Engineering Applications of the Laplace Transform

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PREFACE

The Laplace transform, a technique of transforming a function from one domain to another, has a vital role to play in engineering and science. Laplace transformation methods offer simple and efficient strategies for solving many science and engineering problems, including: control system analysis; heat conduction; analyzing signal transport; mechanical networks; electrical networks; communications systems; and analog and digital filters.

This book is aimed explicitly at undergraduates and graduates in applied mathematics, electrical and electronic engineering, physics, and computer science. The reader can follow the step by step problem solving and derivations presented with minimal instructor assistance. The first two chapters give a straightforward introduction to the Laplace transform, including its functional properties, finding inverse Laplace transforms by different methods, and the operating properties of inverse Laplace transform. Chapter 3 describes transfer function applications for mechanical and electrical networks to develop the input and output relationships. Chapters 4 and 5 demonstrate applications in problem solving, such as the solution of LTI differential equations arising in electrical and mechanical engineering fields, along with the initial conditions. The state-variables approach is discussed in Chapter 6 and explanations of boundary value problems connected with the heat conduction, waves, and vibrations in elastic solids are presented in Chapter 7.

CHAPTER 1

THE LAPLACE TRANSFORM

1.1 Introduction

Named in honor of the French mathematician Pierre Simon Laplace, the Laplace transform plays a vital role in technical approaches to studying and designing engineering problems. The significance of the Laplace transform is its application in many different functions. For example, the Laplace transform enables us to deal efficiently with linear constant-coefficient differential equations with discontinuous forcing functions—these discontinuities comprise simple jumps that replicate the action of a switch. Using Laplace transforms, we can also design a meaningful mathematical model of the impulse force provided by, for example, a hammer blow or an explosion.

It is certainly not a lazy assumption to suggest that differential equations comprise the most important and significant mathematical entity in engineering and technology. The linear, time-invariant differential equation, Eq. (1), is one such design:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{1}$$

With the system parameters a_k and b_k, several systems are represented by this equation, which relates the output $y(t)$, to the input $x(t)$. However, we want to design a system where the state variables vary with time and/or space. The most natural way to describe this behavior is through differential equations. The development of a differential equation model requires a detailed understanding of the system we wish to depict. It is not enough to set up a differential equation model; we also have to solve the equations. Therefore, an essential mathematical method for modeling and analyzing linear systems is the Laplace transform. In terms of the mathematical representation of a physical system, the Laplace transform can simplify the study of its behavior considerably.

The Laplace transform offers tremendous benefits. We model physical systems of continuous-time (linear time-invariant) when appropriate, with linear differential equations having constant coefficients. A clear explanation of the characteristics of the equations and physical structure is given by the Laplace transform of the LTI system. Once transformed, however, these differential equations are algebraic and are thus easier to solve. The solutions are functions of the Laplace transform variable s rather than the time variable t when we use the Laplace transform to solve differential equations. Consequently, we will need a procedure to translate functions from the frequency domain to the time domain, which is called the inverse Laplace transform.

In Section 1.3, we discuss the conditions under which a function's Laplace transform occurs. In Section 1.4, we derive the Laplace transforms of standard signals as examples and present Laplace transform pairs in Table 1.1. In Section 1.5, various operational properties, i.e. linearity, time-domain shift rules, multiplication, division, and scaling practices, are described. The regulations on differentiation and integration are presented in this section also. These are more difficult to demonstrate, but are of great importance in application. The time-domain differentiation rule is essential for applying the differential equations set out in chapters 3 and 4. We also deal with two theorems in this section, i.e. the initial and final value theorems for the Laplace transform. We can see how to evaluate the Laplace transform of a periodic function in Section 1.6. Section 1.7 explains how you can determine the Laplace transform of a piecewise continuous function that uses the unit step function.

LEARNING OBJECTIVES

On reaching the end of this chapter, we expect you to have understood and be able to apply:

- The definition of the Laplace transform.
- The concept of the existence of the Laplace transform.
- The standard examples of the Laplace transform.
- The properties of linearity, shifting, and scaling.
- The rules of differentiation and integration.
- The initial and final value theorems.
- The Laplace transform of a periodic function.
- How to express the piecewise continuous function in terms of the unit step function.

1.2 Definition

Suppose that $f(t)$ is a real or complex-valued function of the (time) variable $t > 0$ and s is a real or complex parameter. We define the Laplace transform as

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\ dt \tag{1.1}$$

where the limit exists as a finite number. s is a fixed parameter (real/complex) when evaluating the integral (1.1). However, the reader should understand that, in advanced applications of Laplace transforms primarily to solve partial differential equations in digital signal processing, it is essential to consider s as a complex number. Before we proceed further, it is worth making a few observations relating to the definition in (1.1).

(i) The Laplace transform is an ***integral transform***.

(ii) The Laplace transform only uses values of $f(t) \quad for \;\; t \geq 0.$

(iii) The symbol L denotes the **Laplace transform operator**. When it operates on the function $f(t)$, it transforms it into the function $F(s)$ of the complex variable s. That is, the operator transforms the function $f(t)$ in the t domain (**time-domain**) into the function $F(s)$ in the s domain (**complex frequency domain**, or **frequency domain**). This relationship is depicted graphically in Figure 1.1.

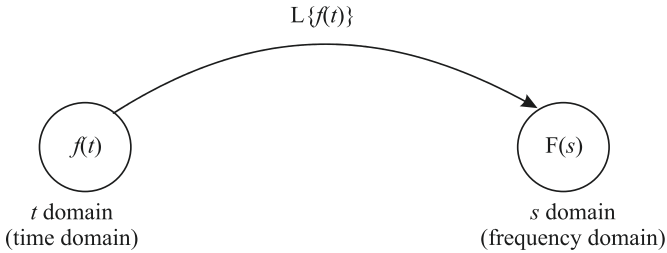


Figure 1.1. The Laplace transform operator

(iv) For the integral (1.1) to exist, any discontinuity of the integrand inside the interval $(0,\infty)$ must be a finite jump so that there are right and left-hand limits at those discontinuous points. An exception is a discontinuity at $t=0$ (if it exists). For instance, the function $f(t)=\dfrac{1}{\sqrt{t}}$ diverges at $t=0$, but the integral (1.1) exists.

(v) The Laplace transform is a mathematical toolbox for solving linear ODEs and related initial value problems, PDEs, and boundary value problems. It is used extensively in electrical engineering, control theory, and the stability of algorithms.

1.3 Existence of Laplace Transforms

A Laplace transform should exist if the magnitude of the transform is finite, that is, $|F(s)|<\infty$.

Piecewise continuous: A function $f(t)$ is piecewise continuous on a finite interval $A \le t \le B$ if f is continuous on $[A,B]$, except possibly at many finite points. Each of these finite points f has a finite limit on both sides.

Sufficient condition: The sufficient condition for a Laplace transform to exist if $f(t)$ is that it is piecewise continuous on $(0,\infty)$ and some constants L and M exist such that $|f(t)|<M\,e^{Lt}$, then $|F(s)|$ exists for $s>L$.

Proof: As $f(t)$ is piecewise continuous on $(0,\infty)$, $f(t)e^{-st}$ is integrable on $(0,\infty)$.

$$\left|L\{f(t)\}\right|=\left|\int_0^\infty f(t)e^{-st}\,dt\right|\leq\int_0^\infty |f(t)|e^{-st}\,dt\leq\int_0^\infty M\,e^{Lt}\,e^{-st}\,dt$$

$$=\frac{M}{L-s}\left[e^{-(s-L)t}\right]_0^\infty=\frac{M}{L-s}[0-1]=\frac{M}{s-L}$$

$$\left|L\{f(t)\}\right|=|F(s)|<\infty\quad for\ s>L.$$

Definition: A function $f(t)$ is said to be of **exponential order** L if positive constants T and M exist such that $|f(t)|<M\,e^{Lt}$, for all $t\geq T$.

The physical significance of the Laplace transform: A Laplace transform has no physical meaning except that it transforms the time domain function to a frequency domain (s). The Laplace transform is applied to simplify mathematical computations and allow the effortless analysis of linear time-invariant systems.

1.4 Laplace Transforms of Some Standard Functions

In this section, we illustrate the procedure to find the Laplace transform of the function $f(t)$. In all expressions, it is assumed that $f(t)$ satisfies the conditions of Laplace transformability.

Example 1.1. Determine the Laplace transform of the constant function $f(t) = a$.

Solution:

Using the definition of the Laplace transform

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st}\ f(t)\ dt$$

$$L\{a\} = \int_0^\infty e^{-st}\ a\ dt$$

$$L\{a\} = \frac{-a}{s}\ e^{-st}\Big|_0^\infty$$

$$L\{a\} = \frac{a}{-s}\left[e^{-\infty} - e^{0}\right] = \frac{a}{-s}[0-1]$$

provided, of course, that $s > 0$ (if s is real). Thus we have

$$L\{a\} = \frac{a}{s} \quad (s > 0).$$

Example 1.2. Find the Laplace transform of $f(t)=e^{at}$, where a is constant.

Solution:

By definition of the Laplace transform

$$F(s)=L\{f(t)\}=\int_0^\infty e^{-st}\ f(t)\ dt$$

$$L\{e^{at}\}=\int_0^\infty e^{-st}\ e^{at}\ \ dt=\int_0^\infty e^{-(s-a)t}\ dt\ .$$

On integrating, we get

$$L\{e^{at}\}=\frac{-1}{(s-a)}e^{-(s-a)t}\Big|_0^\infty$$

$$L\{e^{at}\}=\frac{-1}{(s-a)}\left[e^{-\infty}-1\right]$$

Thus,

$$L\{e^{at}\}=\frac{1}{(s-a)}\quad (for\, s>a)$$

Example 1. 3. Find the Laplace transform of $f(t)=\cosh at$.

Solution:

Let $L\{f(t)\}=L[\cosh at]$

and express the function $\cosh(at)$ in its exponential form

$$\cosh(at) = \frac{1}{2}\left(e^{at} + e^{-at}\right).$$

The Laplace transform becomes

$$L\{f(t)\} = \frac{1}{2}L\left[e^{at} + e^{-at}\right] = \frac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\}\right] = \frac{1}{2}\left[\frac{1}{(s-a)} + \frac{1}{(s+a)}\right]$$

$$L[\cosh at] = \frac{1}{2}\left[\frac{s+a+s-a}{(s^2-a^2)}\right].$$

Thus,

$$L[\cosh at] = \frac{s}{(s^2-a^2)} \quad (for\, s > a).$$

Example 1.4. Find the Laplace transform of $f(t) = \sinh at.$

Solution:

Let $L\{f(t)\} = L[\sinh at]$

and express the function $\cosh(at)$ in its exponential form

$$\sinh(at) = \frac{1}{2}\left(e^{at} - e^{-at}\right).$$

The Laplace transform becomes

$$L[\sinh at] = \frac{1}{2}L\left[e^{at} - e^{-at}\right] = \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right] = \frac{1}{2}\left[\frac{1}{(s-a)} - \frac{1}{(s+a)}\right]$$

$$L[\sinh at] = \frac{1}{2}\left[\frac{s+a-s+a}{(s^2-a^2)}\right]$$

Thus,

$$L[\sinh at] = \frac{a}{(s^2-a^2)} \quad (for\, s > a) \cdot$$

Example 1.5. Find the Laplace transform of $f(t) = \cos at$.

Solution:

Let $L\{f(t)\} = L[\cos at]$

and express the function $\cosh(at)$ in its exponential form

$$\cosh(at) = \frac{1}{2}\left(e^{iat} + e^{-iat}\right).$$

The Laplace transform becomes

$$L\{f(t)\} = \frac{1}{2}L\left[e^{iat} + e^{-iat}\right] = \frac{1}{2}\left[L\{e^{iat}\} + L\{e^{-iat}\}\right] = \frac{1}{2}\left[\frac{1}{(s-ia)} + \frac{1}{(s+ia)}\right]$$

$$L[\cosh at] = \frac{1}{2}\left[\frac{s+ai+s-ai}{(s^2+a^2)}\right].$$

Thus,

$$L[\cosh at] = \frac{s}{(s^2 + a^2)} \quad (for\, s > 0).$$

Example 1.6. Find the Laplace transform of $f(t) = \sin at$.

Solution:

Let $L\{f(t)\} = L[\sin at]$

and express the function $\sin(at)$ in its exponential form

$$\sin(at) = \frac{1}{2i}\left(e^{iat} - e^{-iat}\right).$$

The Laplace transform becomes

$$L[\sinh at] = \frac{1}{2i} L\left[e^{iat} - e^{-iat}\right] = \frac{1}{2}\left[\frac{1}{(s - ia)} - \frac{1}{(s + ia)}\right]$$

$$L[\sinh at] = \frac{1}{2i}\left[\frac{s + ia - s + ia}{(s^2 + a^2)}\right] = \frac{a}{(s^2 + a^2)}.$$

Thus,

$$L[\sinh at] = \frac{a}{(s^2 + a^2)} \quad (for\, s > 0).$$

Example 1.7. Find the Laplace transform of $f(t)=t^n$, where n is a positive integer.

Solution:

Using the definition of the Laplace transform

$$F(s)=L\{f(t)\}=\int_0^\infty e^{-st}\ f(t)\ dt$$

$$L\{t^n\}=\int_0^\infty e^{-st}\ t^n\ dt \qquad \text{Setting}\quad st=x\ \ \&\ dt=\frac{dx}{s}$$

$$L\{t^n\}=\int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n\ \frac{dx}{s}$$

$$L\{t^n\}=\frac{1}{s^{n+1}}\int_0^\infty e^{-x}\ x^n\ \ dx$$

Therefore, $\left(\text{by gamma function } \Gamma(n)=\int_0^\infty e^{-x}\ x^{n-1}dx\right)$

$$L\{t^n\}=\frac{\Gamma(n+1)}{s^{n+1}}$$

Thus,

$$L\{t^n\}=\frac{\Gamma(n+1)}{s^{n+1}}=\frac{n!}{s^{n+1}} \qquad \Bigg|\ \because \Gamma(n+1)=n!$$

Example 1.8. Find the Laplace transform of the Dirac delta function (impulse function) $f(t) = \delta(t)$.

Solution:

An impulse is infinite at $t = 0$ and zero elsewhere. The area under the unit impulse is 1.

The **Dirac delta function** $\delta(t)$ is defined by

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

The **Dirac delta function** $\delta(t-a)$ is characterized by the following two properties

$$(i) \qquad \delta(t-a) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases}$$

$$(ii) \qquad \int_{-\infty}^{\infty} \delta(t-a)\; f(t)\; dt = f(a)$$

Using the definition of the Laplace transform

$$F(s) = L\{f(t)\} = \int_0^{\infty} e^{-st}\; f(t)\; d\,t$$

$$L\{\delta(t-a)\} = \int_0^{\infty} e^{-st}\; \delta(t-a)\; d\,t$$

$$L\{\delta(t-a)\} = e^{-st}\,\Big|_{t=a}\,.$$

Thus,

$$L\{\delta(t-a)\} = e^{-as}$$

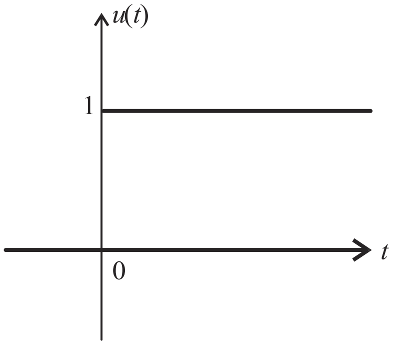
and in particular $L\{\delta(t)\} = 1$.

Example 1.9. Find the Laplace transform of the unit step function $f(t) = u(t)$.

Solution:

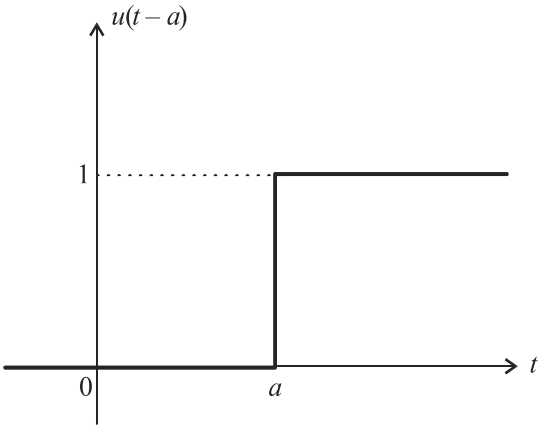
The unit step signal is a typical "engineering signal" in made to measure engineering applications, which often involve functions (mechanical or electrical driving forces).

The **unit step function** $u(t)$ is defined by



$$u(t) = \begin{cases} 1 & for\, t \geq 0 \\ 0 & elsewhere \end{cases}$$

Figure 1.2.



$$u(t-a)=\begin{cases}1 & for\, t\geq a\\ 0 & elsewhere\end{cases}$$

Figure 1.3

The step function is shown in Figure 1.2.

The Laplace transform of $u(t)$, by definition, can be written as

$$F(s)=L\{f(t)\}=\int_0^\infty e^{-st}\ f(t)\ dt$$

$$L\{\ u(t-a)\ \}=\int_0^\infty e^{-st}\ u(t-a)\quad dt$$

$$L\{\ u(t-a)\ \}=\int_a^\infty e^{-st}\ dt=\frac{1}{-s}e^{-st}\Big|_a^\infty$$

Hence, the Laplace transform of the unit step function exists *only* if the real part of s is greater than zero. We denote this by

$$L\left\{\ u(t-a)\ \right\} = \frac{1}{-s}\left[0-e^{-as}\right].$$

Thus,

$$L\left\{\ u(t-a)\ \right\} = \frac{e^{-as}}{s},\ \left(for\, s>0\right)$$

In particular $L\left\{\ u(t)\ \right\} = \dfrac{1}{s} \quad \left(for\ \ a=0\right)$.

Example 1.10. Find the Laplace transform of the ramp function $f(t)=r(t)$.

Solution:

The ramp function $r(t)$ is defined by

$$r(t) = \begin{cases} t & for\, t>0 \\ 0 & for\, t<0 \end{cases}$$

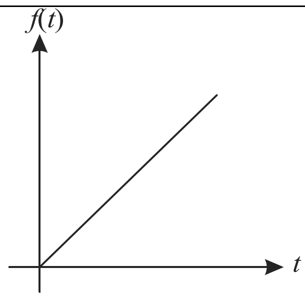


Figure 1.4

Using the definition of the Laplace transform, we have

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st}\ f(t)\ dt$$

$$L\{\ r(t)\ \} = \int_0^\infty e^{-st}\ r(t)\ dt$$

$$L\{\ r(t)\} = \int_0^\infty e^{-st}\ t\, dt$$

Recall from calculus the following formula for integration by parts

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du.$$

$$L\{\ r(t)\} = -t\frac{e^{-st}}{s}\Bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt\ .$$

Integration by parts gives

$$L\{\ r(t)\} = 0 - \frac{1}{s^2}e^{-st}\Big|_0^\infty = \frac{1}{s^2}.$$

Thus,

$$L\{\ r(t)\} = \frac{1}{s^2}.$$

To further progress with the Laplace transform, it is necessary to use a table of Laplace transform pairs for the most commonly occurring functions. Table 1.1 provides a list of the most useful Laplace transform pairs involving elementary functions.

Table 1.1 Laplace Transform Pairs

$L\{1\}=\frac{1}{s}$	$L\{t^n\}=\frac{n!}{s^{n+1}}$, $n=1,2,3.......$
$L\{e^{at}\}=\frac{1}{s-a}$	$L\{e^{-at}\}=\frac{1}{s+a}$
$L\{\sin at\}=\frac{a}{s^2+a^2}$	$L\{\cos at\}=\frac{s}{s^2+a^2}$
$L\{\sinh at\}=\frac{a}{s^2-a^2}$	$L\{\cosh at\}=\frac{s}{s^2-a^2}$
$L\{u(t)\}=\frac{1}{s}$	$L\{u(t-a)\}=\frac{e^{-as}}{s}$
$L\{\delta(t)\}=1$	$L\{\delta(t-a)\}=e^{-as}$

Table 1.1 lists the Laplace transforms of some elementary functions. It would be helpful to familiarize yourself with these expressions. You will frequently encounter problems, solve linear differential equations with constant coefficients, find transfer functions, investigate mechanical systems, and analyze electrical circuits. The Laplace transform function can be derived directly by performing integration. However, it is much simpler to derive the Laplace transform using Laplace transform properties than through direct integration, as shown in the previous examples. The Laplace transform has many interesting operational properties. These properties are why the Laplace transform is considered such a powerful