J. N. REDDY

ENERGY PRINCIPLES AND VARIATIONAL METHODS IN APPLIED MECHANICS

THIRD EDITION

WILEY

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Third Edition

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This edition first published 2017

Registered Offices
John Wiley & Sons Inc., 111 River Street, Hoboken, NJ07030, USA
John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, UK

Editorial Office
The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, UK

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Library of Congress Cataloging-in-Publication Data applied for

Hardback: 9781119087373

Cover design by Wiley

Cover images: (Top) © inakiantonana/Gettyimages;
(Bottom) © VII-photo/Gettyimages

Set in 11/13.5pt, Computer Modern by SPi Global, Chennai, India

10 9 8 7 6 5 4 3 2 1

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The Red, White, and Blue
(The United States of America)
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with sincere gratitude and appreciation

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About the Author

J. N. Reddy is a university distinguished professor, regents' professor, and the holder of Oscar S. Wyatt Endowed Chair in the Department of Mechanical Engineering at Texas A&M University. He is known worldwide for his significant contributions to the field of applied mechanics through the authorship of widely used textbooks on the linear and nonlinear finite element analysis, variational methods, composite materials and structures, applied functional analysis, and continuum mechanics.

Professor Reddy's earlier research focused primarily on mathematics of finite elements, variational principles of mechanics, shear deformation and layerwise theories of laminated composite plates and shells, analysis of bimodular materials, modeling of geological and geophysical phenomena, penalty finite elements for flows of viscous incompressible fluids, and least-squares finite element models of fluid flows and solid continua. His pioneering works on the development of shear deformation theories (that bear his name in the literature as the Reddy third-order plate theory and the Reddy layerwise theory) have had a major impact and have led to new research developments and applications. The finite element formulations and models he developed have been implemented in commercial software like Abaqus, NISA, and HyperXtrude. In recent years, his research involved the development of 7- and 12-parameter shell theories, nonlocal and non-classical continuum mechanics problems, and problems involving couple stresses, surface stress effects, discrete fracture and flow, micropolar cohesive damage, and continuum plasticity of metals from considerations of non-equilibrium thermodynamics.

His most significant awards and honors are: the *Worcester Reed Warner Medal* and *Charles Russ Richards Memorial Award* from the American Society of Mechanical Engineers; *Archie Higdon Distinguished Educator Award* from the Mechanics Division of the American Society for Engineering Education; *Nathan M. Newmark Medal* and *Raymond D. Mindlin Medal* from the American Society of Civil Engineers; *Excellence in the Field of Composites* and *Distinguished Research Award* from the American Society for Composites; *Computational Solid Mechanics* award from the US Association for Computational Mechanics; *The IACM O.C. Zienkiewicz Award* from the International Association of Computational Mechanics; *William Prager Medal* from the Society of Engineering Science; and *ASME Medal* from the American Society of Mechanical Engineers. He is a member of the US National Academy of Engineering and a foreign fellow of the Indian National Academy of Engineering. He is a fellow of all professional societies of his research (e.g., AIAA, ASC, ASCE, ASME, AAM, USACM, IACM).

Professor Reddy is one of the original top 100 ISI Highly Cited Researchers in Engineering around the world, with over 21,000 citations and an h-index of over 68 per Web of Science; the number of citations is over 52,200 with an h-index of 92 according to Google Scholar. He is the founding editor in chief of *Mechanics of Advanced Materials and Structures* and *International Journal for Computational Methods in Engineering Science and Mechanics* and coeditor of *International Journal of Structural Stability and Dynamics*; he also serves on the editorial boards of more than two dozen journals. A more complete resume with links to journal papers can be found at **http://mechanics.tamu.edu**.

"What we have done for ourselves alone dies with us; what we have done for others and the world, remains and is immortal." – *Albert Pike* (American attorney, soldier, and writer)

Note: Quotes by various people included in this book were found at different web sites. For example, visit:

http://naturalscience.com/dsqhome.html,
http://thinkexist.com/quotes/david_hilbert/,
http://www.yalescientific.org/2010/10/from-the-editor-imagination-in-science/,
https://www.brainyquote.com/quotes/topics/topic_science.html. *https://www.wikipedia.org/.*

This author is motivated to include the quotes at various places in his book for their wit and wisdom; The author cannot vouch for their accuracy.

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There you will find valuable material designed to enhance your learning, including:

1. Solutions manual

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Preface to the Third Edition

The development of new material systems paved the way for the development of new devices and improved components of existing structures. Development of mathematical models and their numerical simulations are central to design. While a select number of courses on elasticity and structural mechanics provide engineers with background to formulate a suitable mathematical model using the laws of physics (such as the second law of Newton), refined mathematical models require the tools of variational calculus to formulate and evaluate them. The latest commercial codes like *COMSOL Multiphysics*® and *FEniCS*® require the user to just give the weak forms of the governing equations that they like to solve using the finite element method. It is in this connection that a course on energy principles and variational methods provides the needed background.

Most books on energy principles and variational methods traditionally treat the topics, not incorporating the latest developments and generalizing the classical methods to a broader class of problems. This textbook is unique (i.e., there is no parallel to this book in its class) because most available books in the market are very old (some are even out of print) and do not treat topics that are of current interest. In particular, this book provides a systematic coverage of old and new topics (e.g., finite elements, functionally graded structures, relationships between classical and refined theories) and it contains illustrative examples and problem sets that enable readers to test their understanding of the subject matter.

The third edition of *Energy Principles and Variational Methods* has the same objective as its previous editions, namely, to facilitate an easy and thorough understanding of the concepts and tools necessary to develop variational formulations and associated numerical evaluations. *The book offers easy-to-understand treatment of the subject of energy principles and variational methods.* The new edition is extensively reorganized and contains a substantially large amount of new material. In particular, Chapters 1 and 2 are combined into a single chapter (Chapter 1 in the new edition); Chapter 2 on the review of equations of solid mechanics is an extensively revised version of Chapter 3 from the second edition; Chapters 3 to 9 in the new edition correspond to Chapters 4 to 10 of the second edition but with additional explanations, examples, and exercise problems; Chapter 10 in this new edition is entirely new and devoted to functionally graded beams and plates. In general, all of the chapters of the third edition contain additional explanations, detailed example problems, and fresh exercise problems. Thus the new edition more than replaces the previous editions, and it is hoped that it is acquired by the library of every institution of higher learning by serious structural analysts.

Since the publication of the previous editions, many users of the book communicated their comments and compliments as well as errors they found, for which the author thanks them. The author is grateful to the following professional colleagues for their friendship, encouragement, and support over the years:

Marcilio Alves, University of São Paulo, Brazil
Marco Amabili, McGill University, Canada
Erasmo Carrera, University of Torino, Italy
Antonio Ferreira, University of Porto, Portugal
Somnath Ghosh, Johns Hopkins University
Antonio Grimaldi, University of Rome II, Italy
Yonggang Huang, Northwestern University
S. Kitipornchai, University of Queensland, Australia
K. M. Liew, City University of Hong Kong
C. W. Lim, City University of Hong Kong
Franco Maceri, University of Rome II, Italy
Cristovão Mota Soares, Technical University of Lisbon, Portugal
Antonio Miravete, Zaragoza University, Spain
Glaucio Paulino, Georgia Institute of Technology
Amirtham Rajagopal, Indian Institute of Technology, Hyderabad, India
Jani Romanoff, Aalto University, Finland
Jose Roesset, Texas A&M University
Debasish Roy, Indian Institute of Science, Bangalore, India
Elio Sacco, University of Cassino and Southern Lazio, Italy
E. C. N. Silva, University of São Paulo, Brazil
Arun Srinivasa, Texas A&M University
Karan Surana, University of Kansas
Liqun Tang, South China University of Technology
C. M. Wang, National University of Singapore
Y. B. Yang, National University of Taiwan

Drafts of the manuscript of this book prior to its publication were read by the author's doctoral students, who have made suggestions for improvements. In particular, the author wishes to thank the following former and current students (listed in alphabetical order): Ronald Averill, K. Chandrashekhara, Paul Heyliger, Filis Kokkinos, Filipa Moleiro, Asghar Nosier, Felix Palmerio, Gregory Payette, Grama Praveen, Donald Robbins, Jr., Samit Roy, Ginu Unnikrishnan, Vinu Unnikrishnan, Archana Arbind, Parisa Khodabakhshi, Jinseok Kim, and Namhee Kim. The author requests readers to send their comments and corrections to *jnreddy@tamu.edu*.

J. N. Reddy
College Station, TX
March 2017

Preface to the Second Edition

The increasing use of numerical and computational methods in engineering and applied sciences has shed new light on the importance of energy principles and variational methods. The number of engineering courses that make use of energy principles and variational formulations and methods has also grown very rapidly in recent years. In view of the increase in the use of variational formulations and methods (including the finite element method), there is a need to introduce the concepts of energy principles and variational methods and their use in the formulation and solution of problems of mechanics to both undergraduate and beginning graduate students. This book, an extensively revised version of the author's earlier book *Energy and Variational Methods in Applied Mechanics*, is intended for senior undergraduate students and beginning graduate students of aerospace, civil, and mechanical engineering and applied mechanics who have had a course in fundamental engineering and ordinary and partial differential equations.

The book is organized into 10 chapters and is self-contained as far as the subject matter is concerned. Chapter 1 presents a general introduction to the subject of variational principles. Chapter 2 contains a brief review of the algebra and calculus of vectors and Cartesian tensors, whereas Chapter 3 reviews of the basic equations of linear solid continuum mechanics, which will be frequently referred to in subsequent chapters. Much of the material presented in Chapters 1 to 3 can be assigned as a reading material, especially in a graduate class.

Chapter 4 deals with the concepts of work and energy and basic topics of variational calculus, including the Euler equations, fundamental lemma of calculus of variations, essential and natural boundary conditions, and minimization of functionals with and without equality constraints. Principles of virtual work and energy and energy methods of solid and structural mechanics are presented in Chapter 5. Chapter 6 is devoted to a discussion of Hamilton's principle for dynamical systems. Classical variational methods of approximation (e.g., the methods of Ritz, Galerkin, and Kantorovich) are presented in Chapter 7. All of the concepts and methods presented in Chapters 4 to 7 are illustrated using bars and beams although the methods discussed in Chapter 7 are readily applicable to field problems whose differential equations resemble those of bars and beams. Chapter 8 is dedicated to applications of the energy principles and variational methods developed in earlier chapters to circular and rectangular plates. In the interest of completeness and for use as a reference for approximate solutions, exact solutions are also included. The finite element method is introduced in Chapter 9, with applications to beams and plates. Displacement finite element models of Euler–Bernoulli and Timoshenko beam theories and classical and first-order shear deformation plate theories are presented. A unified approach, more general than that found in most solid mechanics books, is used to introduce the finite element method. As a result, the student can readily extend the method to other subject areas of solid mechanics and other branches of engineering. Lastly, the mixed variational principles of Hellinger and Reissner for elasticity are derived in Chapter 10. Mixed variational formulations, including mixed finite element models of beams and plates, are discussed.

Each chapter of the book contains many example problems and exercises that illustrate, test, and broaden the understanding of the topics covered. A list of references, by no means complete or up to date, is also provided at the end of each chapter. Answers to selective problems are included at the end of the book.

The book is suitable as a textbook for a senior undergraduate course or a first-year graduate course on energy principles and variational methods taught in aerospace, civil, and mechanical engineering and applied mechanics departments. To gain the most from the text, the student should have a senior undergraduate or first-year graduate standing in engineering. Some familiarity with basic courses in differential equations, mechanics of materials, and dynamics would also be helpful.

The author has professionally benefited from the works, encouragement, and support of many colleagues and students who have taught him how to explain complicated concepts in simple terms. While it is not possible to name all of them, without their help and support, it would not have been possible for the author to modestly contribute to the field of mechanics through his teaching, research, and writing. Special thanks are due to his teacher Professor J. T. Oden (University of Texas at Austin) and Professor C. W. Bert (University of Oklahoma, Norman) for their mentorship, advice, and support.

J. N. Reddy
College Station, TX
August 2002

Preface to the First Edition

The increasing use of finite element methods in engineering and applied science has shed new light on the importance of energy and variational methods. The number of engineering courses and research papers that make use of variational and energy methods has also grown very rapidly in recent years. In view of the increased use of variational methods (including the finite element method), there is a need to introduce the concepts of energy and variational methods and their use in the formulation and solution of equations of mechanics to both undergraduate and beginning graduate students. This book is intended for senior undergraduate students and beginning graduate students of aerospace, civil, and mechanical engineering and applied mechanics, who have had a course in ordinary and partial differential equations. The text is organized into four chapters. Chapter 1 is essentially a review, especially for graduate students, of the equations of applied mechanics. Much of the chapter can be assigned as reading material to the student. The equations of bars, beams, torsion, and plane elasticity presented in Section 1.7 are used to illustrate concepts from energy and variational methods. Chapter 2 deals with the study of the basic topics from variational calculus, virtual work and energy principles, and energy methods of mechanics. The instructor can omit Section 2.4 on stationary principles and Section 2.5 on Hamilton's principle if he or she wants to cover all of Chapter 4. Classical variational methods of approximation (e.g., the methods of Ritz, Galerkin, and Kantorovich) and the finite element method are introduced and illustrated in Chapter 3 via linear problems of science and engineering, especially solid mechanics. A unified approach, more general than that found in most solid mechanics books, is used to introduce the variational methods. As a result, the student can readily extend the methods to other subject areas of solid mechanics and other branches of engineering.

The classical variational methods and the finite element method are put to work in Chapter 4 in the derivation and approximate solution of the governing equations of elastic plates and shells. In the interest of completeness, and for use as a reference for approximate solutions, exact solutions of plates and shells are also included. Keeping in mind the current developments in composite material structures, a brief but reasonably complete discussion of laminated plates and shells is included in Sections 4.3 and 4.4. The book contains many example and exercise problems that illustrate, test, and broaden the understanding of the topics covered. A long list of references, by no means complete or up to date, is provided in Bibliography at the end of the book.

The author wishes to acknowledge, with great pleasure and appreciation, the encouragement and support by Professor Daniel Frederick (Head, ESM Department at Virginia Tech) during the course of this writing and the skillful typing of the manuscript by Vanessa McCoy. The author is also thankful to the many students who, through their comments, contributed to the improvement of this book. Special thanks to K. Chandrashekhara, Glenn Creamer, C. F. Liu, and Paul Heyliger for their help in proofreading the galleys and pages and to Dr. Ozden Ochoa for constructive comments on the preliminary draft of the manuscript. It is a pleasure to acknowledge, with many thanks, the cooperation of the technical staff at Wiley, New York (Frank Cerra, Christina Mikulak, and Lisa Morano).

J. N. Reddy
Blacksburg, VA
June 1984

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Introduction and Mathematical Preliminaries

1.1 Introduction

1.1.1 Preliminary Comments

The phrase "energy principles" or "energy methods" in the present study refers to methods that make use of the total potential energy (i.e., strain energy and potential energy due to applied loads) of a system to obtain values of an unknown displacement or force, at a specific point of the system. These include Castigliano's theorems, unit dummy load and unit dummy displacement methods, and Betti's and Maxwell's theorems. These methods are often limited to the (exact) determination of generalized displacements or forces at fixed points in the structure; in most cases, they cannot be used to determine the complete solution (i.e., displacements and/or forces) as a function of position in the structure. The phrase "variational methods," on the other hand, refers to methods that make use of the variational principles, such as the principles of virtual work and the principle of minimum total potential energy, to determine approximate solutions as continuous functions of position in a body. In the classical sense, a *variational principle* has to do with the minimization or finding stationary values of a functional with respect to a set of undetermined parameters introduced in the assumed solution. The functional represents the total energy of the system in solid and structural mechanics problems, and in other problems it is simply an integral representation of the governing equations. In all cases, the functional includes all the intrinsic features of the problem, such as the governing equations, boundary and/or initial conditions, and constraint conditions.

1.1.2 The Role of Energy Methods and Variational Principles

Variational principles have always played an important role in mechanics. Variational formulations can be useful in three related ways. First, many problems of mechanics are posed in terms of finding the extremum (i.e., minima or maxima) and thus, by their nature, can be formulated in terms of variational state-

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ments. Second, there are problems that can be formulated by other means, such as by vector mechanics (e.g., Newton's laws), but these can also be formulated by means of variational principles. Third, variational formulations form a powerful basis for obtaining approximate solutions to practical problems, many of which are intractable otherwise. The principle of minimum total potential energy, for example, can be regarded as a substitute to the equations of equilibrium of an elastic body, as well as a basis for the development of displacement finite element models that can be used to determine approximate displacement and stress fields in the body. Variational formulations can also serve to unify diverse fields, suggest new theories, and provide a powerful means for studying the existence and uniqueness of solutions to problems. In many cases they can also be used to establish upper and/or lower bounds on approximate solutions.

1.1.3 A Brief Review of Historical Developments

In modern times, the term "variational formulation" applies to a wide spectrum of concepts having to do with weak, generalized, or direct variational formulations of boundary- and initial-value problems. Still, many of the essential features of variational methods remain the same as they were over 200 years ago when the first notions of variational calculus began to be formulated.[1]

Although Archimedes (287–212 B.C.) is generally credited as the first to use work arguments in his study of levers, the most primitive ideas of variational theory (the minimum hypothesis) are present in the writings of the Greek philosopher Aristotle (384–322 B.C.), to be revived again by the Italian mathematician/engineer Galileo (1564–1642), and finally formulated into a principle of least time by the French mathematician Fermat (1601–1665). The phrase *virtual velocities* was used by Jean Bernoulli in 1717 in his letter to Varignon (1654–1722). The development of early variational calculus, by which we mean the classical problems associated with minimizing certain functionals, had to await the works of Newton (1642–1727) and Leibniz (1646–1716). The earliest applications of such variational ideas included the classical *isoperimetric problem* of finding among closed curves of given length the one that encloses the greatest area, and Newton's problem of determining the solid of revolution of "minimum resistance." In 1696, Jean Bernoulli proposed the problem of the *brachistochrone*: among all curves connecting two points, find the curve traversed in the shortest time by a particle under the influence of gravity. It stood as a challenge to the mathematicians of their day to solve the problem using the rudimentary tools of analysis then available to them or whatever new ones they were capable of developing. Solutions to this problem were presented by some of the greatest mathematicians of the time: Leibniz, Jean Bernoulli's older brother Jacques Bernoulli, L'Hopital, and Newton.

[1]Many of the developments came from European scientists, whose works appeared in their native language and were not accessible to the whole scientific community.