Engaging all students with "impossible geometry"

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Geometry is an area in which Australian students performed particularly poorly on the 2007 Trends in International Mathematics and Science Study (TIMSS) (Thomson, Wernert, Underwood, & Nicholas, 2008). Scores for Year 8 students fell significantly below the TIMSS scale average. Australian students thus need stronger preparation in geometry. One innovative area of recreational geometry that has rich potential to engage and challenge a wide variety of students is "impossible geometry." An impossible geometric object is a two-dimensional drawing that gives the impression of a viable three-dimensional figure, which in fact cannot exist. It is thus an optical illusion where the misperception requires careful analysis to detect and explain. Whereas individual parts of the figure may be possible, the object in its totality is not. Impossible geometry has been a subject of interest for some time. Necker discussed impossible objects in an 1832 journal article, and Penrose and Penrose particularly popularised the idea in their 1958 publication on the topic. Swedish artist Oscar Reutersvärd, however, is considered to be the "the father of impossible objects" due to his deliberate, prolific work in this area.

Spatial explorations involving two- and three-dimensional shapes are an important part of school curricula. The Linking Two and Three Dimensions component of Queensland's Mathematics A Senior Syllabus (Queensland Board of Senior Secondary School Studies, 2000) addresses the importance of practical knowledge of various construction areas and asserts, "Particular emphasis should be given to the representation of three-dimensional constructions in two dimensions" (p. 14). In Victoria, Shape, Space and Design is a Foundation Mathematics area of study that includes investigation of the properties of two- and three-dimensional shapes and two-dimensional representations of three-dimensional objects (Victorian Curriculum and Assessment Authority, 2005).

In this article, we describe an impossible-geometry lesson where two high school classes in different Northern Nevada school districts (an integrated algebra/geometry/statistics class comprised mainly of freshmen and a tenth-grade geometry class) explore the concept of impossible geometry before

creating their own impossible object. We provide lesson-implementation details, sample student work, and student response to the lesson.

This lesson has the potential to address the mathematics objectives stated above. The intent is to have students explore connections between two- and three-dimensional figures. Students enhance their ability to visualize objects that are both possible and impossible in three-dimensional space, an important skill in real-world construction and design. Students completing this lesson exercise creativity and spatial skills in attempting to draft and verbally justify two-dimensional drawings that give the illusion of three-dimensional objects that in fact cannot be constructed.

Lesson overview

We told students that they would explore a type of recreational geometry that involved seeing sample "impossible figures" and trying to create their own. Students were given the meaning of impossible object, as presented earlier, orally and on an overhead projector. They then watched a nine-minute segment of the film *The Fantastic World of M. C. Escher* (Emner, 1980/2006) on impossible objects, including Penrose's discussion of his impossible triangle. After this, students reviewed four classic impossible objects on handouts, as shown in Figures 1–4. In addition, they and were given brief information

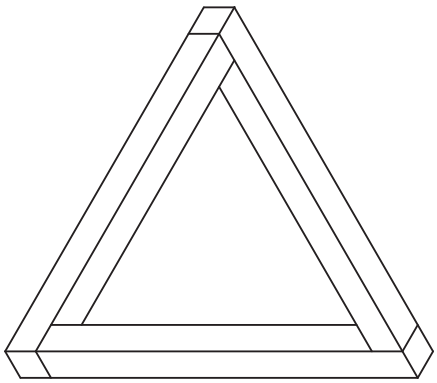

Figure 1. Penrose triangle (tribar).

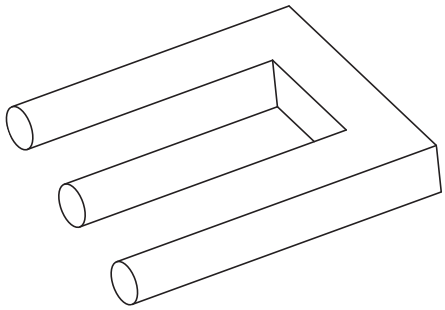

Figure 2. Trident (impossible fork).

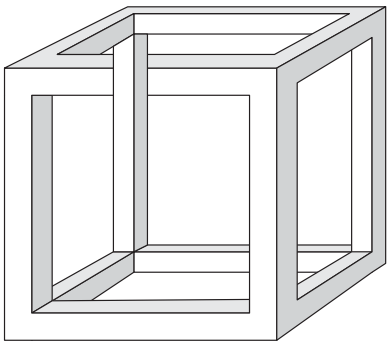

Figure 3. Impossible cube.

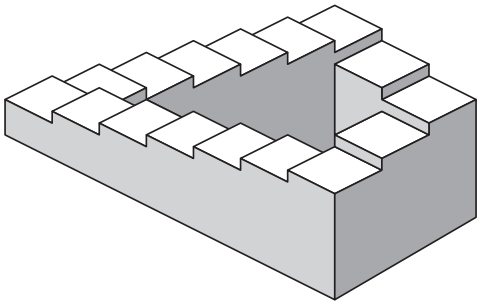

Figure 4. Impossible staircase.

about who developed each object. (Web searches will produce information on these objects; see, for example, http://en.wikipedia.org/wiki/impossible_object.) To reinforce the impossibility of figures, students could for example, try to shade the "outside" of the Penrose triangle.

Students next viewed fantastic artwork that incorporates impossible images, such as the work of Dutch graphic artist M. C. Escher, who used impossible figures (especially those created by others, with particular influence from Penrose) in several of his drawings, including "Belvedere" (e.g., http://en.wikipedia.org/wiki/File:Belvedere.jpg), "Ascending and Descending," and "Waterfall" (e.g., http://en.wikipedia.org/wiki/File:Escher_Waterfall.jpg). For example, the second of these features the impossible staircase, which appears to be both continuously ascending and descending (as shown in Figure 4). Students were asked to explain why they think each object or drawing is considered to be impossible. (Escher's work may be found in various books and web sites. Sample books include *The Graphic Work of M.C. Escher* (Escher, 1971) and *The Magic Mirror of M. C. Escher* (Ernst, 1976). Websites are provided at the end of this article.)

Finally, students attempted to create their own impossible object on their choice of blank or grid paper and to explain why it is impossible. They were given access paper as well as tools they may want (e.g., rulers, protractors, mechanical compasses), to complete the task. They provided evaluative written feedback on the lesson when finished. (Note that the lesson may require two class periods, depending upon class length. Students might be permitted to work on their figures over night.)

Student responses

Upon being presented with the meaning of impossible object and viewing several examples on paper, students displayed minimal enthusiasm, one remarking that she "didn't see anything special about these objects" to make them impossible. Students described what they saw using age-relevant language and examples, one likening the Penrose triangle to "the warning triangle" used in cars. The class gained further insight into impossible geometry and became increasingly engaged as they examined perspective paradoxes in artwork and the film clip. At this point one student accurately stated, "An impossible object is like something you can't build in real life."

Asked to draw their own figures, students mainly showed uncertainty and even frustration. They commented on task difficulty, weak drawing skills, and not knowing where to begin. However, when a student in one class created an "impossible roof" that was shared with the class as an example, others soon followed with their own examples, and enthusiasm grew. Allowed sufficient time to brainstorm ideas and work through initial struggles, all students eventually created their own drawings at various levels of sophistication. Some were correct attempts (Figure 5), and some were not in that they can be created in three-dimensional space (Figure 6). Some students enhanced their drawings appropriately using shading. The student who created the impossi-

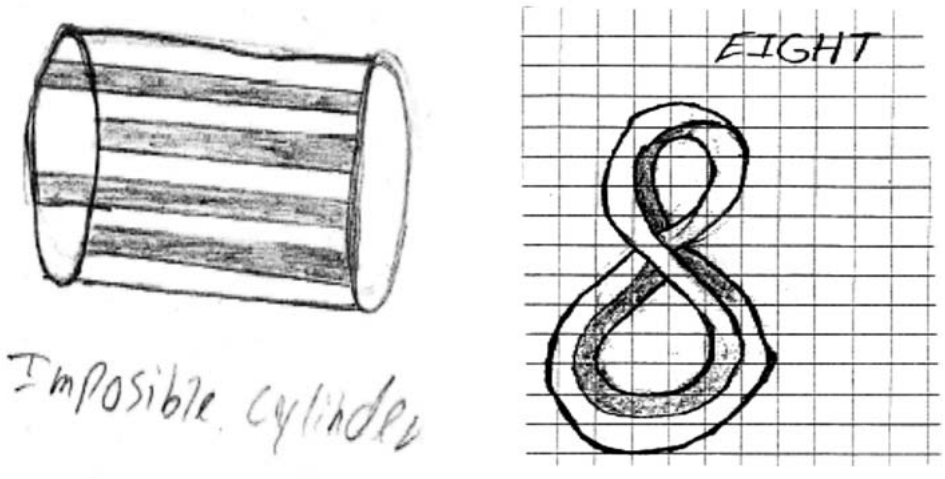

Figure 5. Two correct student attempts to create impossible objects.

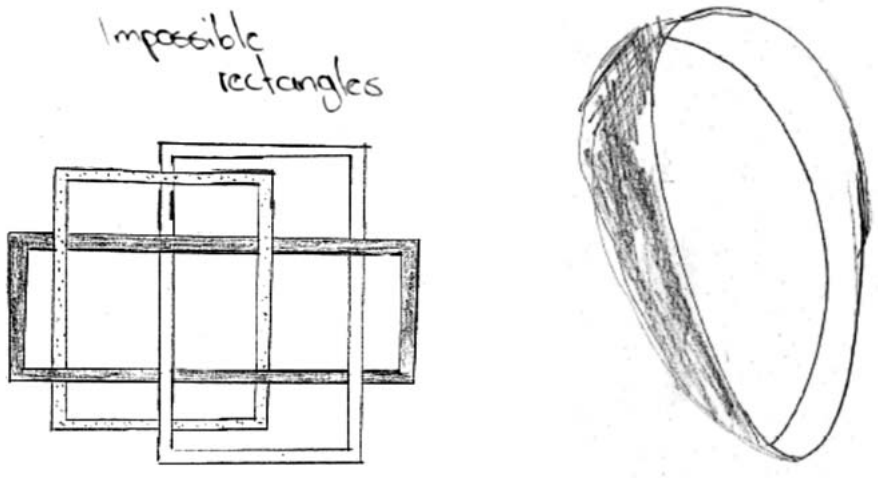

Figure 6. Two incorrect student attempts to create impossible objects.

ble 8 in Figure 5 explained in writing, "This figure is impossible because you can't follow the white without running into the shading. There is no route to get you to end up where you start."

Some students did not make an appropriate attempt at this task in that they either essentially duplicated a figure shown to them earlier or they created an object that was imaginary rather than impossible to create in three-dimensional space. The latter case involved a few nonexistent animals likely inspired by some drawings of impossible animals shared with the class earlier. Of the completed drawings that appropriately observed the task guidelines and were possible to interpret, only about one-third were correct.

In their written feedback, almost all students expressed high interest in the lesson. They especially liked examining the drawings and artwork, and many also liked creating their own drawings. The students said that this topic was completely new to them and both stretched and challenged their thinking (e.g., "made my mind extend"). What they found somewhat confusing and difficult were the very things that interested them most—the drawings they

reviewed and the pictures they themselves drew. In telling how they had approached the task of drawing their own impossible object, students mainly gave a variety of answers indicating use of creativity. One said, "I approached my task of trying to create my impossible ob. [object] by visualizing what I saw and thought made an impossible obj. [object] what it is." Others indicated use of an "open mind" and "looking at different angels [angles]." In terms of whether students believed this lesson had any real-world value, most said no. One elaborated, "No not really because people are just drawing things that you can't draw in real life. It isn't really something important," and another simply stated, "No, we don't have this in real life." A few perceived some value, the two most articulate of which said, "The value of imagination and creating something impossible" and "I think it does because you know what's possible and impossible to construct." Students lacked suggestions for improving the lesson overall. They mainly indicated that they liked the lesson and had no ideas for improvement, one stating that the lesson should be a class offered to everyone and another that the lesson improves everyone's ability "to think and create."

Lesson reflections and suggestions

"Impossible geometry" proved to be a novel lesson that motivated and challenged students while sometimes perplexing them. Some students grasped the concept, whereas others needed additional time and exploration to do so more fully. Nevertheless, the lesson content was reasonably accessible and engaging to all students, regardless of geometry background, placing students on more even ground than may be the case for some geometry topics.

The high school students who participated in this lesson practiced the important and likely underused skill of spatial visualization, such as imagining what is and is not possible in three-dimensional space and imagining objects from different perspectives. They also used reasoning skills by struggling to determine what was and was not possible and how to verbalise their thinking in this regard. Students used artistic skills but may not have fully appreciated the connection of this mathematics lesson to the field of art. This was evident in the fact that most students saw no usefulness of this lesson for the real world. Therefore, we suggest that greater and more explicit attention be given to the real-world value of this lesson in the scientific sense of imagining possibilities in three-dimensional space and in the aesthetic sense of visual intrigue. In the case of the former, the importance of imagination to invention and problem solving in mathematics and science has been well documented. See, for example, Miller (1984) and Shepard, (1988).

Adequate processing time is crucial to student success for this lesson, and this can vary widely among students. Thus, it may be wise to plan the lesson across a two-day period and allow ideas to "simmer" overnight, especially for student-constructed drawings. Students may also need more than one experience with this activity. Further, it may be worthwhile to allow students to tackle this task with a partner, should they so choose.

It is noteworthy that what students reported as most difficult is also what they found most interesting. This lesson thus evoked a most coveted instructional environment—one that simultaneously engaged and challenged students.

Selected Web resources

Articles, Drawings, Information

- Adding Depth to Illusions: http://www.lhup.edu/~dsimanek/3d/illus2.htm
- LookMind.com: Impossible Objects: http://lookmind.com/illusions.php?cat=1
- How To Draw Impossible Shapes: http://www.metacafe.com/watch/385181/how_to_draw_impossible_shapes/ [silent video; one example of others that appear on video sites such as youtube.com and metacafe.com]
- IllusionWorks: Impossible Figures and Objects: http://psylux.psych.tu-dresden.de/i1/kaw/diverses%20Material/www.illusionworks.com/html/hall_of_illusions.html
- Impossible Figures in Perceptual Psychology: http://www.fink.com/papers/impossible.html
- Impossible Objects: http://www.michaelbach.de/ot/cog_imposs1/index.html ["next" button on upper left goes to "Hallucii—An 'Impossible' Movie" at http://www.michaelbach.de/ot/cog_impossHallucii/index.html]
- Impossible World: http://im-possible.info/
- Mighty Optical Illusions: Impossible Objects Category: http://www.moillusions.com/2006/05/impossible-objects-category.html
- Planet Perplex: Impossible Images: http://www.planetperplex.com/en/impossible_images.html
- Visual Illusions: The Principles of Artistic Illusions: http://www.lhup.edu/~dsimanek/3d/illus1.htm

Escher Products (e.g., posters/prints)

- AllPosters.com: http://www.allposters.com/ [search: Escher]
- Art.com: http://www.art.com/ [search: Escher]
- World of Escher: http://www.worldofescher.com/

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