



THE ISLAMIC UNIVERSITY OF GAZA

ENGINEERING FACULTY

DEPARTMENT OF COMPUTER ENGINEERING

DISCRETE MATHEMATICS DISCUSSION – ECOM 2011

Chapter 1

The Foundations: Logic and Proofs



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q: I won the million dollar jackpot.

Express each of these propositions as an English sentence.

a) $\neg p$

b) $p \vee q$

c) $p \rightarrow q$

d) $p \wedge q$

e) $p \leftrightarrow q$

f) $\neg p \rightarrow \neg q$

g) $\neg p \wedge \neg q$

h) $\neg p \vee (p \wedge q)$

Answer:

- a) I **did not** buy a lottery ticket this week.
- b) **Either** I bought a lottery ticket this week **or** I won the million dollar jackpot on Friday.
- c) **If** I bought a lottery ticket this week, **then** I won the million dollar jackpot on Friday.
- d) I bought a lottery ticket this week **and** I won the million dollar jackpot on Friday.
- e) I bought a lottery ticket this week **if and only if** I won the million dollar jackpot on Friday.
- f) **If I did not** buy a lottery ticket this week, **then I did not** win the million dollar jackpot on Friday.
- g) **I did not** buy a lottery ticket this week, **and I did not** win the million dollar jackpot on Friday.
- h) **Either I did not** buy a lottery ticket this week, **or else** I did buy one **and** won the million dollar jackpot on Friday.

14. Let p, q, and r be the propositions

p: You get an A on the final exam.

q: You do every exercise in this book.

r: You get an A in this class.

Write these propositions using p, q, and r and logical connectives (including negations).

- a) You get an A in this class, but you do not do every exercise in this book.
- b) You get an A on the final, you do every exercise in this book, and you get an A in this class.
- c) To get an A in this class, it is necessary for you to get an A on the final.
- d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.
- e) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.
- f) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

Answer:

- a) $r \wedge \neg q$
b) $p \wedge q \wedge r$
c) $p \rightarrow r$
d) $(p \wedge \neg q) \wedge r$
e) $(p \wedge q) \rightarrow r$
f) $r \leftrightarrow (q \vee p)$

17. Determine whether each of these conditional statements is true or false.

- a) If $1 + 1 = 2$, then $2 + 2 = 5$. b) If $1 + 1 = 3$, then $2 + 2 = 4$.
- c) If $1 + 1 = 3$, then $2 + 2 = 5$. d) If monkeys can fly, then $1 + 1 = 3$.

Answer:

- a) False, $T \rightarrow F$
b) True, $F \rightarrow T$
c) True, $F \rightarrow F$
d) True, $F \rightarrow F$

Section 3: Propositional Equivalences

7. Use De Morgan's laws to find the negation of each of the following statements.

- a) Jan is rich and happy.
- b) Carlos will bicycle or run tomorrow.
- c) Mei walks or takes the bus to class.
- d) Ibrahim is smart and hard working.

Answer:

- a) Jan isn't rich or isn't happy.
- b) Carlos won't bicycle and won't run tomorrow.
- c) Mei neither walks nor takes the bus to class.
- d) Ibrahim isn't smart or isn't hard working.

9. Show that each of these conditional statements is a tautology by using truth tables.

a) $(p \wedge q) \rightarrow p$

b) $p \rightarrow (p \vee q)$

Answer:

a)

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

b)

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

10. Show that each of these conditional statements is a tautology by using truth tables.

a) $[\neg p \wedge (p \vee q)] \rightarrow q$

Answer:

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

11. Show that each conditional statement in Exercise 9 is a tautology without using truth tables.

a) $(p \wedge q) \rightarrow p$

b) $p \rightarrow (p \vee q)$

Answer:

$$\begin{aligned} \text{a) } (p \wedge q) \rightarrow p &\equiv \neg(p \wedge q) \vee p \equiv \neg p \vee \neg q \vee p \equiv \neg p \vee (\neg q \vee p) \equiv \neg p \vee T \\ &\equiv T \end{aligned}$$

$$\text{b) } p \rightarrow (p \vee q) \equiv \neg p \vee (p \vee q) \equiv (\neg p \vee p) \vee q \equiv T \vee q \equiv T$$

12. Show that each conditional statement in Exercise 10 is a tautology without using truth tables.

a) $[p \wedge (p \rightarrow q)] \rightarrow q$

Answer:

$$\begin{aligned} \text{a) } [p \wedge (p \rightarrow q)] \rightarrow q &\equiv \neg[p \wedge (p \rightarrow q)] \vee q \equiv \neg[p \wedge (\neg p \vee q)] \vee q \\ &\equiv \neg p \vee \neg(\neg p \vee q) \vee q \equiv \neg p \vee (\neg \neg p \wedge \neg q) \vee q \equiv \neg p \vee (p \wedge \neg q) \vee q \\ &\equiv (\neg p \vee q) \vee (p \wedge \neg q) \equiv (\neg p \vee q) \vee \neg(\neg p \vee q) \equiv \mathbf{T} \end{aligned}$$

40. Find a compound proposition involving the propositional variables p , q , and r that is true when p and q are true and r is false, but is false otherwise. [Hint: Use a conjunction of each propositional variable or its negation.]

Answer:

$$p \wedge q \wedge \neg r$$

Section 4: Predicates and Quantifiers

5. Let $P(x)$ be the statement “ x spends more than five hours every weekday in class,” where the domain for x consists of all students. Express each of these quantifications in English.

- | | |
|--------------------------|--------------------------|
| a) $\exists xP(x)$ | b) $\forall xP(x)$ |
| c) $\exists x \neg P(x)$ | d) $\forall x \neg P(x)$ |

Answer:

- a) **There is a student who** spend more than five hours every weekday in class.
 - b) **All students** spend more than five hours every weekday in class.
 - c) **There is a student who doesn't** spend more than five hours every weekday in class.
- OR
- Some students** spend **five hours or less** every weekday in class.
 - d) **All students** spend **five hours or less** every weekday in class.

6. Let $N(x)$ be the statement “ x has visited North Dakota,” where the domain consists of the students in your school. Express each of these quantifications in English.

- | | | |
|--------------------------|-------------------------|--------------------------|
| a) $\exists xN(x)$ | b) $\forall xN(x)$ | c) $\neg \exists xN(x)$ |
| d) $\exists x \neg N(x)$ | e) $\neg \forall xN(x)$ | f) $\forall x \neg N(x)$ |

Answer:

- a) **Some students** in my school have visited North Dakota.
- b) **All students** in my school have visited North Dakota.
- c) **No students** in my school have visited North Dakota.
- d) **Not all students** in my school have visited North Dakota.
- e) **All students** in my school haven't visited North Dakota.

10. Let $C(x)$ be the statement “ x has a cat,” let $D(x)$ be the statement “ x has a dog,” and let $F(x)$ be the statement “ x has a ferret.” Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

- a) A student in your class has a cat, a dog, and a ferret.
- b) All students in your class have a cat, a dog, or a ferret.
- c) Some student in your class has a cat and a ferret, but not a dog.
- d) No student in your class has a cat, a dog, and a ferret.
- e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

Answer:

- a) $\exists x(C(x) \wedge D(x) \wedge F(x))$
- b) $\forall x(C(x) \vee D(x) \vee F(x))$
- c) $\exists x(C(x) \wedge F(x) \wedge \neg D(x))$
- d) $\neg \exists x(C(x) \wedge D(x) \wedge F(x))$
- e) $\exists xC(x) \wedge \exists yD(y) \wedge \exists zF(z)$ or $(\exists xC(x)) \wedge (\exists xD(x)) \wedge (\exists xF(x))$

23. Translate in two ways each of these statements into logical expressions using predicates, quantifiers, and logical connectives. First, let the domain consist of the students in your class and second, let it consist of all people.

- a) Someone in your class can speak Hindi.
- b) Everyone in your class is friendly.
- c) There is a person in your class who was not born in California.
- d) A student in your class has been in a movie.
- e) No student in your class has taken a course in logic programming.

Answer:

First, the domain consists of all students in your class

- a) $\exists xP(x)$ $P(x)$: x can speak Hindi.
- b) $\forall xF(x)$ $F(x)$: x is friendly.
- c) $\exists x\neg B(x)$ $B(x)$: x was born in California.
- d) $\exists xM(x)$ $M(x)$: x has been in a movie.
- e) $\neg\exists xL(x)$ $L(x)$: x has taken a course in logic programming.

Second, the domain consists of all people, let $C(x)$: x is in your class

- f) $\exists x(C(x) \wedge P(x))$ $P(x)$: x can speak Hindi.
- g) $\forall x(C(x) \rightarrow F(x))$ $F(x)$: x is friendly.
- h) $\exists x(C(x) \wedge \neg B(x))$ $B(x)$: x was born in California.
- i) $\exists x(C(x) \wedge M(x))$ $M(x)$: x has been in a movie.
- j) $\forall x(C(x) \rightarrow \neg L(x))$ $L(x)$: x has taken a course in logic programming.

25. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

- a) No one is perfect.
- b) Not everyone is perfect.
- c) All your friends are perfect.
- d) At least one of your friends is perfect.
- e) Everyone is your friend and is perfect.
- f) Not everybody is your friend or someone is not perfect.

Answer:

Let $P(x)$: x is perfect, and $F(x)$: x is your friend.

- a) $\neg\exists xP(x)$
- b) $\neg\forall xP(x)$

- c) $\forall x(F(x) \rightarrow P(x))$
- d) $\exists x(F(x) \wedge P(x))$
- e) $\forall x(F(x) \wedge P(x))$
- f) $\neg \forall x F(x) \vee \exists x \neg P(x)$

34. Express the negation of these propositions using quantifiers, and then express the negation in English.

- a) Some drivers do not obey the speed limit.
- b) All Swedish movies are serious.
- c) No one can keep a secret.
- d) There is someone in this class who does not have a good attitude.

Answer:

- a) $\exists x \neg P(x)$, x: drivers P(x): x obeys the speed limit
The negation: All drivers obey the speed limit
- b) $\forall x(P(x) \rightarrow Q(x))$, x: movies P(x): x is Swedish Q(x): x is serious
The negation: Some Swedish movies are not serious
- c) $\neg \exists x P(x)$, x: people P(x): x can keep a secret
The negation: Some people can keep a secret."
- d) $\exists x \neg P(x)$, x: someone in this class. P(x): x has a good attitude.
The negation: Everyone in this class has a good attitude.

Section 5: Nested Quantifiers

1. Translate these statements into English, where the domain for each variable consists of all real numbers.

a) $\forall x \exists y (x < y)$

b) $\forall x \forall y ((x \geq 0) \wedge (y \geq 0)) \rightarrow (xy \geq 0)$

c) $\forall x \forall y \exists z (xy = z)$

Answer:

- a) For every real number x , there exists a real number y such that $x < y$. This means that for every real number, there exists a number which is larger than it and the statement is true.
- b) For every real number x and real number y , if x is nonnegative and y is nonnegative, then the multiplication xy is nonnegative (which is true).
- c) For every real number x and real number y , there exists a real number z such that $xy = z$. This is a true statement.

3. Let $Q(x, y)$ be the statement “ x has sent an e-mail message to y ,” where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

a) $\exists x \exists y Q(x, y)$

b) $\exists x \forall y Q(x, y)$

c) $\forall x \exists y Q(x, y)$

d) $\exists y \forall x Q(x, y)$

e) $\forall y \exists x Q(x, y)$

f) $\forall x \forall y Q(x, y)$

Answer:

- a) There is a student in your class has sent an e-mail message to some student in your class.
- b) There is a student in your class who has sent an e-mail message to all students in your class.

- c) Every student in your class has sent an e-mail message to at least one student in your class.
- d) There is a student in your class who has been sent a message by every student in your class .
- e) Every student in your class has been sent a message from at least one student in your class.
- f) Every student in your class has sent an e-mail message to all students in your class.

6. Let $C(x, y)$ mean that student x is enrolled in class y , where the domain for x consists of all students in your school and the domain for y consists of all classes being given at your school. Express each of these statements by a simple English sentence.

- a) $C(\text{Randy Goldberg}, \text{CS 252})$
- b) $\exists x C(x, \text{Math 695})$
- c) $\exists y C(\text{Carol Sitea}, y)$
- d) $\exists x (C(x, \text{Math 222}) \wedge C(x, \text{CS 252}))$
- e) $\exists x \exists y \forall z ((x \neq y) \wedge (C(x, z) \rightarrow C(y, z)))$
- f) $\exists x \exists y \forall z ((x \neq y) \wedge (C(x, z) \leftrightarrow C(y, z)))$

Answer:

- a) Randy Goldberg is enrolled in CS 252.
- b) Someone is enrolled in Math 695.
- c) Carol Sitea is enrolled in some course.
- d) Some student is enrolled simultaneously in Math 222 and CS 252.
- e) There exist two distinct people, the second of whom is enrolled in every course that the first is enrolled in.
- f) There exist two distinct people enrolled in exactly the same courses.

9. Let $L(x, y)$ be the statement “ x loves y ,” where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statements.

- a) Everybody loves Jerry.
- b) Everybody loves somebody.
- c) There is somebody whom everybody loves.
- d) Nobody loves everybody.
- e) There is somebody whom Lydia does not love.
- f) There is somebody whom no one loves.
- g) There is exactly one person whom everybody loves.
- h) There are exactly two people whom Lynn loves.
- i) Everyone loves himself or herself.
- j) There is someone who loves no one besides himself or herself.

Answer:

- a) $\forall xL(x, \text{Jerry})$
- b) $\forall x\exists yL(x, y)$
- c) $\exists y\forall xL(x, y)$
- d) $\neg\exists x\forall yL(x, y)$
- e) $\exists y\neg L(\text{Lydia}, y)$
- f) $\exists y\forall x\neg L(x, y)$.
- g) $\exists!y\forall xL(x, y)$
- h) $\exists y_1\exists y_2 (L(\text{Lynn}, y_1) \wedge L(\text{Lynn}, y_2) \wedge y_1 \neq y_2 \wedge \forall y(L(\text{Lynn}, y) \rightarrow (y = y_1 \vee y = y_2)))$
- i) $\forall xL(x, x)$
- j) $\exists x\forall y(L(x, y) \leftrightarrow x = y)$

11. Let $S(x)$ be the predicate “ x is a student,” $F(x)$ the predicate “ x is a faculty member,” and $A(x, y)$ the predicate “ x has asked y a question,” where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.

- a) Lois has asked Professor Michaels a question.
- b) Every student has asked Professor Gross a question.
- c) Every faculty member has either asked Professor Miller a question or been asked a question by Professor Miller.
- d) Some student has not asked any faculty member a question.
- e) There is a faculty member who has never been asked a question by a student.
- f) Some student has asked every faculty member a question.
- g) There is a faculty member who has asked every other faculty member a question.
- h) Some student has never been asked a question by a faculty member.

Answer:

- a) $A(\text{Lois}, \text{Professor Michaels})$
- b) $\forall x(S(x) \rightarrow A(x, \text{Professor Gross}))$
- c) $\forall x(F(x) \rightarrow (A(x, \text{Professor Miller}) \vee A(\text{Professor Miller}, x)))$
- d) $\exists x(S(x) \wedge \forall y(F(y) \rightarrow \neg A(x, y)))$
- e) $\exists x(F(x) \wedge \forall y(S(y) \rightarrow \neg A(y, x)))$
- f) $\exists x(S(x) \wedge (\forall y F(y) \rightarrow A(x, y)))$
- g) $\exists x(F(x) \wedge \forall y((F(y) \wedge x \neq y) \rightarrow A(x, y)))$
- h) $\exists x(S(x) \wedge \forall y(F(y) \rightarrow \neg A(y, x)))$

23. Express each of these mathematical statements using predicates, quantifiers, logical connectives, and mathematical operators.

- The product of two negative real numbers is positive.
- The difference of a real number and itself is zero.
- Every positive real number has exactly two square roots.
- A negative real number does not have a square root that is a real number.

Answer:

- $\forall x \forall y ((x < 0) \wedge (y < 0) \rightarrow (xy > 0))$
- $\forall x (x - x = 0)$
- $\forall x \exists a \exists b (a \neq b \wedge \forall c (c^2 = x \leftrightarrow (c = a \vee c = b)))$
- $\forall x ((x < 0) \rightarrow \neg \exists y (x = y^2))$

26. Let $Q(x, y)$ be the statement “ $x + y = x - y$.” If the domain for both variables consists of all integers, what are the truth values?

- $Q(1, 1)$
- $Q(2, 0)$
- $\forall y Q(1, y)$
- $\exists x Q(x, 2)$
- $\exists x \exists y Q(x, y)$
- $\forall x \exists y Q(x, y)$
- $\exists y \forall x Q(x, y)$
- $\forall y \exists x Q(x, y)$
- $\forall x \forall y Q(x, y)$

Answer:

- False, since $1 + 1 \neq 1 - 1$.
- This is true, since $2 + 0 = 2 - 0$.
- False, since there are many values of y for which $1 + y \neq 1 - y$.
- False, since the equation $x + 2 = x - 2$ has no solution.
- True, since we can take $x = y = 0$.
- True, since we can take $y = 0$ for each x .

- g) True, since we can take $y = 0$.
- h) False, since part (d) was false.
- i) False.

Section 6: Rules of Inference

2. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If George does not have eight legs, then he is not a spider.

George is a spider.

\therefore George has eight legs.

Answer:

Let p "George does not have eight legs" and q "George is not a spider"

$p \rightarrow q$

$\neg q$

$\therefore \neg p$

This is a valid argument, which is modus tollens.

4. What rule of inference is used in each of these arguments?

a) Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.

b) It is either hotter than 100 degrees today or the pollution is dangerous. It is less than 100 degrees outside today. Therefore, the pollution is dangerous.

c) Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.

d) Steve will work at a computer company this summer. Therefore, this summer Steve will work at a computer company or he will be a beach bum.

e) If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the material. Therefore, if I work all night on this homework, then I will understand the material.

Answer:

$$\begin{array}{l} \text{a) } p \wedge q \\ \hline \therefore q \end{array} \quad \text{Rule of Simplification}$$

$$\begin{array}{l} \text{b) } p \vee q \\ \neg p \\ \hline \therefore q \end{array} \quad \text{Rule of disjunctive syllogism}$$

c) modus ponens

d) addition

e) hypothetical syllogism

6. Use rules of inference to show that the hypotheses “If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on,” “If the sailing race is held, then the trophy will be awarded,” and “The trophy was not awarded” imply the conclusion “It rained.”

Answer:

Let r be the proposition “It rains,” let f be the proposition “It is foggy,” let s be the proposition “The sailing race will be held,” let l be the proposition “The life saving demonstration will go on,” and let t be the proposition “The trophy will be awarded.” We are given premises $(\neg r \vee \neg f) \rightarrow (s \wedge l)$, $s \rightarrow t$, and $\neg t$. We want to conclude r .

Step	Reason
1. $\neg t$	Hypothesis
2. $s \rightarrow t$	Hypothesis
3. $\neg s$	Modus tollens using (1) and (2)
4. $(\neg r \vee \neg f) \rightarrow (s \wedge l)$	Hypothesis
5. $(\neg(s \wedge l)) \rightarrow \neg(\neg r \vee \neg f)$	Contrapositive of (4)
6. $(\neg s \vee \neg l) \rightarrow (r \wedge f)$	De Morgan's law and double negative
7. $\neg s \vee \neg l$	Addition, using (3)
8. $r \wedge f$	Modus ponens using (6) and (7)
9. r	Simplification using (8)

8. What rules of inference are used in this argument? "No man is an island. Manhattan is an island. Therefore, Manhattan is not a man."

Answer:

Let $M(x)$: x is a man. $I(x)$: x is an island.

First we use universal instantiation to conclude from "For all x , if x is a man, then x is not an island"

$$\forall x(M(x) \rightarrow \neg I(x))$$

the special case of interest, "If Manhattan is a man, then Manhattan is not an island."

$$M(\text{Manhattan}) \rightarrow \neg I(\text{Manhattan})$$

Then we form the contrapositive (using also double negative): "If Manhattan is an island, then Manhattan is not a man."

$$I(\text{Manhattan}) \rightarrow \neg M(\text{Manhattan})$$

Finally we use modus ponens to conclude that Manhattan is not a man.

$$\frac{\neg M(\text{Manhattan})}{\therefore I(\text{Manhattan})}$$

$$\therefore I(\text{Manhattan})$$

Alternatively, we could apply modus tollens.

10. For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

a) "If I play hockey, then I am sore the next day." "I use the whirlpool if I am sore." "I did not use the whirlpool."

b) "If I work, it is either sunny or partly sunny." "I worked last Monday or I worked last Friday." "It was not sunny on Tuesday." "It was not partly sunny on Friday."

Answer:

a) Let p: I play hockey, q: I am sore the next day, r: I use the whirlpool.

1. $p \rightarrow q$

2. $q \rightarrow r$

3. $\neg r$

4. $p \rightarrow r$ Rule of hypothetical syllogism 1,2

5. $\neg p$ Rule of modus Tollens 3,4

\therefore I didn't play hockey.

b) Let P(x): I work on x, Q(x): it is sunny on x, R(x): it is partly sunny on x

1. $\forall x(P(x) \rightarrow Q(x) \vee R(x))$
2. $P(\text{Monday}) \vee P(\text{Friday})$
3. $\neg Q(\text{Monday})$
4. $\neg R(\text{Friday})$

We can't conclude anything specific here.

14. For each of these arguments, explain which rules of inference are used for each step.

a) "Linda, a student in this class, owns a red convertible. Everyone who owns a red convertible has gotten at least one speeding ticket. Therefore, someone in this class has gotten a speeding ticket."

Answer:

Let $c(x)$ be "x is in this class," let $r(x)$ be "x owns a red convertible," and let $t(x)$ be "x has gotten a speeding ticket." We are given premises $c(\text{Linda})$, $r(\text{Linda})$, $\forall x(r(x) \rightarrow t(x))$, and we want to conclude $\exists x(c(x) \wedge t(x))$.

Step	Reason
1. $\forall x(r(x) \rightarrow t(x))$	Hypothesis
2. $r(\text{Linda}) \rightarrow t(\text{Linda})$	Universal instantiation using (1)
3. $r(\text{Linda})$	Hypothesis
4. $t(\text{Linda})$	Modus ponens using (2) and (3)
5. $c(\text{Linda})$	Hypothesis
6. $c(\text{Linda}) \wedge t(\text{Linda})$	Conjunction using (4) and (5)
7. $\exists x(c(x) \wedge t(x))$	Existential generalization using (6)

c) "All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners."

Answer:

Let $s(x)$ be "x is a movie produced by Sayles," let $c(x)$ be "x is a movie about coal miners," and let $w(x)$ be "movie x is wonderful." We are given premises $\forall x(s(x) \rightarrow w(x))$ and $\exists x(s(x) \wedge c(x))$, and we want to conclude $\exists x(c(x) \wedge w(x))$. In our proof, y represents an unspecified particular movie.

Step	Reason
1. $\exists x(s(x) \wedge c(x))$	Hypothesis
2. $s(y) \wedge c(y)$	Existential instantiation using (1)
3. $s(y)$	Simplification using (2)
4. $\forall x(s(x) \rightarrow w(x))$	Hypothesis
5. $s(y) \rightarrow w(y)$	Universal instantiation using (4)
6. $w(y)$	Modus ponens using (3) and (5)
7. $c(y)$	Simplification using (2)
8. $w(y) \wedge c(y)$	Conjunction using (6) and (7)
9. $\exists x(c(x) \wedge w(x))$	Existential generalization using (8)

Section 7: Proof Methods and Strategy

Remember:

To prove $P \rightarrow Q$, you can do the following:

- Prove directly, that is assume P and show Q .
- Prove by contradiction, that is assume P and $\neg Q$ and derive a contradiction.
- Prove the contrapositive, that is assume $\neg Q$ and show $\neg P$.

2. Use a direct proof to show that the sum of two even integers is even.

Answer:

Suppose that a and b are two even integers. Then there exist integers s and t such that $a = 2s$ and $b = 2t$. Adding, we obtain $a + b = 2s + 2t = 2(s + t)$.

Since this represents $a + b$ as 2 times the integer $s + t$, we conclude that $a + b$ is even.

4. Show that the additive inverse, or negative, of an even number is an even number using a direct proof.

Answer:

Suppose that a is an even integer. Then there exists an integer s such that $a = 2s$. Its additive inverse is $-2s$, which by rules of arithmetic and algebra equals $2(-s)$. Since this is 2 times the integer $-s$, it is even.

6. Use a direct proof to show that the product of two odd numbers is odd.

Answer:

Suppose that x and y are odd numbers. Then there exist integers a and b such that $x = 2a + 1$ and $y = 2b + 1$.

Their product is $(2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1$. This last expression shows that the product is odd, since it is of the form $2n + 1$, with $n = 2ab + a + b$.

8. Prove that if n is a perfect square, then $n + 2$ is not a perfect square.

Note: A perfect square is a number that can be expressed as the product of two equal integers. For example 9 is a perfect square because it can be expressed as $3 * 3$ (the product of two equal integers)

Answer:

Let $n = m^2$. If $m = 0$, then $n + 2 = 2$, which is not a perfect square, so we can assume that $m \geq 1$.

The smallest perfect square greater than n is $(m + 1)^2$, and we have $(m + 1)^2 = m^2 + 2m + 1 = n + 2m + 1 > n + 2 \cdot 1 + 1 > n + 2$. Therefore $n + 2$ cannot be a perfect square.

10. Use a direct proof to show that the product of two rational numbers is rational.

Answer:

A rational number is a number that can be written in the form x/y where x and y are integers and $y \neq 0$.

Suppose that we have two rational numbers, say a/b and c/d . Then their product is, by the usual rules for multiplication of fractions, $(ac)/(bd)$. Note that both the numerator and the denominator are integers, and that $bd \neq 0$ since b and d were both nonzero. Therefore the product is, by definition, a rational number.

12. Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

Answer:

Suppose that a/b is a nonzero rational number and that x is an irrational number. We must prove that the product xa/b is also irrational.

By contradiction, suppose that xa/b is rational. Since $a/b \neq 0$, we know that $a \neq 0$, so b/a is also a rational number. Let us multiply this rational number b/a by the assumed rational number xa/b . By Exercise 10, the product is rational. But the product is $(b/a)(xa/b) = x$, which is irrational by hypothesis. This is a contradiction, so in fact xa/b must be irrational.

14. Prove that if x is rational and $x \neq 0$, then $1/x$ is rational.

Answer:

If x is rational and not zero, then by definition we can write $x = p/q$, where p and q are nonzero integers. Since $1/x$ is then q/p and $p \neq 0$, we can conclude that $1/x$ is rational.

16. Prove that if m and n are integers and mn is even, then m is even or n is even.

Answer:

We give a proof by contraposition. If it is not true that m is even or n is even, then m and n are both odd. By Exercise 6, this tells us that mn is odd, and our proof is complete.

18. Prove that if n is an integer and $3n + 2$ is even, then n is even using

a) a proof by contraposition.

b) a proof by contradiction.

Answer:

a) We must prove the contrapositive: If n is odd, then $3n + 2$ is odd.

Assume that n is odd. Then we can write $n = 2k + 1$ for some integer k .

Then $3n + 2 = 3(2k + 1) + 2 = 6k + 5 = 2(3k + 2) + 1$. Thus $3n + 2$ is two times some integer plus 1, so it is odd.

b) Suppose that $3n + 2$ is even and that n is odd. Since $3n + 2$ is even, so is $3n$. If we add subtract an odd number from an even number, we get an odd number, so $3n - n = 2n$ is odd. But this is obviously not true.

Therefore our supposition was wrong, and the proof by contradiction is complete.