ELEMENTS OF CIVIL ENGINEERING & ENGINEERING MECHANICS

Subject Code: 10CIV 13/10CIV 23 IA Marks: 25
Hours/Week: 04 Exam Hours: 03
Total Hours: 52 Exam Marks: 100

PART - A

Unit-I

1. **Introduction to Civil Engineering**, Scope of different fields of Civil Engineering Surveying, Building Materials, Construction Technology, Geotechnical Engineering, Structural Engineering, Hydraulics, Water Resources and Irrigation Engineering, Transportation Engineering, Environmental Engineering. .

Infrastructure: Types of infrastructure. Role of Civil Engineer in the Infrastructural Development,Effect of infrastructural facilities on socio- economic development of a country. (04 Hours)

2. Roads: Type of roads, Components and their functions. (02 Hours)
3. Bridges and Dams; Different types with simple sketches. (01 Hour)

Unit - II

4. **Introduction to Engineering mechanics**: Basic idealizations - Particle, Continuum, Rigid body and Point force; Newton's laws of motion, Definition of force, Introduction to SI units. Elements of a force, Classification of force and force systems; Principle of physical independence of forces, Principle of superposition of forces, Principle of transmissibility of forces; Moment of a force, couple, moment of a couple, characteristics of couple. Equivalent force - couple system;

Resolution of forces, composition of forces; Numerical problems on moment of forces and couples, on equivalent force – couple system. (07 Hours)

Unit -III

5. **Composition of forces** - Definition of Resultant; Composition of coplanar – concurrent force system. Principle of resolved parts; Numerical problems on composition of coplanar concurrent force systems. (03 Hours)

6. Composition of coplanar-non-concurrent force system, Varignon's principle of moments; Numerical problems on composition of coplanar non-concurrent force systems. (05 Hours)

Unit -IV

7. **Centroid of plane figures**; Locating the centroid of triangle, semicircle, quadrant of a circle and sector of a circle using method of integration, Centroid of simple built up sections; Numerical problems. (06 Hours)

PART B

Unit -V

8. **Equilibrium of forces** - Definition of Equilibriant; Conditions of static equilibrium for different force systems, Lami's theorem; Numerical problems on equilibrium of coplanar - concurrent force system. (06 Hours)

Unit -VI

9. **Beams**: Types of supports, statically determinate beams, Numerical problems on equilibrium of coplanar non-concurrent force system and support reactions for statically determinate beams. Analysis of plane trusses by method of joints and method of sections. (06 Hours)

Unit -VII

10-**Friction** - Types of friction, Laws of static friction, Limiting friction. Angle of friction, angle of repose; Impending motion on horizontal and inclined planes; Wedge friction; Ladder friction; Numerical problems. (06 Hours)

Unit -VIII

11. **Moment of inertia of an area**, polar moment of inertia, Radius of gyration, Perpendicular axis theorem and Parallel axis theorem; Moment of Inertia of rectangular, circular and triangular areas from method of integration; Moment of inertia of composite areas; Numerical problems. (06 Hours)

Text Books:

1. Elements of Civil Engineering by Jagadeesh T.R. and Jayaram, Sapna Book House, Bangalore.
2. Elements of Civil Engineering (IV Edition) by S.S. Bhavikatti, Vikas Publishing House Pvt. Ltd., New Delhi.

Reference Books:

1. Mechanics for Engineers: Statics by Ferdinand P. Beer and E. Russet Johnston Jr., McGraw-Hill
Book Company, New York.
2. Engineering Mechanics by K.L. Kumar, Tata McGraw-Hill Publishing Company, New Delhi.
3. Engineering Mechanics by Timoshenko and Young, McGraw-Hill Book Company, New Delhi.

CONTENTS

Unit I: Introduction to Civil Engineering

1.1 Introduction

Engineering: It is a profession of converting scientific knowledge into useful practical applications, where the materials & forces in nature are effectively used for the benefit of mankind.
An Engineer is a person who plays a key role in such activities.

1.1.1 Civil Engineering: It is the oldest branch of professional engineering, where the civil engineers are concerned with projects for the public or civilians.
The role of civil engineers is seen in every walk of life or infrastructure development activity such as follows:-

1. Providing shelter to people in the form of low cost houses to high rise apartments.
2. Laying ordinary village roads to express highways.
3. Constructing irrigation tanks, multipurpose dams & canals for supplying water to agricultural fields.
4. Supplying safe and potable water for public & industrial uses.
5. Protecting our environment by adopting sewage treatment & solid waste disposal techniques.
6. Constructing hydro-electric & thermal-power plants for generating electricity.
7. Providing other means of transportation such as railways, harbour & airports.
8. Constructing bridges across streams, rivers and also across seas.
9. Tunneling across mountains & also under water to connect places easily & reduce distance.

As seen above, civil engineering is a very broad discipline that incorporates many activates in various fields. However, civil engineers specialize themselves in one field of civil engineering. The different fields of civil engineering and the scope of each can be briefly discussed as follows.

1. **Surveying**: It is a science and art of determining the relative position of points on the earth's surface by measuring distances, directions and vertical heights directly or indirectly. Surveying helps in preparing maps and plans, which help in project implementation. (setting out the alignment for a road or railway track or canal, deciding the location for a dam or airport or harbour) The cost of the project can also be estimated before implementing the project. Now-a-days, using data from remote sensing satellites is helping to prepare maps & plans & thus cut down the cost of surveying.
2. **Geo-Technical Engineering** (**Soil Mechanics**): Any building, bridge, dam, retaining wall etc. consist of components like foundations. The foundation is laid from a certain depth below the ground surface till a hard layer is reached. The soil should be thoroughly checked for its suitability for construction purposes. The study dealing with the properties & behaviour of soil under loads & changes in environmental conditions is called geo-technical engineering. The knowledge of the geology of an area is also very much necessary.

3. **Structural Engineering**: A building or a bridge or a dam consists of various elements like foundations, columns, beams, slabs etc. These components are always subjected to forces. It becomes important to determine the magnitude & direction the nature of the forces and acting all the time. Depending upon the materials available or that can be used for construction, the components or the parts of the building should be safely & economically designed. A structured engineer is involved in such designing activity. The use of computers in designing the members, is reducing the time and also to maintain accuracy.

4. **Transportation Engineering**: The transport system includes roadways, railways, air & waterways. Here the role of civil engineers is to construct facilities related to each one. Sometimes crucial sections of railways & roads should be improved. Roads to remote places should be developed. Ports & harbours should be designed to accommodate, all sizes of vehicles. For an airport, the runway & other facilities such as taxiways, terminal buildings, control towers etc. should be properly designed.

5. **Irrigation & Water resources engineering (Hydraulics Engineering)**: Irrigation is the process of supplying water by artificial means to agricultural fields for raising crops. Since rainfall in an area is insufficient or unpredictable in an area, water flowing in a river can be stored by constructing dams and diverting the water into the canals & conveyed to the agricultural fields. Apart from dams & canals other associated structures like canals regulators, aqua ducts, weirs, barrages etc. are also necessary. Hydro electric power generation facilities are also included under this aspect.

6. **Water Supply and Sanitary Engineering (Environmental Engineering)**: People in every village, town & city need potable water. The water available (surface water & ground water) may not be fit for direct consumption. In such cases, the water should be purified and then supplied to the public. For water purification, sedimentation tanks, filter beds, etc. should be designed. If the treatment plants are for away from the town or city, suitable pipelines for conveying water & distributing it should also be designed. In a town or city, a part of the water supplied returns as sewage. This sewage should be systematically collected and then disposed into the natural environment after providing suitable treatment. The solid waster that is generated in a town or locality should be systematically collected and disposed off suitably. Before disposal, segregation of materials should be done so that any material can be recycled & we can conserve our natural resources.

7. **Building Materials & Construction Technology:** Any engineering structure requires a wide range of materials known as building materials. The choice of the materials is wide & open. It becomes important for any construction engineer to be well versed with the properties & applications of the different materials. Any construction project involves many activities and also required many materials, manpower, machinery & money. The different activities should be planned properly; the manpower, materials & machinery should be optimally utilized, so that the construction is completed in time and in an economical manner. In case of large construction projects management techniques of preparing bar charts & network diagrams, help in completing the project orderly in time.

1.1.1 Effects of Infrastructure development on the Socio-economic development of a country:

The term infrastructure is widely used to denote the facilities available for the socio-economic development of a region. The infrastructure facilities to be provided for the public include:

1. Transport facilities
2. Drinking water and sanitation facilities
3. Irrigation facilities
4. Power generation & transmission facilities
5. Education facilities
6. Health care facilities
7. Housing facilities
8. Recreation facilities

The well being of a nation is dependent on the quality & the quantity of the above services that are provided to the public. Development of infrastructure has number of good effects which can be listed as follows.

1. It is a basic necessity for any country or state.
2. It forms a part of business, research & education.
3. It improved health care & Cultural activities.
4. It provided housing & means of communication to people.
5. It provided direct employment to many number of skilled, semiskilled & unskilled laborers.
6. It leads to the growth of associated industries like cement, steel, glass, timber, plastics, paints, electrical goods etc.
7. It helps in increasing food production & protection from famine.
8. Exporting agricultural goods can fetch foreign currency.

Some ill effects of infrastructure development can also be listed as follows:

1. Exploitation of natural resources can lead to environmental disasters.
2. Migration of people from villages to towns & cities in search of job takes place.
3. Slums are created in cities.
4. It becomes a huge financial burden on the government and tax prayers.

1.2 ROADS

Transportation of goods & people can be done by roadways, railways, waterways & airways. Each mode of transportation has advantages and disadvantages of its own in comparison to the others.

Roads play a crucial role in any country's development. They have the following advantages when compared with other modes of transportation.

1. They give maximum service to one and all.
2. They have maximum flexibility with respect to route, direction and speed.
3. They are directly accessible to the users at all destination points.

4. Other modes of transportation are dependent on roads to serve, the people from their terminals.
5. Roads can be used by various types of vehicles but other modes of transportation can cater to a particular locomotive only.
6. Roads can provide door to door service.
7. Construction of roads is easier when compared to other modes.
8. For short distance traveling, road travel is easier.

At the same time, the disadvantages of roads can be listed as follows:-

1. The severity of accidents is related to speed.
2. Frequency of accidents among road users is more.
3. It does not provide much comfort for long distance travel.
4. Roads get easily damaged in heavy rainfall areas and require frequent maintenance.

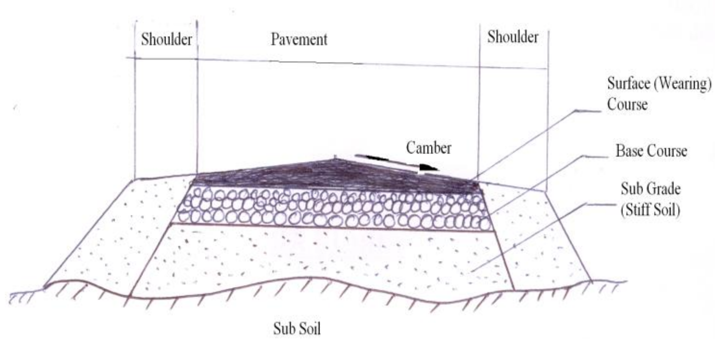
1.2.1: Components or Cross section elements of a road:

All roads should essentially consist of the following components:-

1. Pavement or Carriage way
2. Shoulders

Pavement or Carriage way: It refers to the path over which the vehicles and other traffic can move lawfully. It also includes the path way and other related structures like bridges, flyovers, underpasses which make road traveling easier. The width of the road is designed according to the traffic volume on the road. According to Indian Road Congress (IRC) specification, the maximum width of a vehicle should not be more than 2.44 mts. A side clearance of 0.68 mts for safety should also be provided. Hence, the width of the pavement for a single lane road becomes 3.8 mts. In case of multi lane traffic, the width of each lane should be at least 3.5 mts. Depending upon the number of lanes, that can be provided; the total width of the pavement can be fixed.

The different cross section elements of a road can be represented as follows.



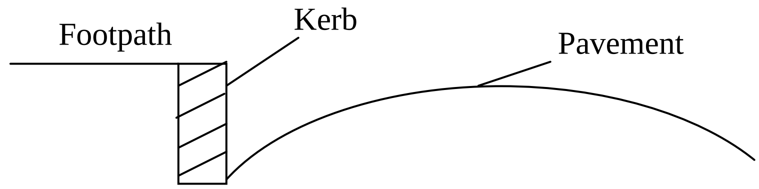
1. **Sub Soil**: It refers to the natural soil or prepared soil on which the loads coming on the road are ultimately transferred. Hence the Sub Soil should be prepared by compacting it properly by rollers.
2. **Sub grade**: This layer gives support to the road structure. This should remain dry and stable throughout the year. Much attention should be given in preparing the sub grade. This layer consists of disintegrated rocks mixed with gravel. Now a day, a lean mixture of concrete with large amounts of sand and stones is used in preparing this layer. The thickness of this layer depends upon, the type of vehicles and traffic volume on the road.
3. **Base Course**: This layer is constructed in one or two layers consisting of stones mixed with gravel. Bigger stones are used at the bottom. At the top level, smaller stones mixed with cohesive soil or cement are provided and thoroughly compacted. This layer provides a proper support for the upper layers.
4. **Surface course/wearing course**: It is the topmost layer of the carriage way. It takes the loads directly. This layer is either made of flexible materials (bitumen or coal tar mixed with stones) or a rigid material (concrete). This layer should be moderately rough to provide good grip for the vehicles. The top surface of the varying course is provided a lateral slope (camber), to drain off the rain water from the road surface quickly and effectively.
5. **Shoulders**: The width of a road is always extended beyond the road on both sides by a width of at least 2 to 5 mts. This space acts as a space for moving away any broken down vehicles or parking vehicles in an emergency. The shoulders should satisfy the following requirements:-

- They should have a sufficient bearing capacity even in wet condition.
- The shoulders should have distinctive colour from the pavement to guide the vehicles users on the pavement only.
- The surface of the shoulder should be rough to avoid the drivers from using the shoulders frequently.

<u>1.2.2: Other components of roads</u>:

The following components are also essential for roads depending upon, the places where the roads are provided.

1. **Traffic separators**: These are provided to separate the traffic moving in opposite directions, thus avoiding head on collision. These can be provided as a yellow colour strip or steel barricade or a permanent median all along the centre lines of the road.
2. **Kerbs**:



Within the city limits, to separate the pavements from the footpath, a raised stone (kerb) is provided at the edge of the pavement. The height of the kerb is normally 15 to 20 cms.

3. **Footpath**: Apart from vehicles, the pedestrians should also be provided some space for moving at the edge of the roads. Footpaths should be provided essentially everywhere. The footpath may be in level with the road surface or slightly raised higher than the road surface.
4. **Parking Lanes**: These are usually provided or reserved on the road edges within a city limit for allowing the vehicles to be parked conveniently. The parking lanes are distinctively separated by white colour strips so that moving vehicles do not enter parking lanes.
5. **Cycle tracks**: In some countries in urban areas, separate cycle tracks of 2 mts. width are provided all along the length of the road.
6. **Guard stones and guard rails**: Whenever, the road formation level is higher than the natural ground level, at the edges of the shoulders, guard stones or guard rails should be provided to avoid accidental fall of vehicles from the earth slope.
7. **Fencing**: Whenever a highway or an expressway passes through urban areas, fencing is provided all along the road to prevent the cattle and people from entering the traffic zone.

1.2.3 Classification or types of roads:

Depending upon various criteria, roads can be classified as follows:

1. **Based on seasonal usage**:

a. **All weather roads**: These are roads which are usable in all seasons including rainy season in a year.

b. **Fair weather roads**: These are roads which are usable during the dry seasons in a year.

2. **Based on the nature of pavement surface provided**:

a. **Surfaced roads**: These are roads in which the topmost layer is covered with a bituminous material or a rigid material like concrete.

b. **Un- surfaced roads**: These are roads in which the topmost layer is not covered by a bituminous material or concrete but covered with a layer of stones mixed with gravel & thoroughly compacted.

3. **Based on the importance of connectivity, function & traffic volume**:

a. **Expressways**: These are roads which are developed to inter connect two important cities only, where the traffic volume is very high. On these roads cross traffic & traffic in opposite direction is not allowed. Throughout the length of the road medians are provided & vehicles can move at high speeds. Fencing is also provided all along the road.

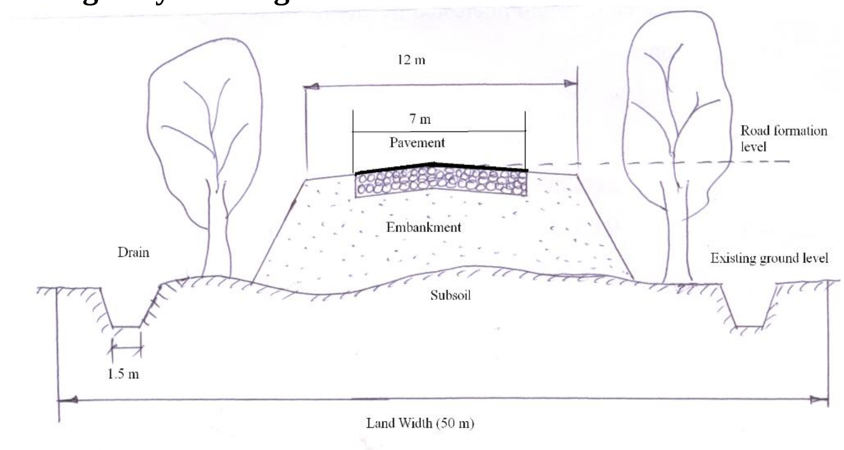
b. **National highways**: These are the main network of roads, running through the length & breadth of a country. These roads inter connect state capitals, union territories, major ports, industrial areas and tourist destinations.

c. **State highways**: These are roads at a state level, which interconnect district headquarters and also interlink national highways running through a state or neighbouring states.

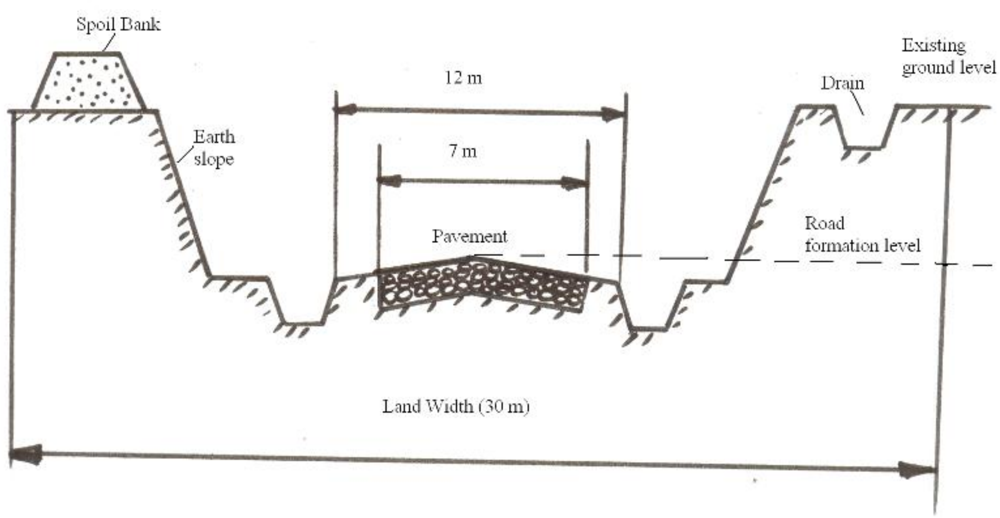
d. **Major district roads**: These are important roads within a district, which help in moving goods from agricultural production areas to market places. These roads may also inter connect state highways and national highways. The permissible speed and traveling comforts on such roads is lesser.
e. **Other district roads**: These are roads which interlink taluk headquarters and other main roads. They also serve as a link between agricultural areas and market places.
f. **Village roads**: These are roads connecting villages & remote habitat groups with major district roads & other district roads. The surface of such roads may not be covered with a bituminous layer, but fairly leveled and covered with stones and gravel.

<u>Typical Cross Sections of a Highway</u>:

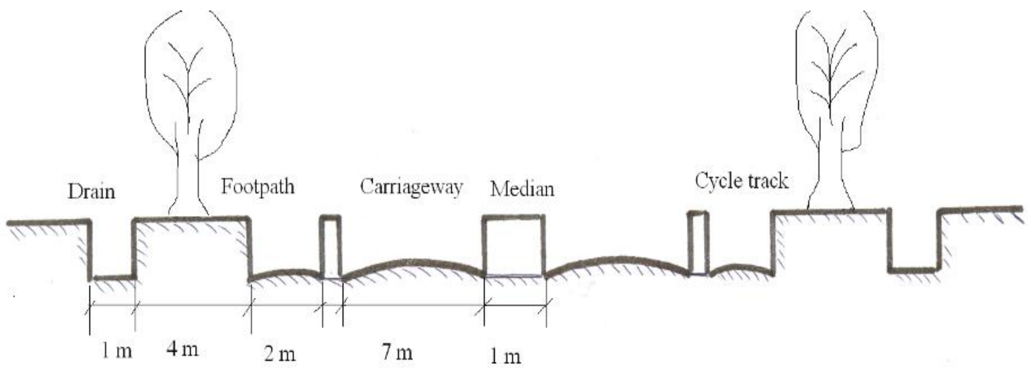
1. **Highway running over an embankment:**



2. **Highway running in cutting**:



3. **Highway running in urban areas**:

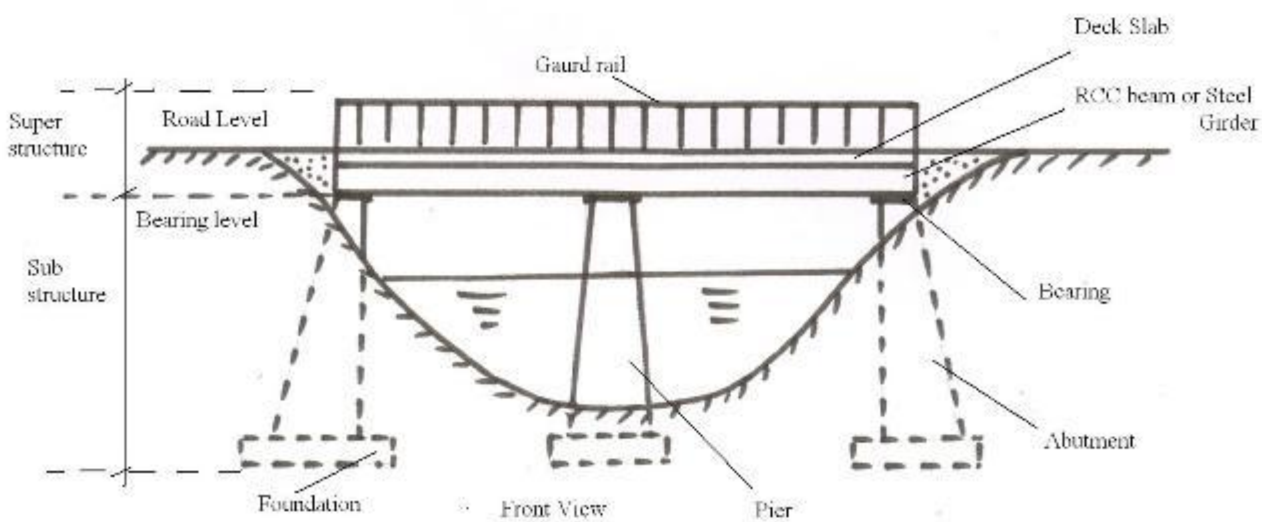


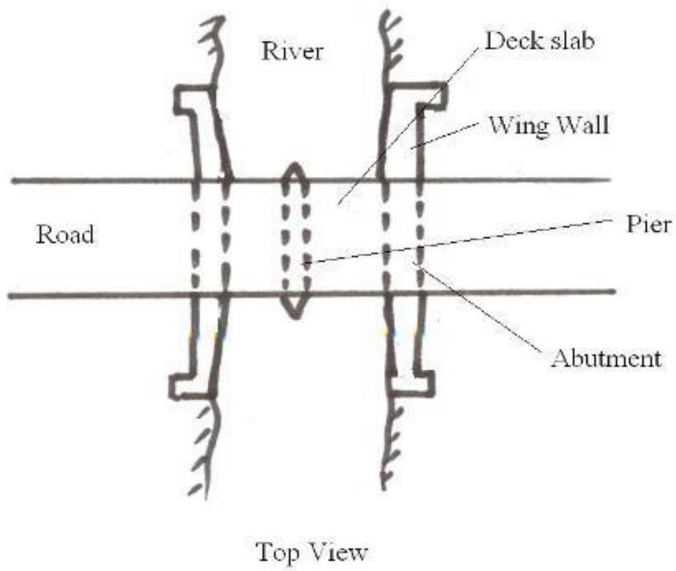
1.3 BRIDGES

A bridge is a structure which provides a safe passage for a road or railway track over obstacles, without closing the obstacle below. The obstacle to be crossed may be a river or stream, a canal, road or a railway track. A bridge may also be built for the safe passage of a canal (aqua duct).

Components of a bridge: A bridge basically consists of following two components.

a. Super Structure
b. Sub Structure





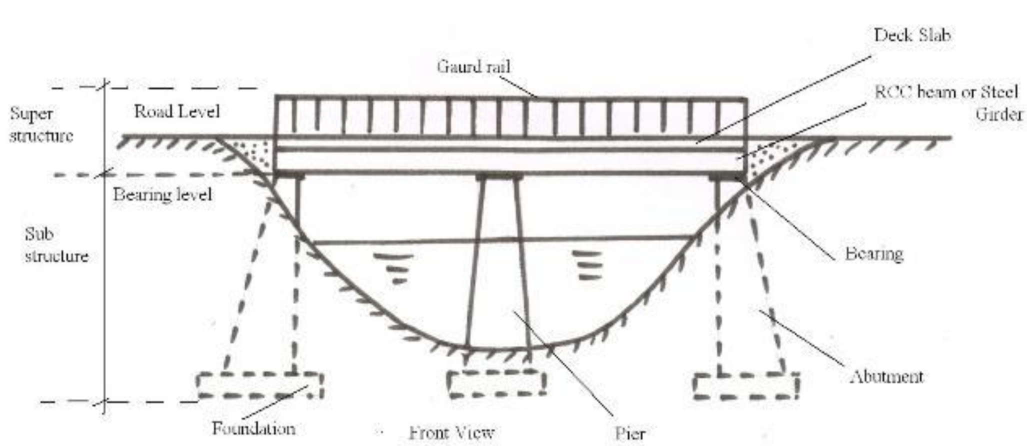
a. **Super Structure**: It refers to the part of the bridge above the bearing level. The components included in the super structure are RCC beam, Deck slab, guard rails, pavement etc.

b. **Sub Structure**: It refers to the part of the bridge below the bearing level. The components included in the substructure are bearings, piers, abutments, wing walls, foundation, etc.

Classification of bridges:

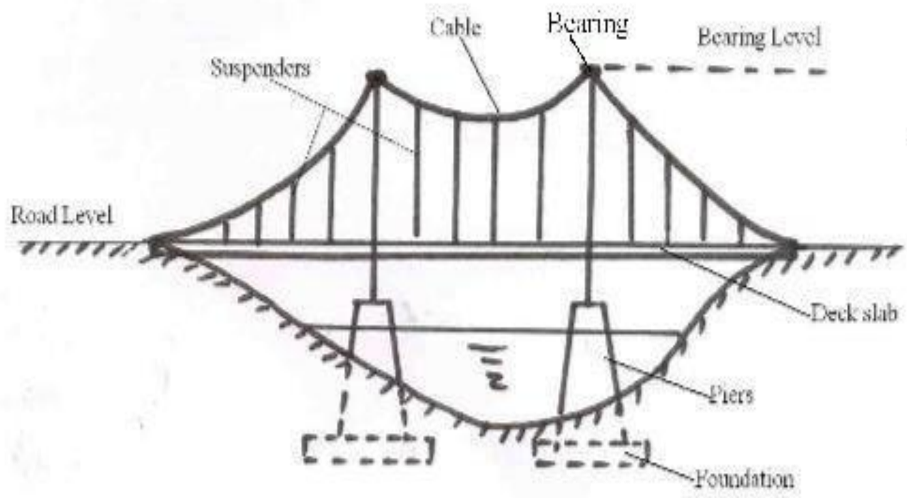
Depending upon the position of the road surface or road formation level with respect to the bearing level in a bridge, bridges are classified as follows.

a. **Deck bridges**: These are bridges in which road formation level or pavement is above the bearing level in a bridge.



In such bridges sufficient head room for all vehicles is available. RCC beam bridges, Steel girder bridges are example for Deck bridges.

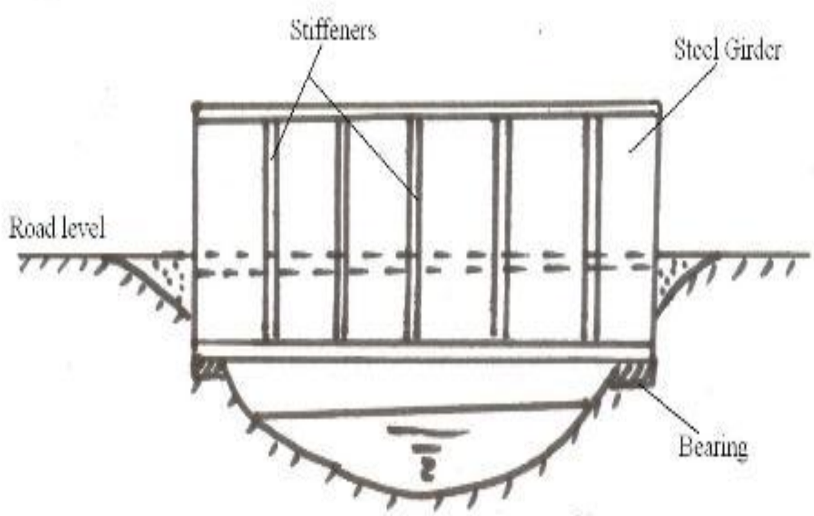
b. **<u>Through bridges</u>**: These are bridges in which the road formation level is lower than the bearing level in the bridge.



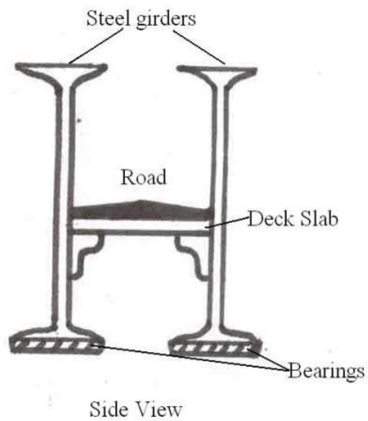
Front View

Cable stayed bridges & truss bridges are example for through bridges. Such bridges may not provide sufficient head room for all vehicles, if the road formation level is increased subsequently.

c. **<u>Semi -through bridges</u>**: These are bridges in which the road formation level is at some intermediate level of the super structure.



Front View

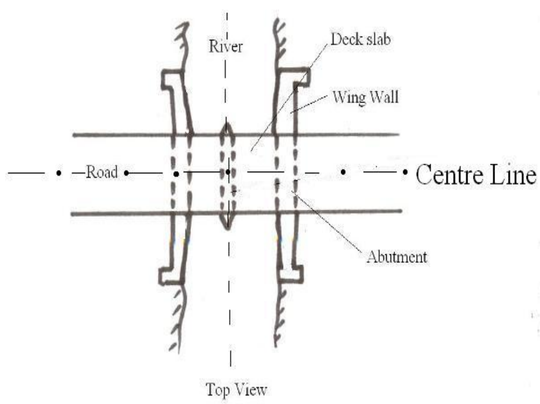


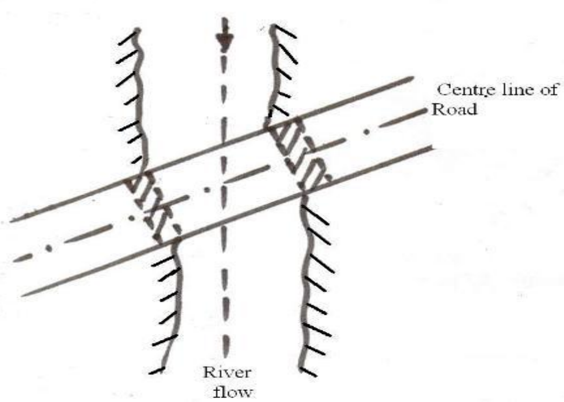
Such bridges do not provide sufficient head room for all vehicles when the road formation level increases subsequently. Steel girder bridges are examples of semi -through bridges.

Square bridges & skew bridges:

Whenever a bridge is to be constructed over a stream or river, the centre line of the bridge should be aligned at right angles to the direction of flowing water in the river. In such cases, the flowing water does not exert excessive forces on the piers and abutments. If the centre line of the bridge is at right angles to the direction of flowing water in the river, the bridge is known as square bridge.

If the centre line of the bridge is not at right angles to the direction of flowing water in the river, the bridge is known as skew bridge.

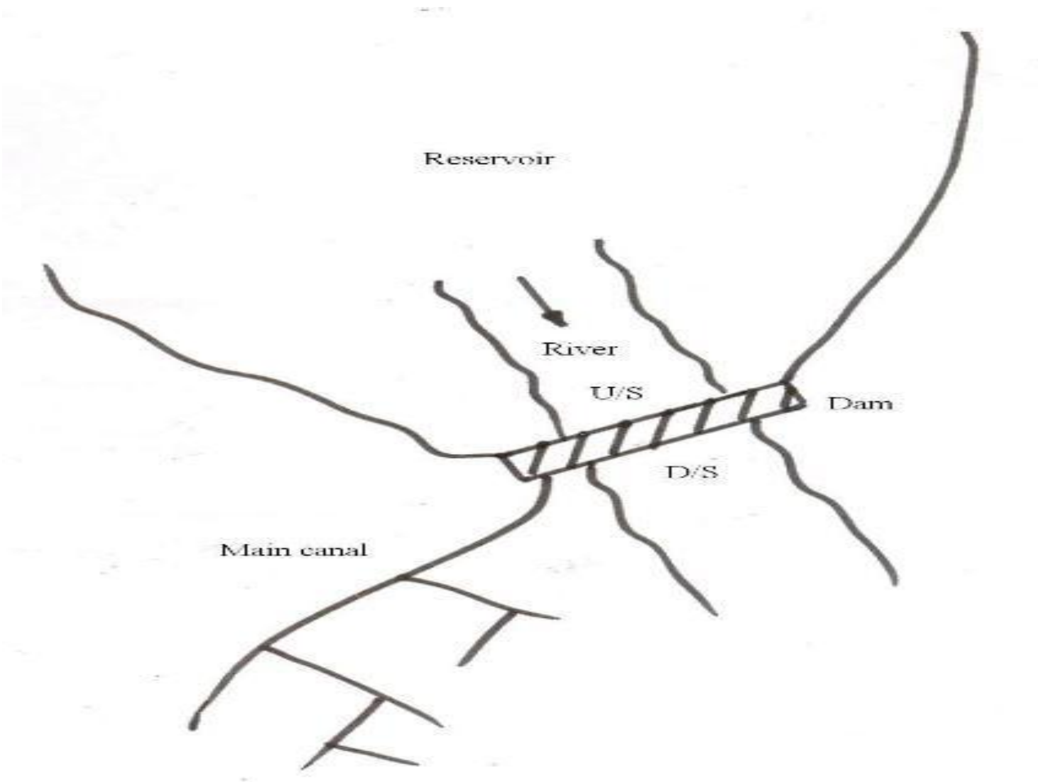




Square Bridge Skew bridge

1.4 DAMS

A Dam is an obstruction or barrier or a hydraulic structure which is constructed across a river or stream to store water on the upstream side as an artificial lake or reservoir.



Dams serve the following purposes:

1. Storing water for irrigation
2. Supplying water for domestic & industrial uses
3. Supplying water for hydroelectric power generation
4. Aquaculture
5. For controlling floods in a river
6. For inland navigation

Classification of Dams:
Depending upon various criteria, dams can be classified as follows.

1. **Based on the purpose served**:

a. **Coffer dams**: These are temporary dams which are constructed during the construction of actual main dam to keep the dam site free from water.

b. **Storage dams**: These are dams built across non perennial rivers to store water in a reservoir during excess flow. The stored water is released to the down stream side through canals and can be utilized for various uses.

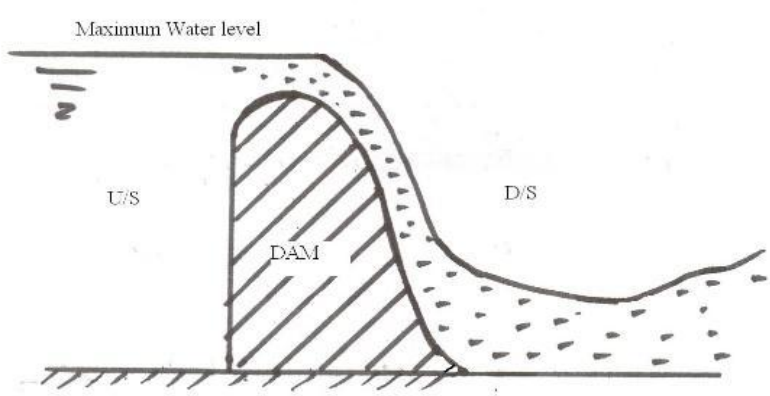
c. **Diversion dams**: These are irrigation structures which are constructed across a river to slightly raise the water level, without making an attempt to store water. By increasing the water level, water is directly diverted into the canals. Barrages are examples of diversion dams.

d. **Detention dams**: These are dams which are constructed to store water temporarily only during floods. The water is then released to the down stream side when the floods recede.

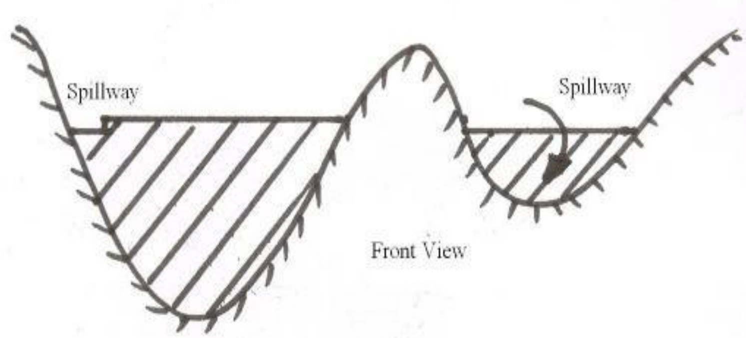
e. **Debris dams**: These are small dams which are built across, the streams or tributaries which join the main river. By constructing these dams across streams or tributaries on the upstream side of the main dam, entry of silt & debris can be controlled & the useful life of the reservoir can be increased.

2. **<u>Based on the hydraulic design</u>**:

a. **Overflow dams**: Any dam is designed or constructed to store water up to a certain maximum height only. When the water level exceeds the maximum level, the excess water should be discharged to the down stream side safely. In overflow dams, the excess water is allowed to overtop the body of the dam.

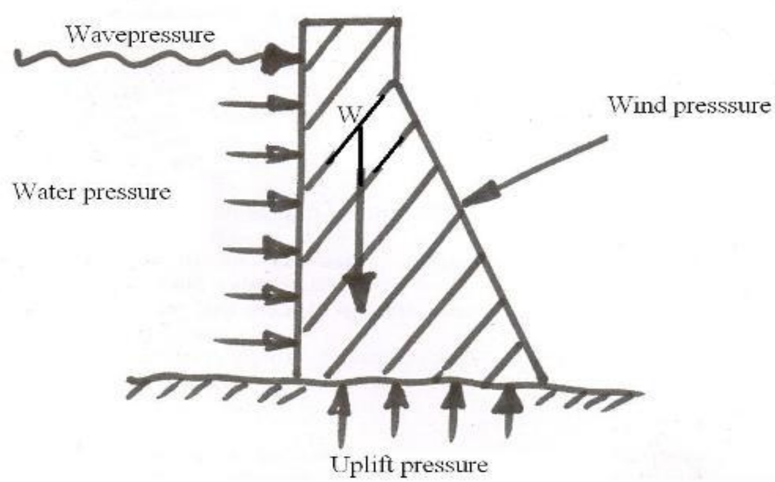


b. **Non-over flow dams**: In majority of the dams, the excess water entering the reservoir is not allowed to overtop the entire length of the dam. The excess water is released to the down stream side through a separate spill way & such dams are called non over flow dams. The spill may be included in the main portion of dam or through a separate spill way section.

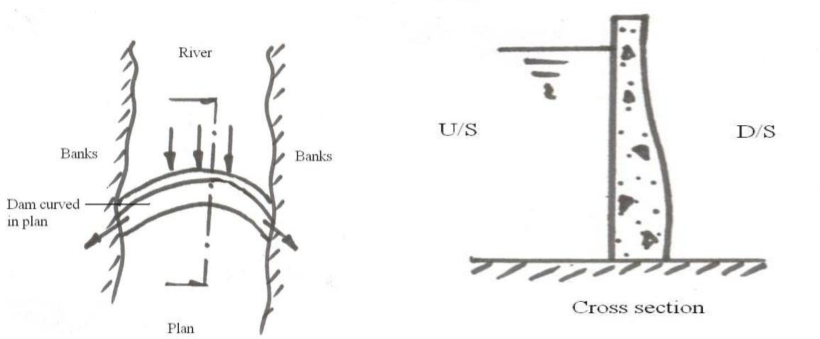


3. **<u>Based on the resisting action to external forces</u>**:
   When water is stored in a dam or reservoir, the stored water exerts forces on the upstream face of the dam. Some amount of water also seeps through the bottom of the dam. This seeping water exerts (applies) uplift pressure on the dam. The down stream face of the dam is exposed to wind pressure. The waves that are generated at the top of the reservoir also exert forces on the dam. The body of the dam should be able to resist all such forces and different dams resist these forces in different ways.

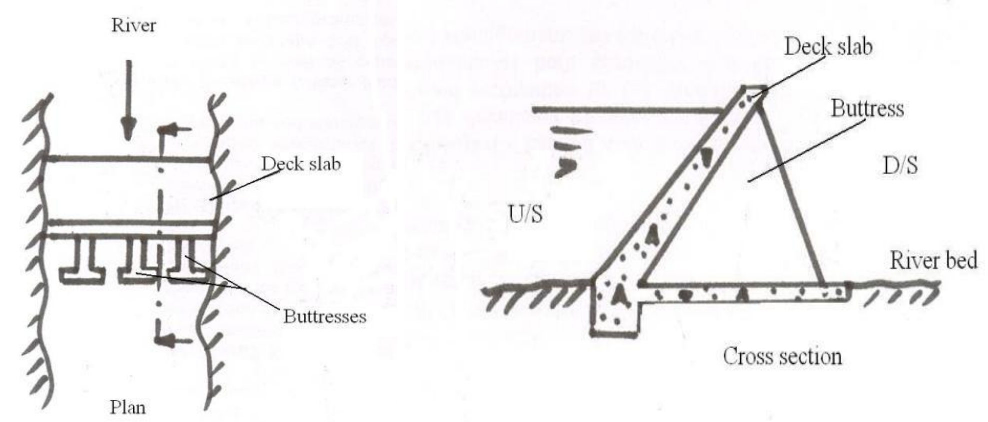
a. **Gravity dams**: These are dams which are built of rigid materials like concrete & stone masonry. The cross sections of such dams are very large & also have enormous self weight. The destabilizing forces like hydrostatic pressure, wind pressure, uplift pressure, and wave pressure are resisted by the self weight of the dam only.



b. **Arch dams**: These are dams which are also constructed of rigid material like concrete or stone masonry. These dams are curved in plan. The cross section of such dams is slender when compared to gravity dams. These dams transfer the water pressure, wind pressure etc. to the banks of the river by arch action.



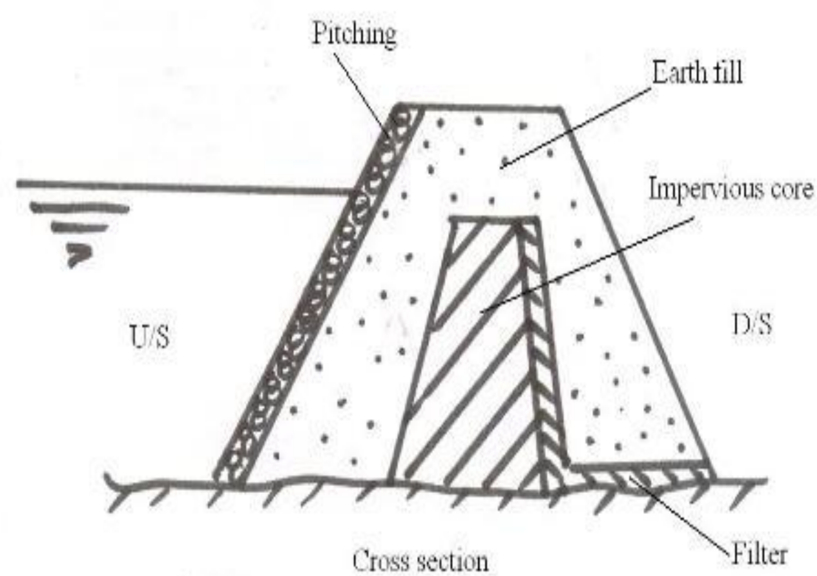
c. **Buttress dams**: These are dams in which the water pressure from the stored water acts on a thin deck slab, which is supported over a series of buttresses (piers or columns). The loads are transferred from the deck slab to the buttresses and then to the bed of the river.



4. **<u>Based on materials used</u>**:

a. **Rigid dams**: These are dams which are constructed of rigid materials like concrete, stone masonry, steel sheets. Gravity dams, Arch dams and Buttress dams are examples of rigid dams.

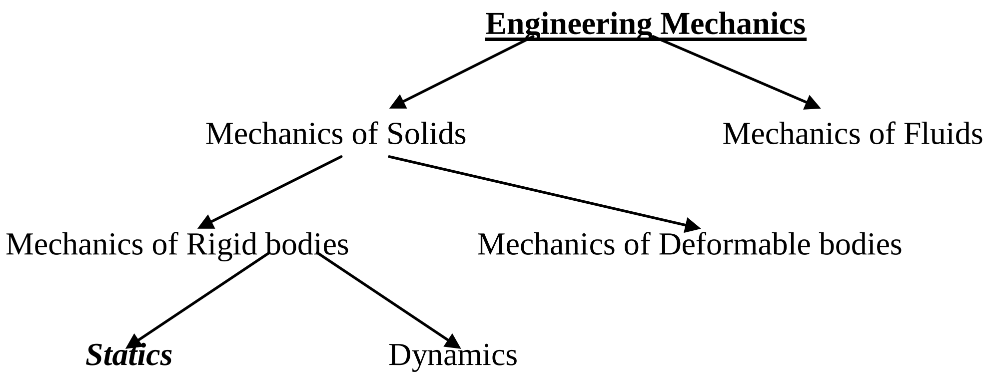
b. **Non rigid dams**: These are dams which are constructed art of non rigid materials like earth fill & rock fill. Such dams cannot exceed 30 mtrs. in height.



Unit II: Engineering Mechanics

2.1 Engineering Mechanics

It is a branch of applied sciences that describes and predicts the state of rest or of uniform motion of ***bodies*** under the action of ***forces.***
Engineering Mechanics deals with the application of principles of mechanics and different laws in a systematic manner.

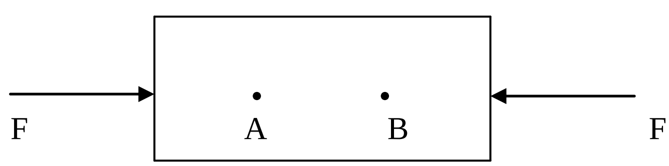


Concepts of: Physical quantity, Scalar quantity, and Vector quantity

Particle: A particle is a body of infinitely small volume and the entire mass of the body is assumed to be concentrated at a point.

Rigid body: It is one, which does not alter its shape, or size or the distance between any two points on the body does not change on the application of external forces.

Deformable body: It is one, which alters its shape, or size or the distance between any two points on the body changes on the application of external forces.



In the above example, the body considered is rigid as long as the distance between the points A and B remains the same before and after application of forces, or else it is considered as a deformable body.

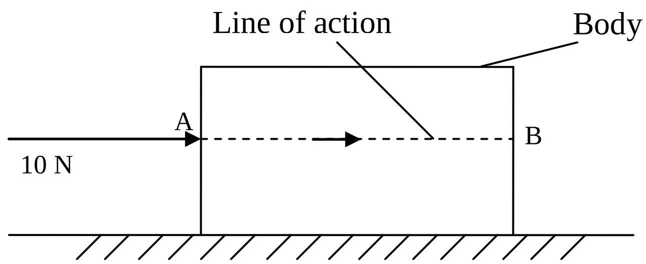
Force: According to Newton's I law, force is defined as an action or agent, which changes or tends to change the state of rest or of uniform motion of a body in a straight line.
Units of force: The gravitational (MKS) unit of force is the kilogram force and is denoted as 'kgf'. The absolute (SI) unit of force is the Newton and is denoted as 'N'.

Note: 1 kgf = 'g' N (But $g = 9.81 m/s^2$) Therefore 1 kgf = 9.81 N or ≅ 10 N.

2.2 Characteristics of a force

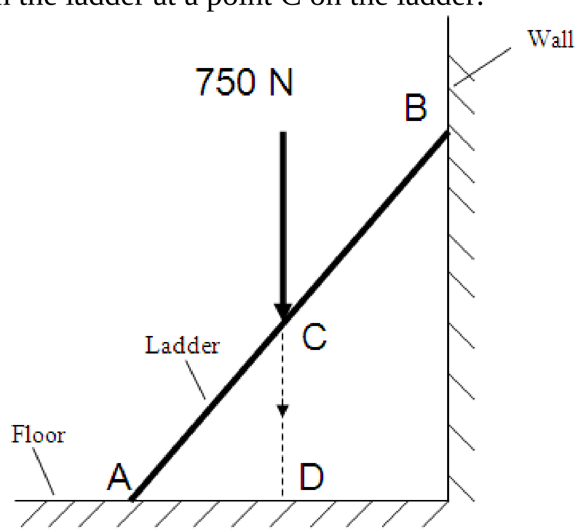
These are ones, which help in understanding a force completely, representing a force and also distinguishing one force from one another.
A force is a vector quantity. It has four important characteristics, which can be listed as follows.
1) **Magnitude**: It can be denoted as 10 kgf or 100 N.
2) **Point of application**: It indicates the point on the body on which the force acts.
3) **Line of action**: The arrowhead placed on the line representing the direction represents it.
4) **Direction**: It is represented by a co-ordinate or cardinal system.

Ex.1: Consider a body being pushed by a force of 10 N as shown in figure below.



The characteristics of the force acting on the body are
1) Magnitude is 10 N.
2) Point of application is A.
3) Line of action is A to B or AB.
4) Direction is horizontally to right.

Ex.2: Consider a ladder AB resting on a floor and leaning against a wall, on which a person weighing 750 N stands on the ladder at a point C on the ladder.

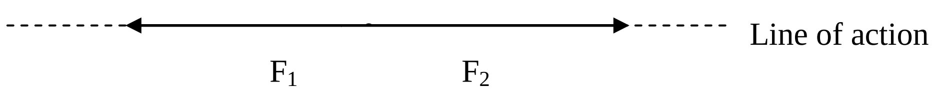


The characteristics of the force acting on the ladder are
1) Magnitude is 750 N.
2) Point of application is C.
3) Line of action is C to D or CD.
4) Direction is vertically downward.

<u>Idealization or assumptions in Mechanics</u>: In applying the principles of mechanics to practical problems, a number of ideal conditions are assumed. They are as follows.
1) A body consists of continuous distribution of matter.
2) The body considered is perfectly rigid.
3) A particle has mass but not size.
4) A force acts through a very small point.

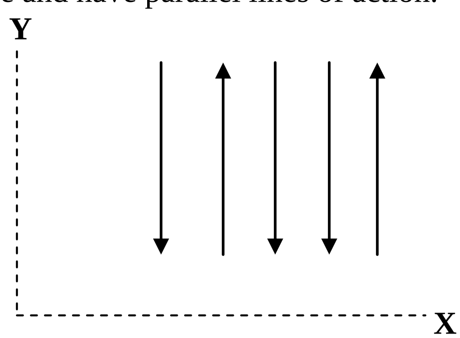
<u>Classification of force systems</u>: Depending upon their relative positions, points of applications and lines of actions, the different force systems can be classified as follows.

1) **<u>Collinear forces</u>**: It is a force system, in which all the forces have the same line of action.



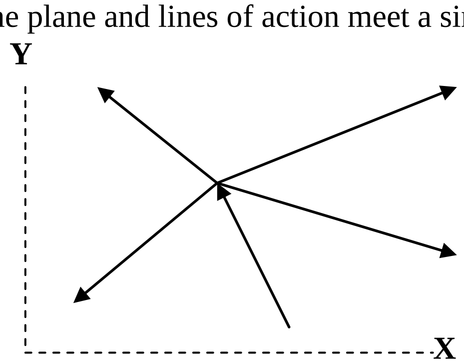
Ex.: Forces in a rope in a tug of war.

2) **<u>Coplanar parallel forces</u>**: It is a force system, in which all the forces are lying in the same plane and have parallel lines of action.



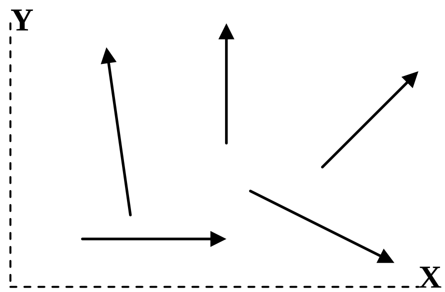
Ex.: The forces or loads and the support reactions in case of beams.

<u>3) Coplanar Concurrent forces:</u> It is a force system, in which all the forces are lying in the same plane and lines of action meet a single point.



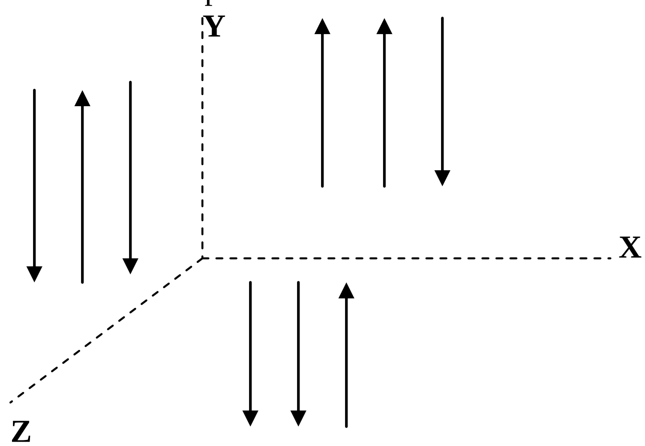
Ex.: The forces in the rope and pulley arrangement.

4) **<u>Coplanar non-concurrent forces:</u>** It is a force system, in which all the forces are lying in the same plane but lines of action do not meet a single point.



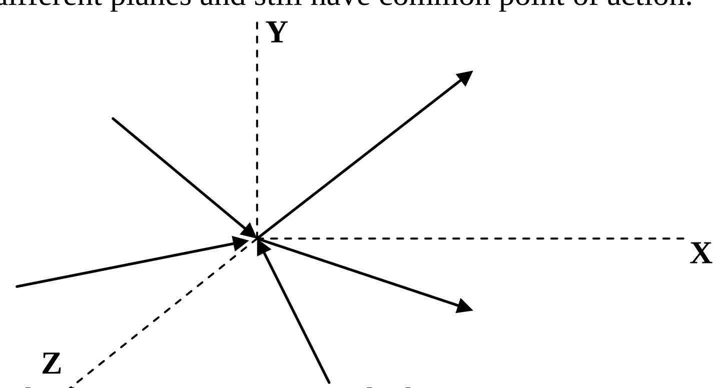
Ex.: Forces on a ladder and reactions from floor and wall, when a ladder rests on a floor and leans against a wall.

5) **<u>Non- coplanar parallel forces</u>**: It is a force system, in which all the forces are lying in the different planes and still have parallel lines of action.



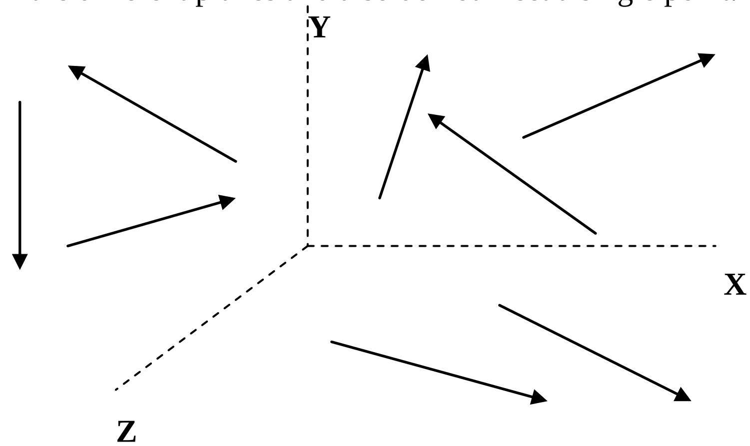
Ex: The forces acting and the reactions at the points of contact of bench with floor in a classroom.

6) <u>Non- coplanar concurrent forces</u> It is a force system, in which all the forces are lying in the different planes and still have common point of action.



Ex.: The forces acting on a tripod when a camera is mounted on a tripod.

7) **Non- coplanar non-concurrent forces**: It is a force system, in which all the forces are lying in the different planes and also do not meet a single point.



Ex.: Forces acting on a building frame.

2.3 Fundamental Laws in Mechanics

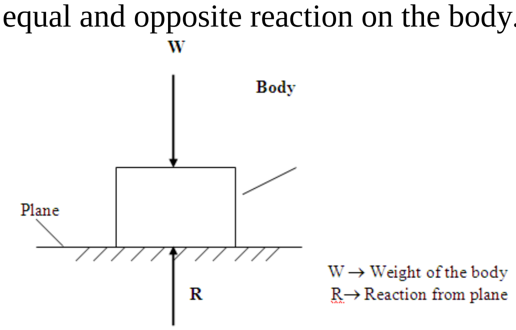
Following are considered as the fundamental laws in Mechanics.

1) Newton's I law
2) Newton's II law
3) Newton's III law
4) Principle or Law of transmissibility of forces
5) Parallelogram law of forces.

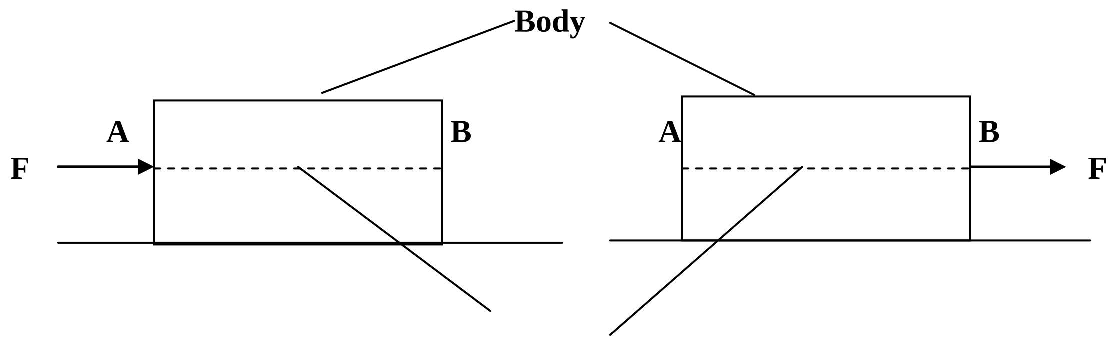
1) Newton's I law: It states, "Every body continues in its state of rest or of uniform motion in a straight line, unless it is compelled to do so by force acting on it."
This law helps in defining a force.

2) Newton's II law: It states, "The rate of change of momentum is directly proportional to the applied force and takes place in the direction of the impressed force."
This law helps in defining a unit force as one which produces a unit acceleration in a body of unit mass, thus deriving the relationship F = m . a

3) Newton's III law: It states, "For every action there is an equal and opposite reaction."
The significance of this law can be understood from the following figure.
Consider a body weighing W resting on a plane. The body exerts a force W on the plane and in turn the plane exerts an equal and opposite reaction on the body.



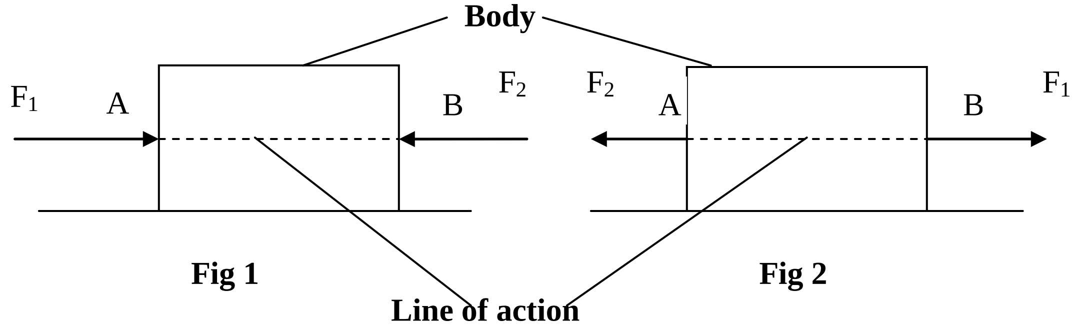
4) <u>Principle or Law of transmissibility of forces</u>: It states, "**The state of rest or of Uniform motion of a <u>rigid body</u> is unaltered if the point of application of the force is Transmitted to any other point along the line of action of the force.**"



From the above two figures we see that the effect of the force F on the body remains the same when the force is transmitted through any other point on the line of action of the force.

This law has a limitation that it is applicable to rigid bodies only.

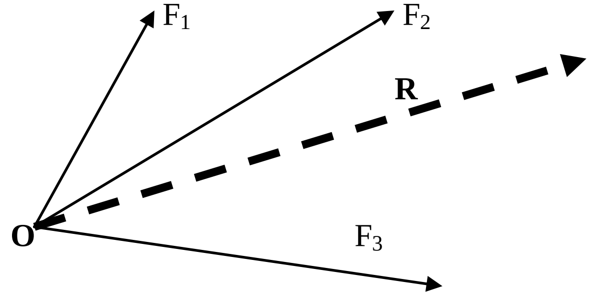
<u>Explanation of limitation</u>:



In the example if the body considered is deformable, we see that the effect of the two forces on the body are not the same when they are shifted by principle of transmissibility. In the first case the body tends to compress and in the second case it tends to elongate. Thus principle of transmissibility is not applicable to deformable bodies or it is applicable to rigid bodies only.

<u>Resultant Force:</u>

Whenever a number of forces are acting on a body, it is possible to find a single force, which can produce the same effect as that produced by the given forces acting together. Such a single force is called as resultant force or resultant.

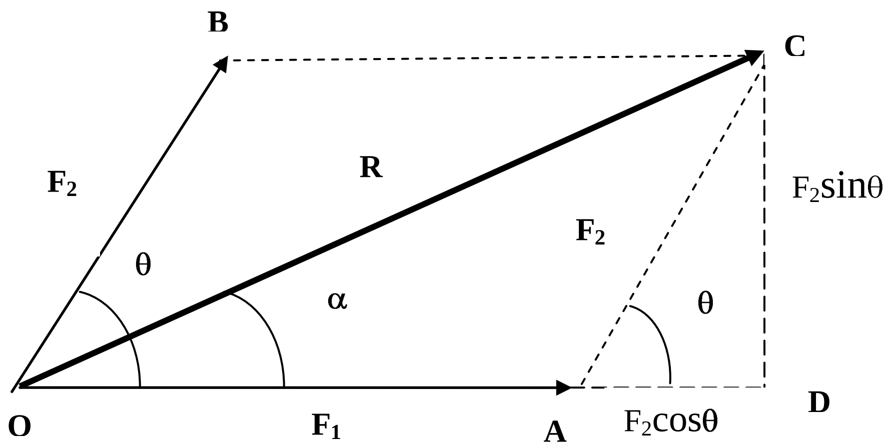


In the above figure R can be called as the resultant of the given forces F_1, F_2 and F_3.

The process of determining the resultant force of a given force system is known as **Composition of forces.**

The resultant force of a given force system can be determining by **Graphical** and **Analytical** methods. In **analytica**l methods two different principles namely: **Parallelogram law** of forces and **Method of Resolution** of forces are adopted.

Parallelogram law of forces: This law is applicable to determine the resultant of two coplanar concurrent forces only. This law states "***If two forces acting at a point are represented both in magnitude and direction by the two adjacent sides of a parallelogram, then the resultant of the two forces is represented both in magnitude and direction by the diagonal of the parallelogram passing through the same point.***"



Let F_1 and F_2 be two forces acting at a point O and θ be the angle between them. Let OA and OB represent forces F_1 and F_2 respectively both in magnitude and direction. The resultant R of F1 and F_2 can be obtained by completing a parallelogram with OA and OB as the adjacent sides of the parallelogram. The diagonal OC of the parallelogram represents the resultant R both magnitude and direction.

From the figure $OC = \sqrt{OD^2 + CD^2}$

$$= \sqrt{(OA + AD)^2 + CD^2}$$

$$= \sqrt{(F_1 + F_2 \cos\theta)^2 + (F_2 \sin \theta)^2}$$

i.e $R = \sqrt{F_1^2 + F_2^2 + 2. F_1. F_2.\cos\theta}$ ---------> 1

Let α be the inclination of the resultant with the direction of the F1, then

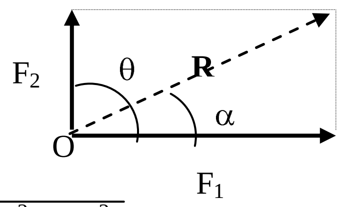
$$\alpha = \tan^{-1} \left[\frac{F_2 \sin\theta}{F_1 + F_2.\cos\theta} \right] \quad \text{------->} \ 2$$

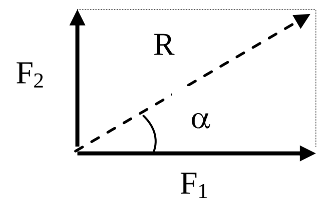
Equation 1 gives the magnitude of the resultant and Equation 2 gives the direction of the resultant.

Different cases of parallelogram law:

For different values of θ, we can have different cases such as follows:

Case 1: When $\theta = 90^0$:

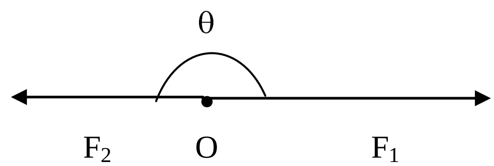




$$R = \sqrt{F_1^2 + F_2^2}$$

$$\alpha = \tan^{-1}\left[\frac{F_2}{F_1}\right]$$

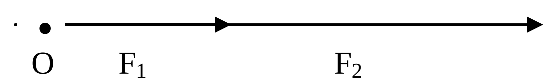
Case 2: When $\theta = 180^0$:



$R = [\, F_1 - F_2 \,]$

$\alpha = 0^0$

Case 3: When $\theta = 0^0$:



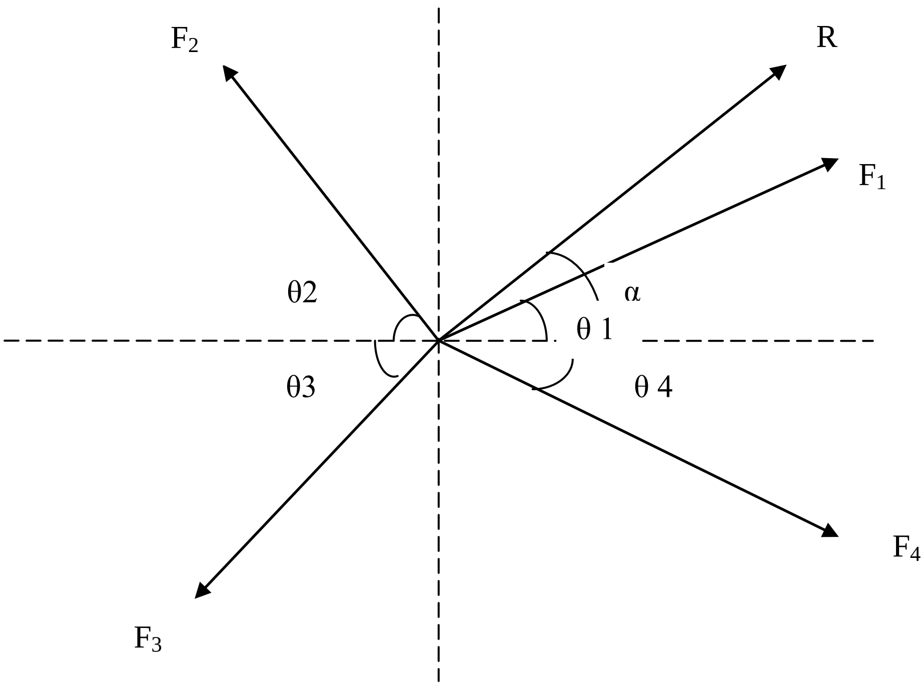
$R = [\, F_1 + F_2 \,]$

$\alpha = 0^0$

Unit III: Composition of forces by method of Resolution

3.1 Introduction

If two or more forces are acting in a single plane and passing through a single point , such a force system is known as a



coplanar concurrent force system

Let F1, F2, F3 , F4 represent a coplanar concurrent force system. It is required to determine the resultant of this force system.

It can be done by first resolving or splitting each force into its component forces in each direction are then algebraically added to get the sum of component forces.

These two sums are then combines using parallelogram law to get the resultant of the force systems.

In the ∑fig, let fx_1, fx_2, fx_3, fx_4 be the components of Fx_1, Fx_2, Fx_3, Fx_4 be the forces in the X-direction.

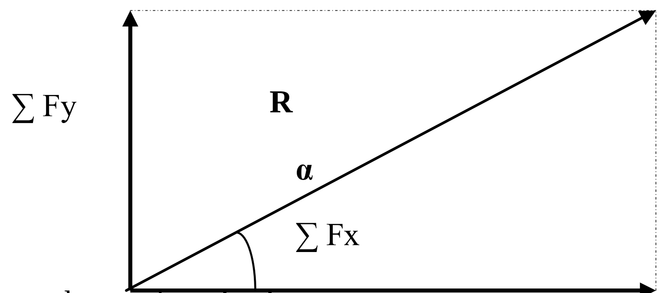
Let ∑ Fx be the algebraic sum of component forces in an x-direction

$$\sum Fx = fx_1 + fx_2 + fx_3 + fx_4$$

Similarly,

$\sum Fy = fy_1 + fy_2 + fy_3 + fy_4$

By parallelogram law,



The magnitude os the resultant is given as

$$R = \sqrt{(\sum Fx)^2 + (\sum Fy)^2}$$

The direction of resultant can be obtained if the angle α made by the resultant with x direction is determined here,

$$\alpha = \tan^{-1}\left(\frac{\sum Fy}{\sum Fx}\right)$$

The steps to solve the problems in the coplanar concurrent force system are, therefore as follows.

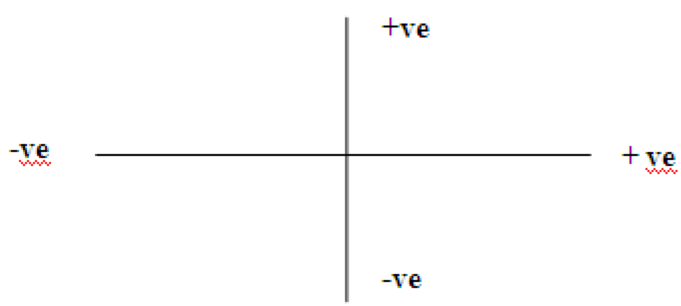
1. Calculate the algebraic sum of all the forces acting in the x-direction (ie. $\sum Fx$) and also in the y- direction (ie. $\sum Fy$)
2. Determine the direction of the resultant using the formula

$$R = \sqrt{(\sum Fx)^2 + (\sum Fy)^2}$$

3. Determine the direction of the resultant using the formula

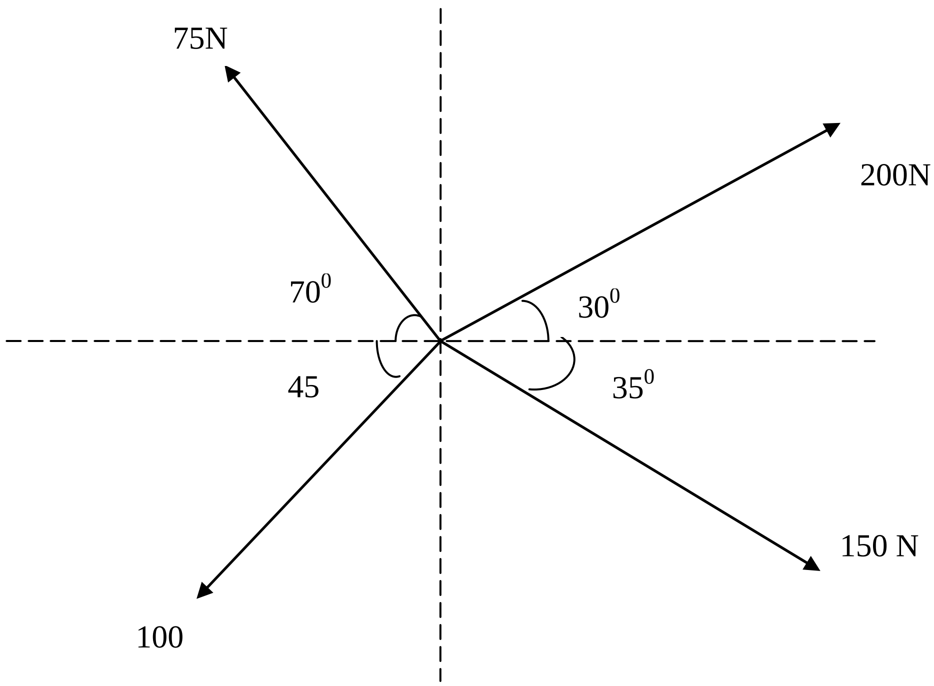
$$\alpha = \tan^{-1}\left(\frac{\sum Fy}{\sum Fx}\right)$$

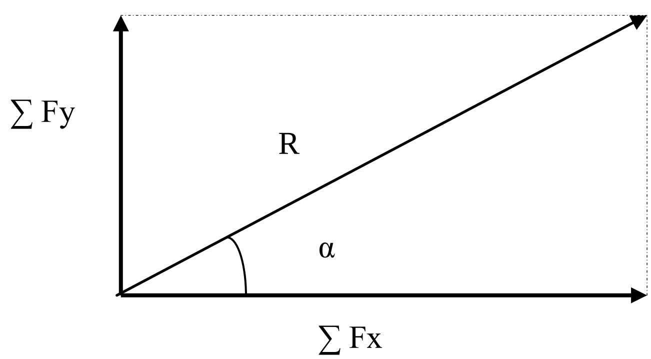
3.2 Sign Conventions:



Problems

Determine the magnitude & direction of the resultant of the coplanar concurrent force system shown in figure below.





Let R be the given resultant force system

Let α be the angle made by the resultant with x- direction.

The magnitude of the resultant is given as

$$R = \sqrt{(\sum Fx)^2 + (\sum Fy)^2} \quad \text{and} \quad \alpha = \tan^{-1}\left(\frac{\sum Fy}{\sum Fx}\right)$$

$\sum Fx = 200\cos 30^0 - 75\cos70^0 - 100\cos45^0 + 150\cos 35^0$

$\sum Fx = 199.7N$

$\sum Fy = 200\sin 30^0 + 75\sin70^0 - 100\sin45^0 - 150\sin 35^0$

$\sum Fy = 13.72\ N$

$R = \sqrt{(\sum Fx)^2 + (\sum Fy)^2}$

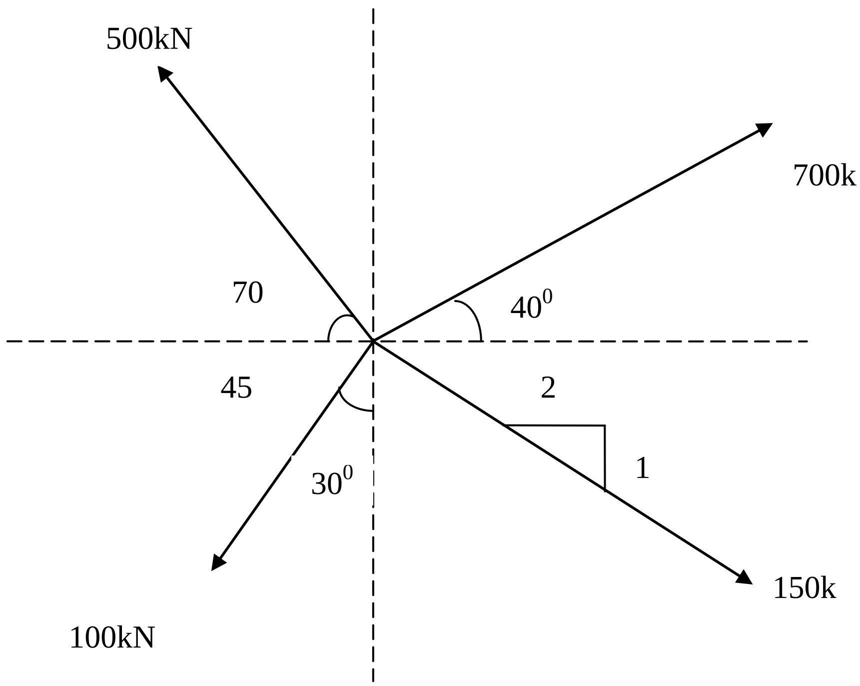
$R = 200.21N$

$\alpha = \tan^{-1}\left(\frac{\sum Fy}{\sum Fx}\right)$

$\alpha = \tan^{-1}(13.72/\ 199.72) = 3.93^0$

---

2. Determine the resultant of the concurrent force system shown in figure.



Let R be the given resultant force system

Let α be the angle made by the resultant with x- direction.

The magnitude of the resultant is given as

$$R = \sqrt{(\sum Fx)^2 + (\sum Fy)^2} \quad \text{and} \quad \alpha = \tan^{-1}\left(\frac{\sum Fy}{\sum Fx}\right)$$

$\sum Fx = 700\cos 40^0 - 500\cos70^0 - 800\cos60^0 + 200\cos 26.56^0$

$\sum Fx = 144.11$ kN

$\sum Fy = 700\sin 40^0 + 500\sin70^0 - 800\sin60^0 - 200\sin 26.56^0$

$\sum Fy = 137.55$kN

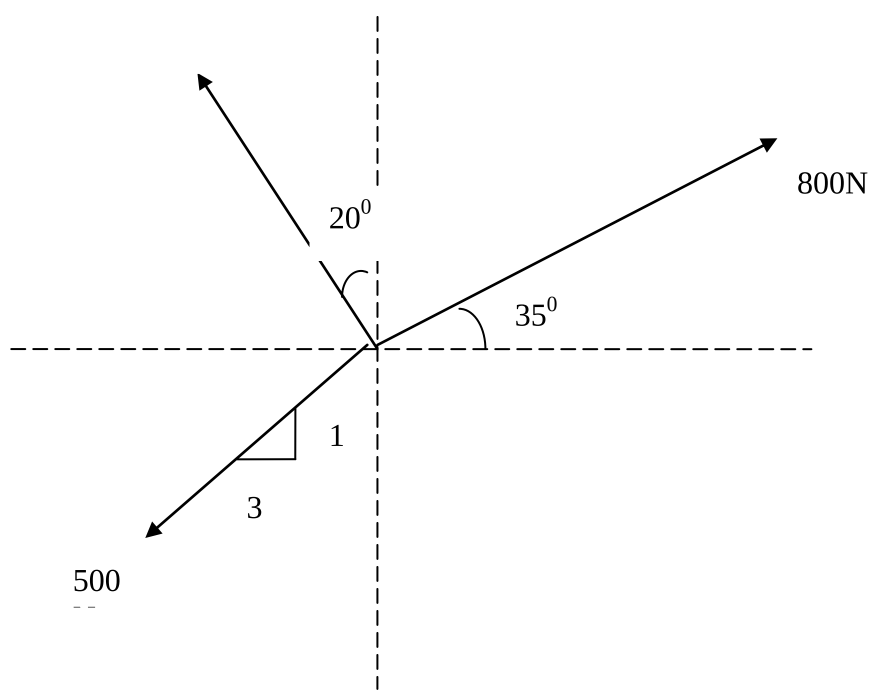
$R = \sqrt{(\sum Fx)^2 + (\sum Fy)^2}$

R= 199.21N

$\alpha = \tan^{-1}\left(\frac{\sum Fy}{\sum Fx}\right)$

$\alpha = \tan^{-1}(137.55/ 144.11) = 43.66$

<u>3. Determine the resultant of a coplanar concurrent force system shown in figure below</u>



Let R be the given resultant force system

Let α be the angle made by the resultant with x- direction.

The magnitude of the resultant is given as

$R= \sqrt{(\sum Fx)^2 +(\sum Fy)^2}$ and $\alpha = \tan^{-1}(\frac{\sum Fy}{\sum Fx})$

$\sum Fx = 800\cos 35^0 - 100\cos70^0 + 500\cos60^0 + 0$

$\sum Fx = 1095.48$ N

$\sum Fy = 800\sin 35^0 + 100\sin70^0 + 500\sin60^0 - 600$

$\sum Fy = 110.90$ N

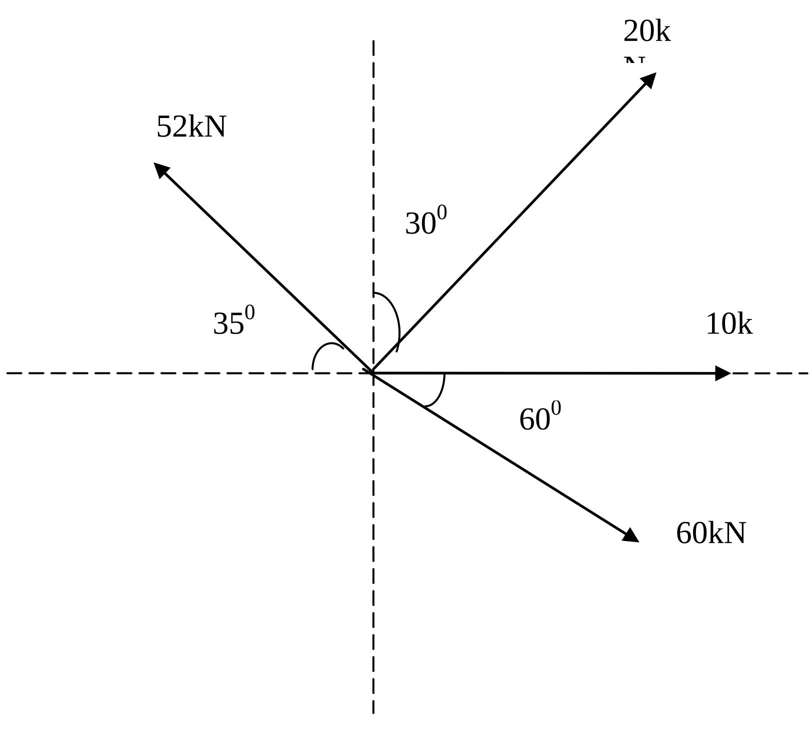
$R= \sqrt{(\sum Fx)^2 +(\sum Fy)^2}$

R= 1101.08 N

$\alpha = \tan^{-1}(\frac{\sum Fy}{\sum Fx})$

$\alpha = \tan^{-1}(110.90/ 1095.48) = 5.78^0$

<u>4. The Magnitude and direction of the resultant of the resultant of the coplanar concurrent force system shown in figure.</u>



Let R be the given resultant force system

Let α be the angle made by the resultant with x- direction.

The magnitude of the resultant is given as

$$R = \sqrt{(\sum Fx)^2 + (\sum Fy)^2} \text{ and } \alpha = \tan^{-1}\left(\frac{\sum Fy}{\sum Fx}\right)$$

$\sum Fx = 20\cos 60^0 - 52\cos 30^0 + 60\cos 60^0 + 10$

$\sum Fx = 7.404$ kN

$\sum Fy = 20 \sin 60^0 + 52\sin 30^0 - 60\sin 60^0 + 0$

$\sum Fy = -8.641$ kN

$R = \sqrt{(\sum Fx)^2 + (\sum Fy)^2}$

$R = 11.379$ N

$\alpha = \tan^{-1}\left(\frac{\sum Fy}{\sum Fx}\right)$

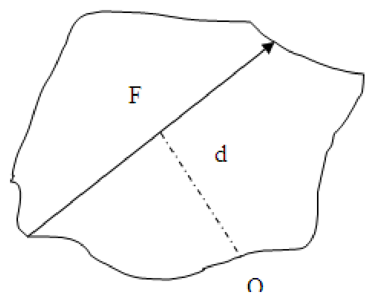
$\alpha = \tan^{-1}(-8.641 / 7.404) = -49.40$

3.3 COMPOSITION OF COPLANAR NONCONCURRENT FORCE SYSTEM

If two or more forces are acting in a single plane, but not passing through the single point, such a force system is known as coplanar non concurrent force system.

3.3.1 Moment of Force:

It is defined as the rotational effect caused by a force on a body. Mathematically Moment is defined as the product of the magnitude of the force and perpendicular distance of the point from the line of action of the force from the point.

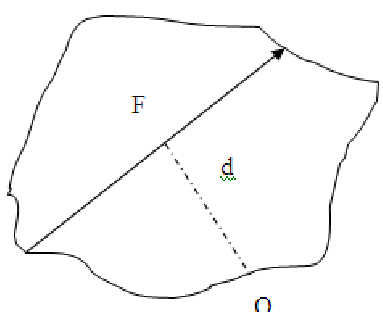


Let “ F” be a force acting in a plane. Let” O” be a point or particle in the same plane. Let “d ” be the perpendicular distance of the line of action of the force from the point “O” . Thus the moment of the force about the point “O” is given as

Mo= F x d

Moment or rotational effect of a force is a physical quantity dependent on the units for force and distance. Hence the units for moment can be “Nm” or “KNm” or “ N mm” etc.

The moment produced by a force about differences points in a plane is different. This can be understood from the following figures.



Let " F" be a force in a plane and O_1, O_2, and O_3 be different points in the same plane
Let moment of the force " F" about point O_1 is Mo,
$Mo_1 = F \times d_1$
Let moment of the force " F" about point O_2 is Mo,
$Mo_2 = F \times d_2$

Let moment of the force " F" about point O_3 is Mo,
Mo= 0x F
The given force produces a clockwise moment about point O1 and anticlockwise moment about O_2 . A clockwise moment ($\circlearrowright$) is treated as positive and an anticlockwise moment ($\circlearrowleft$) is treated as negative.

Note; The points O_1, O_2, O_3 about which the moments are calculated can also be called as moment centre.

3.3.2 Couple

Two forces of same magnitude separated by a definite distance, (acting parallely) in aopposite direction are said to form a couple.
A couple has a tendency to rotate a body or can produce a moment about the body. As such the moment due to a couple is also denoted as M.
Let us consider a point O about which a couple acts. Let S be the distance separating the couple. Let d1 & d2 be the perpendicular distance of the lines of action of the forces from the point o.

Thus the magnitude of the moment due to the couple is given a s
Mo = (Fx d1) + (F x d2)

Mo = F x d
i.e The magnitude of a moment due to a couple is the product of force constituting the couple & the distance separating the couple . Hence the units for magnitude of a couple can be N m, kN m , N mm etc.

3.3.3 Varignon's principle of moments:

If a number of coplanar forces are acting simultaneously on a particle, the algebraic sum of the moments of all the forces about any point is equal to the moment of their resultant force about the same point.
PROOF:
For example, consider only two forces F_1 and F_2 represented in magnitude and direction by AB and AC as shown in figure below.

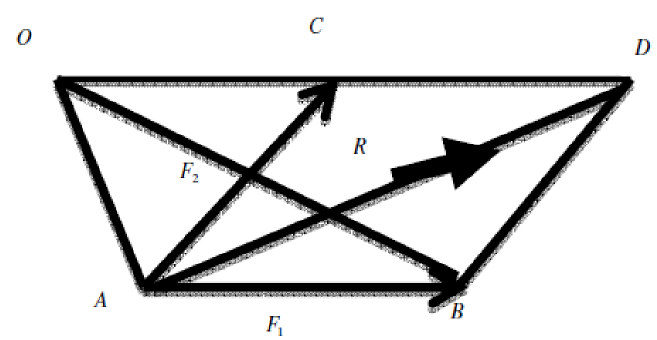
Let O be the point, about which the moments are taken. Construct the parallelogram ABCD and complete the construction as shown in fig.

By the parallelogram law of forces, the diagonal AD represents, in magnitude and

Direction, the resultant of two forces F_1 and F_2, let R be the resultant force.
By geometrical representation of moments
the moment of force about O=2 Area of triangle AOB
the moment of force about O=2 Area of triangle AOC
the moment of force about O=2 Area of triangle AOD

But,
Area of triangle AOD=Area of triangle AOC + Area of triangle ACD
Also, Area of triangle ACD=Area of triangle ADB=Area of triangle AOB
Area of triangle AOD=Area of triangle AOC + Area of triangle AOB

Multiplying throughout by 2, we obtain
2 Area of triangle AOD =2 Area of triangle AOC+2 Area of triangle AOB
i.e., Moment of force R about O=Moment of force F_1 about O + Moment of force F_2 about O
Similarly, this principle can be extended for any number of forces.



By using the principles of resolution composition & moment it is possible to determine Analytically the resultant for coplanar non-concurrent system of forces.
The procedure is as follows:

1. Select a Suitable Cartesian System for the given problem.
2. Resolve the forces in the Cartesian System
3. Compute ∑fxi and ∑fyi
4. Compute the moments of resolved components about any point taken as the moment Centre O. Hence find ∑M0

$$R = \sqrt{\left(\Sigma f x_i\right)^2 + \left(\Sigma f y_i\right)^2} \qquad \alpha_R = \tan^{-1}\left(\frac{\sum f y_i}{\sum f x_i}\right)$$

5. Compute moment arm $d_R = \left| \frac{\sum M_o}{R} \right|$

6. Also compute x- intercept as $x_R = \left| \frac{\sum M_o}{\sum f_{x_i}} \right|$

7. And Y intercept as $y_R = \left| \frac{\sum M_o}{\sum f_{x_i}} \right|$

Problems

Example 1: Compute the resultant for the system of forces shown in Fig 2 and hence compute the Equilibriant.

$$\sum f_{x_i} = 44.8 - 32\cos 60^\circ$$
$$= 28.8 \text{ KN}$$

$$\sum f_{y_i} = 8 - 14.4 - 32\sin 60^\circ$$
$$= -34.11 \text{ KN}$$
$$R = 44.6 \text{ KN}$$
$$\alpha_R = 49.83^\circ$$
$$\circlearrowleft + \sum M_o = -14.4(3) + 32\cos 60^o(4) - 32\sin 60^o(3)$$
$$= -62.34 KNM$$
$$d_R = \frac{62.34}{44.64} = 1.396 \text{ m}$$
$$x_R = \frac{62.34}{34.11} = 1.827 \text{ m}$$
$$y_R = \frac{62.34}{28.8} = 2.164 \text{ m}$$

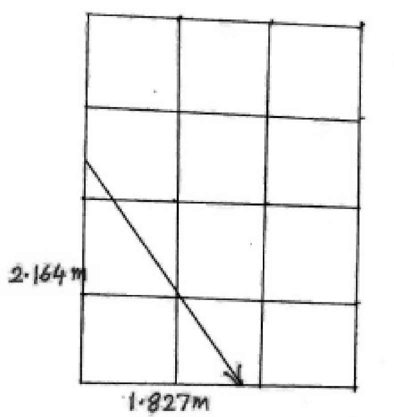


Fig. 2(a) Example 1

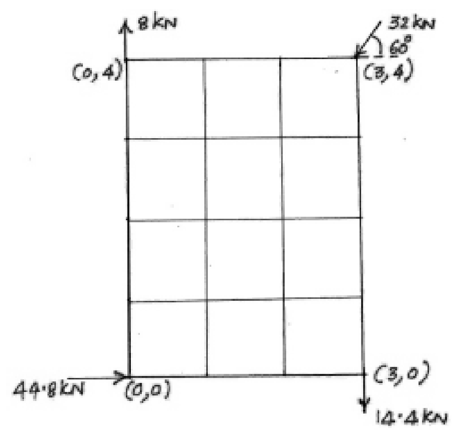


Fig. 2 Example 1

Example 2: Find the Equilibriant for the rigid bar shown in Fig 3 when it is subjected to forces.

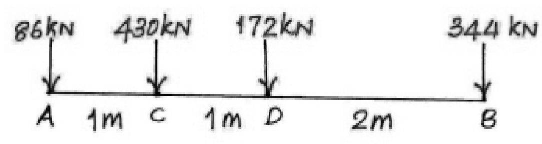


Fig. 3 Example 2

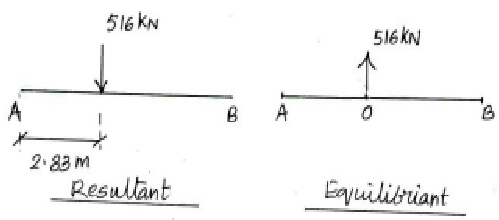


Fig. 3(a) Example 2

$$\circlearrowleft + \sum M_A = -430(1) + 172(2) - 344(4)$$
$$= -1462 \text{ KNM}$$

- Resultant and Equilibriant

$$\sum f_{x_i} = 0$$
$$\sum f_{y_i} = -516KN$$
$$\alpha_R = 90^o;$$

Example 3: A bar AB of length 3.6 m and of negligible weight is acted upon by a vertical force F1 = 336kN and a horizontal force F2 = 168kN shown in Fig 4. The ends of the bar are in contact with a smooth vertical wall and smooth incline. Find the equilibrium position of the bar by computing the angle θ.

$$\tan\alpha = 0.9/1.2$$
$$\alpha = 36.87^o$$

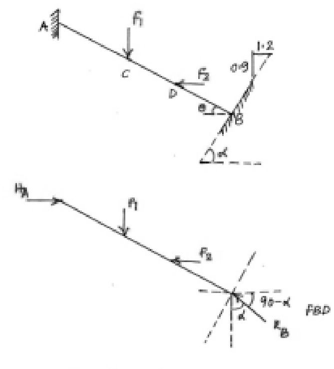


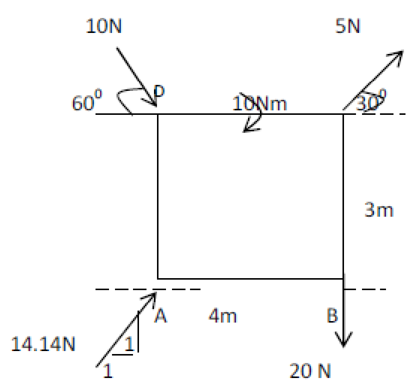
Fig 4 Example 3

$$\sum f_{x_i} = 0$$
$$H_A - F_2 - R_B\cos 53.13^o = 0............(1)$$
$$\sum f_{y_i} = 0$$
$$R_B\sin 53.13^o - F_1 = 0$$
$$R_B = 420KN;$$

- Eq. 1 gives HA=420 KN

$$\varsigma + \sum M_B = 0;$$
$$-H_A(3.6\sin\theta) + 336(2.1\cos\theta) - 168\,(1.2\sin\theta) = 0$$
$$-1310.4\sin\theta + 705.6\cos\theta = 0$$
$$\tan\theta = 0.538$$
$$\theta = 28.3^o$$

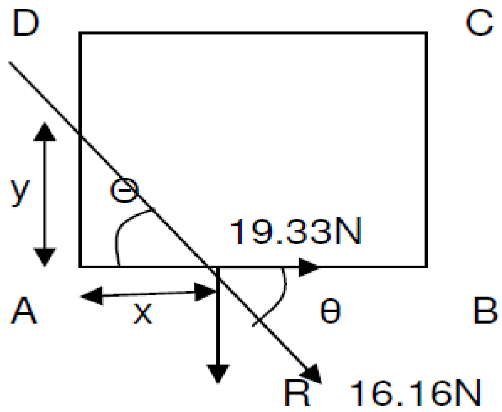
2. <u>Determine the resultant of the force system acting on the plate. As shown in figure given below with respect to AB and AD.</u>

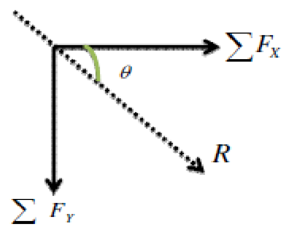


$\sum$Fx = 5cos30^0 + 10cos60^0 + 14.14cos45^0
= 19.33N

$\sum Fy = 5\sin30^0 - 10\sin60^0 + 14.14\sin45^0$
$= -16.16N$
$R = \sqrt{(\sum Fx^2 + \sum Fy^2)} = 25.2N$

$\theta = Tan^{-1}(\sum Fy/\ \sum Fx)$
$\theta = Tan^{-1}(16.16/19.33) = 39.89^0$





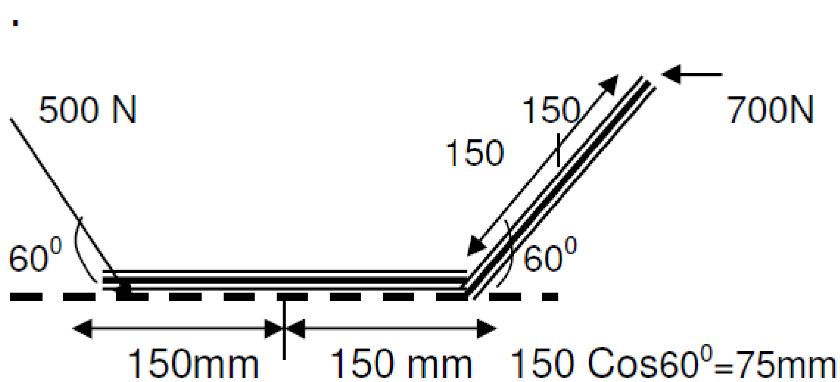
Tracing moments of forces about A and applying varignon's principle of moments we get
$+16.16X = 20x4 + 5\cos30^0x3 - 5\sin30^0x4 + 10 + 10\cos60^0x3$

$x = 107.99/16.16 = 6.683m$

Also $\tan39.89 = y/6.83$
$y = 5.586m.$

---

3. The system of forces acting on a crank is shown in figure below. Determine the magnitude , direction and the point of application of the resultant force.



$\sum Fx = 500\cos60^0 - 700$

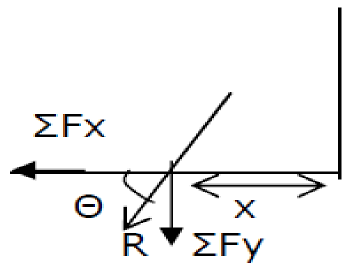
$= 450N$

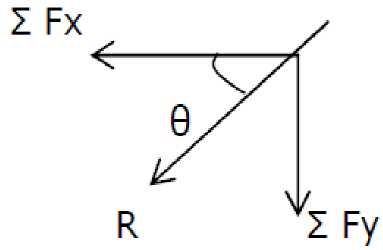
$\sum Fy = 500\sin 60^0$

$= -26.33N$

$R = \sqrt{(\sum Fx^2 + \sum Fy^2)} = (-450)^2 + (-2633)^2$

$R = 267.19N$ (Magnitude)





$\theta = \text{Tan}^{-1}(\sum Fy / \sum Fx)$

$\theta = \text{Tan}^{-1}(2633/450)$

$\theta = 80.30^0$ (Direction)

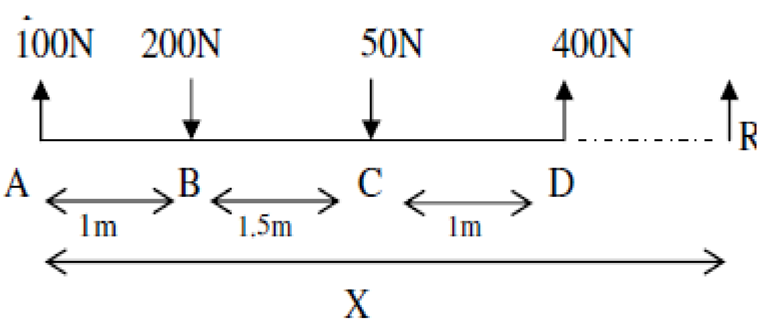
Tracing moments of forces about O and applying varignon's principle of moments we get

$-2633x\ x = -500x \sin 60^0 x300 - 1000x150 + 1200x150\cos 60^0 - 700x300\sin 60^0$

$x = -371769.15/-2633$

$x = 141.20$mm from O towards left (position).

4.For the system of parallel forces shown below, determine the magnitude of the resultant and also its position from A .



$\sum Fy = +100 -200 -50 +400$

$= +250N$

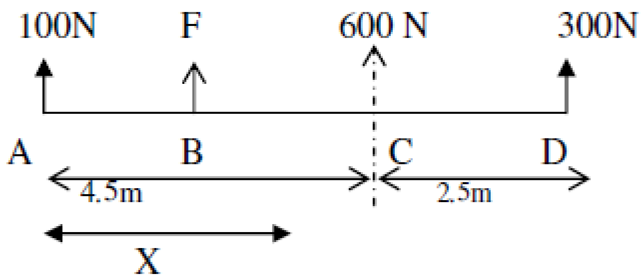
ie. $R = \sum Fy = 250N$ (↑)

Since $\sum Fx = 0$

Taking moments of forces about A and applying varignon's principle of moments
-250 x = -400 x 3.5 + 50 x 2.5 + 200 x 1 – 100 x 0
X = -1075/ -250 = 4.3m

---

5. The three like parallel forces 100 N,F and 300 N are acting as shown in figure below. If the resultant R=600 N and is acting at a distance of 4.5 m from A ,find the magnitude of force F and position of F with respect to A



Let x be the distance from A to the point of application of force F
Here R = ∑ Fy
600=100+F+300
F = 200 N

Taking moments of forces about A and applying varignon's principle of moments,
We get
600 x 4.5 = 300 x 7 + F x
200 x = 600 x 4.5 -300 x 7.
X = 600/200 = 3m from A

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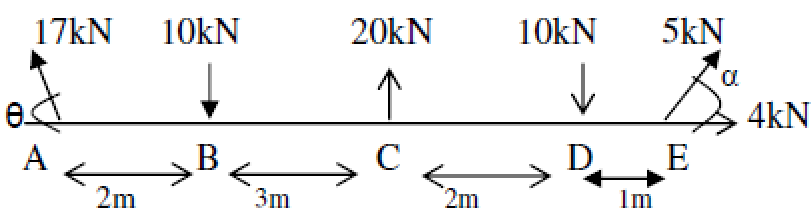
6. A beam is subjected to forces as shown in the figure given below. Find the magnitude, direction and the position of the resultant force.



Given tan θ= 15/8 sin θ = 15/ 17 cos θ = 8/17
tanα= 3/4 sin α = 3/5 cos α = 4/5
∑ Fx = 4 +5 cos α – 17 cos θ
      = 4 + 5 x 4/5 – 17 x 8/17
∑ Fx = 0

∑ Fy = 5 sin α -10 + 20 – 10 + 17 sin θ

$= 5 \times 3/5 - 10 + 20 - 10 + 17 \times 15/17$

$\sum F_y = 18$ kN (↑)

Resultant force $R = \sqrt{(\sum F_x)^2 + (\sum F_y)^2} = \sqrt{0 + 18^2}$

$R = 18$ kN(↑)

Let x = distance from A to the point of application R

Taking moments of forces about A and applying Varignon's theorem of moments

$-18x = -5 \times \sin\alpha \times 8 + 10 \times 7 - 20 \times 5 + 10 \times 2$

$= -3 \times 8 + 10 \times 7 - 20 \times 5 + 10 \times 2$

$X = -34/-18 = 1.89$m from A (towards left)

UNIT - IV: CENTROID OF PLANE FIGURES

4.1 Centre of Gravity:

Everybody is attracted towards the centre of the earth due gravity. The force of attraction is proportional to mass of the body. Everybody consists of innumerable particles, however the entire weight of a body is assumed to act through a single point and such a single point is called centre of gravity.

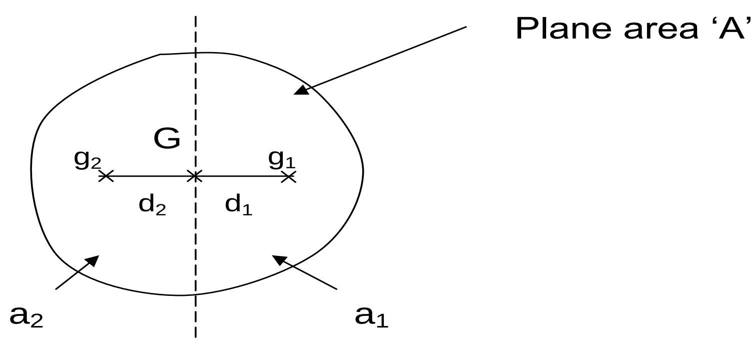
Every body has one and only centre of gravity.

4.2 Centroid:

In case of plane areas (bodies with negligible thickness) such as a triangle quadrilateral, circle etc., the total area is assumed to be concentrated at a single point and such a single point is called centroid of the plane area.

The term centre of gravity and centroid has the same meaning but the following differences.

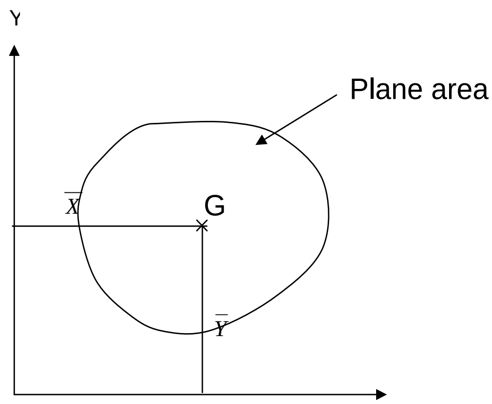
1. Centre of gravity refer to bodies with mass and weight whereas, centroid refers to plane areas.
2. centre of gravity is a point is a point in a body through which the weight acts vertically downwards irrespective of the position, whereas the centroid is a point in a plane area such that the moment of areas about an axis through the centroid is zero



Note: In the discussion on centroid, the area of any plane figure is assumed as a force equivalent to the centroid referring to the above figure G is said to be the centroid of the plane area A as long as

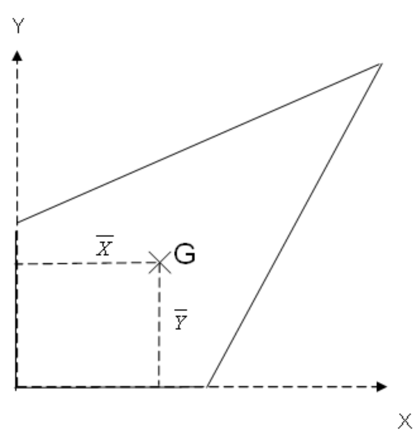
$a_1d_1 - a_2 d_2 = 0.$

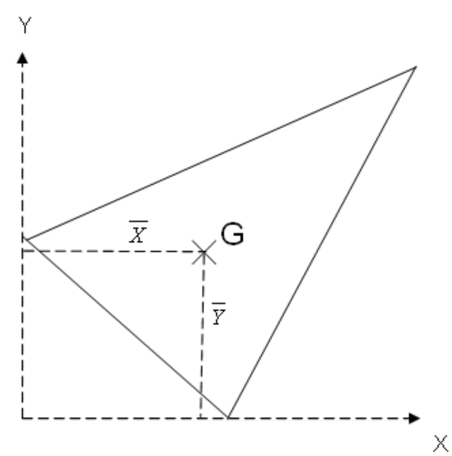
4.3 Location of centroid of plane areas



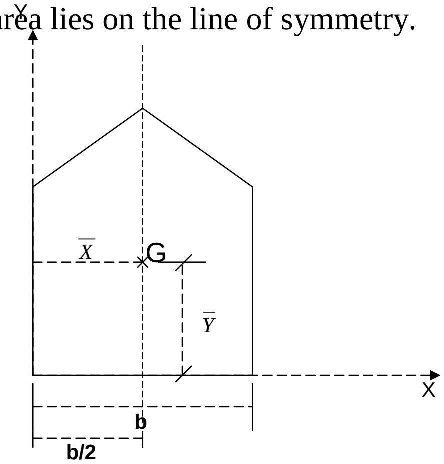
The position of centroid of a plane area should be specified or calculated with respect to some reference axis i.e. X and Y axis. The distance of centroid G from vertical reference axis or Y axis is denoted as X and the distance of centroid G from a horizontal reference axis or X axis is denoted as Y.

While locating the centroid of plane areas, a bottommost horizontal line or a horizontal line through the bottommost point can be made as the X – axis and a leftmost vertical line or a vertical line passing through the leftmost point can be made as Y- axis.





In some cases the given figure is symmetrical about a horizontal or vertical line such that the centroid of the plane area lies on the line of symmetry.



The above figure is symmetrical about a vertical line such that G lies on the line of symmetry. Thus

X= b/2.

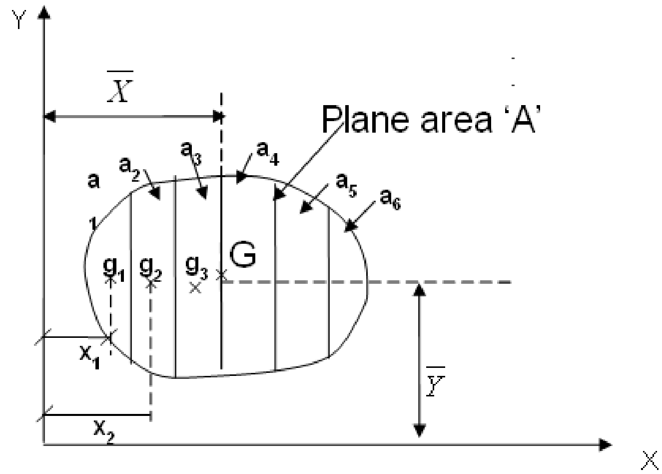
Y=?

The centroid of plane geometric area can be located by one of the following methods

a) Graphical methods
b) Geometric consideration
c) Method of moments

The centroid of simple elementary areas can be located by geometric consideration. The centroid of a triangle is a point, where the three medians intersect. The centroid of a square is a point where the two diagonals bisect each other. The centroid of a circle is centre of the circle itself.

4.4 METHOD OF MOMENTS TO LOCATE THE CENTROID OF PLANE AREAS



Let us consider a plane area A lying in the XY plane. Let G be the centroid of the plane area. It is required to locate the position of centroid G with respect to the reference axis like Y- axis and Xi- axis i.e, to calculate X and Y. Let us divide the given area A into smaller elemental areas a_1, a_2, a_3 …….. as shown in figure. Let g_1 ,g_2, g_3…… be the centroids of elemental areas a_1, a_2, a_3 …….. etc.

Let x_1, x_2, x_3 etc be the distance of the centroids g_1 g_2 g_3 etc. from Y- axis is =A $\overline{X}$ -----(1)

The sum of the moments of the elemental areas about Y axis is

$a_1 . x_1 + a_2 . x_2 + a_3 . x_3 +$ ……………….**(2)**

Equating (1) and (2)

$$A . \overline{X} = a_1 . x_1 + a_2 . x_2 + a_3 . x_3 + \ldots\ldots$$

$$\overline{X} = \frac{a_1 . x_1 + a_2 . x_2 + a_3 . x_3 + \ldots\ldots}{A}$$

$$\overline{X} = \frac{\sum(ax)}{A} \quad \text{or} \quad \overline{X} = \frac{\int x.dA}{A}$$

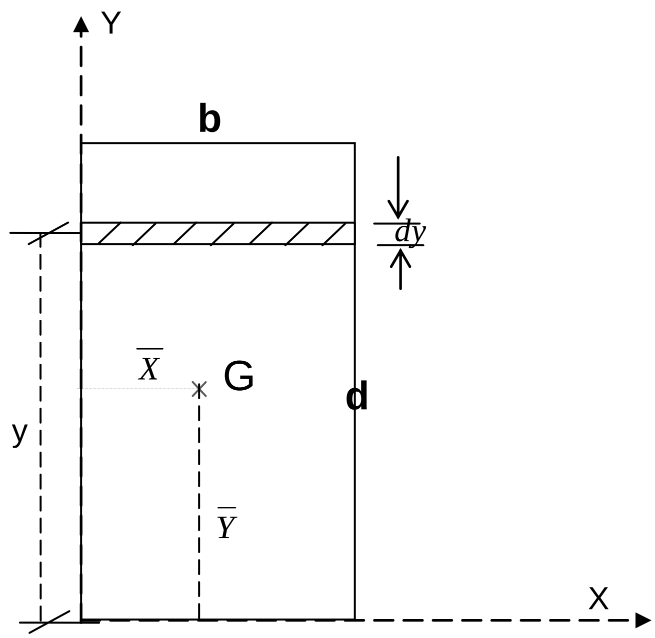
Where a or dA represents an elemental area in the area A , x is the distance of elemental area from Y axis.

Similarly

$$\bar{Y} = \frac{a1 \cdot y1 + a2 \cdot y2 + a3 \cdot y3 + \ldots\ldots\ldots..}{A}$$

$$\bar{Y} = \frac{\sum (a.y)}{A} \quad \text{or} \quad \bar{Y} = \frac{\int y.dA}{A}$$

TO LOCATE THE CENTROID OF A RECTANGLE FROM THE FIRST PRINCIPLE (METHOD OF MOMENTS)



Let us consider a rectangle of breadth b and depth d. let g be the centroid of the rectangle. Let us consider the X and Y axis as shown in the figure.

Let us consider an elemental area dA of breadth b and depth dy lying at a distance of y from the X axis.

W.K.T

$$\overline{Y} = \frac{\int_0^d y.dA}{A}$$

A = b . d
dA = b . dy

$$\overline{Y} = \frac{\int_0^d y.(b.dy)}{b.d}$$

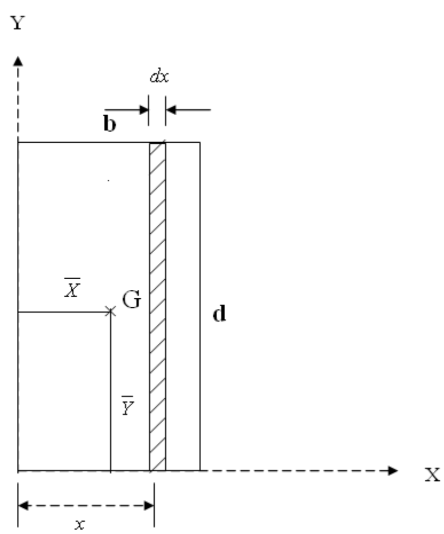
$$\overline{Y} = \frac{1}{d}\int_0^d y.dy$$

$$\overline{Y} = \frac{1}{d}\left[\frac{y^2}{2}\right]_0^d$$

$$\overline{Y} = \frac{1}{d}\left[\frac{d^2}{2}\right]$$

$$\overline{Y} = \frac{d}{2}$$

Similarly



$$\overline{X} = \frac{\int_0^b x.dA}{A}$$

A = b . d

dA = dx . d

$$\overline{X} = \frac{\int_0^b x.(dx.d)}{b.d}$$

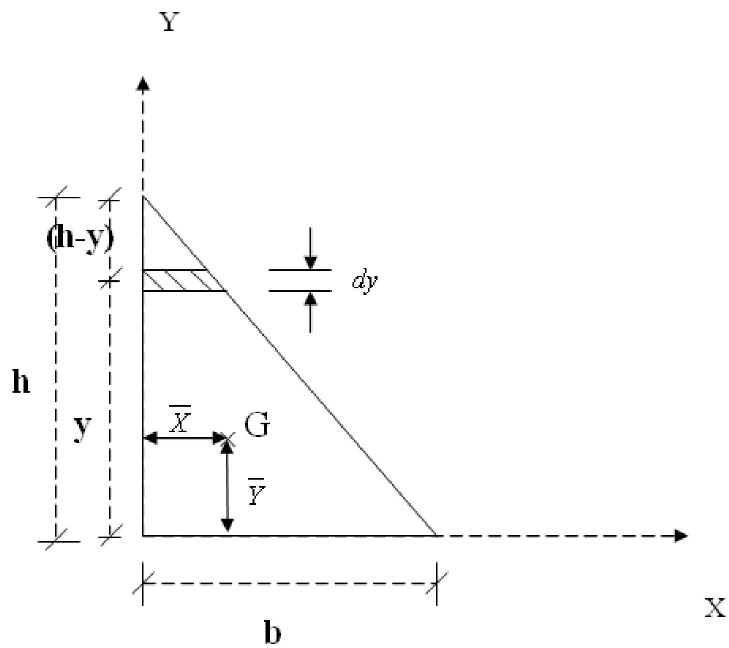
$$\overline{X} = \frac{1}{d} \cdot \int_0^b x.dx$$

$$\overline{X} = \frac{1}{b} \cdot \left[\frac{x^2}{2}\right]_0^b$$

$$\overline{X} = \frac{1}{b} \cdot \left[\frac{b^2}{2}\right]$$

$$\overline{X} = \frac{b}{2}$$

Centroid of a triangle



Let us consider a right angled triangle with a base b and height h as shown in figure. Let G be the centroid of the triangle. Let us consider the X- axis and Y- axis as shown in figure.

Let us consider an elemental area dA of width b1 and thickness dy, lying at a distance y from X-axis.

W.K.T

$$\overline{Y} = \frac{\int_0^h y.dA}{A}$$

$$A = \frac{b.h}{2}$$

$dA = b_1 \, . \, dy$

$$\overline{Y} = \frac{\int_0^h y.(b_1 \, . \, dy)}{\frac{b.h}{2}}$$ [as x varies b1 also varies]

$$\overline{Y} = \frac{2}{h}.\int_0^h \left(y - \frac{y^2}{h} \right) dy$$

$$\overline{Y} = \frac{2}{h}\left[\frac{y^2}{2} - \frac{y^3}{3.h} \right]_0^h$$

$$\overline{Y} = \frac{2}{h}\left[\frac{h^2}{2} - \frac{h^3}{3.h} \right]$$

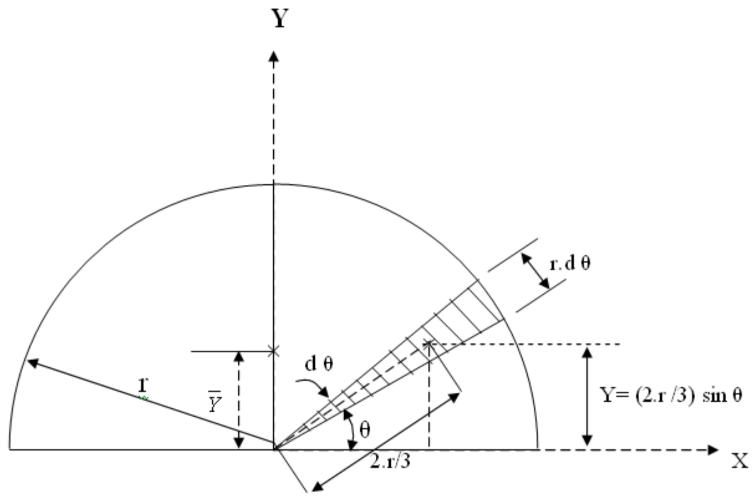
$$\overline{Y} = \frac{2}{h}\left[\frac{h^2}{2} - \frac{h^2}{3} \right]$$

$$\overline{Y} = 2h\left[\frac{1}{2} - \frac{1}{3} \right]$$

$$\overline{Y} = \frac{2.h}{6}$$

$\overline{Y} = \dfrac{h}{3}$ similarly $\overline{X} = \dfrac{b}{3}$

Centroid of a semi circle



$$\overline{X} = 0$$

Let us consider a semi-circle, with a radius ‘r’.

Let ‘O’ be the centre of the semi-circle .let ‘G’ be centroid of the semi-circle. Let us consider the x and y axes as shown in figure.

Let us consider an elemental area ‘dA’ with centroid ‘g’ as shown in fig. Neglecting the curvature, the elemental area becomes an isosceles triangle with base r.dθ and height ‘r’.

Let y be the distance of centroid ‘g’ from x axis.

Here $y = \frac{2r}{3}.\sin\theta$

W K T

$$\overline{Y} = \frac{\int y.dA}{A}$$

$$A = \frac{\pi.r^2}{2}$$

$$\overline{Y} = \frac{\int y.dA}{A}$$

$$\overline{Y} = \frac{\int \frac{2r}{3}.\sin\theta\,.dA}{A}$$

$$dA = \frac{1}{2}.r.d\theta.r$$

$$dA = \frac{r^2}{2}.d\theta$$

$$= \frac{2}{3\pi}\int r\sin\theta.d\theta$$

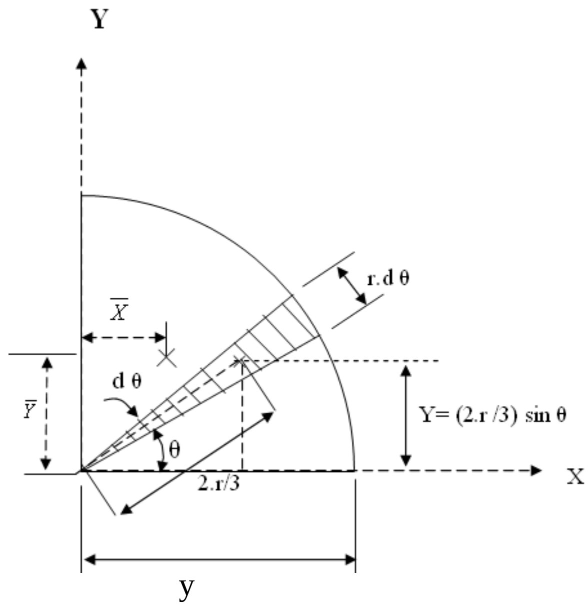
$$= \frac{2r}{3\pi}\int_0^{\pi}\sin\theta.d\theta$$

$$= \frac{2r}{3\pi}\left[-\cos\theta\right]_0^{\pi}$$

$$= \frac{2r}{3\pi}[1+1]$$

$$\overline{Y} = \frac{4r}{3\pi}$$

Centroid of a quarter circle



Let us consider a quarter circle with radius r. Let ‘O’ be the centre and ‘G’ be the centroid of the quarter circle. Let us consider the x and y axes as shown in figure.

Let us consider an elemental area ‘dA’ with centroid ‘g’ as shown in fig.

Let ‘y’ be the distance of centroid ‘g’ from x axis. Neglecting the curvature, the elemental area becomes an isosceles triangle with base r.dθ and height ‘r’.

Here $y = \frac{2r}{3}.\sin\theta$

W K T

$$\bar{Y} = \frac{\int y.dA}{A}$$

$$A = \frac{\pi.r^2}{2}$$

$$\bar{Y} = \frac{\int y.dA}{A}$$

$$\bar{Y} = \frac{\int \frac{2r}{3}.\sin\theta.dA}{A}$$

$$dA = \frac{1}{2}.r.d\theta.r$$

$$dA = \frac{r^2}{2}.d\theta$$

$$\bar{Y} = \frac{\int \frac{2r}{3}.\sin\theta.\frac{r^2}{2}.d\theta}{\frac{\pi.r^2}{4}}$$

$$= \frac{4r}{3\pi}\int_0^{\pi/2}\sin\theta.d\theta$$

$$= \frac{2r}{3\pi}\left[-\cos\theta\right]_0^{\pi/2}$$

$$= \frac{4r}{3\pi}[0+1]$$

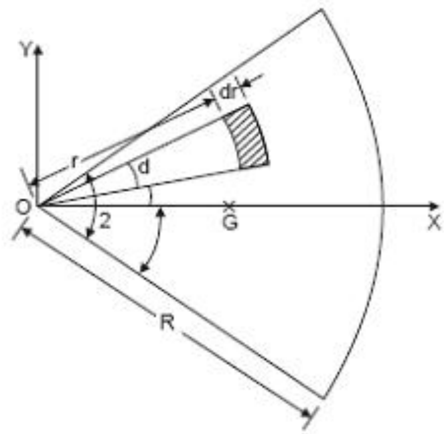
$$\bar{Y} = \frac{4r}{3\pi}$$

Similarly

$$\bar{X} = \frac{4r}{3\pi}$$

Centroid of Sector of a Circle

Consider the sector of a circle of angle 2α as shown in Fig. Due to symmetry, centroid lies on x axis. To find its distance from the centre O, consider the elemental area shown.



Area of the element $= rd\theta\, dr$

Its moment about y axis

$$= rd\theta \times dr \times r\cos\theta$$
$$= r^2 \cos\theta\, drd\theta$$

$\therefore$ Total moment of area about y axis

$$= \int_{-\alpha}^{\alpha}\int_{0}^{R} r^2 \cos\theta\, drd\theta$$

$$= \left[\frac{r^3}{3}\right]_0^R [\sin\theta]_{-\alpha}^{\alpha}$$

$$= \frac{R^3}{3}\, 2\sin\alpha$$

Total area of the sector

$$= \int_{-\alpha}^{\alpha}\int_{0}^{R} rdrd\theta$$

$$= \int_{-\alpha}^{\alpha}\left[\frac{r^2}{2}\right]_0^R d\theta$$

$$= \frac{R^2}{2}[\theta]_{-\alpha}^{\alpha}$$

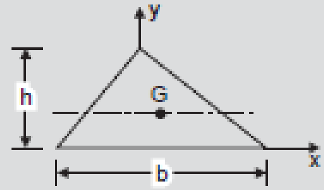
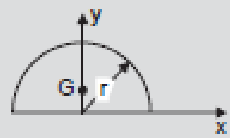
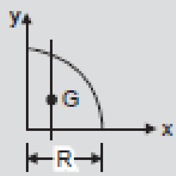
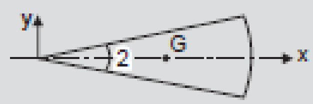
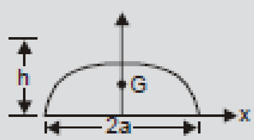
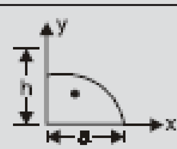
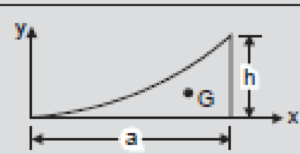
$$= R^2\alpha$$

$\therefore$ The distance of centroid from centre O

$$= \frac{\text{Moment of area about } y \text{ axis}}{\text{Area of the figure}}$$

$$= \frac{\frac{2R^3}{3}\sin\alpha}{R^2\alpha} = \frac{2R}{3\alpha}\sin\alpha$$

Centroid of Some Common Figures

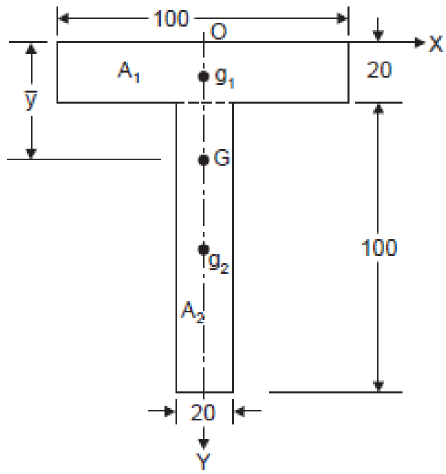
Shape	*Figure*	$\bar{x}$	$\bar{y}$	*Area*
Triangle		—	$\frac{h}{3}$	$\frac{bh}{2}$
Semicircle		0	$\frac{4R}{3\pi}$	$\frac{\pi R^2}{2}$
Quarter circle		$\frac{4R}{3\pi}$	$\frac{4R}{3\pi}$	$\frac{\pi R^2}{4}$
Sector of a circle		$\frac{2R}{3\alpha}$ sin a	0	αR^2
Parabola		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Semi parabola		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$

4.5 Centroid of Composite Sections

In engineering practice, use of sections which are built up of many simple sections is very common. Such sections may be called as built-up sections or composite sections. To locate the centroid of composite sections, one need not go for the first principle (method of integration). The given composite section can be split into suitable simple figures and then the centroid of each simple figure can be found by inspection or using the standard formulae listed in the table above. Assuming the area of the simple figure as concentrated at its centroid, its moment about an axis can be found by multiplying the area with distance of its centroid from the reference axis. After determining moment of each area about reference axis, the distance of centroid from the axis is obtained by dividing total moment of area by total area of the composite section.

PROBLEMS:

Q) Locate the centroid of the T-section shown in fig.



All dimensions in mm

Solution. Selecting the axis as shown in Fig. we can say due to symmetry centroid lies on y axis, *i.e.* $\bar{x} = 0$. Now the given T-section may be divided into two rectangles A_1 and A_2 each of size 100×20 and 20×100. The centroid of A_1 and A_2 are $g_1(0, 10)$ and $g_2(0, 70)$ respectively.

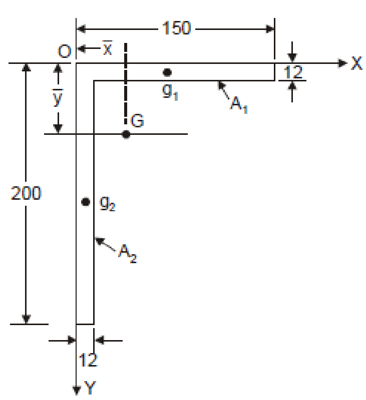
$\therefore$ The distance of centroid from top is given by:

$$\bar{y} = \frac{100 \times 20 \times 10 + 20 \times 100 \times 70}{100 \times 20 + 20 \times 100}$$

$$= 40 \text{ mm}$$

Hence, *centroid of T-section is on the symmetric axis at a distance 40 mm from the top.* **Ans.**

Q) Find the centroid of the unequal angle 200 ×150 × 12 mm, shown in Fig.



All dimensions in mm

Solution. The given composite figure can be divided into two rectangles:

$$A_1 = 150 \times 12 = 1800 \text{ mm}^2$$

$$A_2 = (200 - 12) \times 12 = 2256 \text{ mm}^2$$

Total area $\quad A = A_1 + A_2 = 4056 \text{ mm}^2$

Selecting the reference axis x and y as shown in Fig. 2.30. The centroid of A_1 is g_1 (75, 6) and that of A_2 is:

$$g_2\left[6, 12+\frac{1}{2}(200-12)\right]$$

i.e., $\quad g_2$ (6, 106)

$$\therefore \quad \bar{x} = \frac{\text{Movement about } y \text{ axis}}{\text{Total area}}$$

$$= \frac{A_1x_1 + A_2x_2}{A}$$

$$= \frac{1800 \times 75 + 2256 \times 6}{4056} = 36.62 \text{ mm}$$

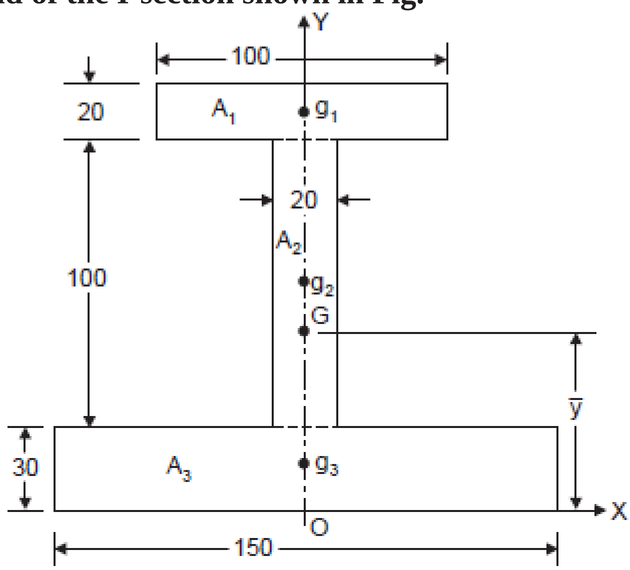
$$\bar{y} = \frac{\text{Movement about } x \text{ axis}}{\text{Total area}}$$

$$= \frac{A_1y_1 + A_2y_2}{A}$$

$$= \frac{1800 \times 6 + 2256 \times 106}{4056} = 61.62 \text{ mm}$$

Thus, the centroid is at $\bar{x}$ = 36.62 mm and $\bar{y}$ = 61.62 mm as shown in the figure **Ans.**

Q) Locate the centroid of the I-section shown in Fig.



All dimensions in mm

Solution. Selecting the co-ordinate system as shown in Fig. due to symmetry centroid must lie on y axis,

i.e., $\bar{x} = 0$

Now, the composite section may be split into three rectangles

$$A_1 = 100 \times 20 = 2000 \text{ mm}^2$$

Centroid of A_1 from the origin is:

$$y_1 = 30 + 100 + \frac{20}{2} = 140 \text{ mm}$$

Similarly $A_2 = 100 \times 20 = 2000 \text{ mm}^2$

$$y_2 = 30 + \frac{100}{2} = 80 \text{ mm}$$

$$A_3 = 150 \times 30 = 4500 \text{ mm}^2, \text{ and}$$

$$y_3 = \frac{30}{2} = 15 \text{ mm}$$

$$\therefore \qquad \bar{y} = \frac{A_1y_1 + A_2y_2 + A_3y_3}{A}$$

$$= \frac{2000 + 140 + 2000 \times 80 + 4500 \times 15}{2000 + 2000 + 4500}$$

$$= 59.71 \text{ mm}$$

Thus, *the centroid is on the symmetric axis at a distance 59.71 mm from the bottom* as shown in Fig. **Ans.**

UNIT – V: EQUILIBRIUM OF FORCES

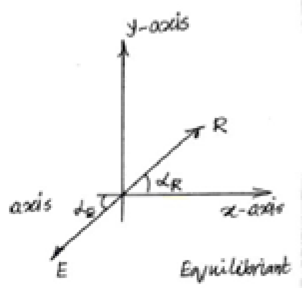
5.1 Concepts

Equilibrium: Equilibrium is the status of the body when it is subjected to a system of forces. We know that for a system of forces acting on a body the resultant can be determined. By Newton's 2nd Law of Motion the body then should move in the direction of the resultant with some acceleration. If the resultant force is equal to zero it implies that the net effect of the system of forces is zero this represents the state of equilibrium. For a system of coplanar concurrent forces for the resultant to be zero, hence

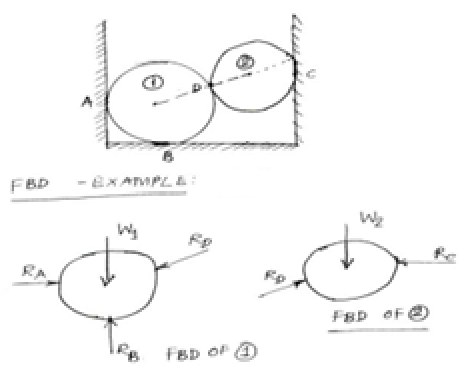
$$\sum f_{x_i} = 0$$

$$\sum f_{y_i} = 0$$

Equilibriant : Equilbriant is a single force which when added to a system of forces brings the status of equilibrium . Hence this force is of the same magnitude as the resultant but opposite in sense. This is depicted in Fig 4.



Free Body Diagram: Free body diagram is nothing but a sketch which shows the various forces acting on the body. The forces acting on the body could be in form of weight, reactive forces contact forces etc. An example for Free Body Diagram is shown below.

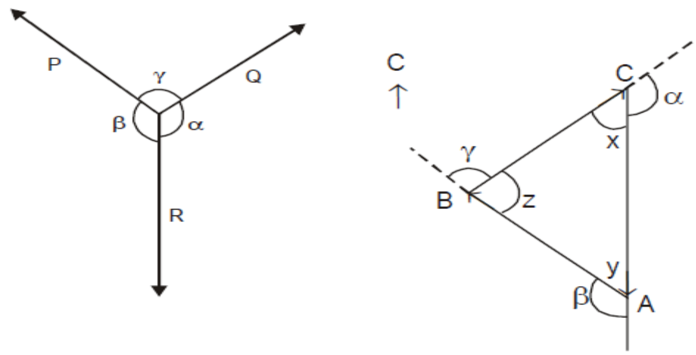


Lami's Theorem

If three forces acting on a particle keep it in equilibrium, each force is proportional to the sine of the angle between the other two.

P, Q and *R* three forces acting at a point keeping it in equilibrium, Fig. If α, β and γ are the angles apposite to each of them respectively,

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$



This law is a direct consequence of the triangle law. Since the forces are in equilibrium, they can be represented by the sides of the triangle *ABC* taken in order. A general property of any triangle is that each side is proportional to the sine of the angle opposite to it. Thus in the triangle *ABC* drawn with the sides parallel to the forces *P, Q*, and *R*,

$$\frac{AB}{\sin x} = \frac{BC}{\sin y} = \frac{CA}{\sin z}$$

Here *x, y* and *z* are the angles of the triangle *ABC*. But by the triangle law of forces, the sides of the triangle are proportional to the respective force. From the Fig. 1.2

$$\sin x = \sin(180 - \alpha) = \sin\alpha$$

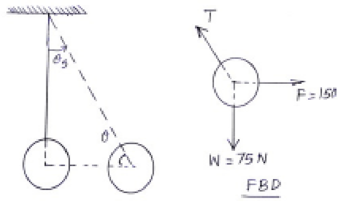
$$\sin y = \sin(180 - \beta) = \sin\beta$$

$$\sin z = \sin(180 - \gamma) = \sin\gamma$$

Hence $$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{Q}{\sin\gamma} = \text{constant}$$

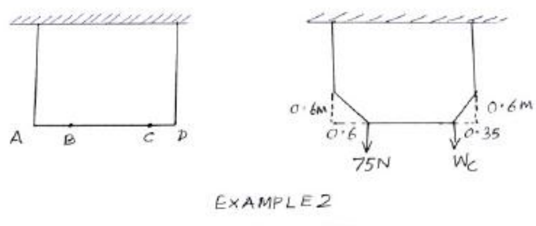
Thus, if 3 forces acting on a particle are in equilibrium, each force is proportional to the sine of the angle between the other two.

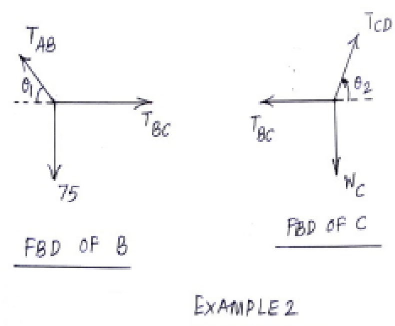
Example 1 : A spherical ball of weight 75N is attached to a string and is suspended from the ceiling. Compute tension in the string if a horizontal force F is applied to the ball. Compute the angle of the string with the vertical and also tension in the string if F =150N



$$\Sigma f_{x_i} = 0$$
$$f - T\cos\theta = 0$$
$$150 - T\cos\theta = 0$$
$$T\cos\theta = 150$$

Example 2: A string or cable is hung from a horizontal ceiling from two points A and D. The string AD, at two points B and C weights are hung. At B, which is 0.6 m from a weight of 75 N is hung. C, which is 0.35 m from D, a weight of wc is hung. Compute wc such that the string portion BC is horizontal.





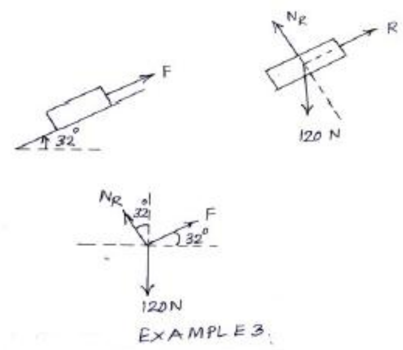
FBD of B

$$\sum f_{x_i} = 0$$
$$T_{BC} - T_{AB}\cos\theta_1 = 0$$
$$\sum f_{y_i} = 0$$
$$T_{AB}\sin\theta_1 - 75 = 0$$
$$T_{AB} = 75\sqrt{2}\,N, T_{BC} = 75N$$

FBD of C

$$\sum f_{x_i} = 0$$
$$-T_{BC} + T_{CD}\cos\theta_2 = 0$$
$$T_{CD} = 148.85N$$
$$\sum f_{y_i} = 0$$
$$T_{CD}\sin\theta_2 - W_c = 0$$
$$W_c = 128.57N$$

Example 3: A block of weight 120N is kept on a smooth inclined plane. The plane makes an angle of 320 with horizontal and a force F allied parallel to inclined plane. Compute F and also normal reaction.



- LAMI'S Theorem

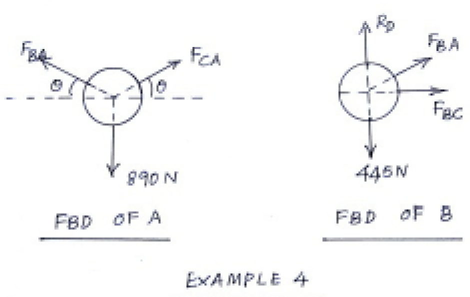
$$\frac{120}{Sin90^o} = \frac{F}{Sin(180-32)^o} = \frac{NR}{Sin(90+32)^o}$$

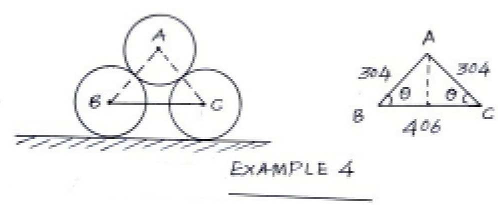
$F = 63.59N$

$N_R = 101.76N$

Example 4: Three smooth circular cylinders are placed in an arrangement as shown. Two cylinders are of radius 052mm and weight 445 N are kept on a horizontal surface. The centers of these cylinders are tied by a string which is 406 mm long. On these two cylinders, third cylinder of weight 890N and of same diameter is kept. Find the force S in the string and also forces at points of contact.

- LAMI'S Theorem





FBD of A

$F_{AC} = 598N$

$F_{BA} = 598\ N$

FBD of B

$\sum f_{x_i} = 0$

$\sum f_{y_i} = 0$

$F_{BC} = 399.5N$

$R_D = 890N$

UNIT – VI: BEAMS

6.1 Introduction

A beam is a structural member or element, which is in equilibrium under the action of a non-concurrent force system. The force system is developed due to the loads or forces acting on the beam and also due to the support reactions developed at the supports for the beam.

For the beam to be in equilibrium, the reactions developed at the supports the should be equal and opposite to the loads.

In a beam, one dimension (length) is considerably larger than the other two dimensions (breath & depth). The smaller dimensions are usually neglected and as such a beam is represented as a line for theoretical purposes or for analysis.

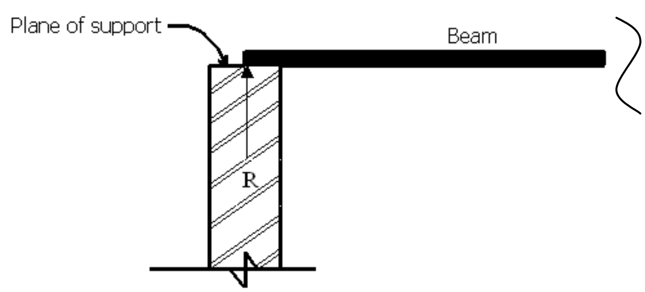
6.2 Types of Supports for beams:

Supports are structures which prevent the beam or the body from moving and help to maintain equilibrium.

A beam can have different types of supports as follows. The support reactions developed at each support are represented as follows.

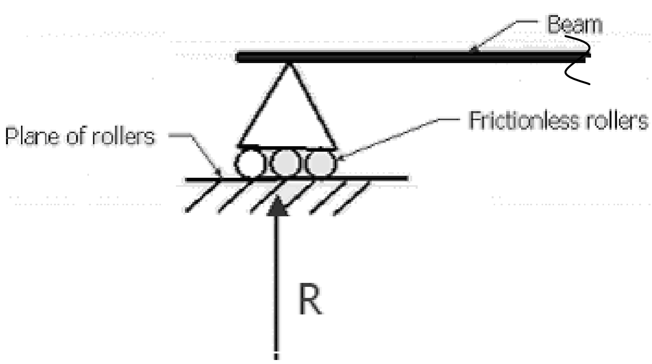
1) <u>Simple support</u>:

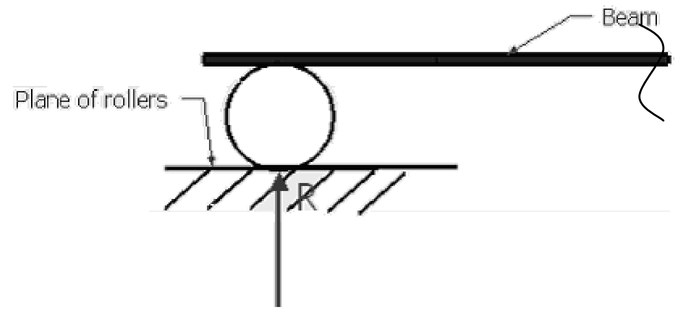
This is a support where a beam rests freely on a support. The beam is free to move only horizontally and also can rotate about the support. In such a support one reaction, which is perpendicular to the plane of support, is developed.



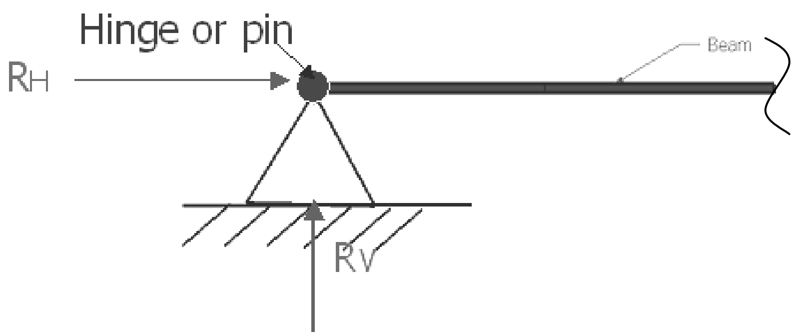
2) <u>Roller support</u>:

This is a support in which a beam rests on rollers, which are frictionless. At such a support, the beam is free to move horizontally and as well rotate about the support. Here one reaction which is perpendicular to the plane of rollers is developed.



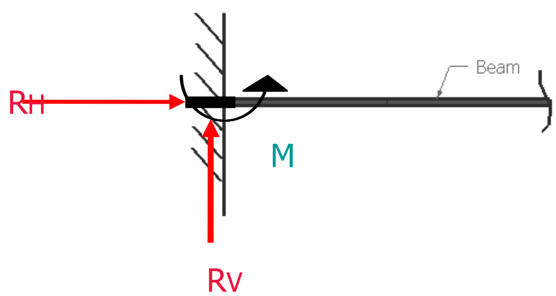


direction but can rotate about the support. In such a support a horizontal reaction and a vertical reaction will develop.



4) <u>Fixed support</u>:

This is a support which prevents the beam from moving in any direction and also prevents rotation of the beam. In such a support a horizontal reaction, vertical reaction and a Fixed End Moment are developed to keep the beam in equilibrium.

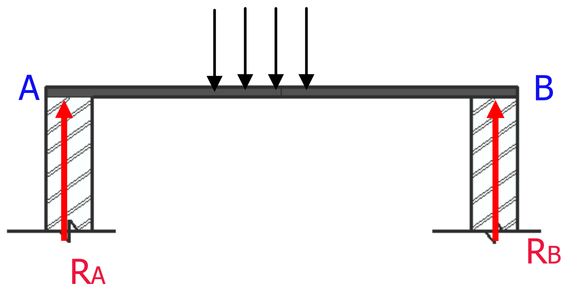


6.3 Types of beams

Depending upon the supports over which a beam can rest (at its two ends), beams can be classified as follows.

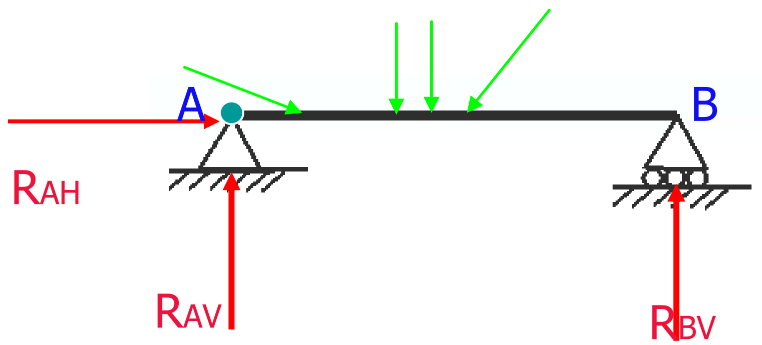
1) <u>Simply supported beam</u>.

A beam is said to be simply supported when both ends of the beam rest on simple supports. Such a beam can carry or resist vertical loads only.



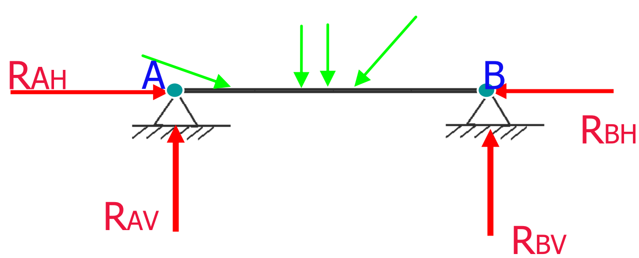
2) Beam with one end hinged & other on rollers.

It is a beam where one end of the beam is hinged to a support and the other end rests on a roller support. Such a beam can carry any type of loads.



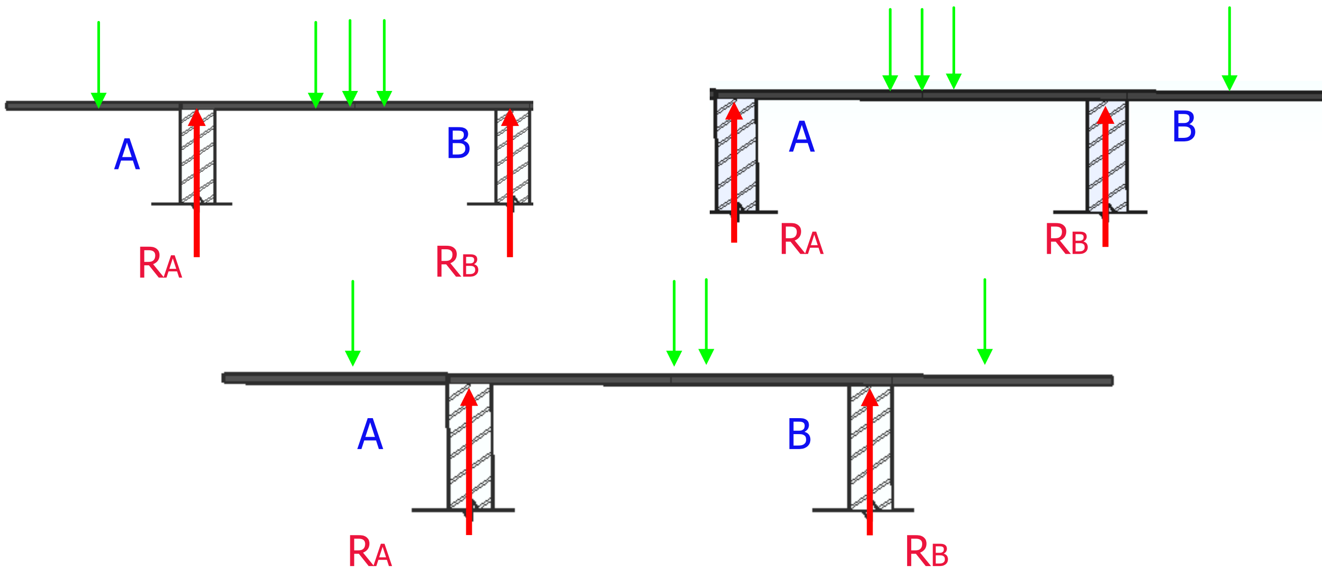
3) Hinged Beam:

It is a beam which is hinged to supports at both ends. Such a beam can carry loads is any direction.



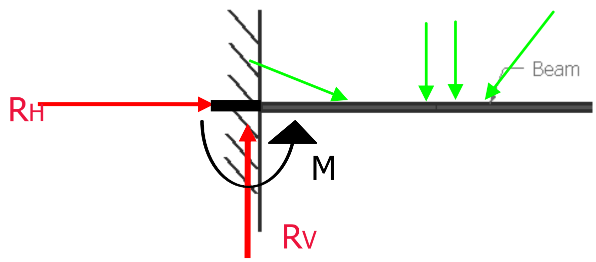
4) Over hanging beam :

It is a beam which projects beyond the supports. A beam can have over hanging portions on one side or on both sides.



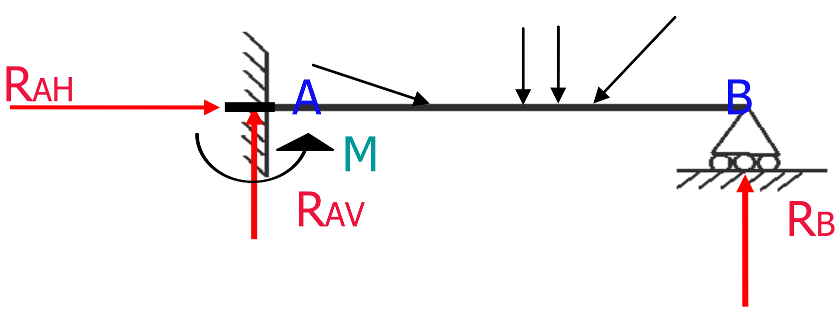
5) <u>Cantilever Beams</u>:

It is a beam, with one end fixed and other and free. Such a beam can carry loads in any directions.



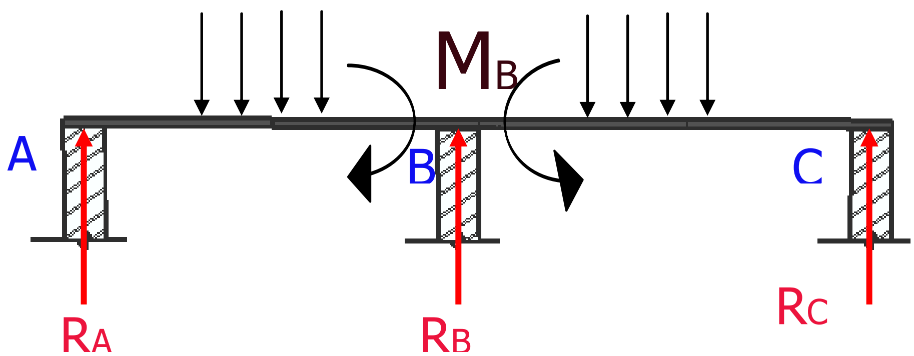
6) <u>Propped cantilever</u>:

It is a beam which has a fixed support at one end and a simple support at the other end.



7) Continuous beam:

It is a beam which rests over a series of supports at more than two points.



Note:

The support reactions in case of simply supported beams, beam with one end hinged and other on rollers, over hanging beams, and cantilever beams, can be determined by conditions of equilibrium only ($\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M = 0$). As such, such beams are known as Statically Determinate Beams.

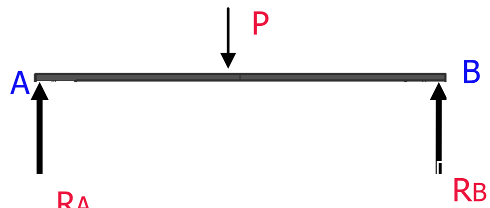
In beams such as Hinged Beams, Propped Cantilever and Continuous Beams the support reactions cannot be determined using conditions of equilibrium only. They need additional special conditions for analysis and as such, such beams are known as Statically Indeterminate Beams

6.4 Types of loads:

The various types of loads that can act over a beam can e listed as follows.

1) <u>Point load or Concentrated load:</u>

If a load acts over a very small length of the beam, it is assumed to act at the mid point of the loaded length and such a loading is termed as Point load or Concentrated load.

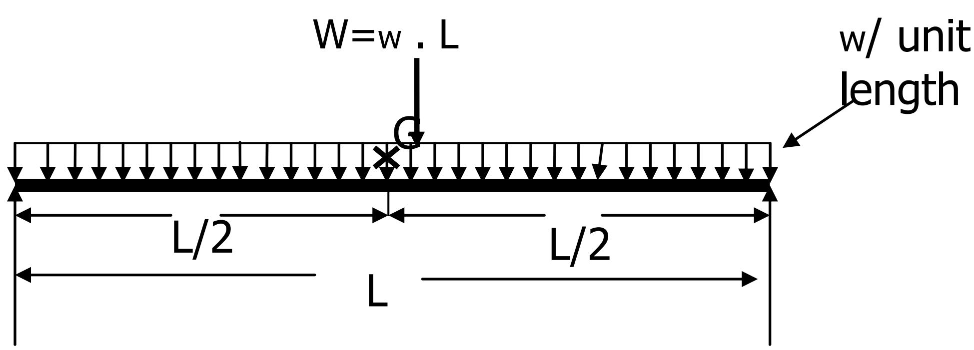


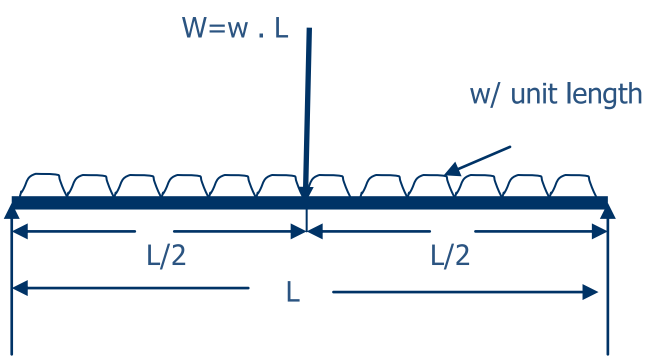
2) Uniformly distributed load (UDL):

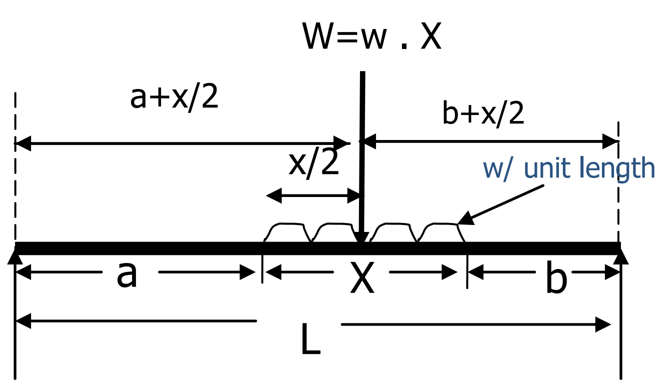
If a beam is loaded in such a manner that each unit length of the beam carries the same intensity of loading, then such a loading is called UDL.

A UDL cannot be considered in the same manner for applying conditions of equilibrium on the beam. The UDL should be replaced by an equivalent point load or total load acting through the mid point of the loaded length.

The magnitude of the point load or total load is equal to the product of the intensity of loading and the loaded length (distance).



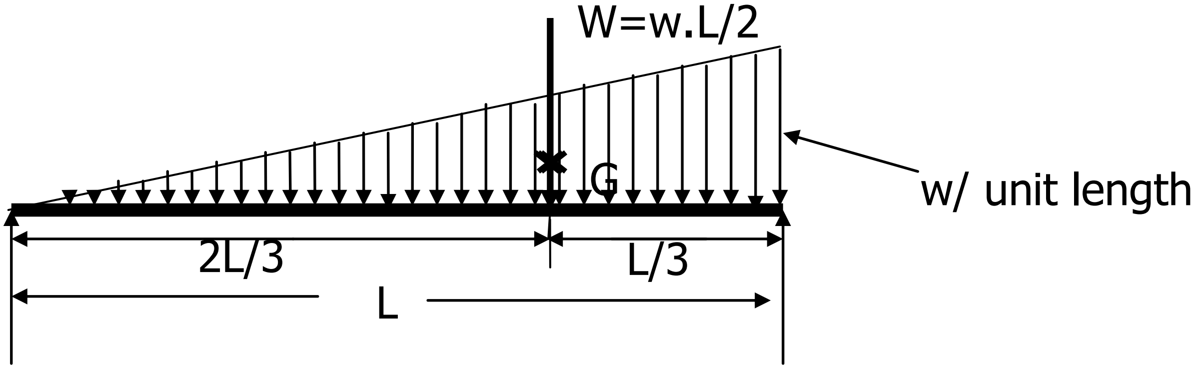


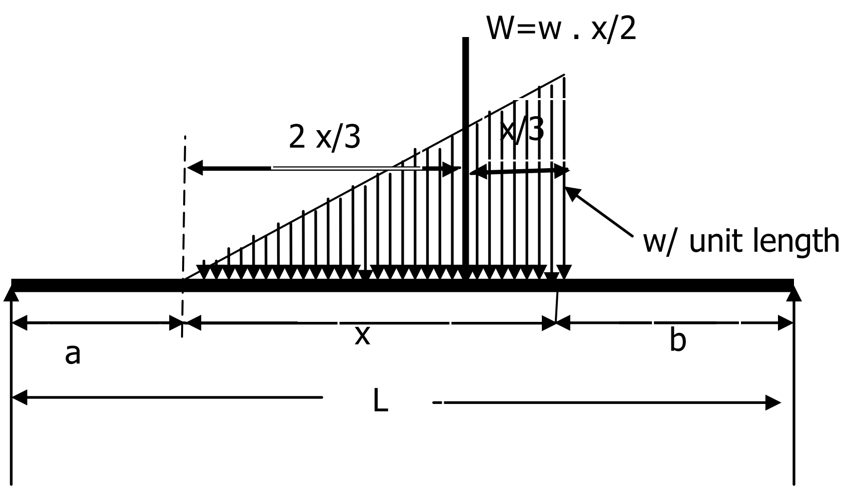


3) Uniformly varying load (UVL):

If a beam is loaded in such a manner, that the intensity of loading varies linearly or uniformly over each unit distance of the beam, then such a load is termed as UVL.

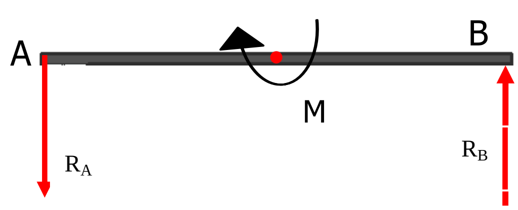
In applying conditions of equilibrium, a given UVL should be replaced by an equivalent point load or total load acting through the centroid of the loading diagram (right angle triangle). The magnitude of the equivalent point load or total load is equal to the area of the loading diagram.



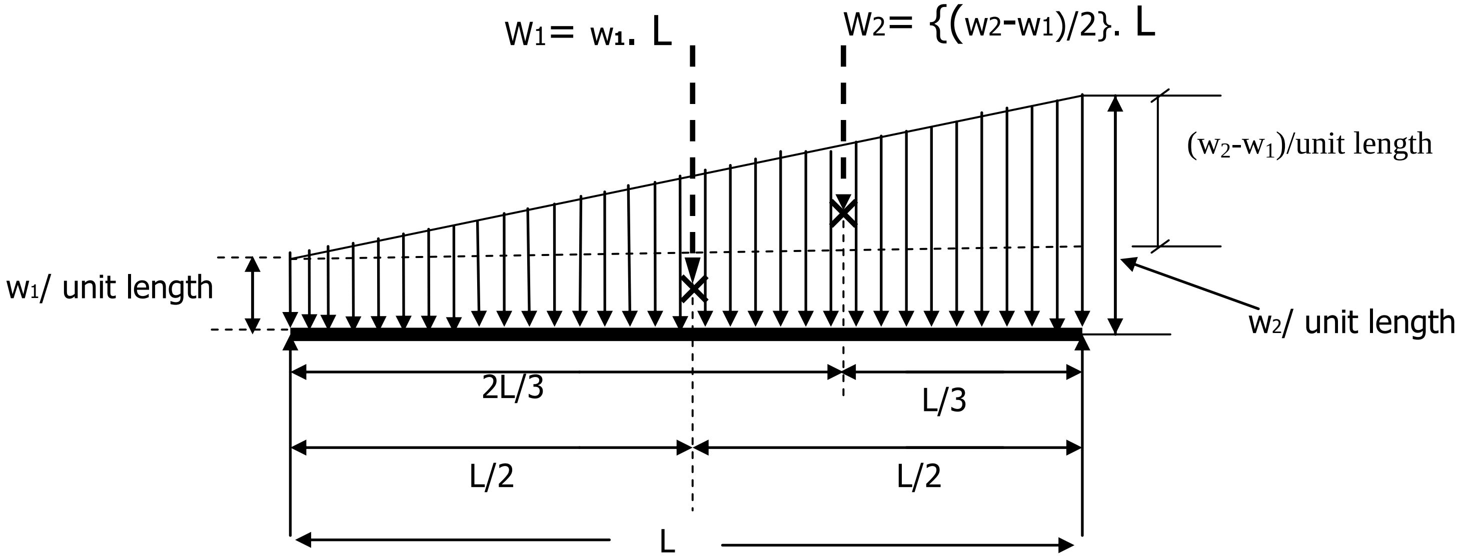


4) <u>External moment:</u>

A beam can also be subjected to external moments at certain points as shown in figure. These moments should be considered while calculating the algebraic sum of moments of forces about a point on the beam



Note : A beam can also be subject to a load as shown in figure below.



In such a case, the UVL can be split into a UDL with a uniform intensity of w_1/unit length another UVL with a maximum intensity of (w_2-w_1) /unit length.

Example 4: Determine the support reactions for the beam shown in Fig 7 at A and B.

$$\sum f_{x_i} = 0;$$
$$\sum f_{y_i} = 0;$$
$$\sum M_o = 0;$$

$$V_A - 10 - 25 - 32 + V_B = 0$$
$$V_A + V_B = 67KN;$$
$$\circlearrowleft + \sum M_A = 0$$
$$-10(2) - 25(5) - 32(9) + V_B(10) = 0$$
$$V_B = 43.3KN$$
$$V_A = 23.7KN$$

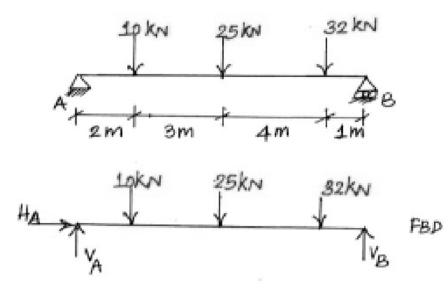


Fig.7 Example 4

Example 5: Determine the support reactions for the beam shown in Fig 8 at A and B.

$$\sum f_{x_i} = 0;\ H_A = 0$$
$$\sum f_{y_i} = 0; V_A - 40 - 40 + V_B = 0$$
$$V_A + V_B = 80$$
$$\varsigma M_A = 0 - 40(2) - 40(7) + V_B(8) = 0$$
$$V_B = 45KN$$
$$V_A = 35KN;$$

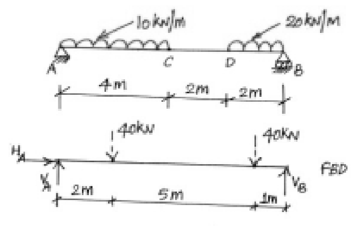


Fig. 8 Example 5

Example 6: Determine the support reactions for the beam shown in Fig 9 at A and B.

$$\sum f_{x_i} = 0;$$
$$H_A - 17.32 = 0$$
$$H_A = 17.32KN$$

$$\sum f_{y_i} = 0$$
$$V_A - 10 - 20 - 15 - 10 + V_B = 0$$
$$V_A + V_B = 55$$
$$\varsigma + \sum M_A = 0$$
$$-10 \times 2 + 25 - 20(6) + V_B(8) - 15(9) - 10(11) = 0$$
$$V_B = 45 \text{ KN}; V_A = 10 \text{ KN}$$

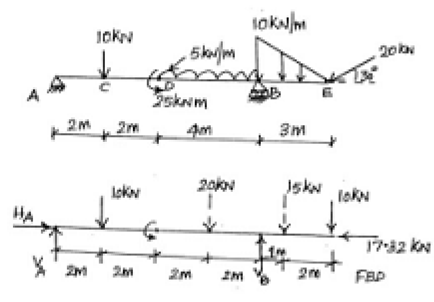


Fig. 9 Example 6

Example 7: Determine the support reactions for the beam shown in Fig 10 at A and B.

$$\sum f_{x_i} = 0;$$
$$H_A - R_B \sin 30^0 = 0$$
$$H_A = 0.5 R_B$$
$$\sum f_{y_i} = 0; V_A - 20 + R_B \cos 30^0 = 0$$
$$V_A + 0.866 R_B = 20$$
$$\varsigma + \sum M_B = 0;$$
$$-V_A(10) + 20(6) = 0$$
$$-V_A = 12KN;$$
$$R_B = 9.24KN;$$
$$H_A = 4.62KN;$$

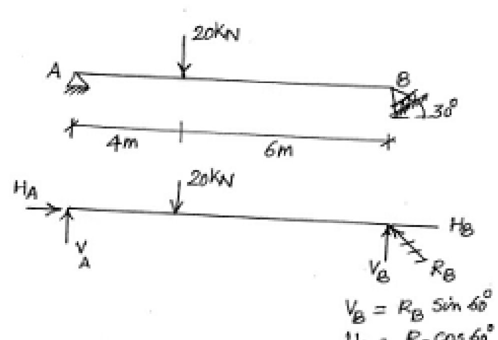


Fig. 10 - Example 7

UNIT VII: FRICTION

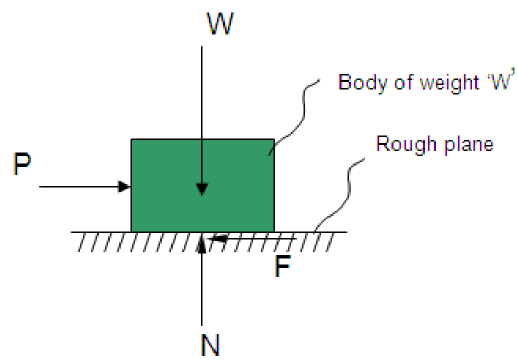
7.1 Introduction

Whenever a body moves or tends to move over another surface or body, a force which opposes the motion of the body is developed tangentially at the surface of contact, such on opposing force developed is called friction or frictional resistance**.**

The frictional resistance is developed due to the interlocking of the surface irregularities at the contact surface b/w two bodies

Consider a body weighing W resting on a rough plane & subjected to a force 'P'

to displace the body.

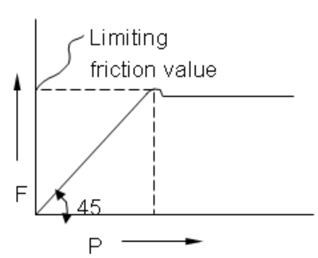


Where

P = Applied force
N = Normal reaction from rough surface
F = Frictional resistance
W = Weight of the body

The body can start moving or slide over the plane if the force 'P' overcomes the frictional 'F'

The frictional resistance developed is proportional to the magnitude of the applied force which is responsible for causing motion upto a certain limit.



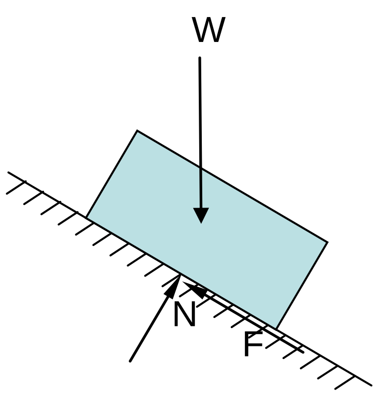
From the above graph we see that as P increases, F also increases. However F cannot increase beyond a certain limit. Beyond this limit (Limiting friction value) the frictional resistance becomes constant for any value of applied force. If the magnitude of the applied force is less than the limiting friction value, the body remains at rest or in equilibrium. If the magnitude of the applied force is greater than the limiting friction value the body starts moving over the surface.

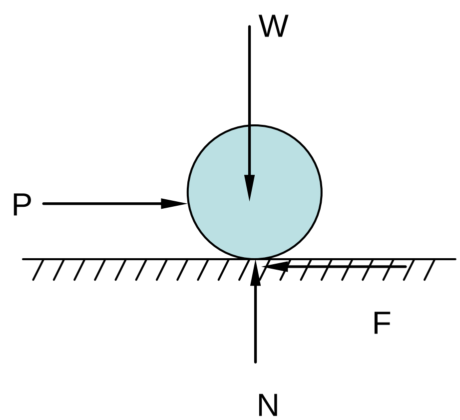
The friction experienced by a body when it is at rest or in equilibrium is known as static friction. It can range between a zero to limiting fraction value.

The friction experienced by a body when it is moving is called **dynamic friction**.

The dynamic friction experienced by a body as it slides over a plane as it is shown in figure is called **sliding friction.**

The dynamic friction experienced by a body as it roles over surface as shown in figure is called **rolling friction**.





7.2 CO-EFFICIENT OF FRICTION: It ha been experimentally proved that between two contacting surfaces, the magnitude of limiting friction bears a constant ratio to normal reaction between the two this ratio is called as co-efficient of friction.

It is defined by the relationship $\mu = \dfrac{F}{N}$

Where

μ – Represents co-efficient of friction
F – Represents frictional resistance
N – Represents normal reaction.

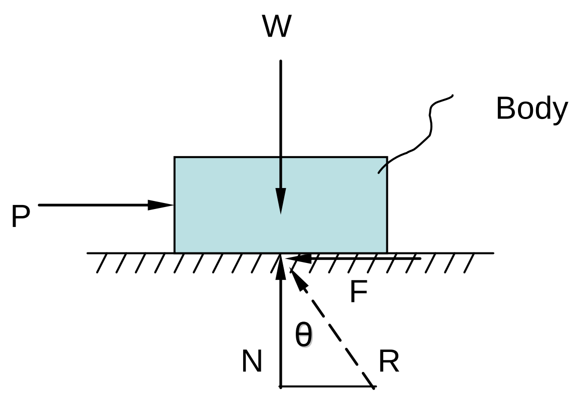
Note: Depending upon the nature of the surface of contact i.e. dry surface & wet surface, the frictional resistance developed at such surface can be called dry friction & wet friction (fluid friction) respectively. In our discussion on friction all the surface we consider will be dry sough surfaces.

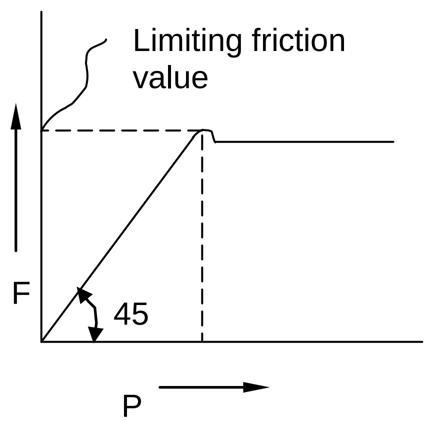
7.3 LAWS OF DRY FRICTION: (COLUMB'S LAWS)

The frictional resistance developed between bodies having dry surfaces of contact obey certain laws called laws of dry friction. They are as follows.

1) The frictional resistance depends upon the roughness or smoothness of the surface.
2) Frictional resistance acts in a direction opposite to the motion of the body.
3) The frictional resistance is independent of the area of contact between the two bodies.
4) The ratio of the limiting friction value (F) to the normal reaction (N) is a constant (co-efficient of friction, μ)
5) The magnitude of the frictional resistance developed is exactly equal to the applied force till limiting friction value is reached or where the bodies is about to move.

7.4 ANGLE OF FRICTION





Consider a body weighing 'W' placed on a horizontal plane. Let 'P' be an applied force required to just move the body such that, frictional resistance reaches limiting friction value. Let 'R' be resultant of F & N. Let 'θ' be the angle made by the resultant with the direction of N. such an angle 'θ' is called the Angle of friction

As P increases, F also increases and correspondingly 'θ' increases. However, F cannot increase beyond the limiting friction value and as such 'θ' can attain a maximum value only.

Let $\theta_{max} = \alpha$

Where α represents angle of limiting friction

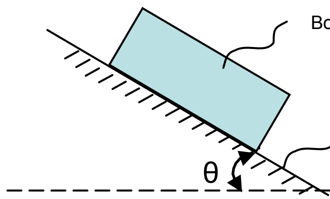
$$\tan\theta_{max} = \tan\alpha = \frac{F}{N}$$

But $\frac{F}{N} = \mu$

Therefore $\mu = \tan\alpha$

i.e. co-efficient of friction is equal to the tangent of the angle of limiting friction

7.5 ANGLE OF REPOSE:



Consider a body weighing 'w' placed on a rough inclined plane, which makes an angle 'θ' with the horizontal. When 'θ' value is small, the body is in equilibrium or rest without sliding. If 'θ' is gradually increased, a stage reaches when the body tends to slide down the plane

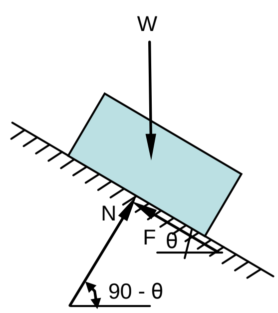
The maximum inclination of the plane with the horizontal, on which a body free from external forces can rest without sliding is called angle of repose.

Let $\theta_{max} = \Phi$

Where Φ = angle of repose

When = angle of repose.

Let us draw the free body diagram of the body before it slide.



Applying conditions of equilibrium.

$$\sum F_x = 0$$

$$N \cos(90 - \theta) - F \cos\theta = 0$$

$$N \sin\theta = F \cos\theta$$

$$\tan\theta = \frac{F}{N}$$

$$\tan\theta_{max} = \tan\Phi$$

$$\text{but } \frac{F}{N} = \mu$$

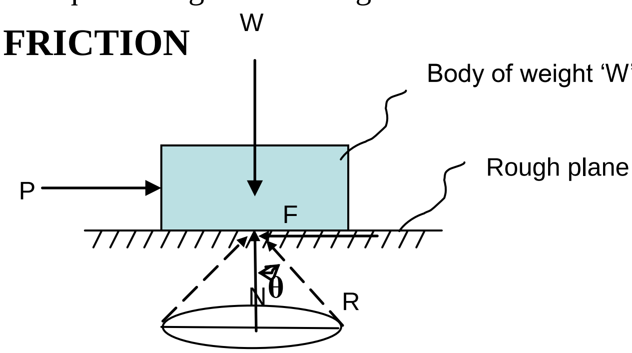
$$\mu = \tan\alpha$$

$$\tan\Phi = \tan\alpha$$

$$\Phi = \alpha$$

i.e. angle of repose is equal to angle of limiting friction

7.6 CONE OF FRICTION



Consider a body weighting 'W' resting on a rough horizontal surface. Let 'P' be a force required to just move the body such that frictional resistance reaches limiting value. Let 'R' be the resultant of 'F' & 'N' making an angel with the direction of N.

If the direction of 'P' is changed the direction of 'F' changes and accordingly 'R' also changes its direction. If 'P' is rotated through 360^o, R also rotates through 360^o and generates an imaginary cone called cone of friction.

Note: In this discussion, all the surface that bee consider are rough surfaces, such that, when the body tends to move frictional resistance opposing the motion comes into picture tangentially at the surface of contact in all the examples, the body considered is at the verge of moving such that frictional resistance reaches limiting value. We can consider the body to be at rest or in equilibrium & we can still apply conditions of equilibrium on the body to calculate unknown force.

Ex. Block *A* weighing 1000 N rests over block *B* which weighs 2000 N as shown in Fig. Block *A* is tied to wall with a horizontal string. If the coefficient of friction between *A* and *B* is 1/4 and between *B* and the floor is 1/3, what should be the value of *P* to move the block *B* if (a) *P* is horizontal? (b) *P* acts 30° upwards to horizontal?

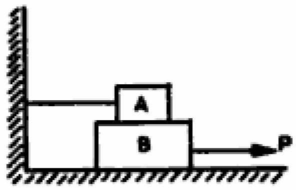
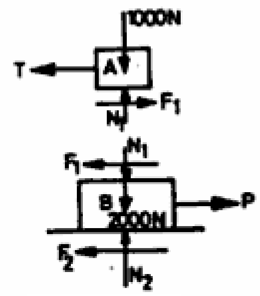


Fig. 5.5(a)



(a) *When P is horizontal*:

The free body diagrams of the two blocks are shown in Fig. Note the frictional forces F_1 and F_2 are to be marked in the opposite direction of impending relative motion. Considering block *A*,

$$\Sigma V = 0$$

$$N_1 = 1000 \text{ N}$$

since F_1 is limiting friction, $\dfrac{F_1}{N_1} = \dfrac{1}{4}$

$$\therefore \quad F_1 = 250 \text{ N}$$

$$\Sigma H = 0$$

$$T = F_1$$

$$= 250 \text{ N}$$

Considering block *B*,

$$\Sigma V = 0$$

$$N_2 - 2000 - N_1 = 0$$

$$N_2 = 3000 \text{ N} \quad \text{since } N_1 = 1000 \text{ N}$$

Since F_2 is the limiting friction $F_2 = \mu_2 N_2$

$$= \frac{1}{3} \times 3000 = 1000 \text{ N}$$

$$\Sigma H = 0$$

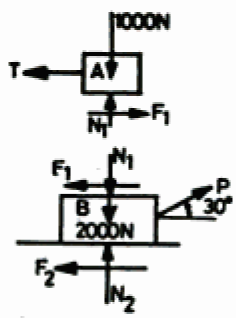
$$P - F_1 - F_2 = 0$$

$$\mathbf{P} = F_1 + F_2 = 250 + 1000$$

$$= \mathbf{1250\ N}$$

(b) *When P is enclined*

Free body diagram for this case is shown in Fig.



As in the previous case, here also N_1 = 1000 N and F_1 = 250 N. Consider the equilibrium of block *B*.

$$\Sigma V = 0$$

$$N_2 - 2000 - N_1 + P \sin 30° = 0$$

$$N_2 + 0.5\,P = 3000 \text{ N} \qquad \text{since } N_1 = 1000 \text{ N}$$

From law of friction,

$$F_2 = \frac{1}{3} N_2$$

$$= \frac{1}{3}(3000 - 0.5\,P)$$

$$= 1000 - \frac{0.5}{3} P$$

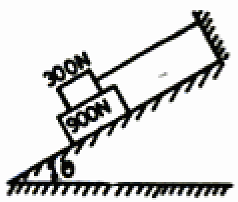
$$\Sigma H = 0$$

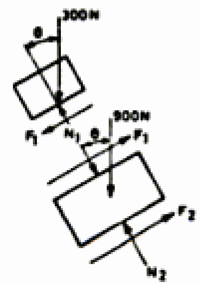
$$P\cos 30^{\circ} - F_1 - F_2 = 0$$

$$P\cos 30^{0} - 250 - (1000 - \frac{0.5}{3}P) = 0$$

$\therefore$ $\quad$ **P = 1210.43 N** $\quad$ *Ans.*

Ex. What should be the value of θ in Fig. which will make the motion of 900 N block down the plane to impend? The coefficient of friction for all contact surfaces is $\frac{1}{3}$.





900 N block is on the verge of moving downward. Hence frictional forces F_1 and F_2 act up the plane on 900 N block. Free body diagram of the blocks is as shown in Fig.

For 300 N block:

Σ forces normal to plane = 0

$N_1 - 300\cos\theta = 0$

or $N_1 = 300\cos\theta$ ...(1)

From law of friction $F_1 = \frac{1}{3}N_1 = 100\cos\theta$ ...(2)

For 900 N block:

Σ forces normal to the plane = 0

$N_2 - N_1 - 900\cos\theta = 0$

or $N_2 = N_1 + 900\cos\theta$

Substituting the value of N_1 from (1) we get

$N_2 = 1200\cos\theta$ ...(3)

From law of friction

$F_2 = \frac{1}{3}N_2 = 400\cos\theta$ ...(4)

Σ forces parallel to the plane = 0

$$F_1 + F_2 - 900 \sin\theta = 0$$

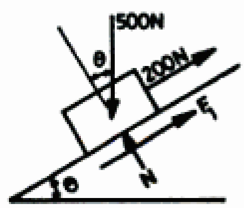
i.e. $100 \cos\theta + 400 \cos\theta = 900 \sin\theta$

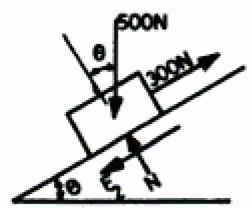
$$\tan\theta = \frac{5}{9}$$

$\therefore$ $\theta = \mathbf{29.05°}$ *Ans.*

Ex. A weight 500 N just starts moving down a rough inclined plane supported by a force of 200 N acting parallel to the plane and it is at the point of moving up the plane when pulled by a force of 300 N parallel to the plane. Find the inclination of the plane and the coefficient of friction between the inclined plane and the weight.

Free body diagram of the block when it just starts moving down is shown in Fig. and when just starts moving up is shown in the Fig. Now, frictional forces oppose the direction of the movement of the block and since it is limiting case $\frac{F}{N} = \mu$.





When the block just starts moving down

Σ forces perpendicular to the plane = 0 ...(1)

$$N = 500 \cos\theta$$

From law of friction, $F_1 = \mu N$

i.e. $F_1 = 500\,\mu \cos\theta$ ...(2)

Σ forces parallel to the plane = 0

$$500 \sin\theta - F_1 - 200 = 0$$

i.e $500 \sin\theta - 500 \;\mu\; \cos\theta = 200$ ...(3)

When the block just starts moving up the plane

Σ forces perpendicular to the plane = 0

$$N = 500 \cos\theta \qquad \ldots(4)$$

From the law of friction, $F_2 = 500\,\mu \cos\theta$ ...(5)

Σ forces parallel to the plane = 0

$$500 \sin\theta + F_2 - 300 = 0$$

i.e $\quad 500 \sin\theta + 500\,\mu \cos\theta = 300 \qquad \ldots(6)$

Adding (3) and (6) we get

$$1000 \sin\theta = 500$$

$$\sin\theta = \frac{1}{2}$$

or $\theta = 30°$ *Ans.*

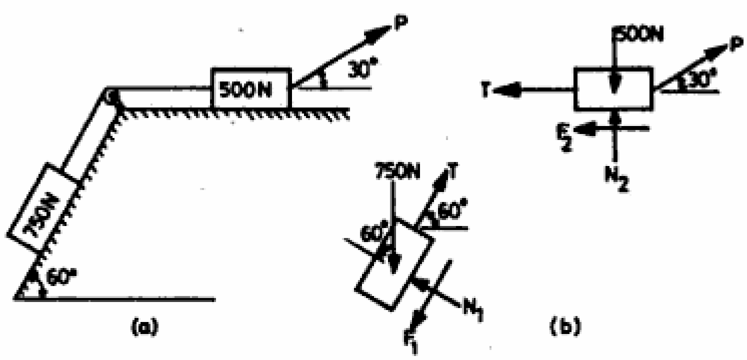
Substituting it in Eqn. (6) we get:

$$500\,\mu \cos 30° = 300 - 500 \sin 30°$$

$$= 50$$

$$\therefore \quad \mu = 0.11547 \qquad Ans.$$

Ex. **What is the value of *P* in the system shown in Fig. to cause the motion to impend? Assume the pulley is smooth and coefficient of friction between the other contact surfaces is 0.20.**



Free body diagrams of the blocks are as shown in Fig. Considering 750 N block:

Σ forces normal to the plane = 0

$$N_1 - 750 \cos 60° = 0$$

$$N_1 = 375 \text{ N}$$

Since the motion is impending, from law of friction,

$$F_1 = \mu N_1 = 0.2 \times 375$$

$$= 75 \text{ N}$$

Σ forces parallel to the plane = 0

$$T - F_1 - 750 \sin 60° = 0$$

$$T = 75 + 750 \sin 60°$$

$$= 724.52 \text{ N}$$

Considering 500 N body:

$$\Sigma V = 0$$

$$N_2 - 500 + P \sin 30° = 0$$

$$N_2 + 0.5 P = 500$$

From law of friction,

$$F_2 = 0.2 N_2$$

$$= 0.2(500 - 0.5P)$$

$$= 100 - 0.1 P$$

$$\Sigma H = 0$$

$$P \cos 30° - T - F_2 = 0$$

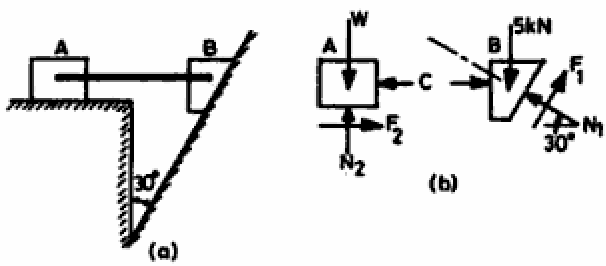
$$P \cos 30° - 724.52 - 100 + 0.1 P = 0$$

$$\mathbf{P = 853.52 \text{ N}}$$

Ans.

Ex. Two blocks connected by a horizontal link *AB* are supported on two rough planes as shown in Fig. The coefficient of friction for the block on the horizontal plane is 0.4. The limiting angle of friction for block *B* on the inclined plane is 20°. What is the smallest weight *W* of the block *A* for which equilibrium of the system can exist if weight of block *B* is 5 kN?

Free body diagrams for block *A* and *B* are as shown in Fig.
Consider block *B*.
From law of friction,



$F_1 = N_1 \tan 20°$ [Since $\mu = \tan 20°$]

$\Sigma V = 0$

$N_1 \sin 30° + F_1 \sin 60° - 5 = 0$

$0.5N_1 + N_1 \tan 20° \sin 60° = 5$

$\therefore \quad N_1 = 6.133 \text{ kN}$

$\therefore \quad F_1 = 6.133 \tan 20° = 2.232 \text{ kN}$

$\Sigma H = 0$

$C + F_1 \cos 60° - N_1 \cos 30° = 0$

$C = 6.133 \cos 30° - 2.232 \cos 60°$

$= 4.196 \text{ kN}$

Now consider the equilibrium of block *A*.

$\Sigma H = 0$

$F_2 = C = 4.196 \text{ kN}$

$F_2 = C = 4.196 \text{ kN}$

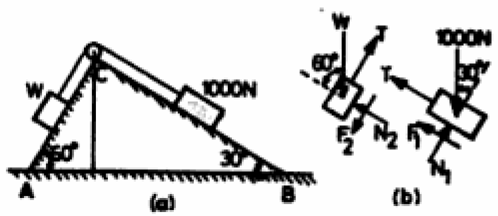
From law of friction $F_2 = \mu N_2$

i.e $N_2 = \dfrac{4.196}{0.4} = 10.49\text{ kN}$

$\Sigma V = 0$

W = N2 = 10.49 kN *Ans.*

Ex. Two identical planes AC and BC inclined at 60° and 30° to the horizontal meet at C. A load of 1000 N rests on the inclined plane BC and is tied by a rope passing over a pulley to a block weighing W Newtons and resting on the plane AC as shown in Fig. If the coefficient of friction between the load and the plane BC is 0.28 and that between the block and the plane AC is 0.20, find the least and the greatest value of W for the equilibrium of the system.



For the least value of W for equilibrium, the motion of 1000 N block is impending downward. For such a case the free body diagram of blocks are shown in Fig.

Considering the 1000 N block:

Σ forces normal to plane = 0

$N_1 = 1000 \cos 30° = 866.03\text{ N}$

From the law of friction $F_1 = 0.28\,N_1$

$= 242.49\text{ N}$

Σ forces parallel to the plane = 0

$T = -F_1 + 1000 \sin 30°$

$= 257.51\text{ N}$

Now consider the equilibrium of block of weight W:

Σ Forces normal to the plane = 0

$N_2 = W \cos 60° = 0.5\,W$

$\therefore \quad F_2 = 0.2\,N_2 = 0.1\,W$

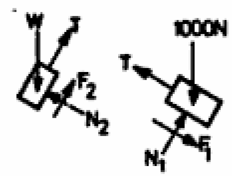
Σ forces parallel to the plane = 0

$$F_2 + W \sin 60° = T$$

$$0.1\, W + W \sin 60° = 257.51$$

$$\therefore\ \mathbf{W = 266.57\ N}$$ *Ans.*

For the greatest value of W, the 1000 N block is on the verge of moving up the plane. For such a case, the free body diagrams of the blocks are as shown in Fig.



Considering block of 1000 N,

$$N_1 = 866.03\ \text{N}$$

$$F_1 = 242.49\ \text{N}$$

$$T = 1000 \sin 30° + F_1 = 742.49\ \text{N}$$

Considering block of weight W,

$$N_2 = W \cos 60° = 0.5\, W$$

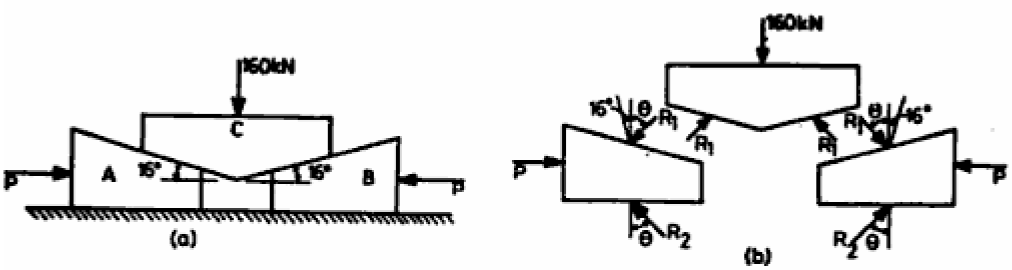
$$F_2 = 0.2\, N_2 = 0.1\, W$$

and $W \sin 60° - F_2 = T$

$$W (\sin 60° - 0.1) = 742.49$$

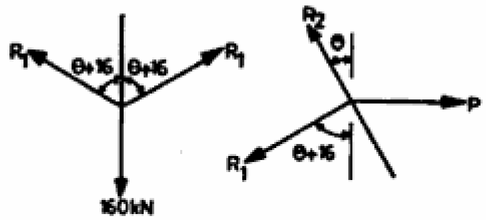
$$\mathbf{W = 969.28\ N}$$ *Ans.*

Ex. A weight of 160 kN is to be raised by means of the widges A and B as shown in Fig. Find the value of force P for impending motion of block C upwards, if coefficient of friction is 0.25 for all surfaces. Weights of the block C and the wedges may be neglected.



Let α be angle of limiting friction. Then,

$$\theta = \tan^{-1}(0.25) = 14.036°$$



The free body diagrams of A, B and C are as shown in Fig. The problem being symmetric, the forces R_1 and R_2 on wedges A and B are the same. The system of forces on block C and on wedge A are shown in the form convenient for applying Lami's theorem

Consider the equilibrium of block C.

$$\frac{R_1}{\sin(180-16-\theta)} = \frac{160}{\sin 2(\theta+16)}$$

i.e. $$\frac{R_1}{\sin 149.96} = \frac{160}{\sin 60.072°}$$

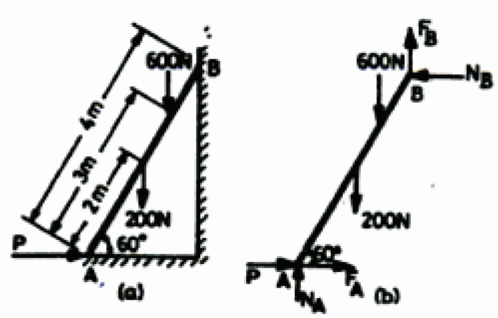
$$R_1 = 92.41 \text{ kN}$$

Consider the equilibrium of wedge A:

$$\frac{P}{\sin(180-\theta-\theta-16)} = \frac{R_1}{\sin(90+\theta)}$$

$$\mathbf{P = 66.256\ kN}$$ *Ans.*

Ex. A ladder of length 4 m weighing 200 N is placed against a vertical wall as shown in Fig. The coefficient of friction between the wall and the ladder is 0.2 and that between the floor and the ladder is 0.3. The ladder in addition to its own weight has to support a man weighing 600 N at a distance of 3 m from A. Calculate the minimum horizontal force to be applied at A to prevent slipping.



The free body diagram of the ladder is as shown in Fig.

$$\Sigma M_A = 0$$

$$N_B 4 \sin 60° + F_B 4 \cos 60° - 600 \times 3 \cos 60° - 200 \times 2 \cos 60° = 0$$

Dividing throughout by 4 and rearranging,

$$N_B 0.866 + 0.5 F_B = 275$$

From the law of friction, $F_B = 0.2 N_B$

$$\therefore \quad N_B (0.866 + 0.5 \times 0.2) = 275$$

$$N_B = 284.68 \text{ N}$$

$$\therefore \quad F_B = 56.934 \text{ N}$$

$$\Sigma V = 0$$

$$N_A - 200 - 600 + 56.934 = 0$$

$$N_A = 743.066 \text{ N}$$

$$\therefore F_A = 0.3 N_A$$

$$\therefore F_A = 222.92 \text{ N}$$

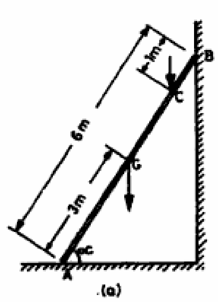
$$\Sigma H = 0$$

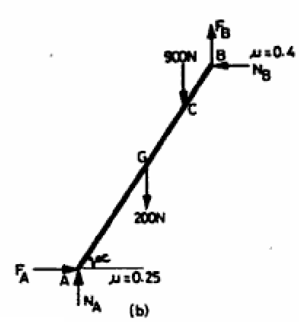
$$P + F_A - N_B = 0$$

$$P = N_B - F_A = 284.68 - 222.92$$

P = 61.76 N *Ans.*

Ex. The ladder shown in Fig. is 6 m long and is supported by a horizontal floor and vertical wall. The coefficient of friction between the floor and the ladder is 0.25 and between wall and ladder is 0.4. The weight of ladder is 200 N and may be considered as concentrated at *G*. The ladder also supports a vertical load of 900 N at *C* which is at a distance of 1m from *B*. Determine the least value of α at which the ladder may be placed without slipping. Determine the reaction at that stage.





From the law of friction,

$$F_A = 0.25\, N_A \qquad \ldots(1)$$

and

$$F_B = 0.4\, N_B \qquad \ldots(2)$$

$\Sigma V = 0$

$N_A - 200 - 900 + F_B = 0$

i.e. $N_A + 0.4\, N_B = 1100$ ...(3)

$\Sigma H = 0$

$F_A - N_B = 0$

$0.25\, N_A = N_B$ ...(4)

From (3) and (4) we get:

$N_A\,(1 + 0.4 \times 0.25) = 1100$

$N_A = 1000\ \text{N}$ *Ans.*

$\mathbf{F_A = 250\ N}$ *Ans.*

$\mathbf{N_B = 250\ N}$ *Ans.*

$\mathbf{F_B = 0.4 \times 250 = 100\ N}$ *Ans.*

$\Sigma M_A = 0$

$N_B \times 6 \sin\alpha + F_B \times 6 \cos\alpha - 200 \times 3 \cos\alpha - 900 \times 5 \cos\alpha = 0$

$250 \times 6 \sin\alpha = (-100 \times 6 + 600 + 4500) \cos\alpha$

$$\tan\alpha = \frac{4500}{1500} = 3$$

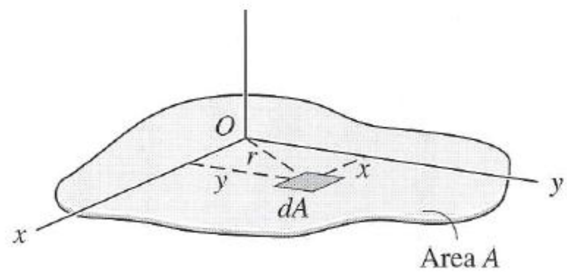
$$\alpha = 71.565^\circ$$ *Ans.*

UNIT – VIII: MOMENT OF INERTIA

8.1 Introduction

The **Moment of Inertia (I)** is a term used to describe the capacity of a cross-section to resist bending. It is always considered with respect to a reference axis such as X-X or Y-Y. It is a mathematical property of a section concerned with a surface area and how that area is distributed about the reference axis (*axis of interest*). The reference axis is usually a centroidal axis.

The moment of inertia is also known as the **Second Moment of the Area** and is expressed mathematically as:



$$I_x = \int_A y^2 dA$$

$$I_y = \int_A x^2 dA$$

Where

y = distance from the x axis to area dA
x = distance from the y axis to area dA

RADIUS OF GYRATION k

The **radius of gyration** of an area with respect to a particular axis is the square root of the quotient of the moment of inertia divided by the area. It is the distance at which the entire area must be assumed to be concentrated in order that the product of the area and the square of this distance will equal the moment of inertia of the actual area about the given axis. In other words, the radius of gyration describes the way in which the total cross-sectional area is distributed around its centroidal axis. If more area is distributed further from the axis, it will have greater resistance to buckling. The most efficient column section to resist buckling is a circular pipe, because it has its area distributed as far away as possible from the centroid.

Rearranging we have:

$$I_x = k_x^2 A$$
$$I_y = k_y^2 A$$

The radius of gyration is the distance *k* away from the axis that all the area can be concentrated to result in the same moment of inertia.

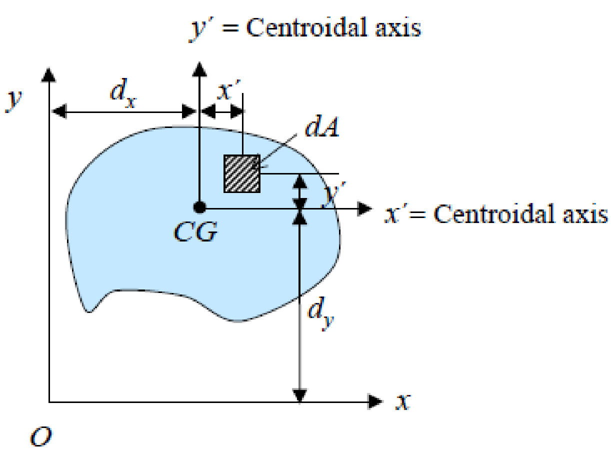
$$k_x = \sqrt{\frac{I_x}{A}}$$

$$k_y = \sqrt{\frac{I_y}{A}}$$

Parallel Axis Theorem

The moment of inertia of an area with respect to any given axis is equal to the moment of inertia with respect to the centroidal axis plus the product of the area and the square of the distance between the 2 axes.

The parallel axis theorem is used to determine the moment of inertia of composite sections.



$$I_x = \int_A (y'+d_y)^2\, dA$$

$$= \int_A [(y')^2 + 2(y')(d_y) + (d_y)^2]\, dA$$

$$= \int_A (y')^2\, dA + \int_A 2(y')(d_y)\, dA + \int_A (d_y)^2\, dA$$

$0,\ \bar{y}' = 0$

$$= \bar{I}_x + \cancel{2d_y \int_A y'\, dA} + d_y{}^2 \int_A dA$$

$$\boxed{I_x = \bar{I}_x + 0 + d_y{}^2 A}$$

$$\boxed{I_y = \bar{I}_y + 0 + d_x{}^2 A}$$

$$\boxed{J_O = \bar{J}_C + Ad^2}$$

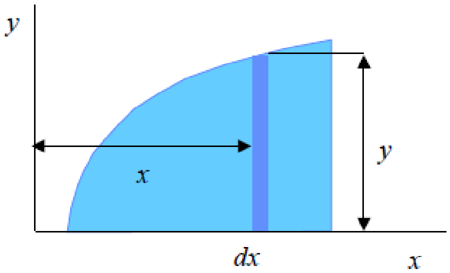
Perpendicular Axis Theorem

Theorem of the perpendicular axis states that if I_{XX} and I_{YY} be the moment of inertia of a plane section about two mutually perpendicular axis X-X and Y-Y in the plane of the section, then the moment of inertia of the section I_{ZZ} about the axis Z-Z, perpendicular to the plane and passing through the intersection of X-X and Y-Y is given by:

$$\mathbf{I_{ZZ} = I_{XX} + I_{YY}}$$

The moment of inertia I_{ZZ} is also known as polar moment of inertia.

8.2 Determination of the moment of inertia of an area by integration



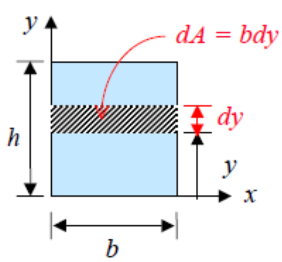
The ***rectangular moments of inertia*** $\boldsymbol{I_x}$ ***and*** $\boldsymbol{I_y}$ ***of an area*** are defined as

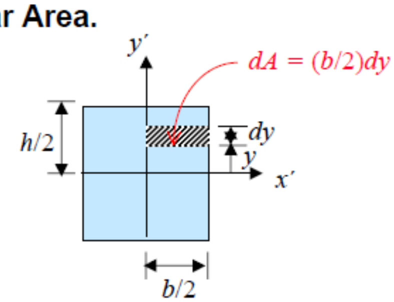
$$I_x = \int y^2\, dA \qquad I_y = \int x^2\, dA$$

These computations are reduced to single integrations by choosing dA to be a thin strip parallel to one of the coordinate axes. The result is

$$dI_x = \frac{1}{3} y^3\, dx \qquad dI_y = x^2 y\, dx$$

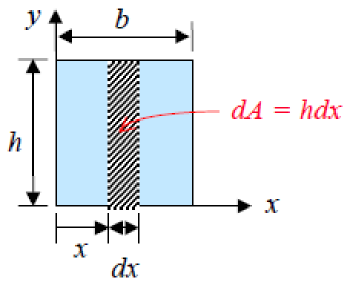
- **Moment of Inertia of a Rectangular Area.**





$$I_x = \int_A y^2 dA = \int_0^h y^2 (bdy) = \left.\frac{(by^3)}{3}\right|_0^h = \frac{bh^3}{3}$$

$$\bar{I}_x = I_{x'} = \int_A y^2 dA = 4\int_0^h y^2 \left(\frac{b}{2} dy\right) = 4\left(\frac{b}{2}\right)\left.\frac{y^3}{3}\right|_0^{h/2} = \frac{bh^3}{12}$$

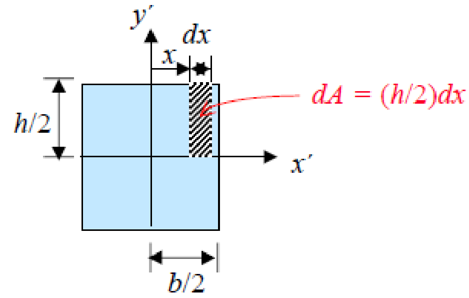


$$I_y = \int_A x^2 dA$$

$$= \int_0^b x^2 (hdx)$$

$$= \left.\frac{(hx^3)}{3}\right|_0^b$$

$$= \frac{hb^3}{3}$$

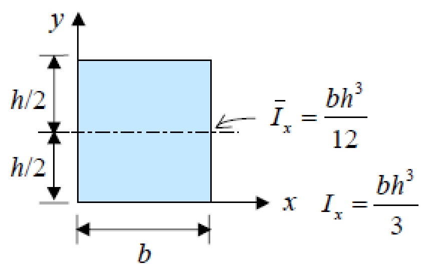


$$\bar{I}_y = I_{y'} = \int_A x^2 dA$$

$$= 4\int_0^h x^2 (\frac{h}{2}dx)$$

$$= 4(\frac{h}{2})\left.\frac{x^3}{3}\right|_0^{b/2}$$

$$= \frac{hb^3}{12}$$



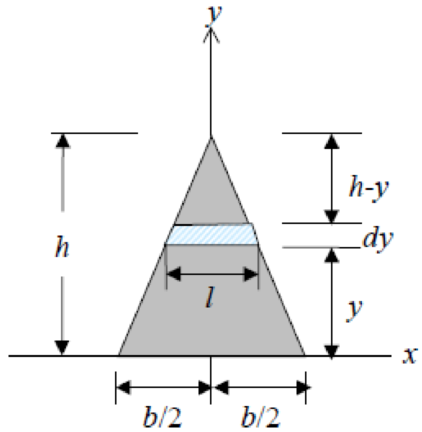
$$I_x = \bar{I}_x + Ad^2$$

$$= \frac{bh^3}{12} + (bh)(\frac{h}{2})^2$$

$$= \frac{bh^3}{12} + \frac{bh^3}{4}$$

$$I_x = \frac{bh^3}{3}$$

• Moment of Inertia of a Triangular Area.



$dI_x = y^2\, dA \qquad dA = l\, dy$

Using similar triangles, we have

$$\frac{l}{b} = \frac{h-y}{h} \qquad l = b\frac{h-y}{h} \qquad dA = b\frac{h-y}{h}dy$$

Integrating dI_x from $y = 0$ to $y = h$, we obtain

$$I_x = \int y^2\, dA$$

$$= \int_0^h y^2 b \frac{h-y}{h} dy = \frac{b}{h}\int_0^h (hy^2 - y^3) dy$$

$$= \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h = \frac{bh^3}{12}$$

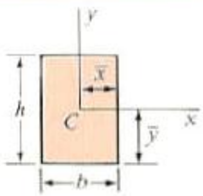
$$I_x = \bar{I}_x + Ad^2$$

$$\bar{I}_x = I_x - Ad^2$$

$$= \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36}$$

Properties of plane areas

1



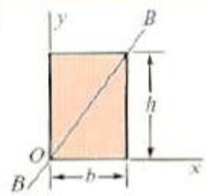
Rectangle (Origin of axes at centroid.)

$$A = bh \qquad \bar{x} = \frac{b}{2} \qquad \bar{y} = \frac{h}{2}$$

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12}$$

$$I_{xy} = 0 \qquad I_p = \frac{bh}{12}(h^2 + b^2)$$

2

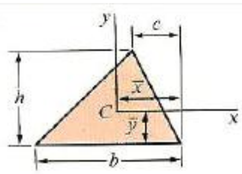


Rectangle (Origin of axes at corner.)

$$I_x = \frac{bh^3}{3} \qquad I_y = \frac{hb^3}{3}$$

$$I_{xy} = \frac{b^2h^2}{4} \qquad I_p = \frac{bh}{3}(h^2 + b^2) \qquad I_{BB} = \frac{b^3h^3}{6(b^2 + h^2)}$$

3



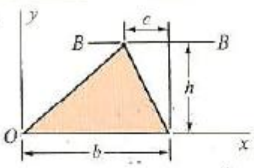
Triangle (Origin of axes at centroid.)

$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b + c}{3} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{bh}{36}(b^2 - bc + c^2)$$

$$I_{xy} = \frac{bh^2}{72}(b - 2c) \qquad I_p = \frac{bh}{36}(h^2 + b^2 - bc + c^2)$$

4

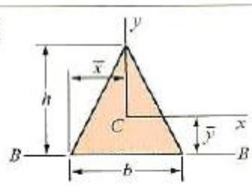


Triangle (Origin of axes at vertex.)

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{bh}{12}(3b^2 - 3bc + c^2)$$

$$I_{xy} = \frac{bh^2}{24}(3b - 2c) \qquad I_{BB} = \frac{bh^3}{4}$$

5



Isosceles triangle (Origin of axes at centroid.)

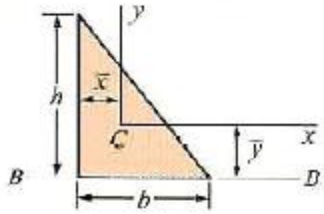
$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b}{2} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{hb^3}{48} \qquad I_{xy} = 0$$

$$I_p = \frac{bh}{144}(4h^2 + 3b^2) \qquad I_{BB} = \frac{bh^3}{12}$$

(Note: For an equilateral triangle, $h = \sqrt{3}b/2$.)

6



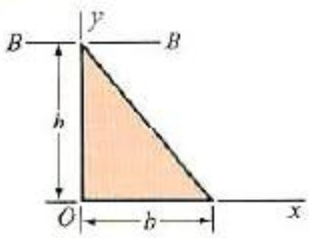
Right triangle (Origin of axes at centroid.)

$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b}{3} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{hb^3}{36} \qquad I_{xy} = -\frac{b^2h^2}{72}$$

$$I_p = \frac{bh}{36}(h^2 + b^2) \qquad I_{BB} = \frac{bh^3}{12}$$

7

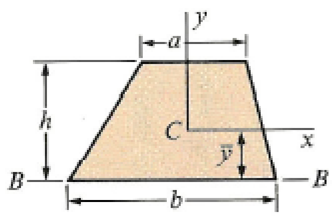


Right triangle (Origin of axes at vertex.)

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12} \qquad I_{xy} = \frac{b^2h^2}{24}$$

$$I_p = \frac{bh}{12}(h^2 + b^2) \qquad I_{BB} = \frac{bh^3}{4}$$

8

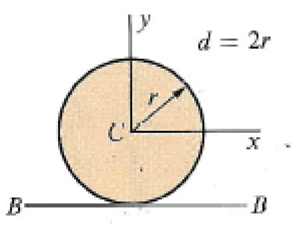


Trapezoid (Origin of axes at centroid.)

$$A = \frac{h(a + b)}{2} \qquad \bar{y} = \frac{h(2a + b)}{3(a + b)}$$

$$I_x = \frac{h^3(a^2 + 4ab + b^2)}{36(a + b)} \qquad I_{BB} = \frac{h^3(3a + b)}{12}$$

9

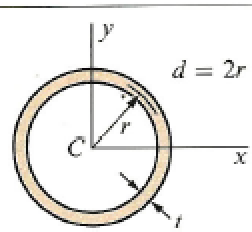


Circle (Origin of axes at center.)

$$A = \pi r^2 = \frac{\pi d^2}{4} \qquad I_x = I_y = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$$

$$I_{xy} = 0 \qquad I_p = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \qquad I_{BB} = \frac{5\pi r^4}{4} = \frac{5\pi d^4}{64}$$

10

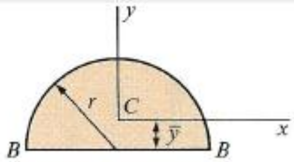


Circular ring (Origin of axes at center.)
Approximate formulas for case when t is small.

$$A = 2\pi rt = \pi dt \qquad I_x = I_y = \pi r^3 t = \frac{\pi d^3 t}{8}$$

$$I_{xy} = 0 \qquad I_p = 2\pi r^3 t = \frac{\pi d^3 t}{4}$$

11



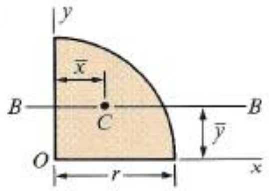
Semicircle (Origin of axes at centroid.)

$$A = \frac{\pi r^2}{2} \qquad \bar{y} = \frac{4r}{3\pi}$$

$$I_x = \frac{(9\pi^2 - 64)r^4}{72\pi} \approx 0.1098 r^4 \qquad I_y = \frac{\pi r^4}{8}$$

$$I_{xy} = 0 \qquad I_{BB} = \frac{\pi r^4}{8}$$

12



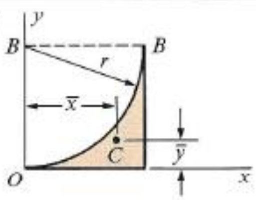
Quarter circle (Origin of axes at center of circle.)

$$A = \frac{\pi r^2}{4} \qquad \bar{x} = \bar{y} = \frac{4r}{3\pi}$$

$$I_x = I_y = \frac{\pi r^4}{16} \qquad I_{xy} = \frac{r^4}{8}$$

$$I_{BB} = \frac{(9\pi^2 - 64)r^4}{144\pi} \approx 0.05488 r^4$$

13



Quarter-circular spandrel (Origin of axes at vertex.)

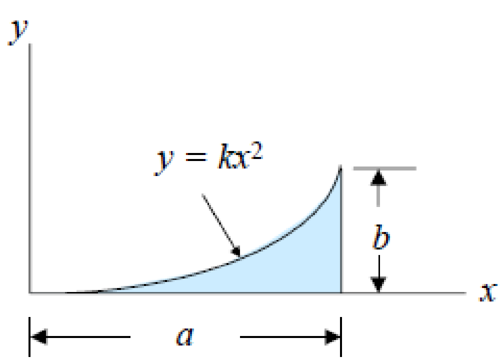
$$A = \left(1 - \frac{\pi}{4}\right) r^2$$

$$\bar{x} = \frac{2r}{3(4 - \pi)} \approx 0.7766 r \qquad \bar{y} = \frac{(10 - 3\pi)r}{3(4 - \pi)} \approx 0.2234 r$$

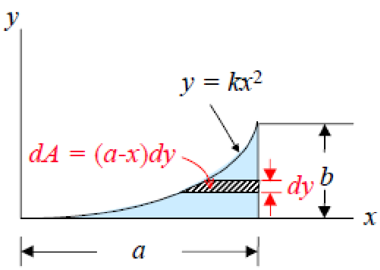
$$I_x = \left(1 - \frac{5\pi}{16}\right) r^4 \approx 0.01825 r^4 \qquad I_y = I_{BB} = \left(\frac{1}{3} - \frac{\pi}{16}\right) r^4 \approx 0.1370 r^4$$

Example

Determine the moment of inertia of the shaded area shown with respect to each of the coordinate axes.



• **Moment of Inertia I_x.**



Substituting $x = a$ and $y=b$

$$y = kx^2$$

$$b = ka^2$$

$$k = \frac{b}{a^2}$$

$$y = \frac{b}{a^2}x^2 \quad or \quad x = \frac{a}{b^{1/2}}y^{1/2}$$

$$I_x = \int_A y^2 dA$$

$$= \int_0^b y^2 (a - x) dy$$

$$= \int_0^b y^2 (a - \frac{a}{b^{1/2}} y^{1/2}) dy$$

$$= a\int_0^b y^2 dy - \frac{a}{b^{1/2}} \int_0^b y^{5/2} dy$$

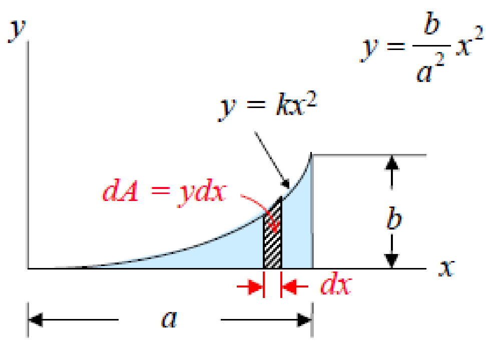
$$= \frac{ay^3}{3}\Big|_0^b - \frac{a}{b^{1/2}} (\frac{2}{7} y^{7/2})\Big|_0^b$$

$$= \frac{ab^3}{3} - \frac{a}{b^{1/2}} (\frac{2}{7} b^{7/2})$$

$$= \frac{ab^3}{3} - \frac{2ab^3}{7}$$

$$= \frac{ab^3}{21}$$

• **Moment of Inertia I_y.**



$$I_y = \int_A x^2 dA$$

$$= \int_0^a x^2 y dx$$

$$= \int_0^a x^2 (\frac{b}{a^2} x^2) dx$$

$$= \frac{b}{a^2} \int_0^a x^4 dx$$

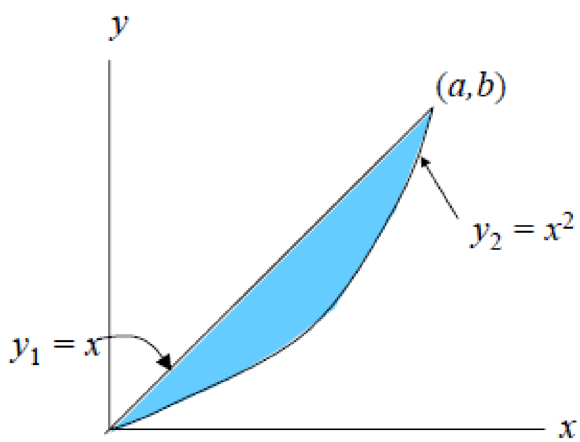
$$= (\frac{b}{a^2})(\frac{x^5}{5})\Big|_0^a$$

$$= (\frac{b}{a^2})(\frac{a^5}{5})$$

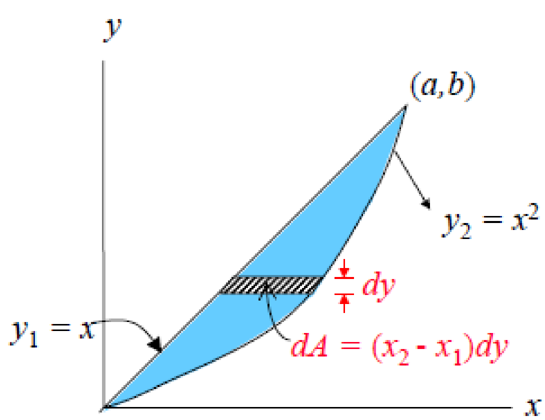
$$= \frac{a^3 b}{5}$$

Example

Determine the moment of inertia of the shaded area shown with respect to each of the coordinate axes.



- **Moment of Inertia I_x.**



$$I_x = \int_A y^2 dA$$

$$= \int_0^b y^2 (x_2 - x_1) dy$$

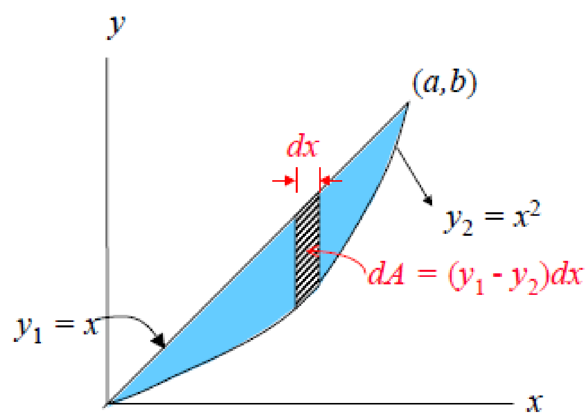
$$= \int_0^b y^2 (y^{1/2} - y) dy$$

$$= \int_0^b (y^{5/2}) dy - \int_0^b (y^3) dy$$

$$= \frac{2}{7} y^{7/2} \Big|_0^b - \frac{y^4}{4} \Big|_0^b$$

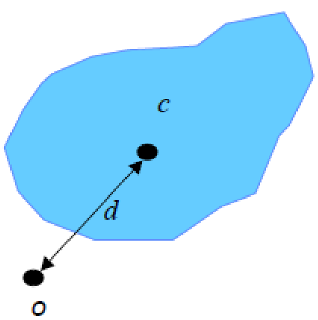
$$= \frac{2}{7} b^{7/2} - \frac{b^4}{4}$$

• **Moment of Inertia I_y.**



$$I_y = \int_A x^2 dA$$
$$= \int_0^a x^2 (y_1 - y_2) dx$$
$$= \int_0^a x^2 (x - x^2) dx$$
$$= \int_0^a (x^3) dx - \int_0^a (x^4) dx$$
$$= \frac{x^4}{4}\Big|_0^a - \frac{x^5}{5}\Big|_0^a$$
$$= \frac{a^4}{4} - \frac{a^5}{5}$$

Moment of inertia of composite areas



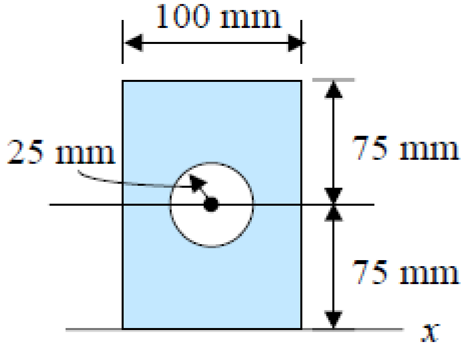
A similar theorem can be used with the polar moment of inertia. The polar moment of inertia $\overline{J}_O$ of an area about O and the polar moment of inertia J_C of the area about its centroid are related to the distance d between points C and O by the relationship

$$\boxed{J_O = \overline{J}_C + Ad^2}$$

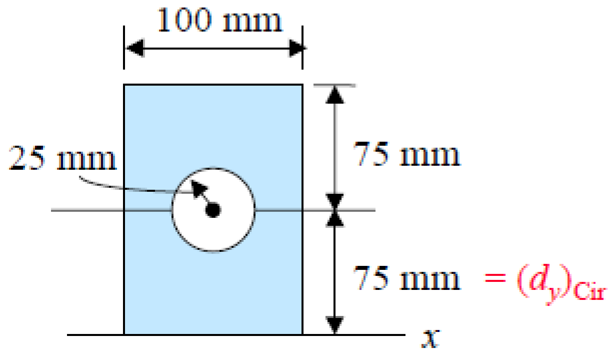
The parallel-axis theorem is used very effectively to compute the ***moment of inertia of a composite area*** with respect to a given axis.

Example

Compute the moment of inertia of the composite area shown.



SOLUTION



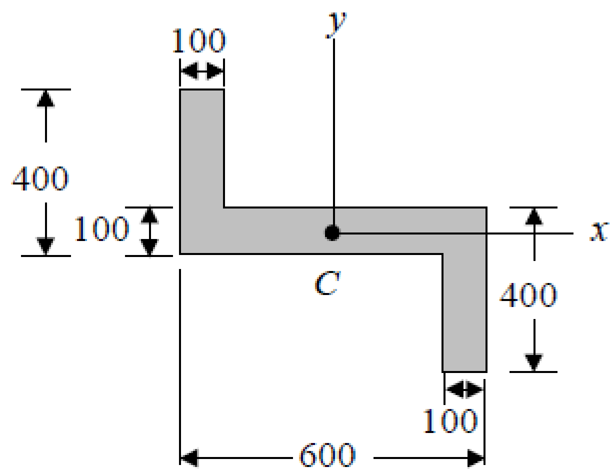
$$I_x = \left(\frac{bh^3}{3}\right)_{Rect} - (\bar{I}_x + Ad_y{}^2)_{Cir}$$

$$= \left[\frac{1}{3}(100)(150)^3\right]_{Rect} - \left[\frac{1}{4}\pi(25)^4 + (\pi \times 25^2)(75)^2\right]_{Cir}$$

$$= 101\text{x}10^6 \text{ mm}^4$$

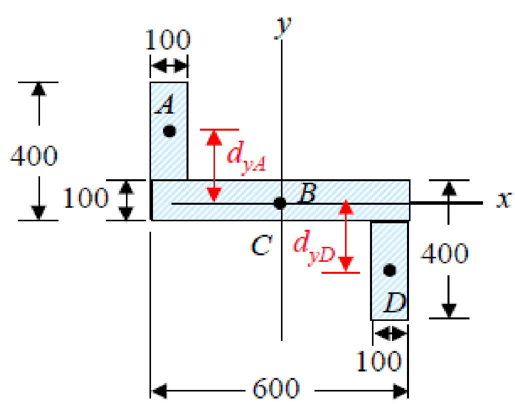
Example

Determine the moments of inertia of the beam's cross-sectional area shown about the *x* and *y* centroidal axes.



Dimension in mm

SOLUTION



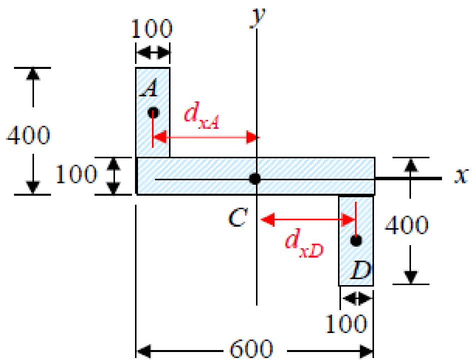
Dimension in mm

$$I_x = (\bar{I}_x + Ad_y^{\,2})_A + (\bar{I}_x + A\overset{0}{d_y^{\,2}})_B + (\bar{I}_x + Ad_y^{\,2})_C$$

$$= [\frac{1}{12}(100)(300)^3 + (100\times 300)(200)^2] + [\frac{1}{12}(600)(100)^3 + 0]$$

$$+ [\frac{1}{12}(100)(300)^3 + (100\times 300)(200)^2]$$

$$= 2.9\text{x}10^9 \text{ mm}^4$$



Dimension in mm

$$I_y = (\bar{I}_y + Ad_x^{\ 2})_A + (\bar{I}_y + A\overset{0}{\cancel{d_x^{\ 2}}})_B + (\bar{I}_y + Ad_x^{\ 2})_C$$

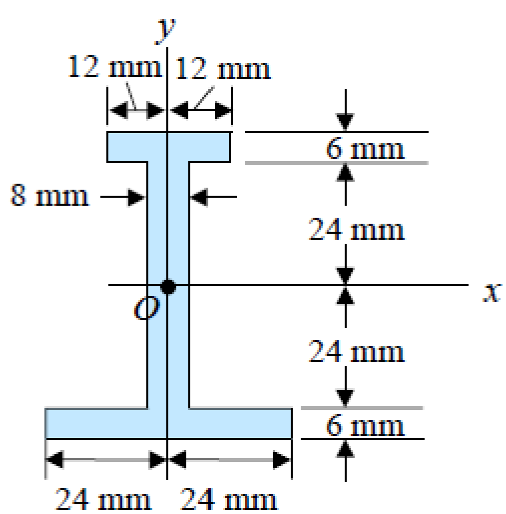
$$= [\frac{1}{12}(300)(100)^3 + (100\times300)(250)^2]_A + [\frac{1}{12}(100)(600)^3 + 0]_B$$

$$+ [\frac{1}{12}(300)(100)^3 + (100\times300)(250)^2]_C$$

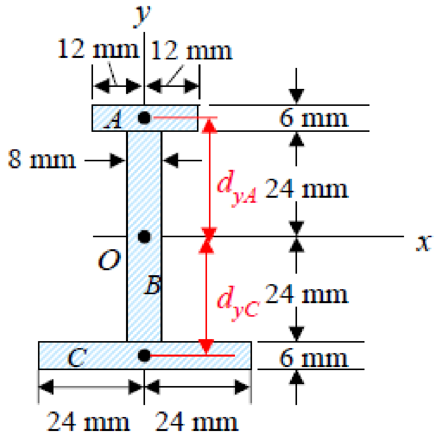
$$= 5.6\text{x}10^9 \text{ mm}^4$$

Example

Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes.



SOLUTION



$$I_x = (\bar{I}_x + A d_y^{\,2})_A + (\bar{I}_x + A \cancelto{0}{d_y}^{\,2})_B + (\bar{I}_x + A d_y^{\,2})_C$$

$$= [\frac{1}{12}(24)(6)^3 + (24\times 6)(27)^2]_A$$

$$+ [\frac{1}{12}(8)(48)^3 + 0]_B$$

$$+ [\frac{1}{12}(48)(6)^3 + (48\times 6)(27)^2]_C$$

$$I_x = 390\text{x}10^3 \text{ mm}^4 \Leftarrow$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{390\times 10^3}{[(24\times 6)+(8\times 48)+(48\times 6)]}} = 21.9\ mm \Leftarrow$$

$$I_y = (\bar{I}_y + A \cancelto{0}{d_x}^{\,2})_A + (\bar{I}_y + A \cancelto{0}{d_x}^{\,2})_B + (\bar{I}_y + A \cancelto{0}{d_x}^{\,2})_C$$

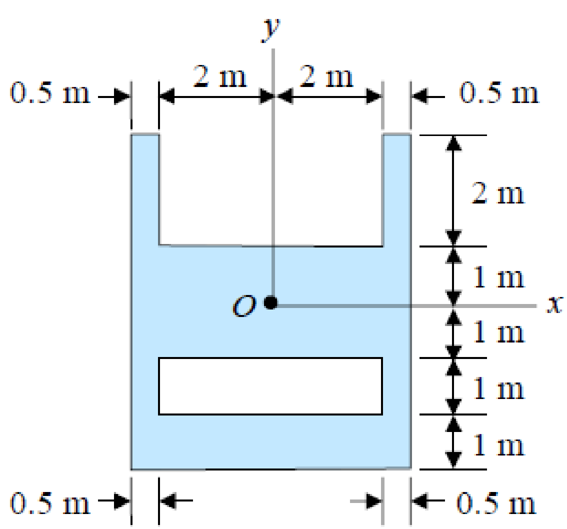
$$= [\frac{1}{12}(6)(24)^3]_A + [\frac{1}{12}(48)(8)^3]_B + [\frac{1}{12}(6)(48)^3]_C$$

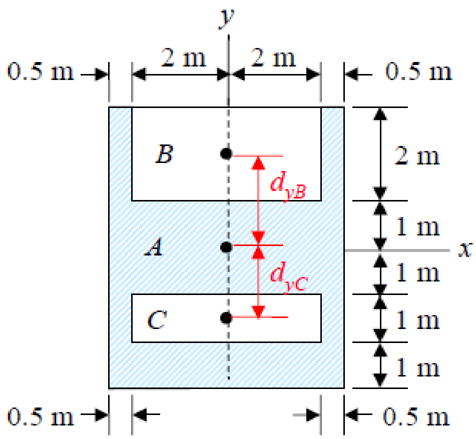
$$I_y = 64.3\text{x}10^3 \text{ mm}^4 \Leftarrow$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{64.3\times 10^3}{[(24\times 6)+(8\times 48)+(48\times 6)]}} = 8.87\ mm \Leftarrow$$

Example

Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes.





$$I_x = (\bar{I}_x + \cancelto{0}{Ad_y^{\,2}})_{A5\times6} - (\bar{I}_x + Ad_y^{\,2})_{B4\times2} - (\bar{I}_x + Ad_y^{\,2})_{C4\times1}$$

$$= [\frac{1}{12}(5)(6)^3 + 0]_A - [\frac{1}{12}(4)(2)^3 + (2\times4)(2)^2]_B$$

$$-[\frac{1}{12}(4)(1)^3 + (4\times1)(1.5)^2]_C$$

$$I_x = 46 \text{ m}^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{46}{[(5\times6)-(4\times2)-(4\times1)]}} = 1.599\ m$$

$$I_y = (\bar{I}_y + \cancelto{0}{Ad_x^{\,2}})_A - (\bar{I}_y + \cancelto{0}{Ad_x^{\,2}})_B - (\bar{I}_y + \cancelto{0}{Ad_x^{\,2}})_C$$

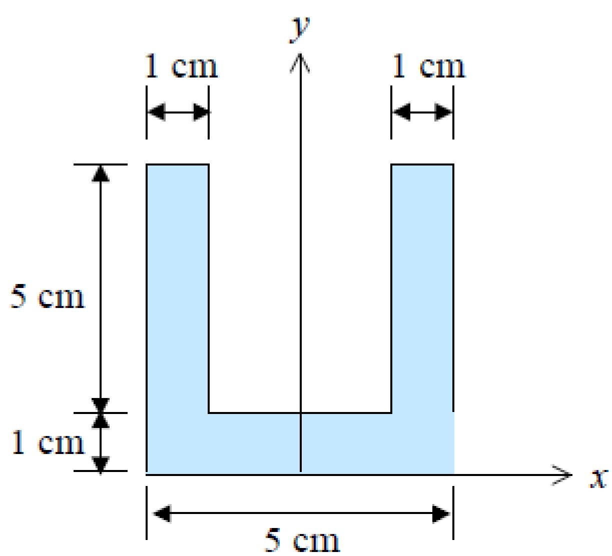
$$= [\frac{1}{12}(6)(5)^3]_A - [\frac{1}{12}(2)(4)^3]_B - [\frac{1}{12}(1)(4)^3]_C$$

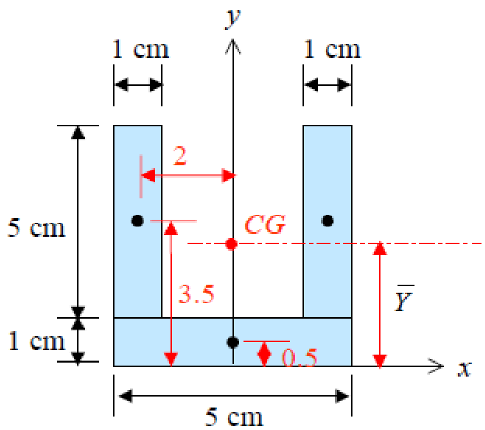
$$I_y = 46.5 \text{ m}^4$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{46.5}{[(5\times6)-(4\times2)-(4\times1)]}} = 1.607\ m$$

Example

Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes and at the centroidal axes.





$$\bar{Y}\sum A = \sum \bar{y}A$$

$$\bar{Y} = \frac{2[(3.5)(5\times1)]+(0.5)(1\times5)}{3(5\times1)}$$

$$= 2.5\ cm$$

• **Moments of inertia about *x* axis**

$$I_x = 2[(\frac{1}{12}(1)(5)^3+(5\times1)(3.5)^2]+\frac{1}{3}(5)(1)^3$$

$$= 145\ cm^4$$

• **Moments of inertia about centroid**

$$\bar{I}_x = I_x - Ad_y^{\ 2}$$

$$= 145-(15)(2.5)^2$$

$$= 51.25\ cm^4$$

OR

$$\bar{I}_x = 2[(\frac{1}{12}(1)(5)^3+(5\times1)(1)^2]$$

$$+[(\frac{1}{12}(5)(1)^3+(5\times1)(2)^2]$$

$$= 51.25\ cm^4$$

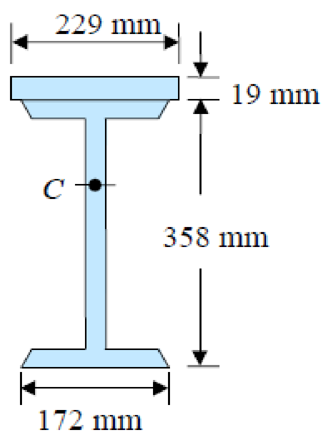
$$\bar{I}_y = I_y = 2[(\frac{1}{12}(5)(1)^3+(5\times1)(2)^2]+\frac{1}{12}(1)(5)^3$$

$$= 51.25\ cm^4$$

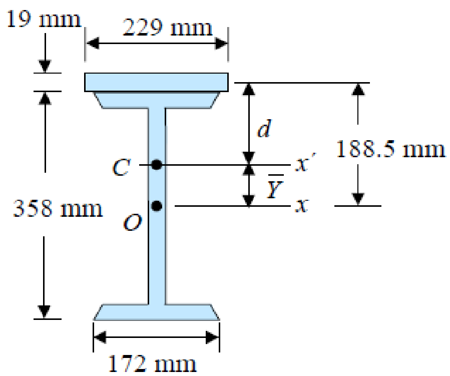
$$\bar{k}_x = \bar{k}_y = \sqrt{\frac{\bar{I}_x}{A}} = \sqrt{\frac{51.25}{15}} = 1.848\ cm$$

Example

The strength of a W360 x 57 rolled-steel beam is increased by attaching a 229 mm x 19 mm plate to its upper flange as shown. Determine the moment of inertia and the radius of gyration of the composite section with respect to an axis which is parallel to the plate and passes through the centroid *C* of the section.



SOLUTION



- **Centroid**

The wide-flange shape of *W*360 x 57 found by referring to Fig. 9.13

$A = 7230\ \text{mm}^2 \qquad \bar{I}_x = 160.2\ mm^4$

$A_{\text{plate}} = (229)(19) = 4351\ \text{mm}^2$

$$\bar{Y}\Sigma A = \Sigma \bar{y} A$$

$$\bar{Y}(4351 + 7230) = (188.5)(4351) + (0)(7230)$$

$$\bar{Y} = 70.8\ mm$$

- **Moment of Inertia**

$$I_{x'} = (I_{x'})_{plate} + (I_{x'})_{wide-flange}$$

$$= (\bar{I}_{x'} + Ad^2)_{plate} + (\bar{I}_{x'} + A\bar{Y}^2)_{wide-flange}$$

$$= \left[\frac{1}{12}(229)(19)^3 + (4351)(188.5 - 70.8)^2\right]$$

$$+ \left[160.2 \times 10^6 + (7230)(70.8)^2\right]$$

$$= 256.8 \times 10^6\ mm^4$$

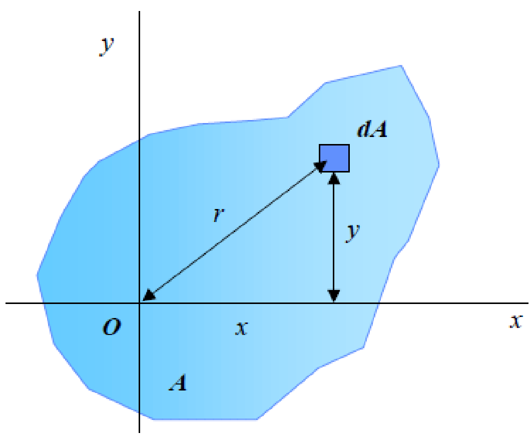
$$I_{x'} = 257 \times 10^6\ mm^4$$

- **Radius of Gyration**

$$k_{x'}^2 = \frac{I_{x'}}{A} = \frac{256.8 \times 10^6}{(4351 + 7230)}$$

$$k_{x'} = 149\ mm$$

Polar Moment of Inertia



The ***polar moment of inertia of an area A*** with respect to the pole *O* is defined as

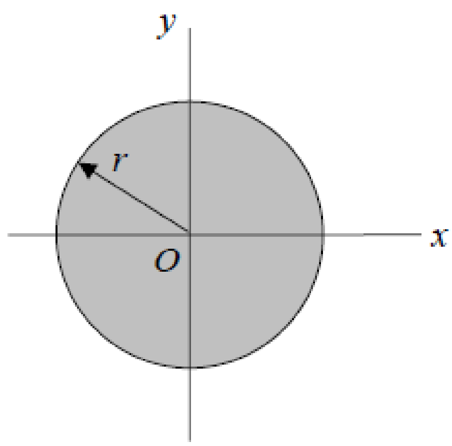
$$\boxed{J_O = \int r^2\, dA}$$

The distance from O to the element of area dA is r. Observing that $r^2 = x^2 + y^2$, we established the relation

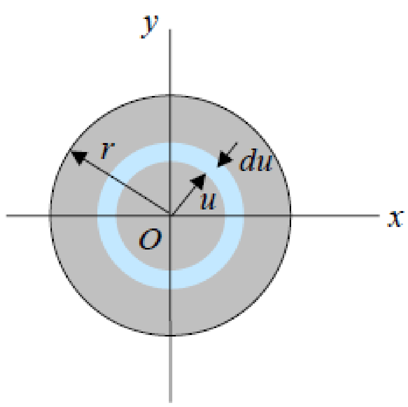
$$\boxed{J_O = I_x + I_y}$$

Example

(a) Determine the centroidal polar moment of inertia of a circular area by direct integration. (b) Using the result of part a, determine the moment of inertia of a circular area with respect to a diameter.



SOLUTION



a. Polar Moment of Inertia.

$$dJ_O = u^2\,dA \qquad dA = 2\pi u\,du$$

$$J_O = \int dJ_O = \int_0^r u^2(2\pi u\,du) = 2\pi\int_0^r u^3 du$$

$$J_O = \frac{\pi}{2}r^4$$

b. Moment of Inertia with Respect to a Diameter.

$$J_O = I_x + I_y = 2I_x$$

$$\frac{\pi}{2}r^4 = 2I_x$$

$$I_{diameter} = I_x = \frac{\pi}{4}r^4$$