.

1.6.5. Suppose $x \in \mathbb{R}$ and $|x| \leq 1$. Prove that $|x^2 - 4| \leq 3|x - 2|$.

1.6.6 ([TBB, Ex. 1.10.8]). Show that

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

1.6.7. This exercise is a continuation of Exercise 1.3.12. Suppose $A \subseteq \mathbb{R}$ and that $\inf A$ and $\sup A$ both exist. Prove that

$$\sup\{|a - b| : a, b \in A\} = \sup A - \inf A.$$

1.7 Constructing the real numbers

In this course, we assume the existence of the real numbers and that they form a complete ordered field. Some may find this approach unsatisfactory. How do we really *know* that the real numbers exist and have these properties? In fact, one can *construct* the real numbers and *prove* that they form a complete ordered field. We will not do so in this course. However, for the interested student, we outline here how such a construction proceeds. (See also this Wikipedia page.)

It is impossible to start from nowhere. Mathematics must be built on *some* axioms that are assumed to be true. The most common set of axioms are called the ZFC axioms of set theory. From these axioms, one can construct the natural numbers using set theory. (In any case, we expect that most students are rather comfortable supposing the existence of the natural numbers.)

Next, we formally enlarge the set $\mathbb{N}$ of natural numbers by including additive inverses. This results in the set $\mathbb{Z}$ of integers. Then we form $\mathbb{Q}$ by considering "formal quotients". Precisely, we define rational numbers to be pairs of natural numbers (think of the numerator and denominator of a fraction) modulo a relation that corresponds to equating equivalent fractions.

The final step is to *complete* the rational numbers. This can be done in several ways. The two most common methods proceed via Dedekind cuts or Cauchy sequences (see Section 2.5). Both procedures are formal ways of "filling in the gaps" that exist in the rational numbers. The resulting number system can then be shown to be a complete ordered field, and one can show that any ordered field is "the same as" (more precisely, *isomorphic* to) this one. We then call this complete ordered field $\mathbb{R}$.

Chapter 2

Sequences

In this chapter, we examine sequences of real numbers. We give the precise definition of a limit of a sequence and some important properties of limits. We also discuss various conditions that ensure the convergence of a sequence, and the concepts of lim inf and lim sup. A good reference for the material in this chapter is [TBB, Ch. 2].

2.1 Limits

Definition 2.1 (Sequence)**.** A *sequence* of real numbers is a function $f\colon \mathbb{N} \to \mathbb{R}$.

We will often think of a sequence in terms of its image. That is, if we define $a_n = f(n)$ for all $n \in \mathbb{N}$, then we will call the list

$$a_1, a_2, a_3, \dots$$

a sequence. We will often denote such a sequence by $\{a_n\}_{n\in\mathbb{N}}$, $\{a_n\}_{n=1}^{\infty}$, $\{a_n\}_{n\geq 1}$, or simply $\{a_n\}$.

We would now like to give a precise definition for what it means for a sequence to converge to some real number. The idea is that the sequence $\{a_n\}_{n\in\mathbb{N}}$ should converge to L if the terms a_n get closer to L as n gets larger. The precise definition is as follows.

Definition 2.2 (Limit of a sequence)**.** Suppose $\{a_n\}$ is a sequence of real numbers. We say that $\{a_n\}$ *converges* to $L \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon \quad \forall\, n \geq N.$$

If $\{a_n\}$ converges to L, we write

$$\lim_{n\to\infty} a_n = L$$

or

$$a_n \to L \text{ as } n \to \infty.$$

(Sometimes we simply write $a_n \to L$.) The number L is called the *limit* of the sequence. If a sequence converges, we say it is *convergent*. A sequence that is not convergent is *divergent*.

Example 2.3. Let $a_n = 6 + \frac{2}{n}$. Let us show that

$$\lim_{n\to\infty} a_n = 6.$$

First note that

$$|a_n - 6| = \left|6 + \frac{2}{n} - 6\right| = \left|\frac{2}{n}\right| = \frac{2}{n}.$$

Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $N > \frac{2}{\varepsilon}$. (Note that we are using the archimedean property here.) Then, for all $n \geq N$, we have

$$|a_n - 6| = \frac{2}{n} \leq \frac{2}{N} < \varepsilon.$$

Example 2.4. Let us show that

$$\lim_{n\to\infty} \frac{3n^2}{4n^2+2} = \frac{3}{4}.$$

First note that

$$\left|\frac{3n^2}{4n^2+2} - \frac{3}{4}\right| = \left|\frac{3n^2}{4n^2+2} - \frac{3n^2+\frac{3}{2}}{4n^2+2}\right| = \left|-\frac{\frac{3}{2}}{4n^2+2}\right| = \frac{3}{8n^2+4} \leq \frac{3}{8n^2}.$$

Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $N > \sqrt{\frac{3}{8\varepsilon}}$. Then, for all $n \geq N$, we have

$$\left|\frac{3n^2}{4n^2+2} - \frac{3}{4}\right| \leq \frac{3}{8n^2} \leq \frac{3}{8N^2} < \varepsilon.$$

Proposition 2.5 (Uniqueness of limits)**.** *If*

$$\lim_{n\to\infty} a_n = L_1 \qquad \textit{and} \qquad \lim_{n\to\infty} a_n = L_2,$$

then $L_1 = L_2$.

Proof. Let $\varepsilon > 0$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that

$$|a_n - L_1| < \varepsilon \quad \forall\, n \geq N_1$$

and

$$|a_n - L_1| < \varepsilon \quad \forall\, n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$, we have

$$|L_1 - L_2| = |L_1 - a_n + a_n - L_2| \leq |L_1 - a_n| + |a_n - L_2| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since ε was an arbitrary positive real number, this implies that $|L_1 - L_2| = 0$, and so $L_1 = L_2$. □

There are many ways that a sequence can diverge. One way is that the terms of the sequence can get arbitrarily large (or large negative).

Definition 2.6 (Divergence to $\pm\infty$)**.** Suppose $\{a_n\}$ is a sequence of real numbers. We say that $\{a_n\}$ *diverges to* ∞ if for all $M \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that

$$a_n \geq M \quad \forall\, n \geq N.$$

In this case we write

$$\lim_{n\to\infty} a_n = \infty$$

or

$$a_n \to \infty \text{ as } n \to \infty.$$

Similarly, we say that $\{a_n\}$ *diverges to* $-\infty$ if for all $M \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that

$$a_n \leq -M \quad \forall\, n \geq N,$$

and we use similar notation as above, replacing ∞ by $-\infty$.

Examples 2.7. (a) $\lim_{n\to\infty} n = \infty$. (Take $N = M$ in Definition 2.6.)

(b) $\lim_{n\to\infty}(-n) = -\infty$. (Take $N = M$ in Definition 2.6.)

(c) The sequence $\{(-1)^n\}_{n\in\mathbb{N}}$ diverges, but does not diverge to $\pm\infty$. (See Exercise 2.1.3.)

Proposition 2.8. *Suppose $\{a_n\}$ is a sequence of real numbers and $r \in \mathbb{N}$. Then the sequence $\{a_n\}_{n=1}^\infty$ converges if and only if the sequence $\{a_{n+r}\}_{n=1}^\infty$ converges. If these sequences converge, then*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+r}.$$

Proof. The proof of this proposition is left as an exercise (Exercise 2.1.2). □

Proposition 2.8 says that we can always ignore some finite number of terms at the beginning of a sequence when computing limits.

Definition 2.9 (Bounded sequence)**.** A sequence is *bounded* if its image is a bounded set. (Recall that a sequence is a function $\mathbb{N} \to \mathbb{R}$.) In order words, the sequence $\{a_n\}$ is bounded if there exists $M \in \mathbb{R}$ such that

$$|a_n| \leq M \quad \forall\, n \in \mathbb{N}.$$

(See Exercise 1.3.1.)

Proposition 2.10. *Every convergent sequence is bounded.*

Proof. Suppose $\{a_n\}$ converges to L. Taking $\varepsilon = 1$ in the definition of a limit, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|a_n - L| < 1 \implies |a_n| - |L| \leq |a_n - L| < 1 \implies |a_n| < |L| + 1.$$

Now let

$$M = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, |L| + 1\},$$

which exists since the maximum of any finite set exists. Then we have $|a_n| \leq M$ for all $n \in \mathbb{N}$, and so the sequence is bounded. □

One of the most useful formulations of Proposition 2.10 is its contrapositive: any sequence which is not bounded is divergent. This gives us one method of proving that sequences diverge. Note that the converse of Proposition 2.10 does *not* hold. For example, the sequence $\{(-1)^n\}_{n\in\mathbb{N}}$ is bounded but divergent.

Exercises.

For the following exercises, you should directly use the definition of a limit (Definition 2.2) and not any properties of limits you may have learned in other courses.

2.1.1. Show that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

2.1.2. Prove Proposition 2.8.

2.1.3. Prove that the sequence $\{(-1)^n\}_{n\in\mathbb{N}}$ diverges, but does not diverge to $\pm\infty$.

2.1.4. Prove that

$$\lim_{n\to\infty} \frac{-n^3 - 3n}{n^2 - 1} = -\infty.$$

2.1.5 ([TBB, Ex. 2.4.12]). Suppose $\{a_n\}$ is a sequence of *integers*. Under what conditions can such a sequence converge?

2.1.6 ([TBB, Ex. 2.4.14]). Show that the statement "$\{a_n\}$ converges to L" is false if and only if there is a positive number c such that the inequality

$$|a_n - L| > c$$

holds for infinitely many values of n.

2.1.7 ([TBB, Ex. 2.4.15]). If $\{a_n\}$ is a sequence of positive numbers converging to 0, show that $\{\sqrt{a_n}\}$ also converges to 0.

2.1.8 ([TBB, Ex. 2.4.16]). If $\{a_n\}$ is a sequence of positive numbers converging to L, show that $\{\sqrt{a_n}\}$ converges to $\sqrt{L}$.

2.1.9 ([TBB, Ex. 2.5.5]). Prove that if $a_n \to \infty$, then $(a_n)^2 \to \infty$ as well.

2.1.10 ([TBB, Ex. 2.5.6]). Prove that if $x_n \to \infty$, then the sequence $\left\{\frac{x_n}{1+x_n}\right\}$ converges. Is the converse true?

2.1.11 ([TBB, Ex. 2.5.7]). Suppose $\{a_n\}$ is a sequence of positive numbers converging to 0. Show that $\lim_{n\to\infty} 1/a_n = \infty$. Is the converse true?

2.1.12 ([TBB, Ex. 2.6.1]). Which of the following statements are true?

(a) If $\{a_n\}$ is unbounded, then either $\lim_{n\to\infty} a_n = \infty$ or $\lim_{n\to\infty} a_n = -\infty$.

(b) If $\{a_n\}$ is unbounded, then $\lim_{n\to\infty} |a_n| = \infty$.

(c) If $\{a_n\}$ and $\{b_n\}$ are both bounded, then so is $\{a_n + b_n\}$.

(d) If $\{a_n\}$ and $\{b_n\}$ are both unbounded, then so is $\{a_n + b_n\}$.

(e) If $\{a_n\}$ and $\{b_n\}$ are both bounded, then so is $\{a_n b_n\}$.

(f) If $\{a_n\}$ and $\{b_n\}$ are both unbounded, then so is $\{a_n b_n\}$.

(g) If $\{a_n\}$ is bounded, then so is $\{1/a_n\}$.

(h) If $\{a_n\}$ is unbounded, then $\{1/a_n\}$ is bounded.

2.1.13 ([TBB, Ex. 2.6.2]). Prove that if $\{a_n\}$ is bounded, then $\{a_n/n\}$ converges.

2.1.14. Prove that the converse of Proposition 2.10 does not hold.

2.1.15. (a) Use the fact that $n = \big(1 + (\sqrt[n]{n} - 1)\big)^n$ and the binomial theorem to prove that

$$n \geq \frac{n(n-1)}{2}(\sqrt[n]{n} - 1)^2.$$

(b) Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

2.2 Properties of limits

Proposition 2.11 (Arithmetic of limits). *Suppose* $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$, *and* $c \in \mathbb{R}$. *(Here* $A, B \in \mathbb{R}$.*)*

(a) $\lim_{n\to\infty} c = c$.

(b) $\lim_{n\to\infty}(ca_n) = cA$.

(c) $\lim_{n\to\infty}(a_n + b_n) = A + B$.

(d) $\lim_{n\to\infty}(a_n b_n) = AB$.

(e) If $A \neq 0$, *then* $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{A}$.

Proof. You should try to prove these statements yourself. To check your proofs, you can compare to [TBB, Th. 2.14–2.17]. □

Proposition 2.12. *Suppose* $\{a_n\}$ *and* $\{b_n\}$ *are convergent sequences and that*

$$a_n \leq b_n \quad \forall\, n \in \mathbb{N}.$$

Then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n.$$

Proof. Let $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$ and suppose $\varepsilon > 0$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that

$$|a_n - A| < \frac{\varepsilon}{2} \quad \forall\, n \geq N_1$$

and

$$|b_n - B| < \frac{\varepsilon}{2} \quad \forall\, n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$, we have

$$0 \leq b_n - a_n = B - A + (b_n - B) + (A - a_n) \leq B - A + |b_n - B| + |a_n - A| < B - A + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = B - A + \varepsilon.$$

Thus

$$A - B < \varepsilon.$$

This this holds for *all* positive real numbers ε, we must have $A - B \leq 0$, and so $A \leq B$ as desired. □

Remark 2.13. Note that the condition $a_n < b_n$ for all $n \in \mathbb{N}$ does *not* necessarily imply that $\lim a_n < \lim b_n$. See Exercise 2.2.1.

Corollary 2.14. *Suppose $\{a_n\}$ is a convergent sequence and that*

$$m \leq a_n \leq M \quad \forall\, n \in \mathbb{N}.$$

Then

$$m \leq \lim_{n\to\infty} a_n \leq M.$$

Proof. Consider the constant sequences $\{m\}_{n\in\mathbb{N}}$ and $\{M\}_{n\in\mathbb{N}}$ and apply Proposition 2.12. □

Theorem 2.15 (Squeeze Theorem)**.** *Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences, that*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

and that

$$a_n \leq x_n \leq b_n \quad \forall\, n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ is also convergent and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Proof. Let $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Let $\varepsilon > 0$ and choose $N_1, N_2 \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon \quad \forall\, n \geq N_1$$

and

$$|b_n - L| < \varepsilon \quad \forall\, n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. For all $n \geq N$, we have

$$-\varepsilon < a_n - L \leq x_n - L \leq b_n - L < \varepsilon.$$

Therefore, for all $n \geq \mathbb{N}$, we have

$$-\varepsilon < x_n - L < \varepsilon \implies |x_n - L| < \varepsilon.$$

□

Example 2.16. For all $n \in \mathbb{N}$, we have $-1 \le \cos n \le 1$. Thus

$$-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}.$$

Since $-\frac{1}{n} \to 0$ and $\frac{1}{n} \to 0$ as $n \to \infty$, we have

$$\frac{\cos n}{n} \to 0 \text{ as } n \to \infty.$$

Example 2.17 (Geometric sequence). Let $r \in \mathbb{R}$ and consider the *geometric sequence* $\{r^n\}_{n=1}^{\infty}$. Using the binomial theorem (or induction), one can show that

$$(1+x)^n > nx \quad \text{for } n \in \mathbb{Z},\ x > 0.$$

If $r > 1$, then $r = 1 + x$ for $x = r - 1 > 0$. Thus

$$r^n = (1+x)^n > nx \to \infty \text{ as } n \to \infty.$$

If $r \le -1$, then the series certainly diverges. If $r = 1$, then $\{r^n\}$ is a constant sequence, so it converges to 1.

Finally, we will show that

$$\lim_{n\to\infty} r^n = 0 \quad \text{if } -1 < r < 1.$$

If $0 < r < 1$, then

$$r = \frac{1}{1+x}, \qquad \text{where } x = \frac{1}{r} - 1 > 0,$$

and so

$$0 < r^n = \frac{1}{(1+x)^n} < \frac{1}{nx} \to 0 \text{ as } n \to \infty.$$

If $-1 < r < 0$, then $r = -t$ for $0 < t < 1$. Thus

$$-t^n \le r^n \le t^n.$$

Since we know from above that $t^n \to 0$, we can conclude from the Squeeze Theorem that $r^n \to 0$. The final remaining case is when $r = 0$, when it is clear that $r^n \to 0$.

Exercises.

2.2.1. Give an example of two sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n < b_n$ for all n, but such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

2.2.2. Suppose $|a_n| \le b_n$ for all $n \in \mathbb{N}$ and that $b_n \to 0$. Prove that $a_n \to 0$.

2.2.3 ([TBB, Ex. 2.7.5]). Which of the following statements are true?

(a) If $\{a_n\}$ and $\{b_n\}$ are both divergent, then so is $\{a_n + b_n\}$.

(b) If $\{a_n\}$ and $\{b_n\}$ are both divergent, then so is $\{a_n b_n\}$.

(c) If $\{a_n\}$ and $\{a_n + b_n\}$ are both convergent, then so is $\{b_n\}$.

(d) If $\{a_n\}$ and $\{a_n b_n\}$ are both convergent, then so is $\{b_n\}$.

(e) If $\{a_n\}$ is convergent, then so is $\{1/a_n\}$.

(f) If $\{a_n\}$ is convergent, then so is $\{(a_n)^2\}$.

(g) If $\{(a_n)^2\}$ is convergent, then so is $\{a_n\}$.

2.2.4 ([TBB, Ex. 2.8.2]). Suppose $\{a_n\}$ is a sequence all of whose values lie in the interval $[a, b]$. Prove that $\{a_n/n\}$ is convergent.

2.2.5 ([TBB, Ex. 2.8.6]). Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers, $a_n \to \infty$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \alpha$$

for some $\alpha \in \mathbb{R}$. What can you conclude?

2.3 Monotonic convergence criterion

Definition 2.18 (Increasing, decreasing, monotonic)**.** Suppose $\{a_n\}$ is a sequence of real numbers. We say that $\{a_n\}$ is *(weakly) increasing* if

$$a_n \le a_{n+1} \quad \forall\, n \in \mathbb{N},$$

and is *strictly increasing* if

$$a_n < a_{n+1} \quad \forall\, n \in \mathbb{N}.$$

We say that $\{a_n\}$ is *(weakly) decreasing* if

$$a_n \ge a_{n+1} \quad \forall\, n \in \mathbb{N},$$

and is *strictly decreasing* if

$$a_n > a_{n+1} \quad \forall\, n \in \mathbb{N}.$$

We say that $\{a_n\}$ is *monotonic* if it is either increasing or decreasing.

Theorem 2.19 (Monotonic Convergence Theorem)**.** *A monotonic sequence converges if and only if it is bounded. More specifically, we have the following:*

(a) If $\{a_n\}$ is increasing, then either $\{a_n\}$ is bounded and converges to $\sup\{a_n\}$ or else $\{a_n\}$ is unbounded and $a_n \to \infty$.

(b) If $\{a_n\}$ is decreasing, then either $\{a_n\}$ is bounded and converges to $\inf\{a_n\}$ or else $\{a_n\}$ is unbounded and $a_n \to -\infty$.

Proof. We will prove part (a) since the proof of part (b) is similar. By Proposition 2.10, any unbounded sequence diverges. Thus, it remains to prove that if $\{a_n\}$ is bounded and increasing, then it converges to $\sup\{s_n\}$.

Suppose $\{a_n\}$ is increasing and bounded. Then

$$L = \sup\{a_n\}$$

exists by the completeness axiom (Axiom 1.14). Thus $a_n \leq L$ for all a. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$L - \varepsilon < a_N \leq L$$

(see Exercise 1.3.10). Then, for all $n \geq N$, we have

$$L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon.$$

Thus, for all $n \geq N$,

$$|a_n - L| < \varepsilon.$$

So $a_n \to L$ as desired. □

Example 2.20. Consider the sequence $\{a_n\}$ with $a_n = 1/\sqrt{n}$. This sequence is decreasing and bounded below by 0. Therefore it converges. To find the limit, we need to do some more work. (See Exercise 2.1.1.)

Proposition 2.21 (Nested interval property)**.** *Suppose a set of intervals* $\{[a_n, b_n] : n \in \mathbb{N}\}$ *satisfies*

- $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ *for all* $n \in \mathbb{N}$ *(i.e. the intervals are* nested*) and*
- $\lim_{n\to\infty}(b_n - a_n) = 0$.

Then there exists $L \in \mathbb{R}$*, such that* $a_n \to L$*,* $b_n \to L$*, and*

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = \{L\}.$$

Proof. The sequence $\{a_n\}$ is increasing and is bounded above by b_1. Similarly, $\{b_n\}$ is decreasing and bounded below by a_1. Therefore, by Theorem 2.19,

$$a_n \to a := \sup\{a_n : n \geq 1\} \qquad \text{and} \qquad b_n \to b := \inf\{b_n : n \geq 1\}.$$

Now, by Proposition 2.11, we have

$$b - a = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty}(b_n - a_n) = 0.$$

Therefore $a = b$. Let $L = a$ and note that

$$a_n \leq a = L = b \leq b_n \quad \forall\, n \in \mathbb{N}.$$

So

$$L \in \bigcap_{n=1}^{\infty} [a_n, b_n].$$

Since $L = a = \sup\{a_n\}$, for any $x < L$, we have $x < a_m$ for some m. Thus $x \notin [a_m, b_m]$, and so $x \notin \bigcap_{n=1}^{\infty} [a_n, b_n]$. Similarly, for any $x > L$, we have $x \notin \bigcap_{n=1}^{\infty} [a_n, b_n]$. So

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = \{L\}. \qquad \square$$

Exercises.

2.3.1. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that $\{a_n\}$ is increasing, and $a_n \leq b_n$ for all n.

(a) Show that if $\{b_n\}$ converges, then $\{a_n\}$ converges.

(b) Show that if $\{a_n\}$ diverges, then $\{b_n\}$ diverges.

2.3.2 ([TBB, Ex. 2.9.2]). Define a sequence $\{t_n\}$ recursively by setting $t_1 = 1$ and

$$t_n = \sqrt{t_{n-1} + 1} \quad \text{for } n \geq 2.$$

Does this sequence converge? If so, to what?

2.4 Subsequences

Definition 2.22. Suppose

$$a_1, a_2, a_3, \ldots$$

is a sequence. A *subsequence* of $\{a_n\}$ is a sequence of the form

$$a_{n_1}, a_{n_2}, a_{n_3}, \ldots \quad \text{where } n_1 < n_2 < n_3 < \cdots.$$

In terms of a functions, a subsequence of a sequence $f \colon \mathbb{N} \to \mathbb{R}$ is a composition $f \circ g$, where $g \colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing function. (The relationship to the above is that g is the function defined by $g(k) = n_k$.)

Remark 2.23. It is important to understand the notation for subsequences. A subsequence of $\{a_n\}_{n\in\mathbb{N}}$ is usually written as $\{a_{n_k}\}_{k\in\mathbb{N}}$. This is really a function

$$\mathbb{N} \to \mathbb{R}, \quad k \mapsto a_{n_k}.$$

So it is k (not n_k, for instance) that ranges over the natural numbers in the subsequence. So, for example, the limit of the subsequence should be written $\lim_{k\to\infty} a_{n_k}$.

Example 2.24. The sequence

$$1, -1, 2, -2, 3, -3, 4, -4, 5, -5, \dots$$

has a subsequence

$$-1, -2, -3, -4, -5, \dots .$$

Here $n_1 = 2$, $n_2 = 4$, $n_3 = 6$, etc.

Example 2.25. Suppose $a_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Consider the strictly increasing function

$$\mathbb{N} \to \mathbb{N}, \quad k \mapsto n_k := 2k.$$

Then we have the corresponding subsequence

$$\{a_{n_k}\}_{k=1}^{\infty} = \left\{\frac{1}{2k}\right\}_{k=1}^{\infty} \quad \text{of} \quad \{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}.$$

Example 2.26. Suppose $a_n = \frac{n}{n+1}$ for $n \in \mathbb{N}$. Consider the strictly increasing functions

$$\mathbb{N} \to \mathbb{N}, \quad k \mapsto n_k := k! \quad \text{and}$$
$$\mathbb{N} \to \mathbb{N}, \quad k \mapsto m_k := k+1.$$

Then we have the corresponding subsequences

$$\{a_{n_k}\}_{k=1}^{\infty} = \left\{\frac{k!}{k!+1}\right\}_{k=1}^{\infty} \qquad \text{and} \qquad \{a_{m_k}\}_{k=1}^{\infty} = \left\{\frac{k+1}{k+2}\right\}_{k=1}^{\infty}.$$

Proposition 2.27 (Existence of monotonic subsequences)**.** *Every sequence contains a monotonic subsequence.*

Proof. We call the m-th element x_m of a sequence $\{x_n\}$ a turn-back point if

$$x_m \geq x_n \quad \forall\, n > m.$$

If there is an infinite subsequence of turn-back points $x_{m_1}, x_{m_2}, x_{m_3}, \dots$ (with $m_1 \leq m_2 \leq m_3 \leq \cdots$), then these form a decreasing sequence since

$$x_{m_1} \geq x_{m_2} \geq x_{m_3} \geq \cdots .$$

It remains to consider the case where there are only finitely many turn-back points. In this case, let x_M be the last turn-back point, so that x_n is not a turn-back point for any $n > M$. Thus, we have $x_m > x_n$ for some $m > n$. So we can choose $m_1 > M+1$ such that $x_{m_1} > x_{M+1}$, then $m_2 > m_1$ such that $x_{m_2} > x_{m_1}$, etc. This yields a strictly increasing subsequence

$$x_{M+1} < x_{m_1} < x_{m_2} < x_{m_3} < \cdots . \qquad \square$$

Theorem 2.28 (Bolzano–Weierstrass Theorem)**.** *Every bounded sequence contains a convergent subsequence.*

Proof. Suppose $\{a_n\}$ is a bounded sequence. By Proposition 2.27, it contains a monotonic subsequence, which is also bounded. By Theorem 2.19, this subsequence converges. □

Note that, for the Bolzano–Weierstrass Theorem, it is important that we are working within the real numbers. For example, it is *not* true that every bounded sequence of rational numbers has a subsequence that converges to a rational number. To see this, consider the sequence

$$1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ \ldots \tag{2.1}$$

formed by truncating the decimal expansion of $\sqrt{2}$. This is a sequence in $\mathbb{Q}$. But no subsequence converges to a rational number, since any subsequence converges in $\mathbb{R}$ to $\sqrt{2}$, which is not rational.

Exercises.

2.4.1. Prove that if a sequence converges to L, then all of its subsequences also converge to L.

2.4.2 ([TBB, Ex. 2.11.3]). If $\{s_{n_k}\}$ is a subsequence of $\{s_n\}$ and $\{t_{m_k}\}$ is a subsequence of $\{t_n\}$, then is it necessarily true that $\{s_{n_k} + t_{m_k}\}$ is a subsequence of $\{s_n + t_n\}$?

2.4.3 ([TBB, Ex. 2.11.6]). Which of the following statements are true?

(a) A sequence is convergent if and only if all of its subsequences are convergent.

(b) A sequence is bounded if and only if all of its subsequences are bounded.

(c) A sequence is monotonic if and only if all of its subsequences are monotonic.

(d) A sequence is divergent if and only if all of its subsequences are divergent.

2.4.4 ([TBB, Ex. 2.11.7]). Which of the following statements are true?

(a) If all monotonic subsequences of sequence $\{s_n\}$ are convergent, then $\{s_n\}$ is bounded.

(b) If all monotonic subsequences of sequence $\{s_n\}$ are convergent, then $\{s_n\}$ is convergent.

(c) If all convergent subsequences of a sequence $\{s_n\}$ converge to 0, then $\{s_n\}$ converges to 0.

(d) If all convergent subsequences of a sequence $\{s_n\}$ converge to 0 and $\{s_n\}$ is bounded, then $\{s_n\}$ converges to 0.

2.4.5 ([TBB, Ex. 2.11.8]). Where possible find subsequences that are monotonic and subsequences that are convergent for the following sequences:

(a) $\{(-1)^n n\}$

(b) $\{\sin(n\pi/8)\}$

(c) $\{n \sin(n\pi/8)\}$

(d) $\left\{\frac{n+1}{n}\sin(n\pi/8)\right\}$

(e) $\{1+(-1)^n\}$

2.4.6 ([TBB, Ex. 2.11.11]). Give an example of a sequence that contains subsequences converging to every natural number (and no other numbers).

2.4.7 ([TBB, Ex. 2.11.14]). Show that if $\{a_n\}$ has no convergent subsequences, then $|a_n| \to \infty$ as $n \to \infty$.

2.5 Cauchy sequences

It would be useful to have a characterization of convergence that does not involve the limit. Such a characterization should involve some sort of notion of the terms in a sequence becoming arbitrarily close together.

Definition 2.29 (Cauchy sequence)**.** A sequence $\{a_n\}$ is a *Cauchy sequence* if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon \quad \forall\, n, m \geq N.$$

Theorem 2.30 (Cauchy convergence criterion)**.** *A sequence $\{a_n\}$ is convergent if and only if it is a Cauchy sequence.*

Proof. First suppose that $\{a_n\}$ converges to L. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$|a_k - L| < \frac{\varepsilon}{2} \quad \forall\, k \geq N.$$

Thus, if $m, n \geq N$, we have

$$|a_n - a_m| \leq |a_n - L| + |L - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and so $\{a_n\}$ is a Cauchy sequence.

Now suppose $\{a_n\}$ is a Cauchy sequence. First we show that $\{a_n\}$ is bounded. Taking $\varepsilon = 1$ in the definition of a Cauchy sequence (Definition 2.29), there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < 1 \quad \forall\, n, m \geq N.$$

Thus, for all $n \geq N$, we have

$$|a_n| \leq |a_N| + |a_n - a_N| \leq |a_N| + 1.$$

Thus, if we set

$$M = \max\{|a_1|, |a_2|, \dots, |a_N|, |a_N| + 1\},$$

we have $|a_n| \le M$ for all $n \in \mathbb{N}$. So $\{a_n\}$ is bounded. (Compare this argument to the proof of Proposition 2.10.)

Now, by the Bolzano–Weierstrass Theorem (Theorem 2.28), $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$. Let $L = \lim_{k\to\infty} a_{n_k}$. We will show that $a_n \to L$ also. Let $\varepsilon > 0$. Since $\{a_n\}$ is a Cauchy sequence, we can choose N such that

$$|a_n - a_m| < \frac{\varepsilon}{2} \quad \forall\, m, n \ge N.$$

We can also choose K such that

$$|a_{n_k} - L| < \frac{\varepsilon}{2} \quad \forall\, k \ge K.$$

Now suppose $n \ge N$. Choose $k \ge K$ such that $n_k \ge N$. (We can do this since the function $k \mapsto n_k$ is strictly increasing.) Then

$$|a_n - L| \le |a_n - a_{n_k}| + |a_{n_k} - L| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $a_n \to L$ as $n \to \infty$. □

Remark 2.31. There is a very general setting, that of a *metric space*, where one has a notion of convergence. In *any* metric space, convergent sequences are Cauchy sequences. But there are metric spaces with Cauchy sequences that do not converge. For instance, this is true in $\mathbb{Q}$: the sequence (2.1) is a Cauchy sequence in $\mathbb{Q}$ but it does not converge in $\mathbb{Q}$. So, in general, convergence is stronger than the Cauchy property. Theorem 2.30, which says that the two notions are equivalent for sequences in $\mathbb{R}$, relies on the completeness axiom (Axiom 1.14).

Exercises.

2.5.1 ([TBB, Ex. 2.12.1]). Show directly that the sequence $\{1/n\}$ is a Cauchy sequence.

2.5.2 ([TBB, Ex. 2.12.2]). Show directly that if $\{a_n\}$ is a Cauchy sequence and $r \in \mathbb{R}$, then $\{ra_n\}$ is a Cauchy sequence.

2.5.3 ([TBB, Ex. 2.12.3]). Show directly that if $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, then so is $\{a_n + b_n\}$.

2.5.4. Show directly that if $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, then so is $\{a_n b_n\}$.

2.5.5 ([TBB, Ex. 2.12.4]). Consider the following condition:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ |a_{n+1} - a_n| < \varepsilon.$$

Show that every Cauchy sequence satisfies this condition. Find a sequence that satisfies this condition that is not a Cauchy sequence.

2.5.6 ([TBB, Ex. 2.12.7]). Show directly that if $\{a_n\}$ is a Cauchy sequence, then so is $\{|a_n|\}$.

2.6 Limit inferior and limit superior

Definition 2.32 (Limit superior)**.** The *limit superior* of a sequence $\{a_n\}$ is defined to be

$$\limsup_{n\to\infty} a_n = \inf\{\beta : \exists\, N \text{ such that } a_n < \beta\ \forall\, n \geq N\}.$$

(Intuitively, the limit superior is the infimum of the set of all β such that the sequence is *eventually* less than β.)

Definition 2.33 (Limit inferior)**.** The *limit inferior* of a sequence $\{a_n\}$ is defined to be

$$\liminf_{n\to\infty} a_n = \sup\{\alpha : \exists\, N \text{ such that } \alpha < a_n\ \forall\, n \geq N\}.$$

(Intuitively, the limit inferior is the supremum of the set of all α such that the sequence is *eventually* greater than α.)

Remark 2.34. In Definitions 2.32 and 2.33, we use the notational conventions introduced in Section 1.3 for suprema and infima of empty or unbounded sets. In particular,

$$\begin{aligned}\limsup_{n\to\infty} a_n = \infty \iff \{\beta : \exists\, N \text{ such that } a_n < \beta\ \forall\, n \geq N\} = \varnothing \\ \iff \{a_n\} \text{ has no upper bound}\end{aligned}$$

and

$$\limsup_{n\to\infty} a_n = -\infty \iff a_n \to -\infty.$$

Similarly

$$\begin{aligned}\liminf_{n\to\infty} a_n = -\infty \iff \{\alpha : \exists\, N \text{ such that } \alpha < a_n\ \forall\, n \geq N\} = \varnothing \\ \iff \{a_n\} \text{ has no lower bound}\end{aligned}$$

and

$$\liminf_{n\to\infty} a_n = \infty \iff a_n \to \infty.$$

Proposition 2.35. *For any sequence $\{a_n\}$, we have*

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

Proof. If $\liminf_{n\to\infty} a_n = -\infty$ or $\limsup_{n\to\infty} a_n = \infty$, the result is trivial. Otherwise, choose $\alpha, \beta \in \mathbb{R}$ such that

$$\limsup_{n\to\infty} a_n \le \beta \quad \text{and} \quad \alpha \le \liminf_{n\to\infty} a_n.$$

Then, by definition, there exists $N, M \in \mathbb{N}$ such that

$$a_n \le \beta \quad \forall\, n \ge N \quad \text{and} \quad \alpha \le a_n \quad \forall\, n \ge M.$$

Thus $\alpha \le \beta$. Since this is true for *all* $\beta \ge \limsup_{n\to\infty} a_n$, this implies that

$$\alpha \le \limsup_{n\to\infty} a_n.$$

And since this holds for *all* $\alpha \le \liminf_{n\to\infty} a_n$, we have

$$\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n.$$ □

The following proposition explains the notation "lim sup" and "lim inf".

Proposition 2.36. *If $\{x_n\}$ is a sequence of real numbers, then*

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}$$

and

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_n, x_{n+1}, x_{n+2}, \dots\}.$$

Proof. We will prove the statement for lim infs, since the statement for lim sups is similar, and can be found in [TBB, Th. 2.47].

Let

$$z_n = \inf\{x_n, x_{n+1}, x_{n+2}, \dots\}.$$

Then $z_n \le x_n$ for all n and so, by Exercise 2.6.4 we have

$$\liminf_{n\to\infty} z_n \le \liminf_{n\to\infty} x_n.$$

But since z_n is a increasing sequence, by Exercise 2.6.5 we have

$$\liminf_{n\to\infty} z_n = \lim_{n\to\infty} z_n.$$

Thus

$$\lim_{n\to\infty} \inf\{x_n, x_{n+1}, x_{n+2}, \dots\} \le \liminf_{n\to\infty} x_n.$$

It remains to show the reverse inequality. If $\liminf_{n\to\infty} x_n = -\infty$, then the sequence is unbounded below. Therefore, for all n, we have

$$\inf\{x_n, x_{n+1}, x_{n+2}, \dots\} = -\infty,$$

and thus

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_n, x_{n+1}, x_{n+2}, \dots\}$$

holds.

If

$$\liminf_{n\to\infty} x_n > -\infty,$$

then choose $\alpha \in \mathbb{R}$ such that $\alpha \le \liminf_{n\to\infty} x_n$. By definition, there exists $N \in \mathbb{N}$ such that $\alpha \le x_n$ for all $n \ge N$. Thus

$$\alpha \le \lim_{n\to\infty} \inf\{x_n, x_{n+1}, x_{n+2}, \dots\}.$$

Since this holds for *all* $\alpha \le \liminf_{n\to\infty} x_n$, we have

$$\liminf_{n\to\infty} x_n \le \lim_{n\to\infty} \inf\{x_n, x_{n+1}, x_{n+2}, \dots\},$$

proving the other inequality. □

Theorem 2.37. *Suppose $\{x_n\}$ is a sequence of real numbers. Then $\{x_n\}$ is convergent if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$ and these are finite. In this case,*

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \lim_{n\to\infty} x_n.$$

Proof. First suppose that $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = L \in \mathbb{R}$. Let $\varepsilon > 0$. Then there exist N_1, N_2 such that

$$x_n < L + \varepsilon \quad \forall\, n \ge N_1$$

and

$$x_n > L - \varepsilon \quad \forall\, n \ge N_2.$$

Let $N = \max\{N_1, N_2\}$. Then, for all $n \ge N$, we have

$$L - \varepsilon < x_n < L + \varepsilon.$$

Therefore $\lim_{n\to\infty} x_n = L$.

Conversely, suppose $\lim_{n\to\infty} x_n = L$. Let $\varepsilon > 0$. Then there exists an N such that

$$L - \varepsilon < x_n < L + \varepsilon \quad \forall\, n \ge N.$$

Therefore

$$L - \varepsilon \le \liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n \le L + \varepsilon.$$

Since ε was an arbitrary positive real number, this implies that

$$L = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n,$$

as required. □

Exercises.

2.6.1 ([TBB, Ex. 2.13.2]). Compute lim sup and lim inf for the following sequences:

(a) $\{(-1)^n n\}$

(b) $\{\sin(n\pi/8)\}$

(c) $\{n\sin(n\pi/8)\}$

(d) $\{1+(-1)^n\}$

2.6.2 ([TBB, Ex. 2.13.3]). Give examples of sequences of rational numbers $\{a_n\}$ with

(a) upper limit $\sqrt{2}$ and lower limit $-\sqrt{2}$,

(b) upper limit $+\infty$ and lower limit $\sqrt{2}$,

(c) upper limit π and lower limit e.

2.6.3 ([TBB, Ex. 2.13.3]). Show that

$$\limsup_{n\to\infty}(-x_n) = -\liminf_{n\to\infty} x_n.$$

2.6.4 ([TBB, Ex. 2.13.5]). If two sequences $\{a_n\}$ and $\{b_n\}$ satisfy the inequality $a_n \le b_n$ for all sufficiently large n (i.e. for $n \ge N$ for some fixed N), show that

$$\limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n \quad \text{and} \quad \liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n.$$

2.6.5 ([TBB, Ex. 2.13.8]). Show that if $\{a_n\}$ is a monotonic sequence, then

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n$$

(including the possibility of infinite limits).

2.6.6 ([TBB, Ex. 2.13.9]). Show that for any bounded sequences $\{a_n\}$ and $\{b_n\}$,

$$\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Give an example to show that the equality need not occur.

Chapter 3

Series

In this chapter, we discuss infinite sums, known as series. We begin with the precise definition of a series. The definition is in terms of limits of sequences and so we can use the properties of sequences we learned in Chapter 2 to deduce properties of series. We then discuss various tests that can be used to conclude that a series is convergent. Finally, we discuss the notion of absolute convergence, which is stronger than convergence. A good reference for the material of this chapter is [TBB, Ch. 3].

3.1 Definition and basic properties

The ordered sum of a sequence is called a *series* and we use the notation

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots .$$

Of course, one cannot in practice actually add an infinite number of real numbers. So we need to define precisely what we mean by such a sum.

Definition 3.1. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of real numbers. We write

$$\sum_{k=1}^{\infty} a_k = c$$

and say that the series *converges*, with sum c, if the sequence $\{s_n\}_{n=1}^{\infty}$, where

$$s_n = \sum_{k=1}^{n} a_k$$

(called the *sequence of partial sums* of the series) converges to c. If the series does not converge, it is said to *diverge*.

Definition 3.1 says that

$$\sum_{k=1}^{\infty} a_k := \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

Since series are defined in terms of sequences, the results of Chapter 2 immediately imply certain results about series.

Proposition 3.2. *If a series $\sum_{k=1}^{\infty} a_k$ converges, then the sum is unique.*

Proof. The proof of this proposition is left as an exercise. □

Proposition 3.3. *If the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge, and $c \in \mathbb{R}$, then the series*

$$\sum_{k=1}^{\infty}(a_k + b_k) \qquad \text{and} \qquad \sum_{k=1}^{\infty} ca_k$$

both converge and

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k.$$

Proof. The proof of this proposition is left as an exercise. □

Proposition 3.4. *If the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge and $a_k \leq b_k$ for all k, then*

$$\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k.$$

Proof. The proof of this proposition is left as an exercise. □

Proposition 3.5. *Let $r \in \mathbb{N}$. The series*

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

converges if and only if the series

$$\sum_{k=r+1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{k+r} = a_{r+1} + a_{r+2} + a_{r+3} + \ldots$$

converges.

Proof. The proof of this proposition is left as an exercise. □

Example 3.6 (Telescoping series). Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

Note that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Thus, the n-th partial sum is

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

(See Exercise 1.4.7.) Therefore

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{n(n+1)}\right) = 1.$$

This is an example of a *telescoping series*.

Example 3.7 (Harmonic series). Consider the *harmonic series*

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots .$$

One can show (Exercise 3.1.7) that

$$s_{2^n} = \sum_{k=1}^{2^n} \frac{1}{k} \geq 1 + \frac{n}{2} \quad \forall\, n.$$

Thus $\lim_{n\to\infty} s_n = \infty$, and so the harmonic series diverges.

Proposition 3.8. *If the series $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$.*

Proof. If s_n is the n-th partial sum of a convergent series $\sum_{k=1}^{\infty} a_k = c$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = c - c = 0.$$ □

The contrapositive of Proposition 3.8 gives us a method of proving that a series diverges. If the terms of a series do not approach zero, then the series diverges. Note, however, that the converse of Proposition 3.8 does *not* hold. For example, the terms of the harmonic series (Example 3.7) approach zero, but the series diverges.

Example 3.9 (Geometric series). When $r \neq 1$, we have

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

By Example 2.17, we know that the sequence $\{r^n\}$ converges to zero when $|r| < 1$. Therefore

$$\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} \sum_{k=0}^{n} r^k = \lim_{n\to\infty} \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r}$$

when $|r| < 1$. For $|r| \geq 1$, the series $\sum_{k=0}^{\infty} r^k$ diverges by Proposition 3.8 since $r^k \not\to 0$. The series $\sum_{k=1}^{\infty} r^k$ is called a *geometric series*.

Exercises.

3.1.1. Prove Propositions 3.2–3.5.

3.1.2 ([TBB, Ex. 3.4.3]). If $\sum_{k=1}^{\infty}(a_k+b_k)$ converges, what can you say about the series

$$\sum_{k=1}^{\infty} a_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} b_k\ ?$$

3.1.3 ([TBB, Ex. 3.4.4]). If $\sum_{k=1}^{\infty}(a_k+b_k)$ diverges, what can you say about the series

$$\sum_{k=1}^{\infty} a_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} b_k\ ?$$

3.1.4 ([TBB, Ex. 3.4.5]). If the series $\sum_{k=1}^{\infty}(a_{2k}+a_{2k-1})$ converges, what can you say about the series $\sum_{k=1}^{\infty} a_k$?

3.1.5 ([TBB, Ex. 3.4.6]). If the series $\sum_{k=1}^{\infty} a_k$ converges, what can you say about the series $\sum_{k=1}^{\infty}(a_{2k}+a_{2k-1})$?

3.1.6 ([TBB, Ex. 3.4.9]). If $\{s_n\}$ is a strictly increasing sequence of positive numbers, show that it is the sequence of partial sums of some series with positive terms.

3.1.7. If

$$s_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n},$$

show that $s_{2^n} \geq 1+n/2$ for all n. (See Example 3.7.)

3.1.8 ([TBB, Ex. 3.4.15]). Does the series

$$\sum_{k=1}^{\infty} \log\left(\frac{k+1}{k}\right)$$

converge or diverge?

3.1.9 ([TBB, Ex. 3.4.16]). Show that

$$\frac{1}{r-1} = \frac{1}{r+1}+\frac{2}{r^2+1}+\frac{4}{r^4+1}+\frac{8}{r^8+1}+\cdots$$

for all $r>1$. *Hint*: Note that

$$\frac{1}{r-1}-\frac{1}{r+1} = \frac{2}{r^2-1}.$$

3.1.10 ([TBB, Ex. 3.4.18]). Find the sum of the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+2)(k+4)}.$$

3.1.11 ([TBB, Ex. 3.4.20]). Find all values of x for which the following series converges and, in the cases where in converges, determine the sum:

$$x + \frac{x}{1+x} + \frac{x}{(1+x)^2} + \frac{x}{(1+x)^3} + \frac{x}{(1+x)^4} + \cdots .$$

3.2 Convergence tests

A sum whose terms alternate between being nonnegative and nonpositive is called an *alternating series*.

Proposition 3.10 (Alternating series test)**.** *Suppose $\{a_n\}$ is a decreasing sequence of positive real numbers converging to zero. Then*

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k$$

converges and the sum of the series lies between any two consecutive partial sums.

Proof. Since the a_k are nonnegative and decrease, we have

$$a_1 - a_2 = s_2 \le s_4 \le s_6 \le \cdots \le s_5 \le s_3 \le s_1 = a_1. \tag{3.1}$$

So the subsequences of even and odd partial sums are bounded monotonic sequences. So

$$\lim_{n\to\infty} s_{2n} \qquad \text{and} \qquad \lim_{n\to\infty} s_{2n-1}$$

exist. Since

$$s_{2n} - s_{2n-1} = -a_{2n} \to 0,$$

we can conclude that $\lim_{n\to\infty} s_n = L$ exists. It is clear from the inequalities (3.1) that the sum of the series lies between any two consecutive partial sums. □

Example 3.11 (Alternating harmonic series). Consider the *alternating harmonic series*

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

By Proposition 3.10, this series converges and the sum is somewhere between $s_1 = 1$ and $s_2 = 1/2$. We could get better approximations by computing further partial sums.

Proposition 3.12 (Boundedness criterion)**.** *Supose $\{a_n\}$ is a sequence of* nonnegative *real numbers. Then the series $\sum_{k=1}^{\infty} a_k$ converges if and only if its partial sums are bounded.*

Proof. Since $a_k \geq 0$ for all $k \in \mathbb{N}$, the partial sums form an increasing sequence. Thus, the result follows from Theorem 2.19. □

Proposition 3.13 (Cauchy convergence criterion)**.** *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$\left|\sum_{k=n}^{m} a_k\right| < \varepsilon \quad \textit{for all } N \leq n < m < \infty.$$

Proof. Let $s_n = \sum_{k=1}^{n} a_k$ be the n-th partial sum of the series. Then, by Theorem 2.30, we have that $\{s_n\}$ converges if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that,

$$\begin{aligned} & |s_k - s_\ell| < \varepsilon \quad \text{for all } k, \ell \geq N - 1 \\ \iff & \left|\sum_{k=n}^{m} a_k\right| = |s_m - s_{n-1}| < \varepsilon \quad \text{for all } N \leq n < m < \infty. \end{aligned}$$ □

Proposition 3.14 (Comparison test)**.** *Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences and that there exists N such that $0 \leq a_n \leq b_n$ for all $n \geq N$.*

(a) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(b) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Proof. By Proposition 2.8, we may assume that $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$.

(a) If $\sum_{k=1}^{\infty} b_k$ converges, then its partial sums are bounded above by some M, by Proposition 3.12. Since $0 \leq a_n \leq b_n$ for all n, the partial sums of $\sum_{k=1}^{\infty} a_k$ are also bounded above by M. Therefore, by Proposition 3.12, the series $\sum_{k=1}^{\infty} a_k$ converges.

(b) The proof of this part is left as an exercise (Exercise 3.2.2).

□

Example 3.15. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}. \tag{3.2}$$

Since

$$0 \leq \frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)} \quad \text{for all } n \in \mathbb{N},$$

and we know

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

converges by Example 3.7, we can conclude that (3.2) converges by the comparison test (Proposition 3.14). It then follows that

$$1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. Then, by the comparison test again, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

converges for all $k \geq 2$.

Remark 3.16. Note that, in the setup of Proposition 3.14, we *cannot* conclude anything if we only know that $\sum_{k=1}^{\infty} a_k$ converges or $\sum_{k=1}^{\infty} b_k$ diverges.

Proposition 3.17 (Ratio test)**.** *Suppose $\{a_n\}$ is a sequence of positive real numbers.*

(a) If

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1,$$

then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1,$$

then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. (a) Suppose

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = q < 1.$$

Let $r = \frac{q+1}{2}$, so that $q < r < 1$. There there are only finitely many n such that $a_{n+1}/a_n > r$. Hence there exists N such that

$$\frac{a_{n+1}}{a_n} \leq r \quad \forall\, n \geq N.$$

Then, for $n \geq N$, we have

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N = r^{n-N} a_N.$$

Then, by the comparison test (Proposition 3.14), comparing to the sequence

$$\sum_{n=1}^{\infty} \frac{a_N}{r^N} r^n = \frac{a_N}{r^N} \sum_{n=1}^{\infty} r^n,$$

which converges by Example 3.9, we see that $\sum_{n=1}^{\infty} a_n$ converges.

(b) Suppose

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = q > 1.$$

Then, as above, there exists $r > 1$ and $N \in \mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} \geq r > 1 \quad \forall\, n \geq N.$$

Then $a_{n+1} > a_n$ for all $n \geq N$, and so $a_n \not\to 0$. Thus $\sum_{n=1}^{\infty} a_n$ diverges by Proposition 3.8.

□

Remark 3.18. Note that Proposition 3.17 says nothing in the case where

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

For example

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \text{ converges, and } \frac{a_{n+1}}{a_n} = \frac{n}{n+2} \to 1,$$

while

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges, and } \frac{a_{n+1}}{a_n} = \frac{n}{n+1} \to 1.$$

Proposition 3.19 (Root test)**.** *Suppose $\{a_n\}$ is a sequence of positive real numbers.*

(a) If

$$\limsup_{n\to\infty} \sqrt[n]{a_n} < 1,$$

then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If

$$\liminf_{n\to\infty} \sqrt[n]{a_n} > 1,$$

then $\sum_{n=1}^{\infty} a_n$ diverges.

Sketch of proof. (a) Suppose

$$\limsup_{n\to\infty} \sqrt[n]{a_n} = r < 1.$$

Then, as in the proof of Proposition 3.17, there exists $s < 1$ and $N \in \mathbb{N}$ such that

$$0 \leq \sqrt[n]{a_n} \leq s < 1 \quad \forall n \geq N.$$

Then compare to the series $\sum_{n=1}^{\infty} s^n$.

(b) The proof of this part is similar.

□

Remark 3.20. As for the ratio test (Proposition 3.17), Proposition 3.19 says nothing in the case where

$$\lim_{n\to\infty} \sqrt[n]{a_n} = 1.$$

For example,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges, and } \sqrt[n]{\frac{1}{n^2}} = \frac{1}{n^{2/n}} \to 1,$$

while

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges, and } \sqrt[n]{\frac{1}{n}} \to 1.$$

Exercises.

3.2.1. Prove that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3+1}$$

converges.

3.2.2. Prove Proposition 3.14(b).

3.2.3 ([TBB, Ex. 3.5.1]). Suppose that $\sum_{k=1}^{\infty} a_k$ is a convergent series of positive terms. Show that $\sum_{k=1}^{\infty} a_k^2$ is convergent. Does the converse hold?

3.2.4. Suppose $s > 0$, and consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^s}.$$

When $s = 1$, this is the harmonic series, which diverges (Example 3.7).

(a) Show that the series diverges when $s < 1$.

(b) Let s_n be the n-th partial sum of the series. Show that

$$s_{2^{n+1}-1} < \sum_{j=0}^{n} \left(\frac{1}{2^{s-1}}\right)^j.$$

(c) Show that the series $\sum_{k=1}^{\infty} \frac{1}{k^s}$ converges when $s > 1$.

3.2.5 ([TBB, Ex. 3.4.7]). If the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge, what can you say about the series $\sum_{k=1}^{\infty} a_k b_k$?

3.2.6 ([TBB, Ex. 3.4.21]). Determine whether the series

$$\sum_{k=1}^{\infty} \frac{1}{a+kb}$$

converges or diverges, where a and b are positive real numbers.

3.2.7. Which of the following series converge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)^2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2+\sin^3(n+1)}{2^n+n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1 + \cos^2 n^3}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^3+1}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5^n}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$

(i) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5n+3n^3}{7n^3+2}\right)^n$

3.3 Absolute convergence

Proposition 3.21. *If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then so does the series $\sum_{k=1}^{\infty} a_k$.*

Proof. Suppose $\sum_{k=1}^{\infty} |a_k|$ converges. Then, by the Cauchy convergence criterion (Proposition 3.13), for every $\varepsilon > 0$, there exists N such that

$$\sum_{k=n}^{m} |a_k| < \varepsilon \quad \text{for all } N \leq n < m < \infty.$$

But then

$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| < \varepsilon \quad \text{for all } N \leq n < m < \infty.$$

Therefore, again by the Cauchy convergence criterion, the series $\sum_{k=1}^{\infty} a_k$ is convergent. □

Definition 3.22 (Absolutely convergent, nonabsolutely convergent)**.** The series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* (or *converges absolutely*) if $\sum_{n=1}^{\infty} |a_n|$ converges. If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ does not, we say that the series $\sum_{n=1}^{\infty} a_n$ is *nonabsolutely convergent*.

Examples 3.23. (a) Any geometric series $\sum_{n=1}^{\infty} r^n$ is absolutely convergent if $|r| < 1$ and divergent if $|r| \geq 1$, by Example 3.9.

(b) The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is nonabsolutely convergent by Examples 3.7 and 3.11.

Definition 3.24 (Rearrangement)**.** A *rearrangement* of the series $\sum_{n=1}^{\infty} a_n$ is a series of the form $\sum_{n=1}^{\infty} a_{f(n)}$ for some bijection $f\colon \mathbb{N} \to \mathbb{N}$.

Proposition 3.25. *(a) If a series is absolutely convergent, then every rearrangement is convergent, with the same sum.*

(b) If a series is nonabsolutely convergent, then for all $s \in \mathbb{R}$, *there exists a rearrangement of the series that converges to* s.

Proof. We will not prove this proposition in class. A proof can be found in [TBB, §3.7]. □

Proposition 3.25 tells us that the order of the terms in a series is very important, unless the series is absolutely convergent.

Exercises.

3.3.1 ([TBB, Ex. 3.5.3]). Suppose that the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are both absolutely convergent. Show that the series $\sum_{k=1}^{\infty} a_k b_k$ is also absolutely convergent. Does the converse hold?

3.3.2 ([TBB, Ex. 3.5.4]). Suppose that the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are both nonabsolutely convergent. Show that it does not follow that the series $\sum_{k=1}^{\infty} a_k b_k$ is convergent.

3.3.3 ([TBB, Ex. 3.5.11]). Show that a series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent if and only if every subseries $\sum_{k=1}^{\infty} a_{n_k}$ converges.

Chapter 4

Topology of $\mathbb{R}^d$

In this chapter will we consider the topology of $\mathbb{R}^d$. In particular, we will discuss the idea of a norm, which replaces the absolute value on $\mathbb{R}$ and allows us to define convergence in $\mathbb{R}^d$. We will then examine the concepts of open sets, closed sets, and compact sets. These concepts will be crucial in our later study of continuity. Many of the topics discussed in this chapter can be found in [TBB, Ch. 4], but that reference only considers the one-dimensional setting $d = 1$.

4.1 Norms

Recall that, for $d \in \mathbb{N}$,

$$\mathbb{R}^d = \{(x_1, x_2, \dots, x_d) : x_1, \dots, x_d \in \mathbb{R}\}.$$

When we write $x \in \mathbb{R}^d$ we will usually denote its components by $x_1, \dots, x_d$. That is, we will implicitly assume that $x = (x_1, \dots, x_d)$.

We have a binary operation of *vector addition* on $\mathbb{R}^d$:

$$(x_1, \dots, x_d) + (y_1, \dots, y_d) = (x_1 + y_1, \dots, x_d + y_d).$$

This addition is associative and commutative, with additive identity

$$0 = (0, 0, \dots, 0).$$

Note that we use the notation 0 to denote the real number zero as well as the element $0 \in \mathbb{R}^d$. The context should make it clear which one we mean.

We also have the operation of *scalar multiplication*

$$\alpha(x_1, \dots, x_d) = (\alpha x_1, \dots, \alpha x_d), \quad (x_1, \dots, x_d) \in \mathbb{R}^d,\ \alpha \in \mathbb{R},$$

and scalar multiplication is distributive over addition.

Definition 4.1. A *norm* on $\mathbb{R}^d$ is a function

$$\|\cdot\| \colon \mathbb{R}^d \to \mathbb{R}_+ := [0, \infty)$$

having the following properties:

N1 For $x \in \mathbb{R}^d$, $\|x\| = 0$ if and only if $x = 0$.

N2 We have $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$.

N3 We have $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{R}^d$ (*triangle inequality*).

Definition 4.2 (Euclidean norm)**.** The *euclidean norm* on $\mathbb{R}^d$ is defined by

$$\|(x_1, x_2, \dots, x_d)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}.$$

Note that $\|x\| = \sqrt{x \cdot x}$, where $x \cdot y$ denotes the *dot product* of $x, y \in \mathbb{R}^d$. Also note that, when $d = 1$, we have $\|x\| = |x|$ for $x \in \mathbb{R}^1 = \mathbb{R}$.

Of course, we have not yet proved that the euclidean norm is actually a norm in the sense of Definition 4.1. To prove this, we first need an important result.

Proposition 4.3 (Cauchy–Schwarz inequality)**.** *For all $x, y \in \mathbb{R}^d$, we have*

$$|x \cdot y| \leq \|x\| \, \|y\|, \tag{4.1}$$

where $\| \cdot \|$ denotes the euclidean norm.

Proof. Consider the quadratic function

$$q(t) = \|x + ty\|^2 = (x + ty) \cdot (x + ty) = (x \cdot x) + 2t(x \cdot y) + t^2(y \cdot y), \quad t \in \mathbb{R}.$$

Since $q(t) \geq 0$ for all $t \in \mathbb{R}$, its discriminant must be less than or equal to zero. Thus

$$4(x \cdot y)^2 - 4(x \cdot x)(y \cdot y) \leq 0 \implies |x \cdot y| \leq \|x\| \, \|y\|. \qquad \square$$

Proposition 4.4. *The euclidean norm satisfies conditions N1–N3.*

Proof. We will leave the verification of N1 and N2 as an exercise (Exercise 4.1.1). To prove that that N3 is satisfied, note that, for $x, y \in \mathbb{R}^d$,

$$\|x + y\|^2 = (x + y) \cdot (x + y) = x \cdot x + 2x \cdot y + y \cdot y \leq \|x\|^2 + 2\|x\| \, \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2,$$

where the inequality follows from the Cauchy–Schwarz inequality (4.1). Taking square roots then yields N3. $\square$

The euclidean norm will be our default norm on $\mathbb{R}^d$. Unless otherwise specified, this is the norm that we use. However, there are other norms on $\mathbb{R}^d$.

Example 4.5 (ℓ^∞-norm)*.* The ℓ^∞*-norm* on $\mathbb{R}^d$ is given by

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_d|\}.$$

We leave it as an exercise to verify that this satisfies N1–N3, and so is indeed a norm (Exercise 4.1.2).

Example 4.6 (ℓ^1-norm). The ℓ^1*-norm* on $\mathbb{R}^d$ is given by

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_d|.$$

We leave it as an exercise to verify that this satisfies N1–N3, and so is indeed a norm (Exercise 4.1.3).

Proposition 4.7. *Suppose $\|\cdot\|$ and $\|\cdot\|'$ are two norms on $\mathbb{R}^d$. Then there are positive real numbers α and β such that*

$$\alpha\|x\| \le \|x\|' \le \beta\|x\| \quad \forall\, x \in \mathbb{R}^d.$$

Proof. We will not prove this proposition in this course. It is typically done (in the more general setting of finite-dimensional vector spaces) in MAT 3120. A proof can be found in [Savb, Th. 5.3.11]. □

Although there are many norms on $\mathbb{R}^d$, Proposition 4.7 says that, in a certain sense, they are all equivalent. For example, it will follow from Proposition 4.7 that a sequence converges with respect to one norm if and only if it converges with respect to another, and a set is open with respect to one norm if and only if it is open with respect to another. See Remarks 4.15 and 4.28. However, for *infinite-dimensional* vector spaces, different norms can lead to vastly different properties.

Exercises.

4.1.1. Show that the euclidean norm (Definition 4.2) satisfies conditions N1 and N2 (see Proposition 4.4).

4.1.2. Show that the ℓ^∞-norm, as defined in Example 4.5, satisfies N1–N3.

4.1.3. Show that the ℓ^1-norm, as defined in Example 4.6, satisfies N1–N3.

4.1.4. Suppose $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^d$ and $\alpha \in \mathbb{R}$, $\alpha > 0$. Prove that $\|\cdot\|'$, defined by

$$\|x\|' = \alpha\|x\|, \quad x \in \mathbb{R}^d,$$

is also a norm on $\mathbb{R}^d$.

4.1.5. Suppose $\|\cdot\|$ and $\|\cdot\|'$ are two arbitrary norms on $\mathbb{R}^d$. Prove that $\|\cdot\|''$, defined by

$$\|x\|'' = \|x\| + \|x\|', \quad x \in \mathbb{R}^d,$$

is also a norm on $\mathbb{R}^d$.

4.2 Convergence in $\mathbb{R}^d$

We can now define convergence of sequences in $\mathbb{R}^d$ just as we did for sequences in $\mathbb{R}$, replacing the absolute value by the euclidean norm (or any other norm).

Definition 4.8 (Convergence in $\mathbb{R}^d$)**.** Let $\{x_k\}_{k=1}^\infty$ be a sequence in $\mathbb{R}^d$. We say that $\{x_k\}_{k=1}^\infty$ *converges* to $y \in \mathbb{R}^d$, and write $\lim_{k\to\infty} x_k = y$ or $x_k \to y$, if

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that } \forall\, k \geq N,\ \|x_k - y\| < \varepsilon.$$

Equivalently,

$$\lim_{k\to\infty} x_k = y \iff \lim_{k\to\infty} \|x_k - y\| = 0.$$

(Note that the statement $\lim_{k\to\infty} \|x_k - y\| = 0$ is a statement about limits of sequences of real numbers, since $\|x_k - y\|$ is a real number for each k.) If this condition is satisfied, we say that y is the *limit* of the sequence.

Example 4.9. Consider the sequence $\{x_n\}_{n=1}^\infty$ in $\mathbb{R}^4$ given by

$$x_n = \left(\frac{1}{n}, 2 - \frac{1}{n}, 1 + \frac{1}{n}, \frac{1}{n^2}\right).$$

We have

$$\begin{aligned}\|x_n - (0,2,1,0)\| = \sqrt{\left(\frac{1}{n}\right)^2 + \left(\frac{-1}{n}\right)^2 + \left(\frac{1}{n}\right)^2 + \left(\frac{1}{n^2}\right)^2} &= \sqrt{\frac{3}{n^2} + \frac{1}{n^4}} \\ &\leq \sqrt{\frac{4}{n^2}} = \frac{2}{n} \to 0.\end{aligned}$$

Thus $x_n \to (0,2,1,0)$.

If $\{x_k\}_{k=1}^\infty$ is a sequence in $\mathbb{R}^d$, then each $x_k \in \mathbb{R}^d$ is of the form

$$x_k = (x_{k,1}, x_{k,2}, \dots, x_{k,d}), \quad x_{k,i} \in \mathbb{R},\ i \in \{1, \dots, d\}.$$

So, for $i \in \{1, \dots, d\}$, we have the corresponding sequence of i-th components $\{x_{k,i}\}_{k=1}^\infty$, which is a sequence of real numbers.

Proposition 4.10. *A sequence $\{x_k\}_{k=1}^\infty$ in $\mathbb{R}^d$ converges to $y = (y_1, \dots, y_d)$ if and only if the sequence of real numbers $\{x_{k,i}\}_{k=1}^\infty$ converges to y_i for each $i \in \{1, \dots, d\}$.*

Proof. Suppose the sequence $\{x_k\}_{k=1}^\infty$ converges to $y \in \mathbb{R}^d$. Let $i \in \{1, \dots, d\}$. Then

$$|x_{k,i} - y_i| = \sqrt{(x_{k,i} - y_i)^2} \leq \sqrt{(x_{k,1} - y_1)^2 + \cdots + (x_{k,d} - y_d)^2} = \|x_k - y\| \to 0. \tag{4.2}$$

Thus, $|x_{k,i} - y_i| \to 0$ as $k \to 0$ by the Squeeze Theorem (Theorem 2.15). So $x_{k,i} \to y_i$ as $k \to \infty$.

Now suppose that, for each $i \in \{1, \ldots, d\}$, the sequence $\{x_{k,i}\}_{k=1}^\infty$ converges to some $y_i \in \mathbb{R}$. Thus

$$|x_{k,i} - y_i| \to 0 \text{ as } k \to \infty \quad \forall\, i \in \{1, \ldots, d\}.$$

Let $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$. Then, by the arithmetic of limits (Proposition 2.11), we have

$$(x_{k,1} - y_1)^2 + \cdots + (x_{k,d} - y_d)^2 \to 0 \text{ as } k \to \infty.$$

Thus, by Exercise 2.1.8, we have

$$\|x_k - y\| = \sqrt{(x_{k,1} - y_1)^2 + \cdots + (x_{k,d} - y_d)^2} \to 0.$$

Hence $x_k \to y$. □

Proposition 4.10 says that a sequence in $\mathbb{R}^d$ converges if and only if its components converge. So it reduces the notion of convergence in $\mathbb{R}^d$ to convergence in $\mathbb{R}$.

Definition 4.11 (Cauchy sequence in $\mathbb{R}^d$)**.** A sequence $\{x_k\}_{k=1}^\infty$ in $\mathbb{R}^d$ is a *Cauchy sequence* if

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that } \forall\, m, n \geq N,\ \|x_m - x_n\| < \varepsilon.$$

The following theorem is a higher dimensional analogue of Theorem 2.30.

Theorem 4.12 (Cauchy convergence criterion for $\mathbb{R}^d$)**.** *A sequence in $\mathbb{R}^d$ converges if and only if it is a Cauchy sequence.*

Proof. Suppose $\{x_k\}_{k=1}^\infty$ is a Cauchy sequence in $\mathbb{R}^d$. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$\|x_m - x_n\| < \varepsilon \quad \forall\, m, n \geq N.$$

Then, for each $i \in \{1, \ldots, d\}$, we have (as in (4.2))

$$|x_{m,i} - x_{n_i}| \leq \|x_m - x_n\| < \varepsilon \quad \forall\, m, n \geq N.$$

Thus, $\{x_{k,i}\}_{k=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$ for each i. Therefore, by Theorem 2.30, each such sequence converges. Thus $\{x_k\}_{k=1}^\infty$ converges by Proposition 4.10.

Now suppose $\{x_k\}_{k=1}^\infty$ is a convergent sequence in $\mathbb{R}^d$. Then, by Proposition 4.10, the sequence of components $\{x_{k,i}\}_{k=1}^\infty$ is a convergent sequence of real numbers for each $i \in \{1, \ldots, d\}$. So, by Theorem 2.30, $\{x_{k,i}\}_{k=1}^\infty$ is a Cauchy sequence for each i. Let $\varepsilon > 0$. Then, for each i, we can choose N_i such that

$$|x_{m,i} - x_{n,i}| < \varepsilon/\sqrt{d} \quad \forall\, n, m \geq N_i.$$

Let $N = \max\{N_1, N_2, \ldots, N_d\}$. Then, for all $n, m \geq N$, we have

$$\|x_m - x_n\| = \sqrt{(x_{m,1} - x_{n,1})^2 + \cdots + (x_{m,d} - x_{n,d})^2} < \varepsilon.$$

Therefore $\{x_k\}_{k=1}^\infty$ is a Cauchy sequence. □

Definition 4.13 (Bounded subset of $\mathbb{R}^d$)**.** A subset $E \subseteq \mathbb{R}^d$ is *bounded* if there exists $M \in \mathbb{N}$ such that

$$\|x\| \leq M \quad \forall\, x \in E.$$

Note that there is no natural notion of bounded *above* or bounded *below* for $\mathbb{R}^d$ since we have no natural order on $\mathbb{R}^d$.

The following theorem is a higher dimensional analogue of Theorem 2.28.

Theorem 4.14 (Bolzano–Weierstrass Theorem for $\mathbb{R}^d$)**.** *Every bounded sequence in $\mathbb{R}^d$ has a convergent subsequence.*

Proof. Suppose $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}^d$. So there exists $M \in \mathbb{N}$ such that

$$\|x_n\| \leq M \quad \forall\, n \in \mathbb{N}.$$

For each $i \in \{1, \dots, d\}$, we have

$$|x_{n,i}| \leq \|x_n\| \leq M \quad \forall\, n \in \mathbb{N}.$$

Thus, the sequences $\{x_{n,i}\}_{n=1}^{\infty}$ of i-th coordinates are bounded.

By the Bolzano–Weierstrass Theorem for $\mathbb{R}$ (Theorem 2.28), we can pick a subsequence $\{x_{n_k,1}\}_{k=1}^{\infty}$ of $\{x_{n,1}\}_{n=1}^{\infty}$ that converges. So $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ whose first coordinates converge. Repeating the above argument, we can now choose of subsequence of $\{x_{n_k}\}$ whose second coordinates converge. Continuing in this manner, we obtain a subsequence of $\{x_n\}$ such that the sequence of i-th coordinates converges for all $i \in \{1, \dots, d\}$. By Proposition 4.10, this subsequence converges in $\mathbb{R}^d$. □

Remark 4.15. We can modify Definition 4.8 by replacing the euclidean norm by any other norm. However, it follows from Proposition 4.7 that a sequence in $\mathbb{R}^d$ converges to y with respect to one norm if and only if it converges to y with respect to any other norm. So our notion of convergence is independent of our choice of norm. See Exercise 4.2.1.

Exercises.

4.2.1. Suppose $\{x_k\}_{k=1}^{\infty}$ is a sequence in $\mathbb{R}^d$ and that $\|\cdot\|'$ is an arbitrary norm on $\mathbb{R}^d$. Prove that $\{x_k\}_{k=1}^{\infty}$ converges to $y \in \mathbb{R}^d$ if and only if

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that } \forall\, k \geq N,\ \|x_k - y\|' < \varepsilon.$$

4.2.2. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are convergent sequences in $\mathbb{R}^d$ with limits w and z, respectively. Let $\alpha \in \mathbb{R}$.

(a) Prove that the sequence $\{x_n + y_n\}_{n=1}^{\infty}$ converges to $w + z$.

(b) Prove that the sequence $\{\alpha x_n\}_{n=1}^{\infty}$ converges to αw.

4.2.3. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^d$ converging to y. Prove that $\lim_{n\to\infty} \|x_n\| = \|y\|$.

4.2.4. Prove that every convergent sequence in $\mathbb{R}^d$ is bounded.

4.2.5. Prove that limits in $\mathbb{R}^d$ are unique.

4.3 Open and closed sets

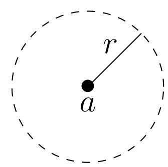
Definition 4.16 (Open ball)**.** Let $a \in \mathbb{R}^d$ and $r \in \mathbb{R}$ with $r > 0$. The *open ball* with centre a and radius r is

$$B(a,r) = \{x \in \mathbb{R}^d : \|x - a\| < r\}.$$

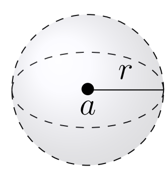
Example 4.17. If $d = 1$, then

$$B(a,r) = \{x \in \mathbb{R} : |x - a| < r\} = (a - r, a + r)$$

is an open interval. If $d = 2$, then $B(a,r)$ is an open disc.



If $d = 3$, then $B(a,r)$ is what you would normally think of as an open "ball". This is where the terminology "open ball" comes from.



If A is a subset of $\mathbb{R}^d$, its *complement* is

$$A^{\complement} := \{x \in \mathbb{R}^d : x \notin A\}.$$

Examples 4.18. (a) $B(a,r)^{\complement} = \{x \in \mathbb{R}^d : \|x - a\| \geq r\}$.

(b) $[0,4)^{\complement} = (-\infty, 0) \cup [4, \infty)$.

(c) For any $A \subseteq \mathbb{R}^d$, we have $\left(A^{\complement}\right)^{\complement} = A$.

Definition 4.19 (Open set, closed set)**.** A set $U \subseteq \mathbb{R}^d$ is *open* if

$$\forall\, x \in U,\ \exists\, r > 0 \text{ such that } B(x,r) \subseteq U.$$

A set $F \subseteq \mathbb{R}^d$ is *closed* if $F^{\complement}$ is open.

Example 4.20 (The open ball is open). Let $a \in \mathbb{R}^d$ and $r \in \mathbb{R}$, $r > 0$. We will show that the open ball $B(a,r)$ is open, justifying the name "open ball". Indeed, let $x \in B(a,r)$. Thus, by definition, $\|x-a\| < r$. Let $s = r - \|x-a\|$. So $s > 0$. It suffices to show that $B(x,s) \subseteq B(a,r)$. Let $y \in B(x,s)$. Thus $\|y-x\| < s$. Then, by the triangle inequality,

$$\|y-a\| \le \|y-x\| + \|x-a\| < s + \|x-a\| = r.$$

So $y \in B(x,s)$, as required.

Any open interval (a,b), $a < b$, is an open ball since

$$(a,b) = B\left(\frac{a+b}{2}, \frac{b-a}{2}\right).$$

Thus, it follows from Example 4.20 that open intervals are open.

Remark 4.21. Most sets are neither open nor closed. This is very important. Do *not* fall into the trap of thinking that if a set is not open, then it is closed, or vice versa. For instance, the sets

$$[a,b), \quad a < b,$$

are neither open nor closed (Exercise 4.3.4).

Proposition 4.22. *(a) $\varnothing$ is open.*

(b) $\mathbb{R}^d$ is open.

(c) If U_i, $i \in I$, are open subsets of $\mathbb{R}^d$ (here I is some arbitrary index set), then $\bigcup_{i\in I} U_i$ is open. In other words, arbitrary unions of open sets are open.

(d) If $U_1, \dots, U_n$ are open subsets of $\mathbb{R}^d$, then $\bigcap_{i=1}^n U_i$ is open. In other words, finite *intersections of open sets are open.*

Proof. We will prove part (c) and leave the other parts as an exercise (Exercise 4.3.5). Suppose I is an index set and $U_i \subseteq \mathbb{R}^d$ is open for each $i \in I$. Let $x \in \bigcup_{i\in I} U_i$. Then, $x \in U_j$ for some $j \in I$. Since U_j is open, there exists some $r > 0$ such that $B(x,r) \subseteq U_j \subseteq \bigcup_{i\in I} U_i$. Thus $\bigcup_{i\in I} U_i$ is open. □

Example 4.23. Suppose $a \in \mathbb{R}$. Then

$$(a,\infty) = \bigcup_{n=1}^{\infty} (a, a+n).$$

Since each open interval $(a, a+n)$ is open, it follows from Proposition 4.22(c) that (a,∞) is open. Similarly, $(-\infty, a)$ is open.

Example 4.24. Note that it is very important in Proposition 4.22(d) that the intersection is *finite*. For example,

$$\bigcap_{n=1}^{\infty} \left(\frac{-1}{n}, 1\right) = [0,1),$$

which is not open (or closed).

Example 4.25 (Closed intervals). Suppose $a, b \in \mathbb{R}$, $a < b$. Then

$$[a,b]^{\complement} = (-\infty, a) \cup (b, \infty),$$

which is open by Example 4.23 and Proposition 4.22(c). Thus $[a, b]$ is closed.

Proposition 4.26. *(a) $\varnothing$ is closed.*

(b) $\mathbb{R}^d$ is closed.

(c) If $F_1, \dots, F_n$ are closed subsets of $\mathbb{R}^d$, then $\bigcup_{i=1}^n F_i$ is closed. In other words, finite *unions of closed sets are closed.*

(d) If F_i, $i \in I$, are closed subsets of $\mathbb{R}^d$ (here I is some arbitrary index set), then $\bigcap_{i\in I} F_i$ is closed. In other words, arbitrary intersections of closed sets are closed.

Proof. We will prove part (d) and leave the other parts as an exercise (Exercise 4.3.6). Suppose F_i, $i \in I$, are closed subsets of $\mathbb{R}^d$. Then each $F_i^{\complement}$ is open. Hence

$$\left(\bigcap_{i\in I} F_i\right)^{\complement} = \bigcup_{i\in I} F_i^{\complement}$$

is open by Proposition 4.22(c). Therefore $\bigcap_{i\in I} F_i$ is closed. □

Remark 4.27. By Propositions 4.22 and 4.26, the sets $\varnothing$ and $\mathbb{R}^d$ are both open *and* closed.

Remark 4.28. We can modify the definition of an open ball by replacing the euclidean norm by any other norm. However, it follows from Proposition 4.7 that this would not change the definition of open sets. See Exercise 4.3.8.

Remark 4.29 (Topology). If X is a nonempty set, then a *topology* on X is a collection $\mathcal{T}$ of subsets of X such that

- $\varnothing \in \mathcal{T}$,
- $X \in \mathcal{T}$,
- arbitrary unions of elements of $\mathcal{T}$ are again elements of $\mathcal{T}$,
- finite intersections of element so $\mathcal{T}$ are again elements of $\mathcal{T}$.

Thus, Proposition 4.22 implies that the collection of open sets of $\mathbb{R}^d$ form a topology on $\mathbb{R}^d$. A pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a topology on X is called a *topological space*.

Theorem 4.30. *A set $F \subseteq \mathbb{R}^d$ is closed if and only if every sequence in F that converges (to some element of $\mathbb{R}^d$) has its limit in F.*

Proof. First suppose that F is closed. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in F and suppose $x_n \to y$. We wish to show that $y \in F$, which we will do by contradiction. Suppose $y \notin F$. Thus y is in $F^{\complement}$, which is open since F is closed. Therefore, there exists $\varepsilon > 0$ such that $B(y, \varepsilon) \subseteq F^{\complement}$. Since $x_n \to y$, there exists $N \in \mathbb{N}$ such that

$$\|x_n - y\| < \varepsilon \quad \forall\, n \geq N.$$

In particular, $\|x_N - y\| < \varepsilon$, and so

$$x_N \in B(y, \varepsilon) \subseteq F^{\complement}.$$

But this contradicts the fact that $x_N \in F$.

Now suppose that every convergent sequence in F has its limit in F. We wish to show that F is closed, which we do by contradiction. Suppose F is *not* closed. (Remember that this does *not* mean that F is open! See Remark 4.21.) Thus $F^{\complement}$ is *not* open. Therefore, there exists $y \in F^{\complement}$ such that

$$\forall\, r > 0,\ B(y, r) \not\subseteq F^{\complement}.$$

That is,

$$\forall\, r > 0,\ B(y, r) \cap F \neq \varnothing.$$

Therefore, for each $n \in \mathbb{N}$, we can choose $x_n \in B(y, 1/n) \cap F$. Then $\{x_n\}_{n=1}^{\infty}$ is a sequence in F and

$$\|x_n - y\| < \frac{1}{n} \to 0.$$

Therefore $x_n \to y \notin F$, which contradicts our assumption on F. □

Example 4.31. The interval $(0, 2)$ is not closed since $\{1/n\}_{n=1}^{\infty}$ is a sequence in $(0, 1)$ converging to $0 \notin (0, 2)$.

Definition 4.32 (Interior point, isolated point, accumulation point, boundary point)**.** Suppose $A \subseteq \mathbb{R}^d$.

(a) A point $x \in A$ is an *interior point* of A if there exists some $r > 0$ such that

$$B(x, r) \subseteq A.$$

(b) A point $x \in A$ is an *isolated point* of A if there exists some $r > 0$ such that

$$B(x, r) \cap A = \{x\}.$$

(c) A point $x \in \mathbb{R}^n$ is an *accumulation point* of A if, for all $r > 0$, the intersection

$$B(x, r) \cap A$$

contains infinitely many points. (Note that we do *not* require that $x \in A$.)

(d) A point $x \in \mathbb{R}^n$ is a *boundary point* of A if, for all $r > 0$, the open ball $B(x, r)$ contains at least one point of A and one point of $A^{\complement}$. (Note that we do *not* require that $x \in A$.)

Examples 4.33. (a) A set U is open if and only if every point of U is an interior point of U.

(b) In the open interval (a, b), $a < b$:

- Every point of (a, b) is an interior point.
- No point is an isolated point.
- The set of accumulation points is $[a, b]$.
- The points a and b are the only boundary points.

(c) In the closed interval $[a, b]$, $a < b$:

- The interior points are those $x \in \mathbb{R}$ satisfying $a < x < b$.
- No point is an isolated point.
- Every point of $[a, b]$ is an accumulation point.
- The points a and b are the only boundary points.

(d) In $\mathbb{Q}$:

- No point is an interior point.
- No point is an isolated point.
- Every point is an accumulation point.
- Every point is a boundary point.

(e) In $\mathbb{N}$:

- No point is an interior point.
- Every point of $\mathbb{N}$ is an isolated point.
- No point is an accumulation point.
- Every point of $\mathbb{N}$ is a boundary point.

(f) Suppose $a \in \mathbb{R}^d$, $r > 0$. Consider the open ball $B(a, r)$ and the *closed ball* $B^{\mathrm{cl}}(a, r) = \{x \in \mathbb{R}^d : \|x - a\| \le r\}$. In both:

- The interior points are the elements of $B(a, r)$.
- No points of $B(a, r)$ are isolated points.
- The accumulation points are the points of $B^{\mathrm{cl}}(a, r)$.
- The boundary points are the elements of the *hypersphere*

$$\{x \in \mathbb{R}^d : \|x - a\| = r\}.$$

Exercises.

4.3.1. Prove that a subset of $\mathbb{R}^d$ is bounded if and only if it is contained in some open ball. (For the reverse implication, remember that the ball need not be centred at the origin.)

4.3.2. (a) Prove that singletons $\{x\}$, $x \in \mathbb{R}^d$, are closed.

(b) Prove that finite sets are closed.

4.3.3. Let $a \in \mathbb{R}^d$ and $r > 0$. The *closed ball* with centre a and radius r is

$$B^{\mathrm{cl}}(a, r) = \{x \in \mathbb{R}^d : \|x - a\| \le r\}.$$

Prove that closed balls are closed.

4.3.4. Let $a, b \in \mathbb{R}$ with $a < b$. Using the definition of open and closed sets directly, prove that the interval $[a, b)$ is neither open nor closed.

4.3.5. Complete the proof of Proposition 4.22.

4.3.6. Complete the proof of Proposition 4.26.

4.3.7. Prove that the intervals $[a, \infty)$ and $(-\infty, a]$ are closed for all $a \in \mathbb{R}$.

4.3.8. Let $\| \cdot \|'$ be an arbitrary norm on $\mathbb{R}^d$ and, for $a \in \mathbb{R}^d$, $r > 0$, define

$$B'(a, r) = \{x \in \mathbb{R}^d : \|x - a\|' < r\}$$

(a) Given an example of $d \in \mathbb{N}$, a norm $\| \cdot \|'$, $a \in \mathbb{R}^d$, and $r > 0$, such that $B(a, r) \neq B'(a, r)$.

(b) Show that changing Definition 4.19 by replacing $B(x, r)$ by $B'(x, r)$ does not change the concept of an open set. More precisely, prove that a set $U \subseteq \mathbb{R}^d$ is open in the modified definition if and only if it is open in the original definition. See Remark 4.28.

4.3.9. Find all the interior points, isolated points, accumulation points, and boundary points of the set

$$\left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

4.3.10. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^d$ converging to y. Show that y is the only accumulation point of the sequence.

4.3.11 ([TBB, Ex. 4.4.10]). Write the closed interval $[0, 1]$ as an intersection of open sets. Can it also be expressed as a union of open sets?

4.3.12 ([TBB, Ex. 4.4.11]). Write the open interval $(0, 1)$ as a union of closed sets. Can it also be expressed as an intersection of closed sets?

4.3.13. Suppose that, for each $i \in \{1, \dots, d\}$, $A_i \subseteq \mathbb{R}$ is closed. Prove that

$$A_1 \times A_2 \times \cdots \times A_d := \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i \in A_i \ \forall \ i \in \{1, \dots, d\}\}$$

is a closed subset of $\mathbb{R}^d$.

4.4 Compact sets

In this section, we will discuss the notion of a *compact* set. We will see later that compact sets behave nicely under continuous functions.

Definition 4.34 (Open cover)**.** If $A \subseteq \mathbb{R}^d$, a collection $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ of open sets $U_\alpha \subseteq \mathbb{R}^d$ is called an *open cover* of A if

$$A \subseteq \bigcup_{\alpha \in I} U_\alpha = \{x \in \mathbb{R}^d : x \in U_\alpha \text{ for some } \alpha \in I\}.$$

Examples 4.35. (a) $\left\{\left(\frac{1}{n}, 1\right) : n \in \mathbb{N}\right\}$ is an open cover of $(0, 1)$.

(b) $\left\{\left(\frac{1}{n}, 2\right) : n \in \mathbb{N}\right\}$ is an open cover of $(0, 1)$.

(c) $\{(-x, x) : x > 0\}$ is an open cover of $\mathbb{R}$.

(d) If $a \in \mathbb{R}^d$, then $\{B(a, r) : r > 0\}$ is an open cover of $\mathbb{R}^d$.

Definition 4.36 (Compact set)**.** A set $K \subseteq \mathbb{R}^d$ is *compact* if for every open cover $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ of K, there exists $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n \in I$ such that

$$K \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_n}.$$

The collection $\{U_{\alpha_i} : i = 1, \dots, n\}$ is called a *finite subcover* of $\mathcal{U}$.

Example 4.37 (Finite sets are compact). Suppose K is a finite subset of $\mathbb{R}^d$. Let $\mathcal{U}$ be an open cover of K. Then, for each $x \in K$, there is some $U_x \in \mathcal{U}$ such that $x \in U_x$. Then

$$\{U_x : x \in K\}$$

is a finite subcover of K. So K is compact. Hence *finite sets are compact.* (Note that the empty set $\varnothing$ is also compact.)

Example 4.38. The set $\mathbb{R}$ is not compact. Indeed, the open cover

$$\{(-x, x) : x > 0\}$$

has no finite subcover. Note, however that $\{(-x, x) : x > 0\} \cup \{\mathbb{R}\}$ is an open cover that *does* have a finite subcover, namely $\{\mathbb{R}\}$. So it is important to remember that, in order for a set to be compact, *every* open cover must have a finite subcover.

Proposition 4.39. *A closed subset of a compact set is compact.*

Proof. Suppose K is a closed subset of a compact set $T \subseteq \mathbb{R}^d$. Let $\mathcal{U}$ be an open cover of K. Since K is closed, $K^{\complement}$ is open. Thus

$$\mathcal{U} \cup \{K^{\complement}\}$$

is an open cover of T. Since T is compact, it has a finite subcover $\mathcal{U}'$. Then $\mathcal{U}' \setminus \{K^{\complement}\}$ is a finite subcover of K. □

Theorem 4.40 (Heine–Borel Theorem)**.** *A set $K \subseteq \mathbb{R}^d$ is compact if and only if it is closed and bounded.*

Proof. Compact ⇒ (closed + bounded): Suppose that K is compact. Then the open cover

$$\{B(0, n) : n \in \mathbb{N}\}$$

of K has a finite subcover. So there exist $n_1, \ldots, n_k \in \mathbb{N}$ such that

$$K \subseteq B(0, n_1) \cup \cdots \cup B(0, n_k).$$

Let $M = \max\{n_1, \ldots, n_k\}$. Then

$$K \subseteq B(0, M),$$

and so K is bounded.

We now show that K is closed by showing that $K^{\complement}$ is open. Let $x \in K^{\complement}$. (Note that $K^{\complement} \neq \varnothing$ since K is bounded.) For all $y \in K$, we have $y \neq x$. Thus, we have

$$B(x, r_y) \cap B(y, r_y) = \varnothing, \quad \text{where } r_y = \frac{1}{2}\|x - y\| > 0.$$

The collection

$$\{B(y, r_y) : y \in K\}$$

is an open cover of K. Since K is compact, it has a finite subcover:

$$K \subseteq B(y_1, r_{y_1}) \cup \cdots \cup B(y_n, r_{y_n}).$$

Now let $r = \min\{r_{y_1}, \ldots, r_{y_n}\} > 0$. Then

$$B(x, r) \cap B(y_i, r_{y_i}) = \varnothing \quad \forall\, i \in \{1, \ldots, n\}.$$

Thus

$$B(x, r) \cap K = \varnothing \implies B(x, r) \subseteq K^{\complement}.$$

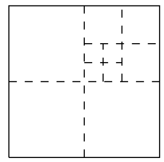
This proves that $K^{\complement}$ is open, hence K is closed.

(Closed + bounded) ⇒ compact: Now suppose K is closed and bounded. Since K is bounded there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in K$. Thus K is contained in a box:

$$K \subseteq T_0 := [-M, M]^d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : -M \leq x_i \leq M \ \forall\, i \in \{1, \ldots, d\}\}.$$

Since K is closed, it follows from Proposition 4.39 that it suffices to show that T_0 is compact. We do this by contradiction.

Assume that T_0 is not compact. Then there is an open cover $\mathcal{U}$ of T_0 that has no finite subcover. Bisecting each side of the box T_0, we can break it up into 2^d sub-boxes, each of which has side-length equal to half the side-length of T_0.



If each sub-box had a finite subcover, the union of these finite subcovers would be a finite subcover of T_0, which is a contradiction. Thus, we can choose one sub-box T_1 that does not have a finite subcover.

Again bisecting the sides of T_1, we can find a sub-box T_2 that does not have a finite subcover. Continuing in this manner, we obtain a sequence of nested sub-boxes:

$$T_0 \supseteq T_1 \supseteq T_2 \supseteq T_3 \supseteq \cdots,$$

where the side length of T_k is $2M/2^k$, which tends to zero as $k \to \infty$. For each k, choose $x_k \in T_k$. The sequence $\{x_k\}_{k=1}^{\infty}$ is Cauchy (since the side-lengths of the boxes tend to zero), so it must converge to some limit L. Since each T_n is closed by Exercise 4.3.13 and the sequence is eventually in T_n (precisely, $x_k \in T_n$ for all $k \geq n$), we see that $L \in T_n$ for all n.

Since $\mathcal{U}$ covers T_0 and $L \in T_0$, there is some $U \in \mathcal{U}$ such that $L \in U$. Since U is open, there exists some $r > 0$ such that $B(L, r) \subseteq U$. For large enough k, we have

$$T_k \subseteq B(L, r) \subseteq U.$$

But then $\{U\} \subseteq \mathcal{U}$ is a finite subcover of T_k, which is a contradiction. Thus T_0 is compact. □

Example 4.41. (a) The interval $[a, b]$, $a < b$, is closed and bounded. Hence it is compact.

(b) The interval $(a, b]$, $a < b$, is not closed. Hence it is not compact.

(c) The interval $[a, \infty)$, $a \in \mathbb{R}$, is not bounded. Hence it is not compact.

(d) For $a \in \mathbb{R}^d$ and $r > 0$, the closed ball $B^{\text{cl}}(a, r)$ is closed (Exercise 4.3.3) and bounded. Hence it is compact.

Definition 4.42 (Sequentially compact)**.** A subset $K \subseteq \mathbb{R}^d$ is *sequentially compact* if every sequence in K has an accumulation point in K. Equivalently, K is sequentially compact if every sequence in K has a subsequence that converges to point in K. (See Exercise 4.4.4.)

Example 4.43. The set

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$

is not sequentially compact. Indeed, consider the sequence $\{1/n\}_{n=1}^{\infty}$. This sequence converges to 0, hence every subsequence also converges to 0. Since $0 \notin A$, the set A is not compact.

Theorem 4.44. *Suppose $K \subseteq \mathbb{R}^d$. The following statements are equivalent:*

(a) K is compact.

(b) K is sequentially compact.

(c) K is closed and bounded.

Proof. By Theorem 4.40 it suffices to show that K is sequentially compact if and only if it is closed and bounded

First suppose K is closed and bounded. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in K. By the Bolzano–Weierstrass Theorem (Theorem 4.14), there is a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. Since K is closed, it then follows from Theorem 4.30 that the limit of this subsequence is an element of K. Hence K is sequentially compact.

For the reverse implication, we will prove the contrapositive. That is, we will show that if K is not closed or not bounded, then K is not sequentially compact. First consider the case where K is not closed. Then, as in the proof of Theorem 4.30, we can construct a sequence in K that converges to some point $y \in K^{\complement}$. But then every subsequence also converges to $y \notin K$. So K is not sequentially compact.

Now consider the case where K is not bounded. Thus, for all $n \in \mathbb{N}$, we can choose $x_n \in K$ such that $\|x_n\| > n$. Then every subsequence of $\{x_n\}_{n=1}^{\infty}$ is unbounded, and hence cannot converge. So K is not sequentially compact. □

In light of Theorem 4.44, we will use the terms *compact* and *sequentially compact* interchangeably.

Proposition 4.45. *(a) Finite unions of compact sets are compact. That is, a union of a finite number of compact sets is compact.*

(b) Arbitrary intersections of compact sets are compact.

Proof. We leave the proof of this proposition as an exercise (Exercise 4.4.2). □

Remark 4.46. In a general topological space (see Remark 4.29), compact is *not* the same as sequentially compact. In fact, neither property implies the other.

Exercises.

4.4.1. Give an open cover of the interval $(0, 2]$ with no finite subcover.

4.4.2. Prove Proposition 4.45.

4.4.3. Prove directly that a finite set is sequentially compact (without using the fact that a set is compact if and only if it is sequentially compact or that a set is compact if and only if it is closed and bounded).

4.4.4. Justify the use of the word "equivalently" in Definition 4.42. That is, show that if $K \subseteq \mathbb{R}^d$ then a sequence in K has an accumulation point in K if and only if it has a subsequence that converges to a point in K.

4.4.5. Give an example of an infinite collection of compact sets whose union is not compact.

4.4.6. Suppose $K_1, \dots, K_d$ are compact subsets of $\mathbb{R}$. Prove that

$$K_1 \times K_2 \times \cdots \times K_d := \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i \in K_i \ \forall \ i \in \{1, \dots, d\}\}$$

is a compact subset of $\mathbb{R}^d$. *Hint*: Use Exercise 4.3.13.

Chapter 5

Continuity

In this chapter we discuss the important concept of continuity of functions. We begin with the definition of a limit of a function at a point. We then give the precise definition of continuity and prove various properties of continuous functions. We conclude with a treatment of the stronger property of *uniform continuity*.

We suppose throughout this chapter that $A \subseteq \mathbb{R}^d$.

5.1 Limits

Definition 5.1 (Limit)**.** Let $f\colon A \to \mathbb{R}^m$ be a function. Suppose that a is an accumulation point of A and $L \in \mathbb{R}^m$. We write

$$\lim_{x\to a} f(x) = L$$

if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall\, x \in A \ \big(0 < \|x - a\| < \delta \implies \|f(x) - L\| < \varepsilon\big).$$

Note that the condition $0 < \|x - a\|$ in Definition 5.1 forces $x \neq a$. So the value of the function at a is *completely irrelevant* when computing the limit of $f(x)$ as $x \to a$. In fact, we do not even require that $a \in A$, as long as a is an accumulation point of A. So f may not even be defined at a!

Example 5.2. Consider the function

$$g\colon \mathbb{R} \to \mathbb{R}, \quad g(x) = x^2.$$

We will show that, for any $a \in \mathbb{R}$, $\lim_{x\to a} g(x) = a^2$. Let $a \in \mathbb{R}$ and $\varepsilon > 0$. First note that

$$|g(x) - a^2| = |x^2 - a^2| = |x + a| \cdot |x - a|.$$

We can force $|x - a|$ to be small, but what about $|x + a|$? Note that, if $|x - a| < 1$, then

$$|x + a| \le |x - a| + |2a| < 1 + 2|a|.$$

Thus, let

$$\delta = \min\left\{1, \frac{\varepsilon}{1 + 2|a|}\right\}.$$

Then,

$$|x-a|<\delta \implies |g(x)-a^2| = |x+a|\cdot|x-a| < (1+2|a|)\,\frac{\varepsilon}{1+2|a|} = \varepsilon.$$

Thus $\lim_{x\to a} g(x) = a^2$.

The following theorem gives an alternative characterization of limits in terms of sequences.

Theorem 5.3. *Let $f\colon A \to \mathbb{R}^m$ be a function and suppose that a is an accumulation point of A. Then*

$$\lim_{x\to a} f(x) = L$$

if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ in $A\setminus\{a\}$, with $x_n \to a$ as $n\to\infty$, we have

$$\lim_{n\to\infty} f(x_n) = L.$$

Proof. Suppose first that

$$\lim_{x\to a} f(x) = L$$

and that $\{x_n\}_{n=1}^{\infty}$ is a sequence in $A\setminus\{a\}$, with $x_n\to a$ as $n\to\infty$. Let $\varepsilon>0$. Then there exists $\delta>0$ such that

$$0<\|x-a\|<\delta \implies \|f(x)-L\|<\varepsilon.$$

Since $x_n\to a$ and $x_n\neq a$ for all n, we can choose $N\in\mathbb{N}$ such that

$$0<\|x_n-a\|<\delta \quad \forall\, n\geq N.$$

It follows that

$$\|f(x_n)-L\|<\varepsilon \quad \forall\, n\geq N.$$

Thus $f(x_n)\to L$.

Now suppose that

$$\lim_{x\to a} f(x) \neq L.$$

We will find a sequence of points $\{x_n\}$ in $A\setminus\{a\}$ converging to a such that $f(x_n)$ does not converge to L. Since $\lim_{x\to a} f(x)\neq L$, there exists some $\varepsilon>0$ such that for all $\delta>0$ there are points $x\in A$ with

$$0<\|x\_a\|<\delta \quad \text{and} \quad \|f(x)-L\|\geq\varepsilon.$$

In particular, taking $\delta=1/n$, $n\in\mathbb{N}$, we obtain a sequence of points $x_n\in A$ such that, for all $n\in\mathbb{N}$,

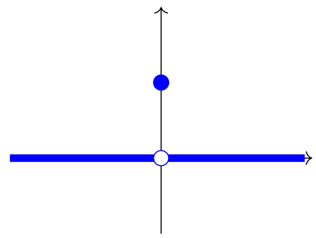
$$0<\|x_n-a\|<\frac{1}{n} \quad \text{and} \quad \|f(x_n)-L\|\geq\varepsilon.$$

So $x_n\to a$ but $f(x_n)\not\to L$. $\square$

Example 5.4. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

So the graph of f looks like:



Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R} \setminus \{0\}$. Then $f(x_n) = 0$ for all $n \in \mathbb{N}$. So $\{f(x_n)\}_{n=1}^{\infty}$ is the constant zero sequence, which converges to zero. Thus

$$\lim_{x\to 0} f(x) = 0.$$

Suppose $f\colon A \to \mathbb{R}^m$. Then the function f is given by

$$f(x) = \left(f_1(x), f_2(x), \dots, f_m(x)\right), \quad x \in A,$$

for some *component functions* $f_1, \dots, f_m\colon A \to \mathbb{R}$. In this case, we write $f = (f_1, \dots, f_m)$.

Proposition 5.5. *If $f = (f_1, \dots, f_m)\colon A \to \mathbb{R}^m$ and a is an accumulation point of A, then $\lim_{x\to a} f(x)$ exists if and only if*

$$\lim_{x\to a} f_i(x) \text{ exists for all } i \in \{1, \dots, m\}.$$

If these limits exist, then

$$\lim_{x\to a} f(x) = \left(\lim_{x\to a} f_1(x), \dots, \lim_{x\to a} f_m(x)\right).$$

Proof. For $y = (y_1, \dots, y_m) \in \mathbb{R}^m$, we have

$$\begin{aligned} \lim_{x\to a} f(x) = y &\iff \lim_{n\to\infty} f(x_n) = y \text{ for all sequences } x_n \to a && \text{(Th. 5.3)} \\ &\iff \lim_{n\to\infty} f_i(x_n) = y_i \text{ for all } x_n \to a \text{ and } i \in \{1, \dots, m\} && \text{(Prop. 4.10)} \\ &\iff \lim_{x\to a} f_i(x) = y_i \text{ for all } i \in \{1, \dots, m\}. && \text{(Th. 5.3)} \end{aligned}$$

□

Proposition 5.6. *Let $f, g\colon A \to \mathbb{R}^m$ and suppose that a is an accumulation point of A. If*

$$\lim_{x\to a} f(x) \qquad \textit{and} \qquad \lim_{x\to a} g(x)$$

both exist, then

$$\lim_{x\to a} \big((f(x) + g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $A \setminus \{a\}$ with $x_n \to a$. Then, by Proposition 2.11 and Theorem 5.3, we have

$$\lim_{n\to\infty} \big(f(x_n) + g(x_n)\big) = \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

Thus, by Theorem 5.3, the proposition follows. □

Proposition 5.7. *Let $f\colon A \to \mathbb{R}$, $g\colon A \to \mathbb{R}^m$, and suppose that a is an accumulation point of A. If*

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

both exist, then

$$\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right).$$

Proof. We leave the proof of this proposition as an exercise (Exercise 5.1.3). □

Proposition 5.8. *Let $f\colon A \to \mathbb{R}$ and suppose that a is an accumulation point of A. If $\lim_{x\to a} f(x)$ exists and is nonzero, then there exists $r > 0$ such that $f(x) \neq 0$ for all $x \in B(a,r) \cap A$, and*

$$\lim_{x\to a} \left(\frac{1}{f(x)}\right) = \frac{1}{\lim_{x\to a} f(x)}.$$

Proof. Suppose $\lim_{x\to a} f(x) = L \neq 0$. Choose $r > 0$ such that

$$\|x - a\| < r \implies |f(x) - L| < \frac{|L|}{2}.$$

Then

$$|L| \leq |L - f(x)| + |f(x)| \implies |f(x)| \geq |L| - |L - f(x)| \geq |L| - \frac{|L|}{2} = \frac{|L|}{2}.$$

Therefore, $f(x) \neq 0$ for $x \in B(a,r) \cap A$.

Now let $\varepsilon > 0$. Choose $\delta > 0$ such that $\delta < r$ and

$$0 < |x - a| < \delta \implies |f(x) - L| < \frac{\varepsilon |L|^2}{2}.$$

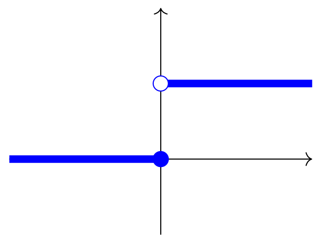
Then, for all x satisfying $0 < |x - a| < \delta$, we have

$$\left|\frac{1}{f(x)} - \frac{1}{L}\right| = \left|\frac{L - f(x)}{Lf(x)}\right| = \frac{|L - f(x)|}{|L| \cdot |f(x)|} \leq \frac{2|f(x) - L|}{|L|^2} < \varepsilon.$$ □

Consider the *step function*

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases} \tag{5.1}$$

So the graph of f looks like



It is not hard to see that $\lim_{x\to 0} f(x)$ does not exist. However, if we restrict f to $(-\infty, 0]$ then the limit *does* exist, and is equal to 0. On the other hand, if we restrict f to $[0, \infty)$, then the limit also exists, and is equal to 1. This inspires the following definitions of *one-sided limits*.

Definition 5.9 (Right-hand limit)**.** Suppose $E \subseteq \mathbb{R}$, $f\colon E \to \mathbb{R}$, and a is an accumulation point of $E \cap (a, \infty)$. Then we write

$$\lim_{x\to a^+} f(x) = L$$

if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x \in E,\ a < x < a + \delta \implies |f(x) - L| < \varepsilon.$$

In this case, we say that L is the *right-hand limit* of f at a.

Definition 5.10 (Left-hand limit)**.** Suppose $E \subseteq \mathbb{R}$, $f\colon E \to \mathbb{R}$, and a is an accumulation point of $E \cap (-\infty, a)$. Then we write

$$\lim_{x\to a^-} f(x) = L$$

if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x \in E,\ a - \delta < x < a \implies |f(x) - L| < \varepsilon.$$

In this case, we say that L is the *left-hand limit* of f at a.

Example 5.11. If f is the step function of (5.1), then

$$\lim_{x\to 0^-} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to 0^+} f(x) = 1.$$

Exercises.

5.1.1. Define

$$f\colon (0, \infty) \to \mathbb{R}, \quad f(x) = \frac{x+2}{x+1}, \quad x \in (-1, \infty).$$

Prove that $\lim_{x\to 1} f(x) = 3/2$ directly using the definition of a limit.

5.1.2. Prove that
$$\lim_{x\to 0} \frac{|x|}{x}$$
does not exist.

5.1.3. Prove Proposition 5.7.

5.1.4. Suppose $f, g\colon A \to \mathbb{R}^m$ and that a is an accumulation point of A. Show that if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then
$$\lim_{x\to a} \big(f(x)\cdot g(x)\big) = \Big(\lim_{x\to a} f(x)\Big)\cdot\Big(\lim_{x\to a} g(x)\Big),$$
where $\cdot$ denotes the dot product on $\mathbb{R}^m$.

5.1.5 ([TBB, Ex. 5.1.5]). Suppose that in Definition 5.1 we do not require that a is accumulation point of A. Show that then the limit statement $\lim_{x\to -2}\sqrt{x} = L$ would be true for *every* real number L.

5.1.6. Prove that the limit $\lim_{x\to 0} \frac{1}{x}$ does not exist.

5.1.7. Consider the *Dirichlet function*
$$f\colon \mathbb{R}\to\mathbb{R}, \quad f(x) = \begin{cases} 1 & \text{if } x\in\mathbb{Q}, \\ 0 & \text{if } x\notin\mathbb{Q}. \end{cases}$$
For which real numbers a does $\lim_{x\to a} f(x)$ exist?

5.1.8. Suppose $E\subseteq\mathbb{R}$, $f\colon E\to\mathbb{R}$, and a is an interior point of E. Prove that $\lim_{x\to a} f(x)$ exists if and only if the two one-sided limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist and are equal.

5.2 Definition of continuity

Definition 5.12 (Continuous)**.** The function $f\colon A\to\mathbb{R}^m$ is *continuous* at $a\in A$ provided that a is an isolated point of A or else that a is an accumulation point of A and
$$\lim_{x\to a} f(x) = f(a).$$
Equivalently, f is continuous at a if for all $\varepsilon > 0$ there exists $\delta > 0$ such that
$$x\in A,\ \|x-a\| < \delta \implies \|f(x)-f(a)\| < \varepsilon. \tag{5.2}$$
If f is not continuous at a, we say it is *discontinuous* at a. The function f is *continuous on* A (or simply *continuous*) if it is continuous at all $a\in A$.

Remark 5.13. (a) If $a\in A$ is an isolated point, then there exists some $\delta > 0$ such that $B(a,\delta)\cap A = \{a\}$. Then we trivially have
$$x\in A,\ |x-a| < \delta \implies x = a \implies |f(x)-f(a)| = 0 < \varepsilon.$$

(b) We will often write (5.2) as

$$|x-a|<\delta \implies |f(x)-f(a)|<\varepsilon,$$

leaving it implied that we only consider $x \in A$. (Otherwise $f(x)$ is not even defined.)

Recall that if $f\colon A \to \mathbb{R}^m$ and $S \subseteq A$, then the *image of S under f* is

$$f(S) := \{f(x) : x \in S\}.$$

(The *image of f* is $f(A)$.) Using this concept, we see that f is continuous at a if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f\left(B(a,\delta)\cap A\right) \subseteq B(f(a),\varepsilon).$$

Example 5.14. Fix $\lambda \in \mathbb{R}$ and $b \in \mathbb{R}^d$ and consider the function

$$f\colon \mathbb{R}^d \to \mathbb{R}^d, \quad f(x) = \lambda x + b.$$

We will show that this function is continuous. We split the proof into two cases: $\lambda = 0$ and $\lambda \neq 0$.

If $\lambda = 0$, then, for all $a \in \mathbb{R}^d$,

$$\|f(x)-f(a)\| = \|b-b\| = \|0\| = 0,$$

and so f is clearly continuous at a.

Now assume $\lambda \neq 0$. Let $a \in \mathbb{R}^d$ and $\varepsilon > 0$. Set $\delta = \varepsilon/|\lambda|$. Then, for all $x \in \mathbb{R}^d$ satisfying $\|x-a\| < \delta$, we have

$$\|f(x)-f(a)\| = \|(\lambda x + b) - (\lambda a - b)\| = \|\lambda(x-a)\| = |\lambda|\cdot\|x-a\| \leq |\lambda|\cdot\frac{\varepsilon}{|\lambda|} = \varepsilon.$$

Thus f is continuous at a. Since a was arbitrary, f is continuous.

Example 5.15. Consider the function

$$g\colon \mathbb{R} \to \mathbb{R}, \quad g(x) = x^2.$$

By Example 5.2, for any $a \in \mathbb{R}$, we have

$$\lim_{x\to a} g(x) = a^2 = g(a).$$

Thus g is continuous.

Example 5.16. Consider the step function of (5.1). We leave it as an exercise to show that f is continuous at a if $a \neq 0$ (Exercise 5.2.1). We will show that f is *not* continuous at 0. Negating the condition in Definition 5.12, we see that we need to show

$$\exists\, \varepsilon > 0 \text{ such that } \forall\, \delta > 0,\ \exists\, x \text{ such that } \big(|x| < \delta \text{ and } |f(x)| \geq \varepsilon\big).$$

Let $\varepsilon = 1$ and $\delta > 0$. Then, choosing $x = \delta/2$, we have

$$|x| < \delta \quad \text{and} \quad |f(x)| = |f(\delta/2)| = |1| \geq 1.$$

Theorem 5.17. *A function $f\colon A \to \mathbb{R}^m$ is continuous at $a \in A$ if and only if for all sequences $\{a_n\}_{n=1}^\infty$ in A with $a_n \to a$, we have $f(a_n) \to f(a)$.*

Proof. This follows immediately from Theorem 5.3. □

Theorem 5.17 gives another characterization of continuous functions. Namely, continuous functions are those functions that take convergent sequences to convergent sequences.

Example 5.18. Consider the square root function

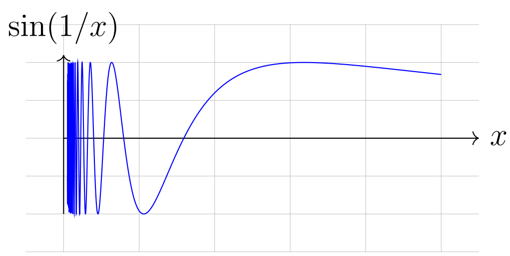
$$f\colon [0,\infty) \to \mathbb{R}, \quad f(x) = \sqrt{x}.$$

We could show that f is continuous using Definition 5.12. Alternatively, by Theorem 5.17, it is enough to show that if $\{a_n\}_{n=1}^\infty$ is a sequence converging to L, then $\{\sqrt{a_n}\}_{n=1}^\infty$ converges to $\sqrt{L}$. But we already did this in Exercise 2.1.8. Thus, the square root function is continuous.

Theorem 5.17 is often useful when we want to show that a function is *not* continuous, as the next example illustrates.

Example 5.19. Consider the function

$$f\colon [0,\infty) \to \mathbb{R}, \quad f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sin\frac{1}{x} & \text{if } x > 0. \end{cases}$$



To prove that this function is discontinuous at 0, it suffices, by Theorem 5.17, to find a sequence $\{a_n\}_{n=1}^\infty$ in $[0,\infty)$ converging to 0, such that $\{f(a_n)\}_{n=1}^\infty$ does not converge. For $n \in \mathbb{N}$, define

$$a_n = \frac{2}{(2n-1)\pi} > 0.$$

Then

$$f(a_n) = \sin\left(\frac{(2n-1)\pi}{2}\right) = (-1)^{n-1}.$$

Thus, the sequence $\{f(a_n)\}_{n=1}^\infty$ diverges, as required.

Exercises.

5.2.1. Prove that the step function of Example 5.16 is continuous at a if $a \neq 0$.

5.2.2. Let

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}.$$

(a) Define $f : A \to \mathbb{R}$ by

$$f(x) = \begin{cases} (-1)^n & \text{if } x = 1/n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that f is discontinuous at 0.

(b) Define $g : A \to \mathbb{R}$ by

$$g(x) = \begin{cases} \frac{(-1)^n}{n} & \text{if } x = 1/n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that g is continuous at 0.

5.2.3. Suppose $f : \mathbb{R}^d \to \mathbb{R}^m$. The *preimage* of a subset $B \subseteq \mathbb{R}^m$ under f is

$$f^{-1}(B) = \{a \in A : f(a) \in B\}.$$

Prove that f is continuous if and only if the preimage of every open subset of $\mathbb{R}^m$ is an open subset of $\mathbb{R}^d$. Prove also that f is continuous if and only if the preimage of every closed subset of $\mathbb{R}^m$ is a closed subset of $\mathbb{R}^d$.

5.2.4. Prove that the set

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^4 \leq 1\}$$

is compact.

5.2.5. At which $a \in \mathbb{R}$ is the function f of Exercise 5.1.7 continuous?

5.2.6. If $f : \mathbb{R} \to \mathbb{R}$, we can define the function $|f| : \mathbb{R} \to \mathbb{R}$ by $|f|(x) = |f(x)|$ for all $x \in \mathbb{R}$. If $|f|$ is continuous, does it follow that f is continuous?

5.3 Properties of continuous functions

The following result says that if a function f is continuous at a, then it is bounded near a.

Proposition 5.20. *If $f : A \to \mathbb{R}^m$ is continuous at $a \in A$, then there exists $r > 0$ and $M > 0$ such that*

$$\|f(x)\| \leq M \quad \forall\, x \in B(a, r) \cap A.$$

Proof. Taking $\varepsilon = 1$ in the definition of continuity, there exists some $\delta > 0$ such that, for all $x \in B(a, \delta) \cap A$, we have

$$\|f(x)\| \leq \|f(x) - f(a)\| + \|f(a)\| \leq 1 + \|f(a)\|.$$

So we can take $M = 1 + \|f(a)\|$. □

Recall that if $f, g\colon A \to \mathbb{R}^m$, then we can define the function

$$f+g\colon A \to \mathbb{R}^m, \quad (f+g)(x) = f(x)+g(x).$$

If $f\colon A \to \mathbb{R}$, $g\colon A \to \mathbb{R}^m$, then we can define

$$fg\colon A \to \mathbb{R}^m, \quad (fg)(x) = f(x)g(x).$$

If $f\colon A \to \mathbb{R}$ and $f(x) \neq 0$ for all $x \in A$, then we can define

$$\frac{1}{f}\colon A \to \mathbb{R}, \quad \left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}.$$

If $f\colon A \to \mathbb{R}^m$ and $B \subseteq A$, then we can define the *restriction*

$$f|_B\colon B \to \mathbb{R}^m, \quad f|_B(x) = x \quad \forall\, x \in B.$$

Sometimes we simply denote the restriction again by f when this should cause no confusion.

Proposition 5.21. *(a) If $f, g\colon A \to \mathbb{R}^m$ are continuous at $a \in A$, then $f+g$ is continuous at a.*

(b) If $f\colon A \to \mathbb{R}$ and $g\colon A \to \mathbb{R}^m$ are continuous at $a \in A$, then fg is continuous at a.

(c) If $f\colon A \to \mathbb{R}$ is continuous at a and $f(a) \neq 0$, then there exists $r > 0$ such that $f(x) \neq 0$ for all $x \in B(a,r) \cap A$, and $\frac{1}{f}$ is continuous at a.

Proof. (a) This follows from Proposition 5.6.

(b) This follows from Proposition 5.7.

(c) This follows from Proposition 5.8.

□

Theorem 5.22. *If $K \subseteq \mathbb{R}^d$ is compact and $f\colon K \to \mathbb{R}^m$ is continuous on K, then $f(K)$ is compact.*

Proof. Suppose $K \subseteq \mathbb{R}^d$ is compact and $f\colon K \to \mathbb{R}^m$ is continuous on K. Let $\{y_n\}_{n=1}^\infty$ be a sequence in $f(K)$. Then, for all $n \in \mathbb{N}$, there exists $x_n \in K$ such that $f(x_n) = y_n$. Since K is compact, the sequence $\{x_n\}_{n=1}^\infty$ in K has a convergent subsequence $\{x_{n_k}\}_{k=1}^\infty$. Let $z = \lim_{k\to\infty} x_{n_k}$. Since f is continuous at z, we have

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f\left(x_{n_k}\right) = f(z) \in f(K).$$

Therefore $\{y_{n_k}\}_{k=1}^\infty$ is a subsequence of $\{y_n\}_{n=1}^\infty$ that converges to a point of $f(K)$. Hence $f(K)$ is compact. □

Theorem 5.23 (Intermediate Value Theorem)**.** *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then for every c between $f(a)$ and $f(b)$, there exists $z \in [a,b]$ such that $f(z) = c$.*

Proof. We assume $f(a) < f(b)$. (If $f(b) < f(a)$, just consider the function $-f$.) If $c = f(a)$, take $z = a$, and if $c = f(b)$, take $z = b$. Therefore, we assume $f(a) < c < f(b)$. Let

$$X = \{s \in [a, b] : \forall\, x \in [a, s],\ f(x) < c\}.$$

Since $a \in X$, we have $X \neq \varnothing$. Also, X is bounded above by b. Thus, $z = \sup X$ exists by the completeness axiom (Axiom 1.14).

Suppose $f(z) < c$. Then $c - f(z) > 0$. Thus, since f is continuous at z, there exists $\delta > 0$ such that

$$|x - z| < \delta \implies |f(x) - f(z)| < c - f(z) \implies f(x) - f(z) < c - f(z) \implies f(x) < c.$$

In particular, we have

$$f(x) < c \quad \forall\, x \in [z, z + \delta/2].$$

But then $z + \delta/2 \in X$, which contradicts the fact that $z = \sup X$.

Now suppose that $f(z) > c$. Then $f(z) - c > 0$. Thus, since f is continuous at z, there exists $\delta > 0$ such that

$$|x - z| < \delta \implies |f(x) - f(z)| < f(z) - c \implies f(z) - f(x) < f(z) - c \implies f(x) > c.$$

In particular, $f(z - \delta/2) > c$, which contradicts the fact that $z = \sup X$.

Since the assumptions $f(z) < c$ and $f(z) > c$ both lead to contradictions, we must have $f(z) = c$, as required. □

Example 5.24. We can use the Intermediate Value Theorem to prove that certain polynomials have roots. For example, consider the polynomial

$$g(x) = x^5 - 15x - 1.$$

Then $g(0) = -1 < 1 = g(2)$. Since g is continuous (see Exercise 5.3.1), it follows from the Intermediate Value Theorem that $g(a) = 0$ for some $a \in (0, 2)$.

Corollary 5.25. *For all $\alpha \in \mathbb{R}_+$ and $n \in \mathbb{N}$, $\sqrt[n]{\alpha}$ exists. That is, there is some $\beta \in \mathbb{R}$ such that $\beta^n = \alpha$.*

Proof. If $n = 1$, then $\sqrt[n]{\alpha} = \alpha$. So we assume $n > 1$. If $\alpha = 0$, we can take $\beta = 0$. If $\alpha = 1$, we can take $\beta = 1$. So we assume $\alpha \neq 0, 1$. Consider the function

$$f(x) = x^n - \alpha.$$

Since f is a polynomial, it is continuous (see Exercise 5.3.1)

If $0 < \alpha < 1$, then

$$f(\alpha) < 0 < f(1),$$

and so, by the Intermediate Value Theorem, there exists $\beta \in (\alpha, 1)$ such that $\beta^n = \alpha$.

On the other hand, if $\alpha > 1$, then

$$f(1) < 0 < f(\alpha)$$

(since $\alpha^n > \alpha$ if $\alpha > 1$ and $n > 1$). Thus, by the Intermediate Value Theorem, there exists $\beta \in (1, \alpha)$ such that $\beta^n = \alpha$. □

Corollary 5.26. *For all $r \in \mathbb{Q}$ and $\alpha \in \mathbb{R}_+$, α^r exists.*

Proof. We leave the proof of this corollary as an exercise (Exercise 5.3.2). □

Definition 5.27 (Bounded function)**.** A function $f\colon A \to \mathbb{R}^m$ is *bounded* on A if its image $f(A)$ is a bounded subset of $\mathbb{R}^m$. That is, f is bounded if there exists $M > 0$ such that

$$\|f(x)\| \le M \quad \forall\, x \in A.$$

Theorem 5.28 (Maximum Theorem)**.** *Suppose K is a compact subset of $\mathbb{R}^d$ and $f\colon K \to \mathbb{R}$ is continuous on K. Then we have the following.*

(a) The function f is bounded on K.

(b) There exist $x_{\min}, x_{\max} \in K$ such that

$$f(x_{\min}) \le f(x) \le f(x_{\max}) \quad \forall\, x \in K.$$

In other words f attains a maximum and minimum*.*

Proof. (a) Since K is compact, Theorem 5.22 implies that $f(K)$ is a compact subset of $\mathbb{R}$. By the Heine–Borel Theorem (Theorem 4.40), $f(K)$ is therefore bounded.

(b) Since the image of f is bounded,

$$M := \sup f(K)$$

exists. By the definition of supremum, for each $n \in \mathbb{N}$ we can choose $x_n \in K$ such that

$$M - \frac{1}{n} < f(x_n) \le M.$$

Then $\{x_n\}_{n=1}^\infty$ is a sequence in K, which is compact. Therefore, it has a subsequence $\{x_{n_k}\}_{k=1}^\infty$ converging to some $x_{\max} \in K$. Then, since f is continuous, we have

$$f(x_{\max}) = f\left(\lim_{k\to\infty} x_{n_k}\right) = \lim_{n\to\infty} f(x_{n_k}) = M,$$

by the Squeeze Theorem. A similar argument shows that there is some $x_{\min} \in K$ such that $f(x_{\min}) = \inf f(K)$.

□

Proposition 5.29. *If $a < b$ and $f\colon [a,b] \to \mathbb{R}$ is continuous, then $f([a,b]) = [c,d]$ for some $c < d$.*

Proof. Since the interval $[a,b]$ is compact, it follows from the Maximum Theorem (Theorem 5.28) that there exist $x_{\min}, x_{\max} \in [a,b]$ such that

$$f(x_{\min}) \le f(x) \le f(x_{\max}) \quad \forall\, x \in [a,b].$$

Let $c = f(x_{\min})$ and $d = f(x_{\max})$. Thus

$$f([a,b]) \subseteq [c,d].$$

Now suppose, $y \in [c,d]$. Then, by the Intermediate Value Theorem, there exists x in between $x_{\min}$ and $x_{\max}$ such that $f(x) = y$. So, in fact,

$$f([a,b]) = [c,d]. \qquad \square$$

Example 5.30. Since polynomials are continuous (Exercise 5.3.1), every polynomial attains a maximum and minimum on a closed interval $[a,b]$, $a < b$.

Remark 5.31. (a) It is crucial that the domain K in the Maximum Theorem (Theorem 5.28) be compact. For instance, the functions

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) = x$$

and

$$g\colon (0,1) \to \mathbb{R}, \quad g(x) = \frac{1}{x},$$

do not attain a maximum or minimum.

(b) The condition that the function be continuous is also necessary. For instance, the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0, \end{cases}$$

does not attain a maximum.

(c) Even if a function is bounded, it need not attain a maximum. For instance, the functions

$$f\colon (0,1) \to \mathbb{R}, \quad f(x) = x,$$

and

$$g\colon [0,\infty) \to \mathbb{R}, \quad g(x) = \frac{-1}{1+x^2},$$

are bounded but do not attain a maximum.

Theorem 5.32 (Composition of continuous functions is continuous)**.** *Let $A \subseteq \mathbb{R}^d$, $B \subseteq \mathbb{R}^m$, and*

$$f\colon A \to \mathbb{R}^m, \qquad g\colon B \to \mathbb{R}^p,$$

be functions such that $f(A) \subseteq B$. Suppose f is continuous at $a \in A$ and g is continuous at $f(a)$. Then the composition

$$g \circ f\colon A \to \mathbb{R}^p, \quad (g \circ f)(x) = g(f(x)), \quad x \in A,$$

is continuous at a. If f is continuous on A and g is continuous on $f(A)$, then $g \circ f$ is continuous on A.

Proof. Let $\varepsilon > 0$. Since g is continuous at $f(a)$, there exists $\delta_1 > 0$ such that

$$y \in B,\ \|y - f(a)\| < \delta_1 \implies \|g(y) - g(f(a))\| < \varepsilon.$$

Since f is continuous at a, there exists $\delta > 0$ such that

$$x \in A,\ \|x - a\| < \delta \implies \|f(x) - f(a)\| < \delta_1.$$

Therefore, for all $x \in A$, we have

$$\|x-a\| < \delta \implies \|f(x)-f(a)\| < \delta_1 \implies \|(g\circ f)(x)-(g\circ f)(a)\| = \|g(f(x))-g(f(a))\| < \varepsilon.$$

Thus $g \circ f$ is continuous at a. The final statement of the theorem follows immediately. □

Remark 5.33. The proof of Theorem 5.32 can easily be modified to prove that if $A \subseteq \mathbb{R}^d$, $B \subseteq \mathbb{R}^m$,

$$f\colon A \to \mathbb{R}^m, \qquad g\colon B \to \mathbb{R}^p,$$

are functions such that $f(A) \subseteq B$, a is an accumulation point of A and $b := \lim_{x\to a} f(x)$ is an accumulation point of B, then

$$\lim_{x\to a}(g \circ f)(x) = \lim_{y\to b} g(y).$$

See Exercise 5.3.6.

Exercises.

5.3.1. Recall that a *polynomial function* is a function $p\colon \mathbb{R} \to \mathbb{R}$ of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

for some $n \in \mathbb{N}$ and $a_0, a_1, \dots, a_n \in \mathbb{R}$. Prove that all polynomial functions are continuous.

5.3.2. Prove Corollary 5.26.

5.3.3. Suppose K is a compact subset of $\mathbb{R}^d$ and, for all $k_1, k_2 \in K$, there exists a continuous function $p\colon [0,1] \to K$ such that $p(0) = k_1$ and $p(1) = k_2$. (A set with this property is said to be *path connected*.) Let $f\colon K \to \mathbb{R}$ be continuous on K. Prove that there exist $k_{\min}$ and $k_{\max}$ such that $f(K) = [f(k_{\min}), f(k_{\max})]$. *Hint*: If $p\colon [0,1] \to K$ and $f\colon K \to \mathbb{R}$ are continuous, then the composition $f \circ p\colon [0,1] \to \mathbb{R}$ is continuous.

5.3.4. Give an example of functions $f, g\colon \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x\to 0} f(g(x)) \qquad \text{and} \qquad f\left(\lim_{x\to 0} g(x)\right)$$

both exist but are not equal.

5.3.5. Suppose you know that the function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$, is continuous at some point a. Prove that it is continuous on $\mathbb{R}$. *Hint*: Use the fact that $e^{x+y} = e^x e^y$ for all $x, y \in \mathbb{R}$.

5.3.6. Prove the statement made in Remark 5.33.

5.4 Inverse functions

Recall that a function $f\colon X \to Y$ is *injective* (or is an *injection*) if, for all $x_1, x_2 \in X$,

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

It is *surjective* (or is a *surjection*) if

$$\forall\, y \in Y,\ \exists\, x \in X \text{ such that } f(x) = y.$$

It is *bijective* (or is a *bijection*) if it is both injective and surjective.

Let $A \subseteq \mathbb{R}$. Recall that a function $f\colon A \to \mathbb{R}$ is *strictly increasing* if, for all $x_1, x_2 \in \mathbb{R}$,

$$x_1 < x_2 \implies f(x_1) < f(x_2).$$

It is *strictly decreasing* if, for all $x_1, x_2 \in \mathbb{R}$,

$$x_1 < x_2 \implies f(x_1) > f(x_2).$$

Lemma 5.34. *If $f\colon [a,b] \to \mathbb{R}$ is continuous and injective, then f is either strictly increasing or strictly decreasing.*

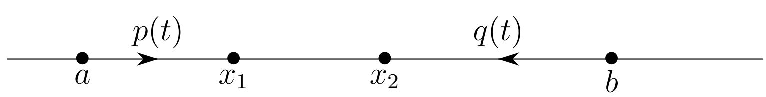
Proof. If $a = b$, there is nothing to prove. So we assume $a < b$. Since f is injective, we must have $f(a) \neq f(b)$. First assume that $f(a) < f(b)$.

Suppose $a < x_1 < x_2 < b$ and define

$$\begin{aligned} p&\colon [0,1] \to [a, x_1], & p(t) &= (1-t)a + t x_1, \\ q&\colon [0,1] \to [x_2, b], & q(t) &= (1-t)b + t x_2. \end{aligned}$$

Note that

$$p(t) \le x_1 < x_2 \le q(t) \quad \forall\, t \in [0,1]. \tag{5.3}$$



Since both p and q are continuous, and their codomains are contained in $[a,b]$, the function

$$k\colon [0,1] \to \mathbb{R}, \quad k(t) = f(q(t)) - f(p(t))$$

is continuous on $[0,1]$. Note that

$$k(0) = f(q(0)) - f(p(0)) = f(b) - f(a) > 0, \quad \text{and}$$

$$k(1) = f(q(1)) - f(p(1)) = f(x_2) - f(x_1).$$

Since $x_1 \neq x_2$ and f is injective, we have $k(1) \neq 0$. So either $k(1) > 0$ or $k(1) < 0$.

If $k(1) < 0$ then, by the Intermediate Value Theorem (Theorem 5.23) applied to k, there exists $c \in [0,1]$ such that

$$0 = k(c) = f(q(c)) - f(p(c)).$$

But since $q(c) > p(c)$ by (5.3), this contradicts the fact that f is injective. Hence $k(1) > 0$, which implies that $f(x_2) > f(x_1)$. So f is strictly increasing.

The proof in the case $f(a) > f(b)$, where one concludes that f is strictly decreasing, is analogous. □

Recall that an *inverse* to a function $f\colon X \to Y$ is a function $f^{-1}\colon Y \to X$ such that

$$f \circ f^{-1}(y) = y \quad \forall\, y \in Y \qquad \text{and} \qquad f^{-1} \circ f(x) = x \quad \forall\, x \in X.$$

A function f has an inverse function if and only if f is bijective. The inverse of f, if it exists, is unique.

Corollary 5.35. *If $f\colon [a,b] \to \mathbb{R}$ is continuous and injective, then there exist $c, d \in \mathbb{R}$, $c \leq d$, such that $f\colon [a,b] \to [c,d]$ is a bijection. Hence, there exists an inverse function $f^{-1}\colon [c,d] \to [a,b]$.*

Proof. Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous and injective. By Lemma 5.34, f is strictly increasing or strictly increasing. If f is strictly increasing, take $c = f(a)$ and $d = f(b)$. If f is strictly increasing, take $c = f(b)$ and $d = f(a)$. □

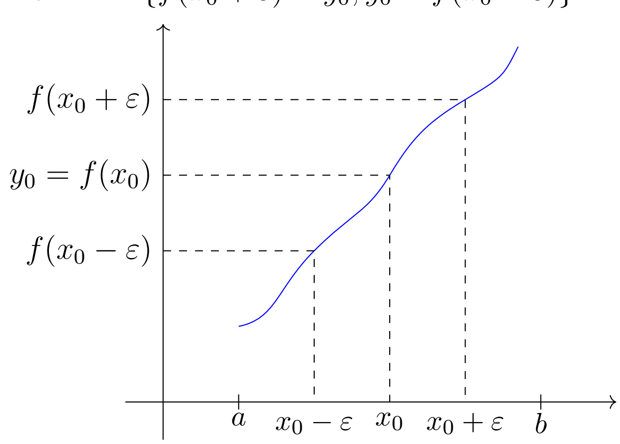
Theorem 5.36. *If $f\colon [a,b] \to [c,d]$ is a continuous bijection, then $f^{-1}\colon [c,d] \to [a,b]$ is also a continuous bijection.*

Proof. Suppose $f\colon [a,b] \to [c,d]$ is a continuous bijection. By Lemma 5.34, f is strictly increasing or strictly decreasing. We will assume f is strictly increasing, since the case where f is strictly decreasing is analogous (or one can consider $-f$).

We leave it as an exercise to prove that f^{-1} is also strictly increasing and bijective (Exercise 5.4.1). It remains to show that f^{-1} is continuous on $[c,d]$.

Let $y_0 \in [c,d]$, $x_0 = f^{-1}(y_0)$. We will show that f^{-1} is continuous at y_0. Let $\varepsilon > 0$, and set

$$\delta = \min\{f(x_0 + \varepsilon) - y_0, y_0 - f(x_0 - \varepsilon)\}.$$



Since f is increasing and $\varepsilon > 0$, we have that $\delta > 0$. We also have

$$\delta \leq f(x_0+\varepsilon) - f(x_0) \iff y_0 + \delta \leq f(x_0+\varepsilon), \qquad \text{and}$$
$$\delta \leq f(x_0) - f(x_0-\varepsilon) \iff f(x_0-\varepsilon) \leq y_0 - \delta.$$

Therefore,

$$|y - y_0| < \delta \implies f(x_0-\varepsilon) \leq y_0 - \delta < y < y_0 + \delta \leq f(x_0+\varepsilon).$$

In particular,

$$|y - y_0| < \delta \implies f(x_0-\varepsilon) < y < f(x_0+\varepsilon).$$

Applying f^{-1} to the terms of the right-hand inequality, and using the fact that f^{-1} is increasing, we have

$$|y - y_0| < \delta \implies x_0 - \varepsilon < f^{-1}(y) < x_0 + \varepsilon \implies |f^{-1}(y) - x_0| < \varepsilon.$$

So f^{-1} is continuous at y_0. □

Example 5.37. Let $p, n \in \mathbb{N}$ and define

$$f\colon [0,n] \to [0,n^p], \quad f(x) = x^p.$$

Then f is continuous and bijective (Exercise 5.4.2). Hence

$$f^{-1}\colon [0,n^p] \to [0,n],$$

which is the function $f^{-1}(x) = \sqrt[p]{x}$, is continuous. Since $n \in \mathbb{N}$ was arbitrary, $f^{-1}\colon [0,\infty) \to [0,\infty)$ is continuous on $[0,\infty)$.

Exercises.

5.4.1. Suppose $f\colon [a,b] \to [c,d]$ is a strictly increasing bijective function with inverse f^{-1}. Prove that f^{-1} is also strictly increasing and bijective.

5.4.2. Let $p, n \in \mathbb{N}$. Prove that the function

$$f\colon [0,n] \to [0,n^p], \quad f(x) = x^p.$$

is continuous and bijective.

5.5 Uniform continuity

Recall (Definition 5.12) that $f\colon A \to \mathbb{R}^m$ is continuous on A if

$$\forall\, a \in A,\ \forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that } \forall\, x \in A,\ \big(\|x - a\| < \delta \implies \|f(x) - f(a)\| < \varepsilon\big).$$

Note here that δ can depend on both ε and a. That is, *given* an $a \in A$ and $\varepsilon > 0$, we need a δ satisfying a certain condition. Different ε and a could require different δ to satisfy the condition. However, *sometimes*, given an $\varepsilon > 0$, the same δ works for *all* $a \in A$.

Example 5.38. Consider the function

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x.$$

Let $\varepsilon > 0$. Set $\delta = \varepsilon/2$. Then, for *any* $a \in A$, we have

$$|x - a| < \delta \implies |f(x) - f(a)| = |2x - 2a| = 2|x - a| < 2\delta = \varepsilon.$$

Note that the choice of δ does *not* depend on a.

Definition 5.39 (Uniformly continuous)**.** A function $f\colon A \to \mathbb{R}^m$ is *uniformly continuous* on A if

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that } \forall\, x, y \in A\ \big(\|x - y\| < \delta \implies \|f(x) - f(y)\| < \varepsilon\big).$$

Remark 5.40. It is immediate from the definitions that uniformly continuous functions are continuous.

Taking the negation of the condition in Definition 5.39, we see that to prove that a function $f\colon A \to \mathbb{R}^m$ is *not* uniformly continuous, we need to prove

$$\exists\, \varepsilon > 0 \text{ such that } \forall\, \delta > 0,\ \exists\, x, y \in \mathbb{R}\ \big(\|x - y\| < \delta \text{ and } \|f(x) - f(y)\| \geq \varepsilon\big). \tag{5.4}$$

Example 5.41. Consider the function

$$g\colon \mathbb{R} \to \mathbb{R}, \quad g(x) = x^2.$$

This function is continuous but *not* uniformly continuous on $\mathbb{R}$. Indeed let us prove that (5.4) is satisfied. Let $\varepsilon = 1$ and $\delta > 0$. Choose

$$x = \frac{1}{\delta} \qquad \text{and} \qquad y = \frac{1}{\delta} + \frac{\delta}{2}.$$

Then

$$|x - y| = \frac{\delta}{2} < \delta \quad \text{and}$$
$$|g(x) - g(y)| = |x^2 - y^2| = |x + y| \cdot |x - y| = \left(\frac{2}{\delta} + \frac{\delta}{2}\right)\frac{\delta}{2} = 1 + \frac{\delta^2}{4} > 1 = \varepsilon.$$

Example 5.42. The function

$$h\colon [a,b] \to \mathbb{R}, \quad g(x) = x^2$$

is uniformly continuous on $[a,b]$. Indeed, let $K = \max\{|a|,|b|\}$. Then

$$|x| \le K \quad \forall\, x \in [a,b].$$

Let $\varepsilon > 0$. Choose $\delta = \varepsilon/2K$. Then, for all $x, y \in [a,b]$ such that $|x-y| < \delta$, we have

$$|x^2 - y^2| = |x+y| \cdot |x-y| \le (|x| + |y|)\,|x-y| < 2K\delta = \varepsilon.$$

So the function $x \mapsto x^2$ is uniformly continuous on $[a,b]$ for *any* $a, b \in \mathbb{R}$, even though it is not uniformly continuous on $\mathbb{R}$ by Example 5.41.

The key difference between Examples 5.41 and 5.42 is that $[a,b]$ is compact, while $\mathbb{R}$ is not. This leads us to the following important theorem.

Theorem 5.43. *If $K \subseteq \mathbb{R}^d$ is compact and $f\colon K \to \mathbb{R}^m$ is continuous on K, then f is* uniformly *continuous on K.*

Proof. Suppose $K \subseteq \mathbb{R}^d$ is compact and $f\colon K \to \mathbb{R}^m$ is continuous on K. Let $\varepsilon > 0$. Since f is continuous on K, for each $k \in K$, there exists $\delta_k > 0$ such that

$$x \in K,\ \|x - k\| < 2\delta_k \implies \|f(x) - f(k)\| < \frac{\varepsilon}{2},$$

or equivalently that

$$f\big(K \cap B(k, 2\delta_k)\big) \subseteq B\big(f(k), \varepsilon/2\big). \tag{5.5}$$

Consider the open cover

$$\{B(k,\delta_k) : k \in K\}$$

of K. Since K is compact, it has a finite subcover. So there exists $k_1, \dots, k_N \in K$ such that

$$K \subseteq B(k_1, \delta_{k_1}) \cup \dots \cup B(k_N, \delta_{k_N}).$$

Let $\delta = \min\{\delta_{k_1}, \dots, \delta_{k_N}\}$.

Now suppose $x, y \in K$ and $\|x - y\| < \delta$. Since $x \in K$, we have

$$x \in B(k_j, \delta_{k_j}) \quad \text{for some } j \in \{1, \dots, N\}.$$

Moreover,

$$\|k_j - y\| = \|k_j - x + x - y\| \le \|k_j - x\| + \|x - y\| < \delta_{k_j} + \delta \le 2\delta_{k_j}.$$

Therefore both points $x, y \in B(k_j, 2\delta_{k_j})$. Therefore, by (5.5),

$$\|f(x) - f(y)\| \le \|f(x) - f(k_j)\| + \|f(k_j) - f(y)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So we have proven that

$$\|x - y\| < \delta \implies \|f(x) - f(y)\| < \varepsilon,$$

as required. □

Exercises.

5.5.1. Generalize Example 5.38 by showing that all linear functions from $\mathbb{R}$ to $\mathbb{R}$ are uniformly continuous. That is, if $a, b \in \mathbb{R}$, show that the function

$$f \colon \mathbb{R} \to \mathbb{R}, \quad f(x) = ax + b,$$

is uniformly continuous.

5.5.2. Suppose that X is a finite subset of $\mathbb{R}^d$. Show that every function $f \colon X \to \mathbb{R}^m$ is uniformly continuous.

5.5.3 ([TBB, Ex. 5.6.8]). Let $f \colon A \to \mathbb{R}^m$ be a uniformly continuous function. Show that if $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in A then $\{f(x_n)\}_{n=1}^\infty$ is a Cauchy sequence in $f(A)$. Show that this need not be true if f is continuous but not uniformly continuous.

5.5.4 ([TBB, Ex. 5.6.11]). Give an example of a function f that is continuous on $\mathbb{R}$ and a sequence of compact intervals $X_1, X_2, \dots,$ on each of which f is uniformly continuous, but for which f is not uniformly continuous on $X = \bigcup_{i=1}^\infty X_i$.

5.5.5 ([TBB, Ex. 5.7.6]). Let $f \colon \mathbb{R} \to \mathbb{R}$ be a continuous function that is periodic in the sense that, for some real number $p > 0$, $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that f has an absolute maximum and absolute minimum.

5.5.6. Suppose $\{U_\alpha : \alpha \in I\}$ is an open cover of a compact set K. Prove that there exists $\delta > 0$ such that

$$\forall\, x \in K, \ \exists\, \alpha \in I \text{ such that } B(x, \delta) \cap K \subseteq U_\alpha.$$

The number δ (which is not unique, of course) is called a *Lebesgue number* for the open cover. This idea was used in the proof of Theorem 5.43.

5.5.7. A function $f \colon A \to \mathbb{R}^m$ is said to be *Lipschitz* if there is a positive number M such that

$$\|f(x) - f(y)\| \le M\|x - y\| \quad \forall\, x, y \in A.$$

(a) Show that a Lipschitz function must be uniformly continuous.

(b) Show that the function

$$h \colon [0,1] \to \mathbb{R}, \quad h(x) = \sqrt{x},$$

is uniformly continuous on $[0, 1]$, but is not Lipschitz.

Chapter 6

Differentiation

In this chapter we explore one of the fundamental concepts in calculus: the derivative. We begin with the definition of the derivative and then deduce some computational rules, including the product rule, quotient rule, and chain rule. We will then investigate the relationship between the derivative and local extrema, as well as the Mean Value Theorem.

Many of the concepts in this chapter you will have seen in previous calculus courses. However, we will focus less on computations and applications, and more on understanding the underlying concepts and *why* the properties you have learned in calculus hold. A good reference for the material in this chapter is [TBB, Ch. 7].

Throughout this chapter, we will only consider real-valued functions of a single variable. That is, all functions will be of the form $f\colon A \to \mathbb{R}$ for some subset $A \subseteq \mathbb{R}$. Often A will be a closed interval $[a, b]$.

6.1 Definition of the derivative

We will use the term *interval* to mean any subset of $\mathbb{R}$ of the form

$$[a,b],\ (a,b],\ [a,b),\ (a,b),\ (-\infty,b],\ (-\infty,b),\ [a,\infty),\ (a,\infty),\ (-\infty,\infty)=\mathbb{R},\quad \text{where } a<b.$$

Throughout this section, I and J will denote intervals.

Definition 6.1 (Derivative)**.** Suppose $f\colon I \to \mathbb{R}$ and $x_0 \in I$. The *derivative* of f at x_0 is

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}, \tag{6.1}$$

provided that this limit exists or is infinite. If $f'(x_0)$ is finite, we say f is *differentiable* at x_0. If f is differentiable at every point of a subset $E \subseteq I$, we say that f is *differentiable on* E. If f is differentiable at every point of I, we say that f is a *differentiable function*.

As you have learned in calculus, the derivative $f'(x_0)$ is the slope of the tangent line to the graph of f at the point $(x_0, f(x_0))$.

If $f\colon I \to \mathbb{R}$ is differentiable, then the derivative yields another function $f'\colon I \to \mathbb{R}$. We will sometimes denote this function by

$$\frac{d}{dx} f(x).$$

Example 6.2. Let

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$$

and choose $x_0 \in \mathbb{R}$. Then

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0} (x + x_0) = 2x_0,$$

where, in the last equality, we used the fact the polynomial function $x \mapsto x + x_0$ is continuous. Note that we were also able to assume that $x \neq x_0$ in our manipulation of the expression above since we are taking the limit $x \to x_0$ and so we never consider $x = x_0$ (see Definition 5.1).

Note that if $f\colon [a, b] \to \mathbb{R}$ and $x_0 = a$ or $x_0 = b$, then the limit in (6.1) is a one-sided limit. Of course, it is also possible that x_0 is an interior point and the one-sided versions of (6.1) exist but are not equal.

Definition 6.3 (Right-hand derivative, left-hand derivative)**.** Suppose $f\colon I \to \mathbb{R}$. The *right-hand derivative* of f at x_0 is

$$f'_+(x_0) = \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0},$$

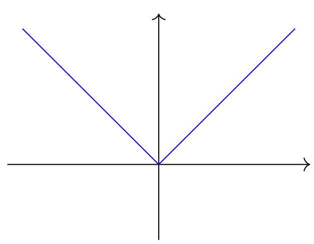
provided that this one-sided limit exists or is infinite. Similarly, the *left-hand derivative* of f at x_0 is

$$f'_-(x_0) = \lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0},$$

provided that this one-sided limit exists or is infinite.

Note that, if x_0 is an interior point of I, then $f'(x_0)$ exists if and only if $f'_+(x_0) = f'_-(x_0)$. (See Exercise 5.1.8.)

Example 6.4. Consider the absolute value function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$. So the graph of f looks like:



Let us compute the two one-sided derivatives at $x_0 = 0$. We have

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x > 0. \end{cases}$$

Thus

$$f'_+(0) = \lim_{x \to 0^+} \frac{|x|}{x} = 1 \qquad \text{and} \qquad f'_-(0) = \lim_{x \to 0^-} \frac{|x|}{x} = -1.$$

Since the one-sided derivatives are not equal, the function f is not differentiable at 0.

Since the absolute value function is continuous, Example 6.4 shows that a function may be continuous at a point, but not differentiable at that point. However, the following theorem states that differentiability implies continuity.

Proposition 6.5. *Suppose $f\colon I \to \mathbb{R}$ and $x_0 \in I$. If f is differentiable at x_0, then f is continuous at x_0.*

Proof. To prove that f is continuous at x_0, it suffices to show that

$$\lim_{x\to x_0} \big(f(x) - f(x_0)\big) = 0.$$

For $x \neq x_0$, we have

$$f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0}(x - x_0).$$

Thus,

$$\begin{aligned}\lim_{x\to x_0} \big(f(x) - f(x_0)\big) &= \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0}(x - x_0)\right) \\ &= \left(\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}\right)\left(\lim_{x\to x_0}(x - x_0)\right) = f'(x_0)\cdot 0 = 0,\end{aligned}$$

as required. □

Remark 6.6. Note that it is *not* necessarily true that the derivative is a continuous function. For example, consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin x^{-1} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

This function is differentiable at every point. However, the derivative f' is discontinuous at 0. See [TBB, §7.4] for details.

Example 6.7. Consider the function $g\colon [0,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \text{ or } x = 0, \\ \frac{1}{q^2} & \text{if } x = \frac{p}{q},\ \gcd(p,q) = 1. \end{cases}$$

Then g is discontinuous at every rational number in $(0,1]$. However, $g'(0)$ exists! To see this, note that $0 \le g(x) \le x^2$ for all $x \in [0,1]$. Thus

$$\left|\frac{g(x) - g(0)}{x - 0} - 0\right| = \frac{|g(x)|}{|x|} \le x \quad \forall\, x \in (0,1].$$

So taking $\delta = \varepsilon$ in the definition of a limit (Definition 5.1), we see that

$$g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0^+} \frac{g(x) - g(0)}{x - 0} = 0.$$

It follows from Proposition 6.5 that g is continuous at 0. (It is also not so hard to see directly from the definition of continuity that g is continuous at 0.)

If a function f is differentiable and the derivative f' is also differentiable, then we say that f is *twice differentiable*. We denote the derivative of f' by f''. The function f'' is called the *second derivative* of f. More generally, we define the *n-th derivative* $f^{(n)}$ of f inductively by

$$f^{(0)} = f, \quad f^{(n)} = \left(f^{(n-1)}\right)', \quad n \in \mathbb{N}.$$

Exercises.

6.1.1 ([TBB, Ex. 7.2.3]). Check the differentiability of each of the functions below at $x_0 = 0$.

(a) $f(x) = x|x|$.

(b) $f(x) = \begin{cases} x \sin x^{-1}, & x \neq 0, \\ 0, & x = 0. \end{cases}$

(c) $f(x) = \begin{cases} x^2 \sin x^{-1}, & x \neq 0, \\ 0, & x = 0. \end{cases}$

(d) $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational}, \\ 0, & \text{if } x \text{ is irrational}. \end{cases}$

6.1.2 ([TBB, Ex. 7.2.4]). Let

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ ax & \text{if } x < 0. \end{cases}$$

(a) For which values of a is f differentiable at $x = 0$?

(b) For which values of a is f continuous at $x = 0$?

6.1.3 ([TBB, Ex. 7.2.5]). For which $p \in \mathbb{N}$ is the function $f(x) = |x|^p$ differentiable at 0?

6.1.4 ([TBB, Ex. 7.2.11]). Give an example of a function with an infinite derivative at some point. Give an example of a function f with $f'_+(x_0) = \infty$ and $f'_-(x_0) = -\infty$ at some point x_0.

6.1.5 ([TBB, Ex. 7.2.14]). Let f be strictly increasing and differentiable on an interval. Does this imply that $f'(x) \geq 0$ on that interval? Does this imply that $f'(x) > 0$ on that interval?

6.1.6 ([TBB, Ex. 7.2.17]). Suppose that a function has both a right-hand and a left-hand derivative at a point. What, if anything, can you conclude about the continuity of that function at that point?

6.2 Computing derivatives

We now deduce some rules of differentiation that you learned in calculus.

Proposition 6.8 (Algebraic rules for differentiation)**.** *Suppose $f, g\colon I \to \mathbb{R}$ and $x_0 \in I$. If f and g are differentiable at x_0, then $f+g$ and fg are differentiable at x_0. If, in addition, $g(x_0) \neq 0$, then f/g is differentiable at x_0. Furthermore, we have the following equalities:*

(a) $(cf)'(x_0) = cf'(x_0)$ *for any* $c \in \mathbb{R}$,

(b) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$,

(c) $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$,

(d) $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$ *if* $g(x_0) \neq 0$.

Proof. We leave the proofs of parts (a) and (b) as exercises (Exercise 6.2.1).

To prove part (c), we compute

$$\begin{aligned}\lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} &= \lim_{x\to x_0} \frac{f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0} \\ &= \lim_{x\to x_0} \left(f(x)\frac{g(x) - g(x_0)}{x - x_0} + \frac{f(x) - f(x_0)}{x - x_0} g(x_0) \right) \\ &= f(x_0)g'(x_0) + f'(x_0)g(x_0),\end{aligned}$$

where we have used Propositions 5.6 and 5.7, and the fact that $\lim_{x\to x_0} f(x) = f(x_0)$ since the differentiability of f at x_0 implies, by Proposition 6.5, that f is also continuous at x_0.

To prove part (d), we let $h = f/g$ and compute

$$\begin{aligned}\frac{h(x) - h(x_0)}{x - x_0} &= \frac{1}{g(x)g(x_0)} \left(\frac{f(x) - f(x_0)}{x - x_0} g(x_0) - f(x) \frac{g(x) - g(x_0)}{x - x_0} \right) \\ &\to \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2} \quad \text{as } x \to x_0. \qquad \square\end{aligned}$$

Example 6.9. Let

$$f(x) = (2x^4 + 2x)^2.$$

Then $f(x) = 4x^8 + 8x^5 + 4x^2$, and so, by Exercise 6.2.3 and Proposition 6.8, parts (a) and (b), we have

$$f'(x) = 32x^7 + 40x^4 + 8x.$$

Alternatively, we can use Proposition 6.8(c) to compute

$$\begin{aligned}f'(x) &= \left(\frac{d}{dx}(2x^4 + 2x) \right)(2x^4 + 2x) + (2x^4 + 2x)\frac{d}{dx}(2x^4 + 2x) \\ &= (8x^3 + 2)(2x^4 + 2x) + (2x^4 + 2x)(8x^3 + 2) = 32x^7 40x^4 + 8x.\end{aligned}$$

Proposition 6.10 (The chain rule)**.** *Suppose $f\colon I \to \mathbb{R}$, $g\colon J \to \mathbb{R}$, $x_0 \in I$, and $f(I) \subseteq J$. If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then the composite function $g \circ f$ is differentiable at x_0 and*

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $I \setminus \{x_0\}$ such that $a_n \to x_0$. For $n \in \mathbb{N}$, let

$$b_n = \frac{g(f(a_n)) - g(f(x_0))}{a_n - x_0}.$$

We split the proof into two cases.

Case 1: There exists $N \in \mathbb{N}$ such that $f(a_n) \neq f(x_0)$ for $n \geq N$. Then, for $n \geq N$, we have

$$b_n = \frac{g(f(a_n)) - g(f(x_0))}{f(a_n) - f(x_0)} \frac{f(a_n) - f(x_0)}{a_n - x_0}. \tag{6.2}$$

Since f is differentiable at x_0, it is continuous at x_0. Thus $f(a_n) \to f(x_0)$. Therefore

$$b_n \to g'(f(x_0))f'(x_0).$$

Case 2: $f(a_n) = f(x_0)$ for infinitely many values of n. Then, by Exercise 6.2.4, $f'(x_0) = 0$. We have that $b_n = 0$ for those n satisfying $f(a_n) = f(x_0)$ and b_n is given by (6.2) otherwise. Thus,

$$b_n \to 0 = g'(f(x_0))f'(x_0).$$

In both cases, we have

$$b_n \to g'(f(x_0))f'(x_0).$$

Then the result follows from Theorem 5.3. □

Proposition 6.11 (Derivative of an inverse function)**.** *Suppose $f\colon I \to J$ is bijective and differentiable at $x_0 \in I$ with $f'(x_0) \neq 0$. Then the inverse function $f^{-1}\colon J \to I$ is differentiable at $f(x_0)$ and*

$$\left(f^{-1}\right)'(f(x_0)) = \frac{1}{f'(x_0)}.$$

Proof. Let $y_0 = f(x_0)$, so that $x_0 = f^{-1}(y_0)$. We know f^{-1} is continuous by Theorem 5.36. We wish to compute

$$\left(f^{-1}\right)'(y_0) = \lim_{k \to 0} \frac{f^{-1}(y_0 + k) - f^{-1}(y_0)}{k}.$$

Define

$$h(k) = f^{-1}(y_0 + k) - f^{-1}(y_0) \to 0 \text{ as } k \to 0.$$

Then

$$f\big(x_0 + h(k)\big) = f\left(f^{-1}(y_0 + k)\right) = y_0 + k,$$

and so

$$k = y_0 - f\big(x_0 + h(k)\big) = f(x_0) - f\big(x_0 + h(k)\big).$$

Thus

$$\frac{f^{-1}(y_0+k)-f^{-1}(y_0)}{k} = \frac{h(k)}{f(x_0)-f\big(x_0+h(k)\big)} = \frac{1}{\frac{f(x_0)-f(x_0+h(k))}{h(k)}}$$

Now,

$$\lim_{k\to 0} h(k) = 0 \quad \text{and} \quad \lim_{y\to 0} \frac{f(x_0+y)-f(x_0)}{y} = f'(y_0).$$

Thus, by Remark 5.33,

$$\left(f^{-1}\right)'(y_0) = \lim_{k\to 0} \frac{f^{-1}(y_0+k)-f^{-1}(y_0)}{k} = \lim_{k\to 0} \frac{1}{\frac{f(x_0)-f(x_0+h(k))}{h(k)}} = \frac{1}{f'(x_0)}. \qquad \square$$

Note, that part of the conclusion in Proposition 6.11 is that the derivative of the inverse exists. If we already knew that this derivative existed, then we could easily derive the given formula using the chain rule

$$f^{-1}(f(x)) = x \implies \left(f^{-1}\right)'(f(x_0))f'(x_0) = 1 \implies \left(f^{-1}\right)'(f(x_0)) = \frac{1}{f'(x_0)}.$$

Exercises.

6.2.1. Prove parts (a) and (b) of Proposition 6.8.

6.2.2 ([TBB, Ex. 7.3.4]). Prove that

$$\frac{d}{dx}(f(x))^2 = 2f(x)f'(x)$$

using Proposition 6.8 and also directly from the definition of the derivative.

6.2.3. Prove by induction that

$$\frac{d}{dx}x^n = nx^{n-1}, \quad n \in \mathbb{N}.$$

6.2.4 ([TBB, Ex. 7.3.11]). Show that if for each open interval U containing x_0 there exists $x \in U$, $x \neq x_0$, for which $f(x) = f(x_0)$, then either $f'(x_0)$ does not exist or else $f'(x_0) = 0$.

6.2.5. Suppose the functions $f, g\colon \mathbb{R} \to \mathbb{R}$ are twice differentiable at a point x_0. Prove that

$$(fg)''(x_0) = f''(x_0)g(x_0) + 2f'(x_0)g'(x_0) + f(x_0)g''(x_0).$$

6.2.6 ([TBB, Ex. 7.3.15]). State and prove a theorem that gives a formula for $f'(x_0)$ when

$$f = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1.$$

Be sure to state all the hypotheses that you need.

6.2.7 ([TBB, Ex. 7.3.17]). If we restrict the domain of $\sin x$ to $[-\pi/2, \pi/2]$, then it is invertible. Find a formula for the derivative of the function $\sin^{-1} x$ assuming that

$$\frac{d}{dx} \sin x = \cos x.$$

6.2.8 ([TBB, Ex. 7.3.18]). If we restrict the domain of $\tan x$ to $(-\pi/2, \pi/2)$, then it is invertible. Find a formula for the derivative of the function $\tan^{-1} x$ assuming that

$$\frac{d}{dx} \tan x = \sec^2 x.$$

6.3 Local extrema

One of the important uses of the derivative is to find maximum or minimum values of functions. We examine this technique in this section.

Definition 6.12 (Local maximum, local minimum)**.** Suppose $f\colon I \to \mathbb{R}$ and x_0 is an interior point of I. We say f has a *local maximum* at x_0 if there exists $\delta > 0$ such that

$$[x_0 - \delta, x_0 + \delta] \subseteq I \quad \text{and} \quad f(x) \le f(x_0)\ \forall\, x \in [x_0 - \delta, x_0 + \delta].$$

We say f has a *local minimum* at x_0 if there exists $\delta > 0$ such that

$$[x_0 - \delta, x_0 + \delta] \subseteq I \quad \text{and} \quad f(x) \ge f(x_0)\ \forall\, x \in [x_0 - \delta, x_0 + \delta].$$

If f has a local maximum or a local minimum at x_0, then we say it has a *local extremum* at x_0.

Theorem 6.13. *Suppose $f\colon I \to \mathbb{R}$. If f has a local extremum at an interior point x_0 of I and f is differentiable at x_0, then $f'(x_0) = 0$.*

Proof. Suppose f has a local minimum at an interior point x_0 of I. (The proof for a local maximum is similar and can be found at [TBB, Th. 7.18].) Then there exists $\delta > 0$ such that

$$[x_0 - \delta, x_0 + \delta] \subseteq I \quad \text{and} \quad f(x) \ge f(x_0)\ \forall\, x \in [x_0 - \delta, x_0 + \delta].$$

Thus

$$\frac{f(x) - f(x_0)}{x - x_0} \ge 0 \text{ for } x \in (x_0, x_0 + \delta) \tag{6.3}$$

and

$$\frac{f(x) - f(x_0)}{x - x_0} \le 0 \text{ for } x \in (x_0 - \delta, x_0). \tag{6.4}$$

If $f'(x_0)$ exists, then

$$f'(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

The right-hand limit is ≥ 0 by (6.3) and the left-hand limit is ≤ 0 by (6.4). Therefore, $f'(x_0) = 0$. ∎

It follows from Theorem 6.13 that if $f\colon [a,b] \to \mathbb{R}$ is continuous, then f must achieve its maximum (or minimum) at one or more of the following types of points:

- Points $x_0 \in (a,b)$ such that $f'(x_0) = 0$.
- Points $x_0 \in (a,b)$ at which f is not differentiable.
- The points a or b.

Exercises.

6.3.1 ([TBB, Ex. 7.5.1]). Give an example of a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ for which $f'(0) = 0$, but 0 is not a local maximum or minimum of f.

6.3.2 ([TBB, Ex. 7.5.2]). Let

$$f(x) = \begin{cases} x^4(2 + \sin x^{-1}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Prove that f is differentiable on $\mathbb{R}$.

(b) Prove that f has an absolute minimum at $x = 0$.

(c) Prove that f' takes on both positive and negative values in every neighbourhood of 0. That is, prove that, for every $\varepsilon > 0$, f' takes on both positive and negative values in the interval $(-\varepsilon, \varepsilon)$.

6.4 The Mean Value Theorem

Theorem 6.14 (Rolle's Theorem)**.** *Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. If f is constant on $[a,b]$, then $f'(x) = 0$ for all $x \in (a,b)$, so we can take c to be any point of (a,b).

Now suppose that f is not constant. Since f is continuous on the compact interval $[a,b]$, it attains a maximum value M and a minimum value m on $[a,b]$ by Theorem 5.28. Because f is not constant, at least one of the values M or m is different from $f(a)$ and $f(b)$. Suppose $m < f(a)$. (The case $M > f(a)$ is similar.) Choose $c \in [a,b]$ such that $f(c) = m$. Since $m < f(a) = f(b)$, we have $c \neq a$ and $c \neq b$. Thus $c \in (a,b)$. By Theorem 6.13, $f'(c) = 0$. □

Note that, in Rolle's Theorem, we do not require that f be differentiable at the end points a and b. For example, the theorem applies to the function $f\colon [0, 1/\pi] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x \sin x^{-1} & \text{if } x \in (0, 1/\pi], \\ 0 & \text{if } x = 0. \end{cases}$$

This function is not differentiable at zero, but it is continuous on $[0, 1/\pi]$ and differentiable on $(0, 1/\pi]$. It has an infinite number of points in the interval $(0, 1/\pi)$ where the derivative is zero.

Theorem 6.15 (Cauchy Mean Value Theorem). *Suppose $f, g\colon [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$\big(f(b) - f(a)\big)g'(c) = \big(g(b) - g(a)\big)f'(c).$$

Proof. Let

$$h(x) = \big(f(b) - f(a)\big)g(x) - \big(g(b) - g(a)\big)f(x).$$

Then h is continuous on $[a, b]$ and differentiable on (a, b). Furthermore,

$$h(a) = f(b)g(a) - f(a)g(b) = h(b).$$

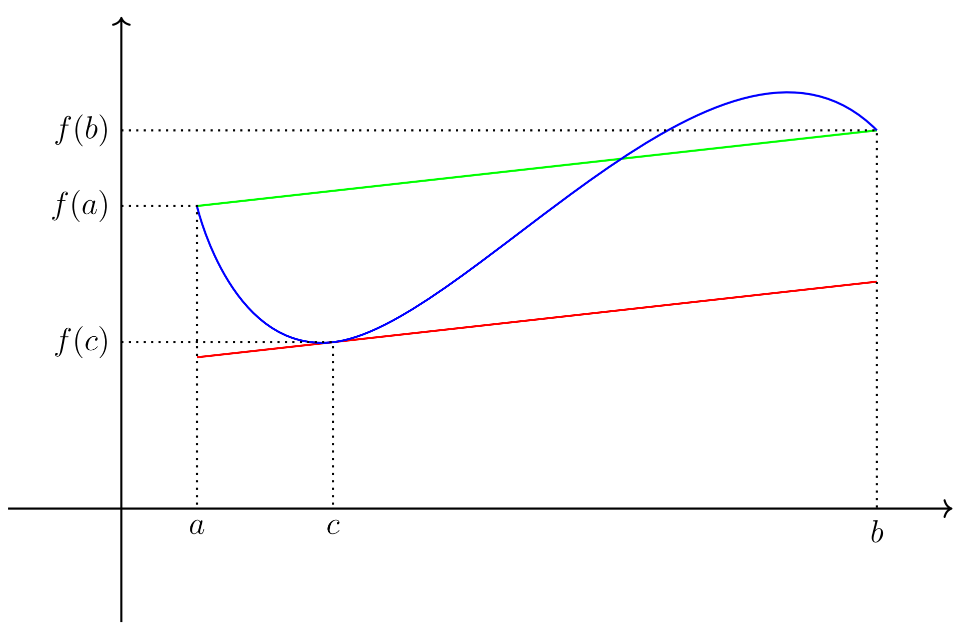
By Rolle's Theorem, there exists $c \in (a, b)$ such that

$$0 = h'(c) = \big(f(b) - f(a)\big)g'(c) - \big(g(b) - g(a)\big)f'(c).$$

The theorem follows. □

Corollary 6.16 (Mean Value Theorem). *Suppose that $f\colon [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Proof. Define

$$g\colon [a,b] \to \mathbb{R}, \quad g(x) = x.$$

Then, by the Cauchy Mean Value Theorem, there exists $c \in (a,b)$ such that

$$\big(f(b) - f(a)\big)g'(c) = (b-a)f'(c).$$

Since $g'(c) = 1$, the result follows. □

Note that the Mean Value Theorem is a generalization of Rolle's Theorem. If $f(a) = f(b)$, then the Mean Value Theorem becomes precisely Rolle's Theorem.

Exercises.

6.4.1 ([TBB, Ex. 7.6.1]). Apply Rolle's Theorem to the function $f(x) = \sqrt{1-x^2}$ on $[-1,1]$. Observe that f fails to be differentiable at the endpoints of the interval.

6.4.2. Use Rolle's Theorem to explain why the cubic equation

$$x^3 + \alpha x + \beta = 0$$

cannot have more than one solution whenever $\alpha > 0$.

6.4.3 ([TBB, Ex. 7.6.4]). Suppose that $f'(x) > c > 0$ for all $x \in [0,\infty)$. Show that $\lim_{x\to\infty} f(x) = \infty$. In other words, show that for all $M > 0$, there exists $N > 0$ such that

$$x > N \implies f(x) > M.$$

6.4.4 ([TBB, Ex. 7.6.5]). Suppose $f\colon \mathbb{R} \to \mathbb{R}$ and both f' and f'' exist everywhere. Show that if f has three zeros, then there must be some point $x_0 \in \mathbb{R}$ such that $f''(x_0) = 0$.

6.4.5 ([TBB, Ex. 7.6.8]). A real-valued function is said to satisfy a *Lipschitz condition* on an interval $[a,b]$ if there exists $M > 0$ such that

$$|f(x) - f(y)| \le M|x-y| \quad \forall\, x,y \in [a,b].$$

Show that if f is assumed to be continuous on $[a,b]$ and differentiable on (a,b), then this condition is equivalent to the derivative f' being bounded on (a,b). *Hint*: First show directly from the definition that the Lipshitz condition will imply a bounded derivative. Then use the Mean Value Theorem to get the converse, that is, apply the Mean Value Theorem to f on the interval $[x,y]$ for any $a \le x < y \le b$.

6.4.6 ([TBB, Ex. 7.6.11]). Give an example to show that the conclusion of the Mean Value Theorem can fail if we drop the requirement that f be differentiable at every point in (a,b). Give an example to show that the conclusion can fail if we drop the requirement of continuity at the endpoints of the interval.

6.4.7. Suppose $f\colon [a,b] \to \mathbb{R}$ is differentiable and that $f'(x) = 0$ for all $x \in [a,b]$. Prove that f is a constant function.

6.4.8. Suppose $f, g\colon [a,b] \to \mathbb{R}$ are differentiable and that $f'(x) = g'(x)$ for all $x \in [a,b]$. Prove that there exists some $k \in \mathbb{R}$ such that $f(x) = g(x) + k$ for all $x \in [a,b]$.

Chapter 7

Integration

In this chapter we discuss the Riemann integral. After defining the integral and the notion of integrability, we prove some properties of the integral that should be familiar from calculus, including the important Fundamental Theorem of Calculus. We conclude with a treatment of improper integrals. A good reference for the material in this section is [Leb, Ch. 5].

7.1 The Riemann Integral

In calculus, you have considered the integral. The intuitive idea was to compute the area under the graph of a function $f\colon [a,b] \to \mathbb{R}$ (with area below the x-axis counted as negative). This was done by approximating the region by rectangles. To do this, we divide the interval $[a,b]$ into subintervals, and the height of the rectangles are given by the value of the function at some point in each subinterval. The sum of the areas of these rectangles (with rectangles below the x-axis counted as negative) is called a *Riemann sum* for f (see Definition 7.9). We then take the limit as the width of the rectangles goes to zero.

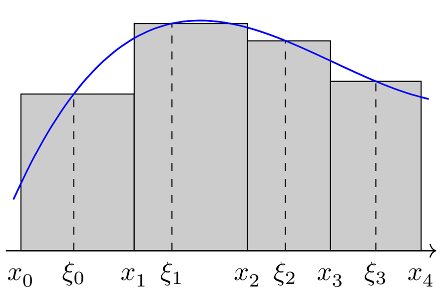


Figure 7.1: Riemann sum

We now make this idea precise. Throughout this section, we will assume that *all functions are bounded*. When we write the interval $[a,b]$, we will always assume $a < b$.

A *partition* of an interval $[a,b]$, $a < b$, is a set

$$P = \{x_0, x_1, \dots, x_n\} \subseteq [a,b]$$

such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

If $P = \{x_0, \dots, x_n\}$ is a partition of $[a, b]$ and $f\colon [a, b] \to \mathbb{R}$ is a bounded function, then we define

$$m_i(P, f) = \inf\{f(x) : x \in [x_{i-1}, x_i]\},$$
$$M_i(P, f) = \sup\{f(x) : x \in [x_{i-1}, x_i]\}.$$

(These exist since f is bounded.)

Definition 7.1 (Upper and lower Darboux sums)**.** Suppose $f\colon [a, b] \to \mathbb{R}$ is bounded and $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$. The *lower Darboux sum* of f for P is

$$L(P, f) = \sum_{i=1}^{n} m_i(P, f)(x_i - x_{i-1}).$$

The *upper Darboux sum* of f for P is

$$U(P, f) = \sum_{i=1}^{n} M_i(P, f)(x_i - x_{i-1}).$$

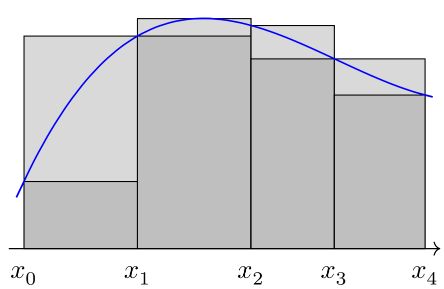


Figure 7.2: Upper and lower Darboux sums

It is clear from the definitions that

$$L(P, f) \le U(P, f)$$

for any f and P.

A partition P' is a *refinement* of a partition P if $P \subseteq P'$.

Lemma 7.2. *If P' is a refinement of P, then for any bounded function $f\colon [a, b] \to \mathbb{R}$, we have*

$$L(P, f) \le L(P', f) \le U(P', f) \le U(P, f).$$

Proof. Let $P = \{x_0, \dots, x_n\}$. We first consider the case where $P' = P \cup \{y\}$, $y \in [a,b] \setminus P$. Then there exists $k \in \{1, \dots, n\}$ such that $y \in (x_{k-1}, x_k)$. We have

$$m_k(P,f) = \inf\{f(x) : x \in [x_{k-1}, x_k]\} \le \inf\{f(x) : x \in [x_{k-1}, y]\} \quad \text{and}$$
$$m_k(P,f) = \inf\{f(x) : x \in [x_{k-1}, x_k]\} \le \inf\{f(x) : x \in [y, x_k]\}.$$

Thus,

$$\begin{aligned}
L(P,f) &= \sum_{i=1}^{n} m_i(P,f)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{k-1} m_i(P,f)(x_i - x_{i-1}) + m_k(P,f)(x_k - x_{k-1}) + \sum_{i=k+1}^{n} m_i(P,f)(x_i - x_{i-1}) \\
&\le \sum_{i=1}^{k-1} m_i(P,f)(x_i - x_{i-1}) + \inf\{f(x) : x \in [x_{k-1}, y]\}(y - x_{k-1}) \\
&\qquad + \inf\{f(x) : x \in [y, x_k]\}(x_k - y) + \sum_{i=k+1}^{n} m_i(P,f)(x_i - x_{i-1}) \\
&= L(P', f).
\end{aligned}$$

The proof that $U(P', f) \le U(P, f)$ is similar.

Now consider an arbitrary refinement P' of P. Thus $P' = P \cup \{y_1, \dots, y_m\}$ for some pairwise distinct $y_1, \dots, y_m \notin P$. By the above, we have

$$\begin{aligned}
L(P,f) \le L(P \cup \{y_1\}, f) \le L(P \cup \{y_1, y_2\}, f) \le \cdots \le L(P', f) \\
\le U(P', f) \le U(P \cup \{y_1, \dots, y_{m-1}\}, f) \le \cdots \le U(P, f).
\end{aligned}$$ □

Corollary 7.3. *If $f\colon [a,b] \to \mathbb{R}$ is bounded and P_1, P_2 are partitions of $[a,b]$, then*

$$L(P_1, f) \le U(P_2, f).$$

Proof. Let $P = P_1 \cup P_2$. Then P is refinement of both P_1 and P_2. Thus, by Lemma 7.2, we have

$$L(P_1, f) \le L(P, f) \le U(P, f) \le U(P_2, f).$$ □

Definition 7.4 (Integrable function)**.** A bounded function $f\colon [a,b] \to \mathbb{R}$ is *integrable* (or *Riemann integrable*) if

$$\sup\{L(P,f) : P \text{ a partition of } [a,b]\} = \inf\{U(P,f) : P \text{ a partition of } [a,b]\}.$$

If f is integrable, we denote this common value by

$$\int_a^b f \qquad \text{or} \qquad \int_a^b f(t)\, dt.$$

If $A \subseteq \mathbb{R}$, $f\colon A \to \mathbb{R}$, and the restriction of f to $[a,b] \subseteq A$ is integrable, we say that f is *integrable on* $[a,b]$. If f is integrable on $[a,b]$, we define

$$\int_b^a f := -\int_a^b f.$$

We also define $\int_a^a f = 0$.

We will often write

$$\sup_P\{L(P,f)\} \text{ instead of } \sup\{L(P,f) : P \text{ a partition of } [a,b]\}, \quad \text{and}$$
$$\inf_P\{U(P,f)\} \text{ instead of } \inf\{U(P,f) : P \text{ a partition of } [a,b]\}.$$

Remark 7.5. By Corollary 7.3, any lower Darboux sum is a lower bound for $\{U(P,f) : P \text{ a partition of } [a,b]\}$, so $\inf_P\{U(P,f)\}$ exists. Similarly $\sup_P\{L(P,f)\}$ exists. Moreover,

$$\sup_P\{L(P,f)\} \le \inf_P\{U(P,f)\}.$$

Example 7.6. Consider the function $f\colon [a,b] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & x \in [a,b] \cap (\mathbb{R} \setminus \mathbb{Q}), \\ 1 & x \in [a,b] \cap \mathbb{Q}. \end{cases}$$

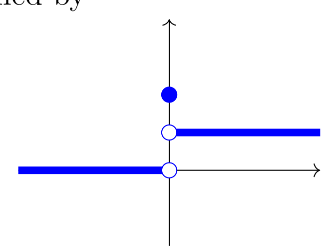
Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$. For each $i \in \{1, \ldots, n\}$, the interval $[x_i, x_{i-1}]$ contains rational and irrational numbers. Thus $m_i(P,f) = 0$ and $M_i(P,f) = 1$. Hence

$$L(P,f) = \sum_{i=1}^n m_i(P,f)(x_i - x_{i-1}) = 0 \qquad \text{and}$$
$$U(P,f) = \sum_{i=1}^n m_i(P,f)(x_i - x_{i-1}) = \sum_{i=1}^n (x_i - x_{i-1}) = b - a \neq 0.$$

Therefore, f is *not* integrable.

Example 7.7. Consider the function $f\colon [-1,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 2 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$



We will show that f is Riemann integrable and $\int_{-1}^1 f = 1$.
Let ε satisfy $0 < \varepsilon < 1$. Consider the partition

$$P = \{-1, -\varepsilon, \varepsilon, 1\}.$$

Then

$$\begin{aligned} m_1(P,f) &= \inf\{f(x) : x \in [-1,-\varepsilon]\} = 0, & M_1(P,f) &= \sup\{f(x) : x \in [-1,-\varepsilon]\} = 0, \\ m_2(P,f) &= \inf\{f(x) : x \in [-\varepsilon,\varepsilon]\} = 0, & M_2(P,f) &= \sup\{f(x) : x \in [-\varepsilon,\varepsilon]\} = 2, \\ m_2(P,f) &= \inf\{f(x) : x \in [\varepsilon,1]\} = 1, & M_2(P,f) &= \sup\{f(x) : x \in [\varepsilon,1]\} = 1. \end{aligned}$$

Thus

$$L(P,f) = \sum_{i=1}^{3} m_i(P,f)(x_i - x_{i-1}) = 0 \cdot (1-\varepsilon) + 0 \cdot 2\varepsilon + 1 \cdot (1-\varepsilon) = 1-\varepsilon,$$
$$U(P,f) = \sum_{i=1}^{3} M_i(P,f)(x_i - x_{i-1}) = 0 \cdot (1-\varepsilon) + 2 \cdot 2\varepsilon + 1 \cdot (1-\varepsilon) = 1+3\varepsilon.$$

Therefore,

$$\inf_Q\{U(Q,f)\} - \sup_Q\{L(Q,f)\} \le U(P,f) - L(P,f) = (1+3\varepsilon) - (1-\varepsilon) = 4\varepsilon.$$

By Remark 7.5, we have $\inf_Q\{U(Q,f)\} - \sup_Q\{L(Q,f)\} \ge 0$. Since ε was arbitrary, we have $\inf_Q\{U(Q,f)\} = \sup_Q\{L(Q,f)\}$. So f is integrable. Finally,

$$1-\varepsilon = L(P,f) \le \int_{-1}^{1} f \le U(P,f) = 1+3\varepsilon.$$

Therefore,

$$-\varepsilon \le \left(\int_{-1}^{1} f\right) - 1 \le 3\varepsilon.$$

Since ε was arbitrary, we have $\int_{-1}^{1} f = 1$. Note that we would obtain the same value for the integral so matter what $f(0)$ is defined to be. See Exercise 7.1.6.

We extract part of the argument in Example 7.7 as a proposition, since it is a useful criterion for integrability.

Proposition 7.8. *A bounded function $f\colon [a,b] \to \mathbb{R}$ is integrable if and only if for all $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$.*

Proof. Suppose that for all $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$. Fix $\varepsilon > 0$ and choose such a partition. By Remark 7.5, we have

$$0 \le \inf_Q\{U(Q,f)\} - \sup_Q\{L(Q,f)\} \le U(P,f) - L(P,f) \le \varepsilon.$$

Since ε was arbitrary, we have $\inf_Q\{U(Q,f)\} = \sup_Q\{L(Q,f)\}$, and so f is integrable. □

Definition 7.9 (Riemann sum)**.** Let $f\colon [a,b] \to \mathbb{R}$ be bounded and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$. For each $i \in \{1, \ldots, n\}$, let $\xi_i \in [x_{i-1}, x_i]$. Let $\xi = \{\xi_1, \ldots, \xi_n\}$. Then

$$R(P,f,\xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

is a *Riemann sum* for f. See Figure 7.1.

If $f\colon [a,b] \to \mathbb{R}$ is bounded and $P = \{x_0, \dots, x_n\}$ is a partition of $[a,b]$, then for all $i \in \{1, \dots, n\}$, we have

$$m_i(P,f) \le f(\xi_i) \le M_i(P,f).$$

Thus,

$$L(P,f) \le R(P,f,\xi) \le U(P,f),$$

for any collection ξ of points in the subintervals of the partition P.

Let $\varepsilon > 0$. If f is integrable, then by Proposition 7.8 there exists a partition P such that $U(P,f) - L(P,f) < \varepsilon$. Then, for any collection ξ of points in the subintervals of the partition P, we have

$$\left| R(P,f,\xi) - \int_a^b f \right| \le U(P,f) - L(P,f) < \varepsilon.$$

In other words, we can force the Riemann sums to be arbitrarily close to the integral by choosing an appropriate partition.

Theorem 7.10. *If $f\colon [a,b] \to \mathbb{R}$ is continuous, then it is integrable.*

Proof. Since f is continuous on $[a,b]$, which is compact, f is uniformly continuous on $[a,b]$ by Theorem 5.43. Let $\varepsilon > 0$ and choose $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}.$$

Now choose a partition $P = \{x_0, \dots, x_n\}$ such that

$$x_i - x_{i-1} < \delta \quad \forall\, i \in \{1, \dots, n\}.$$

Then, for all $i \in \{1, \dots, n\}$, by Exercise 1.6.7 we have

$$M_i(P,f) - m_i(P,f) = \sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\} \le \frac{\varepsilon}{2(b-a)}.$$

Hence

$$U(P,f) - L(P,f) = \sum_{i=1}^n \big(M_i(P,f) - m_i(P,f)\big)(x_i - x_{i-1}) \le \frac{\varepsilon}{2(b-a)} \sum_{i=1}^n (x_i - x_{i-1}) = \frac{\varepsilon}{2} < \varepsilon.$$

Thus f is integrable by Proposition 7.8. □

Remark 7.11. The converse of Theorem 7.10 is false. For example, the function g of Example 7.7 is integrable but not continuous. For another example, see Exercise 7.1.5.

Exercises.

7.1.1 ([Leb, Ex. 5.1.1]). Let $f\colon [0,1] \to \mathbb{R}$ be defined by $f(x) = x^3$, and let $P = \{0, 0.1, 0.4, 1\}$. Compute $L(P,f)$ and $U(P,f)$.

7.1.2. Suppose $c \in \mathbb{R}$. Directly using the definition of the integral, show that $\int_a^b c = c(b-a)$.

7.1.3. Directly using the definition of the integral, show that the function

$$f\colon [1,b] \to \mathbb{R}, \quad f(x) = \frac{1}{x},$$

is integrable for all $b > 1$. *Hint*: Use uniform partitions (i.e. $x_i = 1 + i(b-1)/n$) and Proposition 7.8.

7.1.4. Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded, P is a partition of $[a,b]$, and $c \in \mathbb{R}$. Prove that

$$U(P,cf) = cU(P,f) \qquad \text{and} \qquad L(P,cf) = cL(P,f).$$

7.1.5. Consider the function $g\colon [0,1] \to \mathbb{R}$ given by

$$g(x) = \begin{cases} 0 & x = 0 \text{ or } x \text{ is irrational}, \\ \frac{1}{q} & x = p/q,\ p,q \in \mathbb{N},\ \gcd(p,q) = 1. \end{cases}$$

Prove that g is integrable and that $\int_0^1 g = 0$. *Hint*: Fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$, $N > 2/\varepsilon$. Consider the finite set

$$A = \{p/q : 0 \le p \le q,\ 1 \le q \le N\} \subseteq [0,1].$$

Choose a partition $P = \{x_0, \dots, x_n\}$ of $[0,1]$ containing the points of A in the interiors of subintervals of length $\le \varepsilon/2N^2$. Prove that $U(P,g) < \varepsilon$.

7.1.6. Suppose that $f\colon [a,b] \to \mathbb{R}$ is integrable and that $g\colon [a,b] \to \mathbb{R}$ has the property that $f(x) = g(x)$ for all but finitely many x in the interval $[a,b]$. Prove that g is integrable and $\int_a^b f = \int_a^b g$.

7.1.7 ([Leb, Ex. 5.1.10]). Let $f\colon [0,1] \to \mathbb{R}$ be a bounded function. Let $P_n = \{x_0, x_1, \dots, x_n\}$ be a uniform partition of $[0,1]$, that is, $x_i = i/n$. Is $\{L(P_n,f)\}_{n=1}^{\infty}$ always monotonic? Prove or find a counterexample.

7.2 Properties of the integral

We now prove some important properties of the integral.

Proposition 7.12. *If $f\colon [a,b] \to \mathbb{R}$ is integrable and $c \in (a,b)$, then f is integrable on $[a,c]$ and $[c,b]$, and*

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{7.1}$$

Conversely, if f is integrable on $[a,c]$ and $[c,b]$, then it is also integrable on $[a,b]$ and (7.1) holds.

Proof. Suppose f is integrable on $[a,b]$ and $c \in (a,b)$. Let $\varepsilon > 0$. By Proposition 7.8, there exists a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$. Adding the point c if necessary, we may assume that $c = x_j$ for some $j \in \{1, \ldots, n-1\}$. Then

$$P' = \{x_0, \ldots, x_j\} \qquad \text{and} \qquad P'' = \{x_j, \ldots, x_n\}$$

are partitions of $[a,c]$ and $[c,b]$, respectively. We have

$$L(P,f) = L(P',f) + L(P'',f) \qquad \text{and} \qquad U(P,f) = U(P',f) + U(P'',f).$$

Thus

$$\big(U(P',f) - L(P',f)\big) + \big(U(P'',f) - L(P'',f)\big) = U(P,f) - L(P,f) < \varepsilon.$$

Since each of the summands in parentheses is nonnegative, *both* are less than ε. Since $\varepsilon > 0$ was arbitrary, $\int_a^c f$ and $\int_c^b f$ both exist by Proposition 7.8. Moreover,

$$L(P,f) = L(P',f) + L(P'',f) \le \int_a^c f + \int_c^b f \le U(P',f) + U(P'',f) = U(P,f).$$

Since this is true for all partitions P (containing c), (7.1) holds.

The proof of the converse is left as an exercise (Exercise 7.2.1). □

Proposition 7.13. *Suppose $f, g \colon [a,b] \to \mathbb{R}$ are integrable and $c \in \mathbb{R}$. Then $cf + g$ is also integrable on $[a,b]$ and*

$$\int_a^b (cf + g) = c \int_a^b f + \int_a^b g.$$

Proof. The case $c = 0$ is trivial. First assume $c > 0$. We leave it as an exercise (Exercise 7.2.2) to show that, for any partition $P = \{x_0, \ldots, x_n\}$ and $i \in \{1, \ldots, n\}$,

$$cm_i(P,f) + m_i(P,g) \le m_i(P, cf+g) \qquad \text{and} \qquad M_i(P, cf+g) \le cM_i(P,f) + M_i(P,g). \tag{7.2}$$

Thus

$$cL(P,f) + L(P,g) \le L(P, cf+g) \qquad \text{and} \qquad U(P, cf+g) \le cU(P,f) + U(P,g).$$

Therefore, we have

$$cL(P,f) + L(P,g) \le L(P, cf+g) \le U(P, cf+g) \le cU(P,f) + U(P,g). \tag{7.3}$$

Let $\varepsilon > 0$. Since f and g are integrable on $[a,b]$, by Proposition 7.8, there exist partitions P' and P'' such that

$$U(P',f) - L(P',f) < \frac{\varepsilon}{2c} \qquad \text{and} \qquad U(P'',g) - L(P'',g) < \frac{\varepsilon}{2}.$$

The left-hand inequality implies that

$$cU(P',f) - cL(P',f) < \frac{\varepsilon}{2}.$$

Let $P = P' \cup P''$. Then, by Lemma 7.2,

$$cU(P, f) - cL(P, f) < \frac{\varepsilon}{2} \qquad \text{and} \qquad U(P, g) - L(P, g) < \frac{\varepsilon}{2}.$$

Hence

$$cU(P, f) + U(P, g) - \big(cL(P, f) + L(P, g)\big) < \varepsilon,$$

and so, by (7.3), we have

$$U(P, cf + g) - L(P, cf + g) < \varepsilon.$$

So $cf + g$ is integrable by Proposition 7.8.

By the above, for any partition P, we have

$$cL(P, f) + L(P, g) \leq L(P, cf + g) \leq \int_a^b (cf + g) \leq U(P, cf + g) \leq cU(P, f) + U(P, g). \tag{7.4}$$

And by the integrality of f and g, we have

$$cL(P, f) + L(P, g) \leq c \int_a^b f + \int_a^b g \leq cU(P, f) + U(P, g). \tag{7.5}$$

Let $\varepsilon > 0$. By Proposition 7.8, we can choose partitions P' and P'' such that

$$U(P', f) - L(P', f) < \frac{\varepsilon}{2c} \qquad \text{and} \qquad U(P'', f) - U(P'', g) < \frac{\varepsilon}{2}.$$

Setting $P = P' \cup P''$, (7.4) and (7.5) then imply that

$$\left| \int_a^b (cf + g) - \left(c \int_a^b f_+ \int_a^b g \right) \right| < \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we have

$$\int_a^b (cf + g) = c \int_a^b f + \int_a^b g.$$

We leave the case $c < 0$ as an exercise. □

Remark 7.14. It follows from Proposition 7.13 that the set of all integrable functions is a vector subspace of the vector space of all functions from $[a, b]$ to $\mathbb{R}$. Furthermore, the integral is a linear map from this vector space to $\mathbb{R}$.

If $f \colon [a, b] \to \mathbb{R}$ is bounded, then so is the function

$$|f| \colon [a, b] \to \mathbb{R}, \quad |f|(x) = |f(x)|, \quad x \in [a, b].$$

Lemma 7.15. *If $f \colon [a, b] \to \mathbb{R}$ is integrable, then so is $|f|$.*

Proof. Suppose $f\colon [a,b] \to \mathbb{R}$ is integrable. For any partition $P = \{x_0, \ldots, x_n\}$, we have

$$\begin{aligned} M_i(P,f) - m_i(P,f) &= \sup\{|f(x)-f(y)| : x,y \in [x_{i-1},x_i]\} && \text{(by Exercise 1.6.7)} \\ &\geq \sup\{\big||f(x)|-|f(y)|\big| : x,y \in [x_{i-1},x_i]\} && \text{(by Exercise 1.6.2)} \\ &= M_i(P,|f|) - m_i(P,|f|). && \text{(by Exercise 1.6.7)} \end{aligned}$$

Hence

$$U(P,|f|) - L(P,|f|) \leq U(P,f) - L(P,f).$$

It follows from Proposition 7.8 that $|f|$ is integrable. □

Note that the converse of Lemma 7.15 is *not* true. See Exercise 7.2.4.

Proposition 7.16. *If $f, g\colon [a,b] \to \mathbb{R}$ are integrable and $f(x) \leq g(x)$ for all $x \in [a,b]$, then*

$$\int_a^b f \leq \int_a^b g.$$

Proof. We leave the proof of this proposition as an exercise (Exercise 7.2.5). □

Corollary 7.17. *If $f\colon [a,b] \to \mathbb{R}$ is integrable, then*

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Proof. It follows from Proposition 7.16 and the fact that

$$-|f(x)| \leq f(x) \leq |f(x)| \quad \forall\, x \in [a,b]$$

that

$$-\int_a^b |f| \leq \left| \int_a^b f \right| \leq \int_a^b |f|.$$

The result follows. □

Corollary 7.18. *If $f\colon [a,b] \to \mathbb{R}$ is integrable and $m \leq f(x) \leq M$ for all $x \in [a,b]$, then*

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

Proof. This follows from Proposition 7.16 and Exercise 7.1.2. □

Suppose $f\colon [a,b] \to \mathbb{R}$ is integrable. By Proposition 7.12, f is integrable on $[a,x]$ for all $x \in [a,b]$.

Theorem 7.19. *If $f\colon [a,b] \to \mathbb{R}$ is integrable and we define*

$$F\colon [a,b] \to \mathbb{R}, \qquad F(x) = \int_a^x f,$$

then F is continuous on $[a,b]$.

Proof. Since f is bounded (we defined integrability only for bounded functions), there exists $M > 0$ such that

$$|f(x)| \leq M \qquad \forall, x \in [a, b].$$

Let $\varepsilon > 0$ and choose $\delta = \varepsilon/M$. Suppose $x, y \in [a, b]$ and $|x - y| < \delta$.

First consider the case $x < y$. By Corollary 7.18,

$$-M(y-x) \leq \int_x^y f \leq M(y-x).$$

We also have

$$\int_a^x + \int_x^y f = \int_a^y f,$$

and so

$$\int_x^y f = \int_a^y f - \int_a^x = F(y) - F(x).$$

Therefore,

$$-M(y-x) \leq F(y) - F(x) \leq M(y-x),$$

and so

$$|F(y) - F(x)| \leq M|x - y| < M\delta = \varepsilon.$$

The case $y < x$ is analogous (just interchange x and y everywhere in the above argument). □

Theorem 7.20 (Mean Value Theorem for Integrals)**.** *If $f\colon [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then there exists $c \in (a, b)$ such that*

$$\int_a^b f = f(c)(b-a).$$

Proof. By the Maximum Theorem (Theorem 5.28), there exists $x_{\min}, x_{\max} \in [a, b]$ such that

$$f(x_{\min}) \leq f(x) \leq f(x_{\max}) \quad \forall\, x \in [a, b].$$

Thus, by Corollary 7.18, we have

$$f(x_{\min})(b-a) \leq \int_a^b f \leq f(x_{\max})(b-a).$$

Thus

$$f(x_{\min}) \leq \frac{1}{b-a} \int_a^b f \leq f(x_{\max}).$$

By the Intermediate Value Theorem, there exists a c between $x_{\min}$ and $x_{\max}$ such that

$$\frac{1}{b-a} \int_a^b f = f(c).$$

The result then follows after multiplying both sides by $b - a$. □

Exercises.

7.2.1. Complete the proof of Proposition 7.12.

7.2.2. Prove (7.2).

7.2.3. Complete the proof of Proposition 7.13 by treating the case $c < 0$.

7.2.4. Give an example of a function $f\colon [a,b] \to \mathbb{R}$, such that $|f|$ is integrable, but f is *not* integrable.

7.2.5. Prove Proposition 7.16.

7.2.6. Find another solution to Exercise 7.1.6, this time using Proposition 7.12.

7.2.7 ([Leb, Ex. 5.2.5]). Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous, $f(x) \ge 0$ for all $x \in [a,b]$, and $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

7.2.8 ([Leb, Ex. 5.2.6]). Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous and $\int_a^b f = 0$. Prove that there is a $c \in [a,b]$ such that $f(c) = 0$.

7.2.9 ([Leb, Ex. 5.2.7]). Suppose $f, g\colon [a,b] \to \mathbb{R}$ is continuous and $\int_a^b f = \int_a^b g$. Prove that there is a $c \in [a,b]$ such that $f(c) = g(c)$.

7.2.10. Prove that if $f\colon [a,b] \to \mathbb{R}$ is increasing, then it is integrable. *Hint*: Use a uniform partition, where each subinterval has the same length.

7.3 Fundamental Theorem of Calculus

We now prove one of the most important theorems in calculus.

Theorem 7.21 (Fundamental Theorem of Calculus)**.** *Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous and define*

$$F\colon [a,b] \to \mathbb{R}, \quad F(x) = \int_a^x f.$$

If f is continuous at $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$.

Proof. Suppose f is continuous at c. Let $\varepsilon > 0$. Then we can choose $\delta > 0$ such that, for $x \in [a,b]$,

$$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon \implies f(c) - \varepsilon \le f(x) \le f(c) + \varepsilon.$$

Thus, if $x > c$, we have

$$\big(f(c) - \varepsilon\big)(x - c) \le \int_c^x f \le \big(f(c) + \varepsilon\big)(x - c).$$

When $x > c$, the inequalities are reversed. Therefore, if $c \neq x$, we have

$$f(c) - \varepsilon \leq \frac{\int_c^x f}{x - c} \leq f(c) + \varepsilon.$$

Since

$$F(x) - F(c) = \int_a^x f - \int_a^c f = \int_c^x f,$$

we have

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| \leq \varepsilon.$$

Since the absolute value function is continuous, this yields

$$|F'(c) - f(c)| = \lim_{x \to c} \left| \frac{F(x) - F(c)}{x - c} - f(c) \right| \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $F'(c) = f(c)$. □

Theorem 7.22. *Suppose $F\colon [a, b] \to \mathbb{R}$ is differentiable and F' is continuous on $[a, b]$. Then*

$$\int_a^b F' = F(b) - F(a).$$

Proof. Define

$$G\colon [a, b] \to \mathbb{R}, \quad G(x) = \int_a^x F'.$$

Then, by the Fundamental Theorem of Calculus (Theorem 7.21), for all $x \in [a, b]$, we have

$$F'(x) = G'(x).$$

By Exercise 6.4.8, there exists a $C \in \mathbb{R}$ such that $F(x) = G(x) + C$ for all $x \in [a, b]$. Since $G(a) = 0$, this implies

$$\int_a^b F' = G(b) = G(b) - G(a) = G(b) + C - (G(a) + C) = F(b) - F(a).$$ □

Of course, Theorem 7.22 is the basis of the method of computation of integrals you learned in calculus. To compute the integral $\int_a^b f$, you find an *antiderivative*. That is, you find a function F such that $F' = f$. Then $\int_a^b f = F(b) - F(a)$. Of course, this method only works if $f$is continuous and has an antiderivative. However, using the definition of the integral directly, we can sometimes compute the integrals of discontinuous functions. See Example 7.7 and Exercise 7.1.5.

Exercises.

7.3.1. Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous. Prove that

$$\frac{d}{dx}\int_x^b f = -f(x).$$

7.3.2 ([Leb, Ex. 5.3.1]). Compute $\displaystyle \frac{d}{dx}\int_{-x}^{x} e^{t^2}\,dt$.

7.3.3 ([Leb, Ex. 5.3.2]). Compute $\displaystyle \frac{d}{dx}\int_{0}^{x^2} \sin(t^2)\,dt$.

7.3.4 ([Leb, Ex. 5.3.3]). Suppose $F\colon [a,b] \to \mathbb{R}$ is continuous and differentiable on $[a,b] \setminus S$, where S is a finite set. Suppose that there exists an integrable function $f\colon [a,b] \to \mathbb{R}$ such that $f(x) = F'(x)$ for all $x \in [a,b] \setminus S$. Prove that $\int_a^b f = F(b) - F(a)$.

7.3.5 ([Leb, Ex. 5.3.3]). Prove *integration by parts*. That is, suppose F and G are continuously differentiable functions on $[a,b]$ (that is, they are differentiable, and their derivatives are continuous). Prove that

$$\int_a^b F(x)G'(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b F'(x)G(x)\,dx.$$

7.3.6 ([Leb, Ex. 5.3.3]). Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous and $\int_a^x f = \int_x^b f$ for all $x \in [a,b]$. Show that $f(x) = 0$.

7.4 Improper integrals

So far we have only discussed the integral of bounded functions. What should we mean by integrals such as

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx\ ?$$

Even if we define the function at the point zero (the particular value is irrelevant—see Exercise 7.1.6), the function $1/\sqrt{x}$ is not bounded on $(0,1]$. However, it is bounded on every interval of the form $[\delta,1]$, $0 < \delta < 1$. For such δ, since $2\sqrt{x}$ is an antiderivative of $1/\sqrt{x}$, Theorem 7.22 allows us to compute

$$\int_\delta^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{1} - 2\sqrt{\delta} = 2 - 2\sqrt{\delta}.$$

We therefore define

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\delta\to 0^+}\int_\delta^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\delta\to 0^+}\left(2 - 2\sqrt{\delta}\right) = 2.$$

Definition 7.23. Let $f\colon [a,b] \to \mathbb{R}$ be a function that is Riemann integrable on $[a+\delta,b]$ and unbounded in the interval $(a,a+\delta)$ for every $0 < \delta < b-a$. Then we define

$$\int_a^b f = \lim_{\delta\to 0^+}\int_{a+\delta}^b f$$

if this limit exists, and in this case the integral is said to be *convergent*.

Similarly, suppose $g\colon [a, b] \to \mathbb{R}$ is a function that is Riemann integrable on $[a, b-\delta]$ and unbounded in the interval $(b-\delta, b)$ for every $0 < \delta < b - a$. Then we define

$$\int_a^b g = \lim_{\delta \to 0^+} \int_a^{b-\delta} g$$

if this limit exists, and in this case the integral is said to be *convergent*.

Using a similar technique, we can extend the Riemann integral from bounded intervals to bounded ones. But first we need a definition.

Definition 7.24 (Limit at $\pm\infty$)**.** Suppose $A \subseteq \mathbb{R}$ is unbounded above and $f\colon A \to \mathbb{R}$. We say

$$\lim_{x \to \infty} f(x) = L$$

if for all $\varepsilon > 0$ there exists an $N \in \mathbb{R}$ such that

$$x > N \implies |f(x) - L| < \varepsilon.$$

Similarly, suppose $B \subseteq \mathbb{R}$ is unbounded below and $g\colon B \to \mathbb{R}$. We say

$$\lim_{x \to -\infty} g(x) = L$$

if for all $\varepsilon > 0$ there exists an $N \in \mathbb{R}$ such that

$$x < N \implies |f(x) - L| < \varepsilon.$$

Example 7.25. Define $f\colon [1, \infty) \to \mathbb{R}$ by $f(x) = 1/x$ for all $x \in [1, \infty)$. Let $\varepsilon > 0$ and choose $N = 1/\varepsilon$. Then, for $x > N$, we have

$$|f(x) - 0| = \frac{1}{x} < \frac{1}{N} \leq \varepsilon.$$

So

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

Now, how should we define the integral

$$\int_1^\infty \frac{1}{x^2}\, dx\ ?$$

Note that $-1/x$ is an antiderivative for $1/x^2$. Thus, for any $b > 1$, we have

$$\int_1^b \frac{1}{x^2}\, dx = \frac{-1}{b} - \frac{-1}{1} = 1 - \frac{1}{b}.$$

We therefore define

$$\int_1^\infty \frac{1}{x^2}\, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^2}\, dx = \lim_{b \to \infty} \left(1 - \frac{1}{b}\right) = 1.$$

Definition 7.26. Let f be a function on an interval $[a, \infty)$ that is Riemann integrable on every interval $[a, b]$ for $a < b < \infty$. Then we define

$$\int_a^\infty f = \lim_{b \to \infty} \int_a^b f$$

if this limit exists, and in this case the integral is said to be *convergent*.

Similarly, suppose $g \colon (-\infty, b] \to \mathbb{R}$ is a function that is Riemann integrable on every interval $[a, b]$ for $-\infty < a < b$. Then we define

$$\int_{-\infty}^b g = \lim_{a \to -\infty} \int_a^b g$$

if this limit exists, and in this case the integral is said to be *convergent*.

Remark 7.27. We can also define integrals where both limits of integration are infinite by choosing $c \in \mathbb{R}$ and defining

$$\int_{-\infty}^\infty f = \int_{-\infty}^c f + \int_c^\infty f,$$

provided both integrals on the right-hand side converge. It follows from Proposition 7.12 that the definition does not depend on the choice of c.

The integrals defined in Definitions 7.23 and 7.26 are called *improper integrals*.

Exercises.

7.4.1. Suppose $f \colon \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ both exist. Prove that f is bounded.

7.4.2. For which values of s does the improper integral $\int_0^1 x^s \, dx$ converge? You may use the antiderivatives of x^s, $s \in \mathbb{R}$, that you learned in calculus.

7.4.3. For which values of s does the improper integral $\int_1^\infty x^s \, dx$ converge? You may use the antiderivatives of x^s, $s \in \mathbb{R}$, that you learned in calculus.

7.4.4 ([Leb, Ex. 5.5.5]). Can you interpret

$$\int_{-1}^1 \frac{1}{\sqrt{|x|}} \, dx$$

as an improper integral? If so, compute its value.

7.4.5. Suppose $f \colon [0, \infty) \to \mathbb{R}$ is integrable on every interval $[0, b]$, $b > 0$. Show that $\int_0^\infty f$ converges then for every $\varepsilon > 0$ there exists an M such that

$$M \le a < b \implies \left| \int_a^b f \right| < \varepsilon.$$

(The converse is also true.)

7.4.6 ([Leb, Ex. 5.5.9]). Suppose $f\colon [0,\infty) \to \mathbb{R}$ is nonnegative and decreasing.

(a) Show that if $\int_0^\infty f$ converges, then $\lim_{x\to\infty} f(x) = 0$.

(b) Show that the converse does not hold.

Chapter 8

Sequences and series of functions

In this chapter we consider sequences and series of functions. We begin with the most naive definition of convergence: pointwise convergence. We consider several examples that illustrate that this type of convergence is not very well behaved. We then introduce the concept of uniform convergence and prove that this convergence behaves well with respect to continuity, differentiation, and integration. A good reference for the material in this chapter is [TBB, Ch. 9].

Although most of the examples we will consider involve functions from intervals to $\mathbb{R}$, we will state the main theorems in greater generality, for functions from some subset of $\mathbb{R}^d$ to $\mathbb{R}^m$. Throughout this chapter, $A \subseteq \mathbb{R}^d$.

8.1 Pointwise convergence

We begin with the most naive definition of the convergence of functions.

Definition 8.1 (Pointwise convergence of a sequence of functions)**.** Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of functions with $f_n \colon A \to \mathbb{R}^m$ for all $n \in \mathbb{N}$, and $f \colon A \to \mathbb{R}^m$. We say that the sequence $\{f_n\}$ *converges pointwise* to f on A if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall\, x \in A.$$

In this case, we write

$$\lim_{n\to\infty} f_n = f \qquad \text{or} \qquad f_n \to f.$$

Definition 8.2 (Pointwise convergence of a series of functions)**.** Suppose $\{f_k\}_{k=1}^\infty$ is a sequence of functions with $f_k \colon A \to \mathbb{R}^m$ for all $n \in \mathbb{N}$, and $f \colon A \to \mathbb{R}^m$. For $n \in \mathbb{N}$ and $x \in A$, let

$$S_n(x) = \sum_{k=1}^n f(x).$$

We say that the series $\sum_{k=1}^\infty f_k$ *converges pointwise to f at x* if

$$\lim_{n\to\infty} S_n(x) = f(x).$$

If $\sum_{k=1}^{\infty} f_k$ converges pointwise to f at all $x \in A$, we say that the series $\sum_{k=1}^{\infty} f_k$ *converges pointwise to f on A*. In this case, we write

$$f(x) = \sum_{k=1}^{\infty} f_k(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x).$$

Example 8.3. For $n \in \{0, 1, 2, \dots\}$, define

$$f_n(x) = x^n.$$

For $|x| < 1$, the series

$$\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

converges. It diverges for $|x| \geq 1$. Thus $\sum_{n=0}^{\infty} f_n$ converges pointwise on $(-1, 1)$ to the function f given by $f(x) = \frac{1}{1-x}$.

In the remainder of this section, we will see that pointwise convergence is not very well behaved with respect to some of the properties of functions we have discussed in this course: continuity, differentiability, and integrability.

Example 8.4 (A discontinuous limit of continuous functions). For each $n \in \mathbb{N}$ and $x \in [0, 1]$, let

$$f_n(x) = x^n.$$

See Figure 8.1. Each f_n is continuous on $[0, 1]$. For $x = 0$, it is clear that $\lim_{n\to\infty} f_n(0) = 0$.

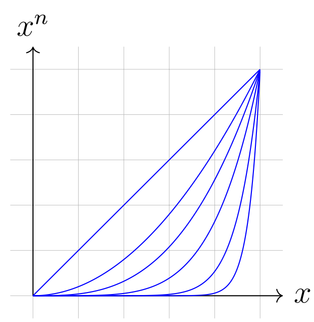


Figure 8.1: Graphs of x^n on $[0, 1]$ for $n = 1, 2, 3, 5, 10, 20$

Similarly, for $x = 1$, it is clear that $\lim_{n\to\infty} f_n(1) = 1$. Now suppose $0 < x < 1$. Let $\varepsilon > 0$ and set $N > (\ln \varepsilon)/(\ln x)$. Then $x^N < \varepsilon$. Hence,

$$n \geq N \implies |f_n(x) - 0| = x^n \leq x^N < \varepsilon.$$

Thus,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Therefore, the pointwise limit f of the sequence of continuous functions $\{f_n\}_{n=1}^{\infty}$ is discontinuous at $x = 1$. So a pointwise limit of continuous functions is not necessarily continuous.

Example 8.5 (The derivative of the limit need not be the limit of the derivative). For $n \in \mathbb{N}$, define

$$f_n\colon [0,1] \to \mathbb{R}, \quad f(x) = \frac{x^n}{n}.$$

Then $f_n \to 0$ on $[0,1]$. By Example 8.4, we have

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} x^{n-1} = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Since the derivative of the limit function is zero everywhere on $[0,1]$, we have

$$\lim_{n\to\infty} \frac{d}{dx} f_n(x) \neq \frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right) \qquad \text{at } x = 1.$$

So the derivative of a pointwise limit of functions is not necessarily the limit of the derivatives.

Example 8.6 (The integral of the limit need not be the limit of the integrals). For each $n \in \mathbb{N}$, define $f_n\colon [0,1] \to \mathbb{R}$ by

$$f_n = \begin{cases} 4n^2x & \text{if } 0 \le x \le 1/2n, \\ 4n - 4n^2x & \text{if } 1/2n < x \le 1/n, \\ 0 & \text{if } 1/n < x \le 1. \end{cases}$$

See Figure 8.2. It is easy to see that $f_n \to 0$ on $[0,1]$ (Exercise 8.1.1). Now, for each $n \in \mathbb{N}$,

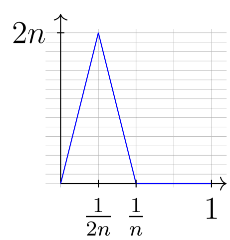


Figure 8.2: Graph of $f_n(x)$ on $[0,1]$ in Example 8.6

$$\int_0^1 f_n(x)\,dx = 1.$$

However,

$$\int_0^1 \left(\lim_{n\to\infty} f_n(x)\right)\,dx = \int_0^1 0\,dx = 0.$$

Thus

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \left(\lim_{n\to\infty} f_n(x)\right)\,dx.$$

So the integral of a pointwise limit of functions is not necessarily the limit of the integrals.

Exercises.

8.1.1. If f_n is defined as in Example 8.6, prove that $f_n \to 0$ on $[0,1]$.

8.1.2 ([TBB, Ex. 9.2.1]). Examine the pointwise limiting behaviour of the sequence of functions

$$f_n\colon \mathbb{R}\setminus\{-1\} \to \mathbb{R}, \qquad f_n(x) = \frac{x^n}{1+x^n}.$$

8.1.3 ([TBB, Ex. 9.2.3]). The rational numbers are *countable*. In particular, there is a sequence $\{x_n\}_{n=1}^{\infty}$, such that $\mathbb{Q} \cap [0,1] = \{x_1, x_2, \dots\}$. Let

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{x_1, \dots, x_n\}, \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f_n converges pointwise to f on $[0,1]$, but $\int_0^1 f_n(x)\,dx = 0$ for all $n \in \mathbb{N}$, while f is not integrable on $[0,1]$.

8.1.4 ([TBB, Ex. 9.2.4]). For $n \in \mathbb{N}$, let

$$f_n(x) = \frac{\sin(nx)}{\sqrt{n}}.$$

Show that $\lim_{n\to\infty} f_n = 0$, but $\lim_{n\to\infty} f_n'(0) = \infty$.

8.1.5 ([TBB, Ex. 9.2.7]). Suppose $f_n \to f$ on the interval $[a,b]$. Which of the following statements is true?

(a) If each f_n is strictly increasing on $[a,b]$, then so is f.

(b) If each f_n is weakly increasing on $[a,b]$, then so is f.

(c) If each f_n is bounded on $[a,b]$, then so is f.

(d) If each f_n is everywhere discontinuous on $[a,b]$, then so is f.

(e) If each f_n is constant on $[a,b]$, then so is f.

(f) If each f_n is strictly positive on $[a,b]$, then so is f.

(g) If each f_n is linear on $[a,b]$, then so is f.

8.2 Uniform convergence

We have seen in Section 8.1 that pointwise convergence does not always behave nicely with respect to continuity, differentiation, and integration. Note that we can reformulate Definition 8.1 by saying that $f_n \to f$ if

$$\forall\, x \in A,\ \forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that } \forall\, n \geq N,\ \|f_n(x) - f(x)\| < \varepsilon.$$

Thus, the choice of N can depend on both ε and x. We now formulate a stronger notion of convergence, where the N depends only on ε. Compare this to the difference between continuity (Definition 5.12) and uniform continuity (Definition 5.39).

Definition 8.7 (Uniform convergence)**.** Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of functions with $f_n \colon A \to \mathbb{R}^m$ for all $n \in \mathbb{N}$, and $f \colon A \to \mathbb{R}^m$. We say that the sequence $\{f_n\}_{n=1}^\infty$ *converges uniformly* to f on A if, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|f_n(x) - f(x)\| < \varepsilon \text{ for all } n \geq N \text{ and } x \in A.$$

In this case, we write

$$f_n \rightrightarrows f.$$

We say that the series $\sum_{k=1}^\infty f_k$ converges uniformly to f if the sequence of partial sums $\{\sum_{k=1}^n f_k\}_{n=1}^\infty$ converges uniformly to f.

Note that it follows immediately from the definitions that if $\{f_n\}_{n=1}^\infty$ converges uniformly to f, then $\{f_n\}_{n=1}^\infty$ also converges pointwise to f.

Example 8.8. Let us modify the functions of Example 8.4 by restricting the domain of the functions. Fix $0 < \eta < 1$ and, for $n \in \mathbb{N}$, define

$$f_n \colon [0, \eta] \to \mathbb{R}, \quad f_n(x) = x^n.$$

We will show that $f_n \rightrightarrows 0$. Indeed, for $0 \leq x \leq \eta$, we have $0 \leq x^n \leq \eta^n$. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} \eta^n = 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies 0 < \eta^n < \varepsilon.$$

Thus, for all $n \geq N$, we have

$$0 \leq x^n \leq \eta^n < \varepsilon.$$

Note that our choice of N works for all $x \in [0, \eta]$ (it does not depend on x).

When discussing convergence of sequences in $\mathbb{R}^d$, we had the Cauchy criterion that told us whether or not a sequence converged, without explicitly referring to a limit point. We have a similar criterion for the uniform convergence of sequences of functions.

Definition 8.9 (Uniformly Cauchy)**.** Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of functions with $f_n \colon A \to \mathbb{R}^m$ for all $n \in \mathbb{N}$. We say this sequence is *uniformly Cauchy* on A if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|f_m(x) - f_n(x)\| < \varepsilon \quad \text{for all } n, m \geq N \text{ and } x \in A.$$

Theorem 8.10 (Cauchy convergence criterion for uniform convergence)**.** *Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of functions with $f_n\colon A \to \mathbb{R}^m$ for all $n \in \mathbb{N}$. Then this sequence converges uniformly to some $f\colon A \to \mathbb{R}^m$ on A if and only if $\{f_n\}_{n=1}^\infty$ is uniformly Cauchy.*

Proof. The proof of this theorem is left as an exercise (Exercise 8.2.1). □

Example 8.11. Consider the geometric series $\sum_{n=0}^\infty x^n$ on $(-1, 1)$. We will use Theorem 8.10 to prove by contradiction that this series does *not* converge uniformly on $(-1, 1)$. Suppose it converged uniformly, and let $f_n(x) = \sum_{k=0}^\infty x^k$ be the n-th partial sum. Then, taking $\varepsilon = 1$ in Theorem 8.10, there exists $N \in \mathbb{N}$ such that, for $N \le m < n$ and $x \in (-1, 1)$, we have

$$1 > |f_n(x) - f_m(x)| = \sum_{k=m+1}^{n} x^k = \frac{x^{m+1}(1 - x^{n-m})}{1 - x} = \frac{x^{m+1} - x^{n+1}}{1 - x}.$$

Thus, for all $x \in (-1, 1)$, we have

$$\frac{x^{m+1}}{1 - x} = \lim_{n\to\infty} \frac{x^{m+1} - x^{n+1}}{1 - x} \le 1.$$

But we know that

$$\lim_{x\to 1^-} \frac{x^{m+1}}{1 - x} = \infty,$$

which contradicts the above inequality.

When we are interested in the uniform convergence of a series of functions, there is a useful simple test, which we now describe.

Theorem 8.12 (Weierstrass M-test)**.** *Suppose $\{f_k\}_{k=1}^\infty$ is a sequence of functions with $f_k\colon A \to \mathbb{R}^m$ for all $k \in \mathbb{N}$, and let $\{M_k\}_{k=1}^\infty$ be sequence of positive real numbers. If*

$$\sum_{k=0}^{\infty} M_k < \infty$$

(i.e. this series converges) and

$$\|f_k(x)\| \le M_k \quad \textit{for all } x \in A,\ k \in \mathbb{N},$$

then the series $\sum_{k=1}^\infty f_k$ converges uniformly on A.

Proof. Let $S_n = \sum_{k=1}^n f_k$ be the n-th partial sum of the series. We will show that the sequence $\{S_n\}_{n=1}^\infty$ is uniformly Cauchy on A. Let $\varepsilon > 0$. For $m < n$ and $x \in A$, we have

$$S_n(x) - S_m(x) = f_{m+1}(x) + \cdots + f_n(x).$$

Thus

$$\|S_n(x) - S_m(x)\| \le M_{m+1} + \cdots + M_n.$$

Since the series $\sum_{k=1}^\infty M_k$ converges by hypothesis, by the Cauchy convergence criterion for $\mathbb{R}^m$, there exists $N \in \mathbb{N}$ such that

$$M_{m+1} + \cdots + M_n < \varepsilon.$$

Thus

$$\|S_n(x) - S_m(x)\| < \varepsilon \quad \text{for all } n > m \geq N \text{ and } x \in A.$$

Hence, the sequence $\{S_n\}_{n=1}^{\infty}$ is uniformly convergent on A. Thus, the series $\sum_{k=1}^{\infty} f_k$ is uniformly convergent on A. □

Example 8.13. Fix $0 < a < 1$ and consider the geometric series $\sum_{n=0}^{\infty} x^n$ on the interval $[-a, a]$. Then

$$|x^k| \leq a^k \quad \text{for all } k \in \{0, 1, 3, \dots, \},\ x \in [-a, a].$$

Since $\sum_{k=0}^{\infty} a^k$ converges, the series $\sum_{k=0}^{\infty} x_k$ converges uniformly on $[-a, a]$ by the Weierstrass M-test. Compare this to Example 8.11.

Exercises.

8.2.1. Prove Theorem 8.10.

8.2.2. For $n \in \mathbb{N}$, define

$$f_n(x) = \frac{x^n}{1 + x^n}.$$

Prove that the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on intervals of the form $(-\infty, -c]$ and $[c, \infty)$ for $c > 1$ and on intervals of the form $[-d, d]$ for $0 < d < 1$.

8.2.3 ([TBB, Ex. 9.3.3]). Prove that if $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f on a finite set A, then the convergence is uniform.

8.2.4 ([TBB, Ex. 9.3.4]). Prove that if $f_n \rightrightarrows f$ on a set A_1 and also on a set A_2, then $f_n \rightrightarrows f$ on $A_1 \cup A_2$.

8.2.5 ([TBB, Ex. 9.3.5]). Prove or disprove that if $f_n \rightrightarrows f$ on each set $A_1, A_2, \dots,$ then $f_n \rightrightarrows f$ on the union $\bigcup_{k=1}^{\infty} A_k$.

8.2.6. Prove or disprove that if $f_n \rightrightarrows f$ on $[a, b]$ for all $a < b$, then $f_n \rightrightarrows f$ on $\mathbb{R}$.

8.2.7 ([TBB, Ex. 9.3.8]). Prove or disprove that if $f_n \rightrightarrows f$ on each closed interval $[a, b]$ contained in an open interval (c, d), then $f_n \rightrightarrows f$ on (c, d).

8.2.8 ([TBB, Ex. 9.3.8]). Prove that if $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ both converge uniformly on A, then so too does the sequence $\{f_n + g_n\}_{n=1}^{\infty}$.

8.2.9 ([TBB, Ex. 9.3.12]). Prove that $f_n \rightrightarrows f$ on A of and only if

$$\lim_{n\to\infty} \sup_{x \in A} \|f_n(x) - f(x)\| = 0.$$

8.2.10. Prove that

$$\sum_{k=1}^{\infty} kx^{k-1}$$

converges uniformly on $[a, b]$ for all $-1 < a < b < 1$.

8.2.11 ([TBB, Ex. 9.3.20]). Prove that the series

$$\sum_{k=0}^{\infty} \frac{x^k}{k}$$

converges uniformly on $[0, b]$ for every $b \in [0, 1)$, but does not converge uniformly on $[0, 1)$.

8.3 Properties of uniform convergence

In Example 8.4 we saw that a pointwise limit of continuous functions may be discontinuous. However, we now see that a *uniform* limit of continuous functions *is* continuous.

Theorem 8.14 (Uniform limit theorem). *Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions with $f_n \colon A \to \mathbb{R}^m$ for all $n \in \mathbb{N}$, and that this sequence converges uniformly to the function $f \colon A \to \mathbb{R}^m$. If f_n is continuous at x_0 for all $n \in \mathbb{N}$, then f is also continuous at x_0. In particular, if f_n is continuous on A for all $n \in \mathbb{N}$, then f is also continuous on A.*

Proof. Let $\varepsilon > 0$. For each $x \in A$ and $n \in \mathbb{N}$, we have

$$\|f(x) - f(x_0)\| \le \|f(x) - f_n(x)\| + \|f_n(x) - f_n(x_0)\| + \|f_n(x_0) - f(x_0)\|.$$

Since f_n converges to f *uniformly*, there exists $N \in \mathbb{N}$ such that

$$\|f_n(x) - f(x)\| < \frac{\varepsilon}{3} \quad \text{for all } x \in A,\ n \ge N.$$

In addition, since f_N is continuous at x_0, there exists $\delta > 0$ such that

$$x \in A,\ \|x - x_0\| < \delta \implies \|f_N(x) - f_N(x_0)\| < \frac{\varepsilon}{3}.$$

Thus, for all $x \in A$ with $|x - x_0| < \delta$, we have

$$\|f(x) - f(x_0)\| \le \|f(x) - f_N(x)\| + \|f_N(x) - f_N(x_0)\| + \|f_N(x_0) - f(x_0)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence f is continuous at x_0. □

Corollary 8.15. *If $\sum_{k=1}^{\infty} f_k$ converges uniformly to f on A, and each of the functions f_k is continuous on A, then f is continuous on A.*

Proof. If each f_k is continuous, then the partial sums $\sum_{k=1}^{n} f_k$ are continuous. Then the result follows from Theorem 8.14. □

Example 8.16. Fix $a \in (0,\infty)$. Then, for all $x \in [-a,a]$, we have

$$\left| \frac{x^n}{n!} \right| \le \frac{a^n}{n!}.$$

Since the series $\sum_{0=1}^\infty \frac{a^n}{n!}$ converges (Exercise 8.3.2), the series

$$\sum_{n=0}^\infty \frac{x^n}{n!}$$

converges uniformly on $[-a,a]$ by the Weierstrass M-test (Theorem 8.12). Thus, by Theorem 8.14, the function

$$e^x := \sum_{n=0}^\infty \frac{x^n}{n!}$$

is continuous on $[-a,a]$. Since a was arbitrary, e^x is continuous on $\mathbb{R}$.

In Example 8.6, we saw that the integral of a pointwise limit is not necessarily the limit of the integrals. We now see that this problem is resolved by requiring *uniform* convergence.

Theorem 8.17. *Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of continuous functions with $f_n \colon [a,b] \to \mathbb{R}$ for all $n \in \mathbb{N}$, and that this sequence converges uniformly to the function $f \colon [a,b] \to \mathbb{R}$. Then*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

Proof. Since f is continuous by Theorem 8.14, the integral $\int_a^b f(x)\,dx$ exists by Theorem 7.10.

Let $\varepsilon > 0$. We have

$$\begin{aligned} \left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| &= \left| \int_a^b \big(f_n(x) - f(x)\big)\,dx \right| \\ &\le \int_a^b |f_n(x) - f(x)|\,dx && \text{(by Cor. 7.17)} \\ &\le (b-a) \left(\max_{x\in[a,b]} |f_n(x) - f(x)| \right) && \text{(by Cor. 7.18).} \end{aligned}$$

Since $f_n \rightrightarrows f$, there exists $N \in \mathbb{N}$ such that

$$\max_{x\in[a,b]} |f_n(x) - f(x)| < \frac{\varepsilon}{b-a} \quad \forall\, n \ge N.$$

(See Exercise 8.2.9.) Thus, for $n \ge N$, we have

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| < (b-a)\frac{\varepsilon}{b-a} = \varepsilon,$$

as required. □

Corollary 8.18. *If an infinite series of continuous functions* $\sum_{k=1}^{\infty} f_k$ *converges uniformly to a function* f *on an interval* $[a, b]$*, then* f *is also continuous and*

$$\int_a^b f(x)\,dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx. \tag{8.1}$$

Proof. This follows from applying Theorem 8.17 to the partial sums of the series. □

Example 8.19. By Example 8.13, the geometric series

$$\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k$$

converges uniformly on the interval $[0, x]$ for all $0 < x < 1$. Thus, by Theorem 8.17, we have

$$\int_0^x \frac{1}{1-t}\,dt = \sum_{k=0}^{\infty} \int_0^x t^k\,dt = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}.$$

Remark 8.20. In Theorem 8.17, we assumed that the functions f_n are continuous. In fact, this hypothesis can be weakened. We need only assume that the f_n are integrable to conclude that f is integrable and (8.1) holds. See [TBB, §9.5.2].

We can now also prove that uniform convergence behaves well with respect to derivatives.

Theorem 8.21. *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of functions, each with a continuous derivative on an interval* $[a, b]$*. If the sequence* $\{f_n'\}_{n=1}^{\infty}$ *of derivatives converges uniformly on* $[a, b]$ *and the sequence* $\{f_n\}_{n=1}^{\infty}$ *converges pointwise to a function* f*, then* f *is differentiable on* $[a, b]$ *and*

$$f'(x) = \lim_{n\to\infty} f_n'(x) \quad \textit{for all } x \in [a, b].$$

Proof. Let $g = \lim_{n\to\infty} f_n'$. Since each f_n' is continuous and the f_n' converge uniformly to g, the function g is continuous by Theorem 8.14. By Theorem 8.17, we have

$$\int_a^x g(t)\,dt = \lim_{n\to\infty} \int_a^x f_n'(t)\,dt \quad \text{for all } x \in [a, b].$$

By Theorem 7.22, we see that, for all $n \in \mathbb{N}$,

$$\int_a^x f_n'(t)\,dt = f_n(x) - f_n(a) \quad \text{for all } x \in [a, b].$$

Thus

$$\int_a^x g(t)\,dt = \lim_{n\to\infty} \big(f_n(x) - f_n(a)\big) = f(x) - f(a).$$

Therefore,

$$f(x) = \int_a^x g(t)\,dt + f(a).$$

Since g is continuous, it follows from the Fundamental Theorem of Calculus (Theorem 7.21) that f is differentiable and that

$$f'(x) = g(x) \quad \text{for all } x \in [a, b].$$

□

Corollary 8.22. *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of functions each with a continuous derivative on $[a,b]$, and suppose $f = \sum_{k=1}^{\infty} f_k$ on $[a,b]$. If the series $\sum_{k=1}^{\infty} f_k'$ converges uniformly on $[a,b]$, then $f' = \sum_{k=1}^{\infty} f_k'$ on $[a,b]$.*

Proof. We apply Theorem 8.21 to the partial sums. □

Example 8.23. Fix $a \in (0,\infty)$. For each $n \in \{0,1,2,\dots\}$, let $f_n(x) = x^n/n!$. Then each f_n has a continuous derivative on $[-a,a]$. As shown in Example 8.16, $e^x = \sum_{n=0}^{\infty} f_n(x)$ converges (uniformly) on $[-a,a]$. Now

$$\sum_{n=0}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

where we let $k = n-1$ in the last equality. By Example 8.16, $\sum_{k=0}^{\infty} x^k/k!$ converges uniformly to e^x on $[-a,a]$. Therefore, by Corollary 8.22, we have

$$\frac{d}{dx} e^x = e^x$$

on $[-a,a]$. Since a was arbitrary, this equality holds for all $x \in \mathbb{R}$.

Exercises.

8.3.1 ([TBB, Ex. 9.4.1]). Can a sequence of discontinuous functions converge uniformly on a interval to a continuous function?

8.3.2. Prove that, for all $a \in \mathbb{R}$, the series $\sum_{n=0}^{\infty} a^n/n!$ converges.

8.3.3 ([TBB, Ex. 9.5.1]). Prove that

$$\lim_{n\to\infty} \int_{\pi/2}^{\pi} \frac{\sin(nx)}{nx}\, dx = 0.$$

8.3.4 ([TBB, Ex. 9.5.3]). Show that if $f_n \rightrightarrows f$ on $[a,b]$, and each f_n is continuous, then the sequence of functions

$$F_n(x) = \int_a^x f_n(t)\, dt$$

also converges uniformly on $[a,b]$.

8.3.5. Prove that

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1}$$

for all $x \in (-1,1)$. *Hint*: Use Corollary 8.22 and the formula for the sum of a geometric series.

8.3.6 ([TBB, Ex. 9.6.2]). Verify that the function

$$y(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$$

is a solution of the differential equation $y' = 2xy$ on $(-\infty, \infty)$ without first finding an explicit formula for $y(x)$.

Chapter 9

Power series

In this final chapter, we investigate some important examples of series of functions. Namely, we examine *power series* and *Fourier series*. A good reference for the material in this section is [TBB, Ch. 10].

9.1 Convergence of power series

Definition 9.1 (Power series)**.** Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers and let $c \in \mathbb{R}$. A series of the form

$$\sum_{k=0}^{\infty} a_k (x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

is called a *power series* centred at c. The numbers a_k are called the *coefficients* of the power series.

A power series clearly converges at the center $x = c$. We now consider some examples that illustrate various possibilities for the sets on which a power series converges.

Example 9.2. The series

$$\sum_{k=1}^{\infty} k^k x^k$$

diverges whenever $x \neq 0$ since the terms $(xk)^k$ do not tend to zero when $k \to \infty$. (For large enough k, we have $xk \geq 1$.) Thus, this power series converges *only* at its center.

Example 9.3. We know that the geometric series

$$\sum_{k=0}^{\infty} x^k$$

converges precisely on the interval $(-1, 1)$.

Example 9.4. Consider the series
$$\sum_{k=1}^{\infty} \frac{x^k}{k}.$$
We have
$$\limsup_{k\to\infty} \sqrt[k]{\frac{|x|^k}{k}} = \limsup_{k\to\infty} \frac{|x|}{\sqrt[k]{k}} = |x|,$$
where we used Exercise 2.1.15 in the last equality. Therefore, by the root test (Proposition 3.19), the series converges on $(-1,1)$ and diverges for $|x|>1$. For $x=1$, the series is the harmonic series, which diverges. For $x=-1$, it is the alternating harmonic series, which converges. Thus, the series converges precisely on the interval $[-1,1)$.

Example 9.5. Consider the series
$$\sum_{k=1}^{\infty} \frac{x^k}{k^2}.$$
For $|x| \le 1$, we have
$$\left|\frac{x^k}{k^2}\right| \le \frac{1}{k^2},$$
and thus the series converges for these values of x. If $|x|>1$, the terms $|x^k/k^2| \to \infty$, and so the series diverges. Hence the series converges precisely on the interval $[-1,1]$.

Example 9.6. We have seen in Exercise 8.3.2 that the series
$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$
converges for all $x \in \mathbb{R} = (-\infty,\infty)$.

In all of the above examples, the series converges on an interval centred at the centre of the series (0 in all the examples above). This, of course, explains the terminology *centre*. We will see that this is *always* the case.

Definition 9.7 (Radius of convergence)**.** Let $\sum_{k=0}^{\infty} a_k(x-c)^k$ be a power series. Then
$$R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}}$$
is called the *radius of convergence* of the series. Here we interpret $R=\infty$ if $\limsup_{k\to\infty}\sqrt[k]{|a_k|} = 0$ and $R=0$ if $\limsup_{k\to\infty}\sqrt[k]{|a_k|}=\infty$.

Theorem 9.8. *Let $\sum_{k=0}^{\infty} a_k(x-c)^k$ be a power series with radius of convergence R.*

(a) If $R=0$, then the series converges only at $x=c$.

(b) If $R=\infty$, then the series converges absolutely for all x.

(c) *If $0 < R < \infty$, then the series converges absolutely for all $x \in (c-R, c+R)$ and diverges for all $x \notin [c-R, c+R]$.*

Proof. For $x \neq c$, we have

$$\limsup_{k\to\infty} \sqrt[k]{|a_k||x-c|^k} = |x-c| \limsup_{k\to\infty} \sqrt[k]{|a_k|} = \begin{cases} \infty & \text{if } R = 0, \\ 0 & \text{if } R = \infty, \\ \frac{|x-c|}{R} & \text{if } 0 < R < \infty. \end{cases}$$

The result then follows from the root test. □

By Theorem 9.8, the set of convergence of a power series $\sum_{k=0}^{\infty} a_k x^k$ with a finite radius of convergence must be one of the four intervals

$$(-R, R), \quad [-R, R], \quad (-R, R], \quad [-R, R).$$

We saw in Chapter 8 that, in order to differentiate and integrate series of functions term-by-term, we need uniform convergence. However, Theorem 9.8 says nothing about uniform convergence. We therefore need the following result.

Theorem 9.9. *Let $\sum_{k=0}^{\infty} a_k(x-c)^k$ be a power series with radius of convergence R.*

(a) *If $R = 0$, then the series converges only at $x = c$.*

(b) *If $R = \infty$, then the series converges absolutely and uniformly on any compact interval $[a, b]$.*

(c) *If $0 < R < \infty$, then the series converges absolutely and uniformly on any interval $[a, b]$ contained entirely inside the interval $(c-R, c+R)$.*

Proof. The case (a) is the same as in Theorem 9.8. To prove cases (b) and (c), choose ρ satisfying $0 < \rho < R$ so that the interval $[a, b] \subseteq (c-\rho, c+\rho)$. Fix ρ_0 such that $\rho < \rho_0 < R$. Then

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} = \frac{1}{R} < \frac{1}{\rho_0}.$$

Thus, there exists $N \in \mathbb{N}$ such that

$$\sqrt[k]{|a_k|} < \frac{1}{\rho_0} \quad \text{for all } k \geq N. \tag{9.1}$$

Then, for $k \geq N$ and $x \in (c-\rho, c+\rho)$, we have

$$\left|a_k(x-c)^k\right| \leq |a_k|\rho^k < \left(\frac{\rho}{\rho_0}\right)^k,$$

where the last inequality follows from (9.1). Since $\rho/\rho_0 < 1$, the series

$$\sum_{k=0}^{\infty} \left(\frac{\rho}{\rho_0}\right)^k$$

converges. Therefore, by the Weierstrass M-test (Theorem 8.12), the given power series converges absolutely on $(c-\rho, c+\rho)$, hence also on the subset $[a, b]$. □

Example 9.10. Consider the geometric series $\sum_{n=0}^{\infty} x^n$, with radius of convergence $R = 1$. This series converges on $(-1, 1)$ (see Example 9.3), but does not converge uniformly on all of $(-1, 1)$ (see Example 8.11). However, it *does* converge uniformly on any $[a, b] \subseteq (-1, 1)$.

Theorem 9.9 is sufficient for examining the interior of the interval of convergence. However, when we wish to make statements about the endpoints of intervals of convergence of the form $[c - R, c + R]$, $(c - R, c + R]$, or $[c - R, c + r)$, we need something further.

Theorem 9.11. *Suppose that the power series $\sum_{k=0}^{\infty} a_k(x - c)^k$ has a finite and positive radius of convergence R and an interval of convergence I.*

(a) If $I = [c-R, c+R]$, then the series converges uniformly (but not necessarily absolutely) on I.

(b) If $I = (c-R, c+R]$, then the series converges uniformly (but not necessarily absolutely) on any interval $[a, c + R]$ for all $c - R < a < c + R$.

(c) If $I = [c-R, c+R)$, then the series converges uniformly (but not necessarily absolutely) on any interval $[c - R, b]$ for all $c - R < b < c + R$.

(d) If $I = (c-R, c+R)$, then the series converges uniformly and absolutely on any interval $[a, b]$ for all $c - R < a < b < c + R$.

Proof. Note that part (d) is a repeat of Theorem 9.9(c). The method of proof of the other statements can be found in [TBB, Th. 10.10]. □

Exercises.

9.1.1. Find the radius of convergence for each of the following series.

(a) $\displaystyle\sum_{k=0}^{\infty} (-1)^k x^{2k}$.

(b) $\displaystyle\sum_{k=0}^{\infty} k x^k$.

(c) $\displaystyle\sum_{k=0}^{\infty} k! x^k$. *Hint*: Note that in the expression $n! = n(n-1)(n-2)\cdots 2 \cdot 1$, at least $n/2$ terms are greater than $n/2$.

9.1.2 ([TBB, Ex. 10.2.8]). Give an example of a power series $\sum_{k=0}^{\infty} a_k x^k$ with interval of convergence exactly $[-\sqrt{2}, \sqrt{2})$.

9.1.3 ([TBB, Ex. 10.2.10]). If the coefficients $\{a_k\}_{k=0}^{\infty}$ of a power series $\sum_{k=0}^{\infty} a_k x^k$ form a bounded sequence, show that the radius of convergence is at least 1.

9.1.4 ([TBB, Ex. 10.2.11]). If the power series $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence R_a, the power series $\sum_{k=0}^{\infty} b_k x^k$ has radius of convergence R_b, and $|a_k| \leq |b_k|$ for all sufficiently large k, what relation must hold between R_a and R_b?

9.1.5 ([TBB, Ex. 10.2.12]). If the power series $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence R, what is the radius of convergence of the series $\sum_{k=0}^{\infty} a_k x^{2k}$?

9.1.6 ([TBB, Ex. 10.2.15]). Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers, and let $x_0 \in \mathbb{R}$. Suppose there exists $M > 0$ such that $|a_k x_0^k| \leq M$ for all $k \in \{0, 1, 2, \dots\}$. Prove that $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely for all x satisfying $|x| < |x_0|$. What can you say about the radius of convergence of the series?

9.1.7 ([TBB, Ex. 10.3.2]). Show that if $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely at a point $x_0 > 0$, then the convergence of the series is uniform on $[-x_0, x_0]$.

9.2 Properties of power series

We now investigate the properties of functions represented by power series. In particular, we will be concerned with their continuity, differentiability, and integrability.

Proposition 9.12 (Continuity of power series)**.** *A function f represented by a power series*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$$

is continuous on its interval of convergence.

Proof. Let R be the radius of convergence. If $R = 0$, then the interval of convergence is a single point and there is nothing to prove. Suppose $R > 0$ and x is in the interval of convergence. By Theorem 9.11 (in the case $R < \infty$) and Theorem 9.9(b) (in the case $R = \infty$), we can find a compact interval $[a, b]$, contained in the interval of convergence, such that $x \in [a, b]$. Then, by Corollary 8.15, f is continuous at x. □

Example 9.13. The series

$$\sum_{k=1}^{\infty} \frac{x^k}{k}$$

converges on the interval $[-1, 1)$ (see Example 9.4). Thus it is continuous on this interval.

Proposition 9.14 (Integration of power series)**.** *Let f be a function represented by a power series*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k,$$

with interval of convergence I. Then, for every point $x \in I$, the function f is integrable on $[c, x]$ (if $x \geq c$) or $[x, c]$ (if $x < c$) and

$$\int_c^x f(t)\, dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x-c)^{k+1}.$$

Proof. Let x be a point in the interval of convergence. By Theorem 9.11, the series converges uniformly on $[c, x]$ (if $x \geq c$) or $[x, c]$ (if $x < c$), so the series can be integrated term-by-term (Theorem 8.17). □

Example 9.15. The geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

has radius of convergence 1. Thus, on the interval $(-1, 1)$, we can integrate term-by-term. Therefore,

$$-\log(1-x) = \int_0^x \frac{1}{1-t}\, dt = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \quad \text{for all } \; -1 < x < 1.$$

Now, the series $\sum_{k=0}^{\infty} x^{k+1}/(k+1)$ converges at $x = -1$ by the alternating series test. Since $-\log(1-x)$ is continuous at $x = -1$, it follows from Proposition 9.12 that the above equality holds on $[-1, 1)$. In particular,

$$\log 2 = -\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}.$$

So we have computed the sum of the alternating harmonic series!

Proposition 9.16 (Differentiation of power series)**.** *Let f be a function represented by a power series*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k,$$

with radius of convergence $R > 0$. Then f is differentiable on $(c - R, c + R)$ and

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x-c)^{k-1} \quad \textit{for all } x \in (c-R, c+R).$$

Proof. Let R' be the radius of convergence of the differentiated series $\sum_{k=1}^{\infty} k a_k (x-c)^{k-1}$. Since $\sqrt[k]{k} \to 1$ as $k \to \infty$ (see Exercise 2.1.15), we have

$$R' = \left(\limsup_{k\to\infty} \sqrt[k]{|k a_k|} \right)^{-1} = \left(\limsup_{k\to\infty} \sqrt[k]{|a_k|} \right)^{-1} = R.$$

Thus, by Theorem 9.9, the differentiated series converges uniformly on any interval $[a, b] \subseteq (c - R, c + R)$. Since each $x \in (c - R, c + R)$ lies inside such an interval, the proposition follows from Corollary 8.22. □

Exercises.

9.2.1 ([TBB, Ex. 10.4.2]). Obtain power series expansions for

$$\frac{x}{1+x^2} \qquad \text{and} \qquad \frac{x}{(1+x^2)^2}.$$

9.2.2 ([TBB, Ex. 10.4.3]). Obtain power series expansions for

$$\frac{x}{1+x^3} \qquad \text{and} \qquad \frac{x^2}{1+x^3}.$$

9.2.3 ([TBB, Ex. 10.4.4]). Find a power series expansion about $x = 0$ for the function

$$f(x) = \int_0^1 \frac{1-e^{-sx}}{s}\, ds.$$

9.3 Taylor series

Note that, in the setting of Proposition 9.16, we have

$$f(c) = a_0, \qquad f'(c) = a_1.$$

This can be generalized as follows.

Proposition 9.17. *Suppose $\sum_{k=0}^{\infty} a_k(x-c)^k$ has radius of convergence $R > 0$. Then the function*

$$f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$$

has derivatives of all orders. Furthermore,

$$a_k = \frac{f^{(k)}(c)}{k!}.$$

Proof. This proof by induction is left as an exercise (Exercise 9.3.1). □

Corollary 9.18 (Uniqueness of power series)**.** *Suppose two power series*

$$f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k \qquad \text{and} \qquad g(x) = \sum_{k=0}^{\infty} b_k(x-c)^k$$

agree on some interval centred at c, that is $f(x) = g(x)$ for $x \in (c-\rho, c+\rho)$ and some $\rho > 0$. Then $a_k = b_k$ for all $k \in \{0, 1, 2, \dots\}$.

Proof. By Proposition 9.17, for $k \in \{0, 1, 2, \dots\}$, we have

$$a_k = \frac{f^{(k)}(c)}{k!} = \frac{g^{(k)}(c)}{k!} = b_k.$$

□

It follows from the above that if a power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges on an interval I (of nonzero length), then the series yields a function f that has derivatives of all orders and

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \quad \text{for all } x \in I.$$

This is called the *Taylor series* for f about the point c.

Example 9.19. From the geometric series, we see that

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \cdots .$$

Note that this is valid *only* for $-1 < x < 1$, even though the function on the left side is defined for all value of x. (This has something to do with the fact that it is not defined at the complex numbers $x = \pm i$, which are a distance 1 from the centre of the series. But that is beyond the scope of this course.) So the Taylor series for $f(x) = 1/(1+x^2)$ represents f only on the interval $(-1, 1)$, and *not* on the full domain of f. There can be no power series that represents f on all of $\mathbb{R}$ since that series would have to agree with the above one on $(-1, 1)$ and hence must be the same series by Corollary 9.18.

Example 9.20. The function

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ e^{-1/x^2} & \text{if } x \neq 0. \end{cases}$$

One can show that this function is infinitely differentiable on $\mathbb{R}$ and that $f^{(k)}(0) = 0$ for $k \in \mathbb{N}$. (See Exercise 9.3.3.) Thus the Taylor series for f about $x = 0$ is

$$\sum_{k=0}^{\infty} 0x^k.$$

This series converges to the zero function on all of $\mathbb{R}$, but it does not represent f, except at the origin.

Definition 9.21 (Analytic)**.** A function f whose Taylor series converges to f in a neighbourhood of c (i.e. an interval $(c-\delta, c+\delta)$ for some $\delta > 0$) is said to be *analytic* at c.

As we see from Example 9.20, to know that a function is analytic, it is *not* enough to show that the Taylor series has a positive radius of convergence. It is possible for the Taylor series to converge everywhere but not agree with the function, so that the function is not analytic.

Exercises.

9.3.1. Prove Proposition 9.17.

9.3.2 ([TBB, Ex. 10.4.6]). Suppose

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

has a positive radius of convergence. If the function f is *even* (i.e., if it satisfies $f(-x) = f(x)$ for all x), what can you deduce about the coefficients a_k? What can you deduce if the function is *odd* (i.e., if $f(-x) = -f(x)$ for all x)?

9.3.3. Consider the function f defined in Example 9.20.

(a) For $k \in \mathbb{N}$, show that $f^{(k)}(x)$ is of the form $R(x^{-1})e^{-1/x^2}$ for $x \neq 0$, where R is a polynomial.

(b) Show that

$$\lim_{x\to 0} \frac{1}{x^n} e^{-1/x^2} = 0$$

for all $n \in \mathbb{N}$.

(c) Conclude that

$$\lim_{x\to 0} f^{(k)}(x) = 0$$

for all $k \in \mathbb{N}$.

9.3.4 ([TBB, Ex. 10.5.8]). Show that if f and g are analytic functions at each point of an interval (a, b), then so too is any linear combination $\alpha f + \beta g$, for $\alpha, \beta \in \mathbb{R}$.

9.4 Fourier series

In this section, we briefly discuss another way of representing functions. Instead of representing them as series of powers of x, we represented them as trigonometric series.

Theorem 9.22. *Suppose that, for some $a_0, a_1, a_2, \ldots \in \mathbb{R}$ and $b_1, b_2, b_3, \ldots \in \mathbb{R}$, we have*

$$f(t) = \frac{1}{2}a_0 + \sum_{j=1}^{\infty} \big(a_j \cos(jt) + b_j \sin(jt)\big),$$

with uniform convergence on $[-\pi, \pi]$. Then the function f is continuous and

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(jt)\,dt \quad \textit{and} \quad b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(jt)\,dt,$$

for all j.

Proof. Fix $j \geq 1$, choose $n > j$, and consider the partial sum

$$s_n(t) = \frac{1}{2}a_0 + \sum_{j=1}^{n} \big(a_k \cos(kt) + b_k \sin(kt)\big).$$

Using Exercise 9.4.1, we have

$$\int_{-\pi}^{\pi} s_n(t)\cos(jt)\,dt = \int_{-\pi}^{\pi} a_j \cos^2(jt)\,dt = a_j\pi \quad \text{for } n > j \geq 1. \tag{9.2}$$

Since s_n converges to f uniformly and

$$|s_n(t)\cos(jt) - f(t)\cos(jt)| = |s_n(t) - f(t)| \cdot |\cos(jt)| \leq |s_n(t) - f(t)|,$$

it follows that $s_n(t)\cos(jt)$ converges uniformly to $f(t)\cos(jt)$ for $t \in [-\pi, \pi]$. Therefore, by Theorem 8.17, we have

$$\int_{-\pi}^{\pi} f(t)\cos(jt)\,dt = \lim_{n\to\infty} \int_{-\pi}^{\pi} s_n(t)\cos(jt)\,dt = a_j\pi,$$

where the last equality follows from (9.2). The proofs of the formulas for a_0 and b_j, $j \geq 1$, are analogous. □

Definition 9.23 (Fourier series)**.** Let $f\colon [-\pi, \pi] \to \mathbb{R}$ be continuous, and let

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(jt)\,dt \quad \text{and} \quad b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(jt)\,dt,$$

for $j \in \{0, 1, 2, \dots\}$. Then the series

$$\frac{1}{2}a_0 + \sum_{j=1}^{\infty} \big(a_j \cos(jt) + b_j \sin(jt)\big)$$

is called the *Fourier series* of f. The a_j and b_j are called the *Fourier coefficients* of f.

Every continuous function f has a Fourier series. The issue of whether/how this Fourier series converges to f is somewhat subtle and beyond the scope of this course. One example of a result in this direction is that if $f\colon [-\pi, \pi] \to \mathbb{R}$ is continuous and

$$f_n(t) = \frac{1}{2}a_0 + \sum_{j=1}^{n} \big(a_j \cos(jt) + b_j \sin(jt)\big)$$

is the n-th truncation of the Fourier series, then

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \big(f(t) - f_n(t)\big)^2\,dt = 0.$$

One says that the Fourier series converges to f in the *L^2-norm*. You can learn more about this in MAT 3120 (see [Savb]). See [TBB, Th. 10.36] for another statement about the convergence of Fourier series.

Example 9.24. Consider the function

$$f\colon [-\pi,\pi] \to \mathbb{R}, \quad f(t) = t^2.$$

Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,dt = \left.\frac{t^3}{3}\right|_{-\pi}^{\pi} = \frac{2\pi^2}{3},$$

and, for $j \in \mathbb{N}$, we have

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\cos(jt)\,dt = \frac{1}{\pi}\left.\left(\frac{2}{j^2}t\cos(jt) + \left(\frac{t^2}{j} - \frac{2}{j^3}\right)\sin(jt)\right)\right|_{-\pi}^{\pi} = (-1)^j\frac{4}{j^2}.$$

(See Exercise 9.4.2.) Furthermore, for $j \in \mathbb{N}$, we have

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\sin(jt)\,dt = 0,$$

since, as you learned in calculus, the integral of an odd function over an interval centred about zero is equal to zero. Thus, the Fourier series for f is

$$\frac{\pi^2}{3} + \sum_{j=1}^{\infty}(-1)^j\frac{4}{j^2}\cos(jt).$$

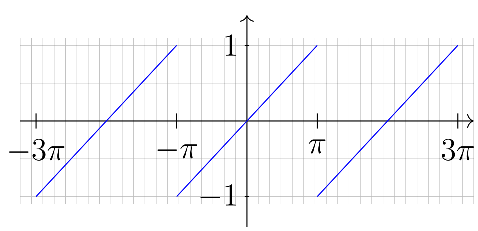
Note that the function

$$f(t) = \frac{1}{2}a_0 + \sum_{j=1}^{\infty}\big(a_j\cos(jt) + b_j\sin(jt)\big),$$

has the property that $f(t+2\pi) = f(t)$ for all $t \in \mathbb{R}$. Thus, if some function $f\colon \mathbb{R} \to \mathbb{R}$ with this property is equal to its Fourier series on $[-\pi,\pi]$, then it is equal to its Fourier series on all of $\mathbb{R}$.

Example 9.25. Consider the *sawtooth wave* function $s\colon \mathbb{R} \to \mathbb{R}$ given by

$$s(x + 2k\pi) = \frac{x}{\pi}, \quad \text{for } x \in [-\pi,\pi),\ k \in \mathbb{Z}.$$



Even though this function is not continuous, we can still compute its Fourier coefficients. They are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{x}{\pi}\cos(nx)\,dx = 0, \quad n \geq 0,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{\pi}\sin(nx)\,dx = \frac{1}{\pi^2}\left(-\frac{1}{n}x\cos(nx) + \frac{1}{n^2}\sin(nx)\right)\Bigg|_{-\pi}^{\pi} = \frac{2(-1)^{n+1}}{n\pi}, \quad n \geq 1,$$

where, for the computation of the a_n, the integral is zero since we integrate an odd function over an interval centred about 0. It can be proven that the Fourier series converges to $s(x)$ at every point where s is differentiable. Therefore, we have

$$s(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx), \quad \text{for } x \notin \{\pi + 2n\pi : n \in \mathbb{N}\}.$$

When $x = \pi$, the Fourier series converges to 0, which is the half-sum of the two one-sided limits of s at $x = \pi$. See https://en.wikipedia.org/wiki/Fourier_series for some nice animations of the partial Fourier series of $s(x)$.

Fourier series have many important applications. For example, in music, they correspond to decomposing an arbitrary sound wave into pure tones.

Exercises.

9.4.1. Show that, for $j, k \in \mathbb{Z}$, we have

$$\int_{-\pi}^{\pi} \cos(jt)\sin(kt)\,dt = 0,$$

$$\int_{-\pi}^{\pi} \sin(jt)\sin(kt)\,dt = \begin{cases} 0 & \text{if } j \neq k, \\ \pi & \text{if } j = k, \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(jt)\cos(kt)\,dt = \begin{cases} 0 & \text{if } j \neq k, \\ \pi & \text{if } j = k, \end{cases}$$

9.4.2. Show that

$$\frac{d}{dt}\left(\frac{2}{j^2}t\cos(jt) + \left(\frac{t^2}{j} - \frac{2}{j^3}\right)\sin(jt)\right) = t^2\cos(jt).$$

(One obtains this antiderivative via integration by parts.)

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