# [Engineering Mathematics]

## [Partial Differential Equations]

[Partial Differentiation and formation of Partial Differential Equations has already been covered in Maths II syllabus. Present chapter is designed as per GGSIPU Applied Maths IV curriculum. ]

## 1.1 Introduction

A differential equation which involves partial derivatives is called partial differential equation (PDE). The order of a PDE is the order of highest partial derivative in the equation and the degree of PDE is the degree of highest order partial derivative occurring in the equation. Thus order and degree of the PDE $\left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^3 + \left(\frac{\partial z}{\partial x}\right)^4 - x\frac{\partial z}{\partial x} = 0$ are respectively 2 and 3.

If 'z' is a function of two independent variables 'x' and 'y', let us use the following notations for the partial derivatives of 'z' :

$$\frac{\partial z}{\partial x} \equiv p\ , \qquad \frac{\partial z}{\partial y} \equiv q\ , \qquad \frac{\partial^2 z}{\partial x^2} \equiv r, \qquad \frac{\partial^2 z}{\partial x\, \partial y} \equiv s, \qquad \frac{\partial^2 z}{\partial y^2} \equiv t$$

## 1.2 Linear Partial Differential Equations of 1st Order

If in a 1st order PDE, both '$p$' and '$q$' occur in 1st degree only and are not multiplied together, then it is called a linear PDE of 1st order, i.e. an equation of the form $Pp + Qq = R;\ \ P, Q, R$ are functions of $x, y, z$ , is a linear PDE of 1st order.

Langrange's Method to Solve a Linear PDE of 1st Order (Working Rule) :

1. Form the auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
2. Solve the auxiliary equations by the method of grouping or the method of multipliers* or both to get two independent solutions: $u = a, v = b$ ; where $a$ and $b$ are arbitrary constants.
3. $\varphi\,(u, v) = 0$ or $u = f(v)$ is the general solution of the equation $Pp + Qq = R$.

| |
|---|
| ***Method of multipliers** : Consider a fraction $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$<br><br>Taking 1,2, 3 as multipliers, each fraction $= \frac{1\times1}{2\times1} = \frac{2\times2}{4\times2} = \frac{3\times3}{6\times3}$ |

**Example 1**. Solve the PDE $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$

Solution: Comparing with general form $P \equiv (z^2 - 2yz - y^2), Q \equiv (xy + zx),$

$R \equiv (xy - zx)$

Step 1.

Auxiliary equations are $\frac{dx}{(z^2-2yz-y^2)} = \frac{dy}{(xy+zx)} = \frac{dz}{(xy-zx)}$

Step 2.

Taking $x, y, z$ as multipliers, each fraction $= \frac{xdx+ydy+zdz}{(xz^2-2xyz-xy^2+xy^2+xyz+xyz-xz^2)}$

$$= \frac{xdx+ydy+zdz}{0}$$

$\Rightarrow \ xdx + ydy + zdz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = a_1$$

$\Rightarrow u = \ x^2 + y^2 + z^2 = a$ -------------- ①

This is 1st independent solution.

Now for 2nd independent solution, taking last two members of auxiliary equations :

$$\frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$$

$\Rightarrow (y-z)dy = (y+z)dz$

$\Rightarrow \ ydy - (zdy + ydz) - \ zdz = 0$

$\Rightarrow \ ydy - d(yz) - \ zdz = 0$

Integrating, we get

$$\frac{y^2}{2} - yz - \frac{z^2}{2} = b_1$$

$\Rightarrow \ \ v = \ y^2 - 2yz - z^2 = b$ --------------- ②

Which is 2nd independent solution

From ① and ②, general solution is :

$$\varphi\,(x^2+y^2+z^2,\ y^2-2yz-z^2) = 0$$

## 1.3 Homogenous Linear Equations with Constant Coefficients

An equation of the form

$$k_0 \frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + - - - - + k_n \frac{\partial^n z}{\partial y^n} = F(x,y) \quad \text{--------------} ③$$

where $k's$ are constant is called a homogeneous linear PDE of n[th] order with constant coefficients. It is homogeneous because all the terms contain derivatives of the same order.

Putting $\frac{\partial}{\partial x} \equiv D$ and $\frac{\partial}{\partial y} \equiv D'$ , ③ may be written as:

$$(k_0 D^n + k_1 D^{n-1} D' + - - - - + k_n D'^n)z = F(x,y)$$

or $f(D, D')$ z = $F(x,y)$

### 1.3.1 Solving Homogenous Linear Equations with Constant Coefficients

Case 1: When $F(x,y) = 0$

i.e. equation is of the form $k_0 \frac{\partial^2 z}{\partial x^2} + k_1 \frac{\partial^2 z}{\partial x \partial y} + k_2 \frac{\partial^2 z}{\partial y^2} = 0$ -------- ④

or $k_0 D^2 + k_1 DD' + k_2 D'^2 = 0$

In this case Z = C.F.

Case 2: When $F(x,y) \neq 0$

i.e. equation is of the form $k_0 \frac{\partial^2 z}{\partial x^2} + k_1 \frac{\partial^2 z}{\partial x \partial y} + k_2 \frac{\partial^2 z}{\partial y^2} = F(x,y)$ -------- ⑤

or $k_0 D^2 + k_1 DD' + k_2 D'^2 = F(x,y)$

In this case Z = C.F. + P.I.

Where C.F. denotes complimentary function and P.I. denotes Particular Integral.

## Rules for finding C.F. (Complimentary Function)

Step 1: Put $D = m$ and $D' = 1$ in ④ or ⑤ as the case may be

Then A.E. (Auxiliary Equation)is :$k_0 m^2 + k_1 m + k_2 = 0$

Step 2: Solve the A.E. ( Auxiliary Equation):

i. If the roots of A.E. are real and different say $m_1$ and $m_2$, then

C.F. = $f_1(y + m_1x) + f_2(y + m_2x)$

ii. If the roots of A.E. are equal say $m$, then

C.F. = $f_1(y + mx) + x f_2(y + mx)$

**Example 1.2** Solve $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} - 5\frac{\partial^2 z}{\partial y^2} = 0$

Solution:⇒ $(D^2 - 4DD' - 5D'^2)z = 0$

Auxiliary equation is:$m^2 - 4m - 5 = 0$

$\Rightarrow (m-5)(m+1) = 0$

$\Rightarrow m = 5, -1$

C.F. = $f_1(y + 5x) + f_2(y - x)$

⇒ Z = $f_1(y + 5x) + f_2(y - x)$

**Example 1.3** Solve $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

Solution:⇒ $(D^2 + 2DD' + D'^2)z = 0$

Auxiliary equation is:$m^2 + 2m + 1 = 0$

$\Rightarrow (m+1)^2 = 0$

$\Rightarrow m = -1, -1$

C.F. = $f_1(y - x) + x f_2(y - x)$

⇒z=$f_1(y - x) + x f_2(y - x)$

## Rules for finding P.I. (Particular Integral)

* Applicable only if $F(x,y) \neq 0$

Let the given PDE be $f(D,D')$ z = $F(x,y)$

P.I = $\dfrac{F(x,y)}{f(D,D')}$

Case I : When $F(x,y) = e^{ax+by}$

Put $D = a$ and $D' = b$

If $f(a,b) \neq 0$, P.I = $\dfrac{e^{ax+by}}{f(a,b)}$

and if $f(a,b) = 0$, P.I = $\dfrac{x\, e^{ax+by}}{\frac{d}{dD} f(D,D')}$

Now put $D = a$ and $D' = b$

**Example 1.4** Solve $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{x+y}$

Solution:⇒ $(D^2 - 5DD' + 6D'^2)z = e^{x+y}$

Auxiliary equation is: $m^2 - 5m + 6 = 0$

$\Rightarrow (m-2)(m-3) = 0$

$\Rightarrow m = 2,3$

C.F. = $f_1(y + 2x) + f_2(y + 3x)$

P.I = $\dfrac{e^{x+y}}{D^2 - 5DD' + 6D'^2}$

Put $D = 1,\ D' = 1$

P.I = $\frac{e^{x+y}}{1-5+6} = \frac{e^{x+y}}{2}$

$\Rightarrow$ Z = $f_1(y+\ 2x) + \ f_2(y+3x) + \frac{e^{x+y}}{2}$

**Example 1.5** Solve $r - 4s + 4t = e^{2x+y}$

Solution:$\Rightarrow \frac{\partial^2 z}{\partial x^2} - \ 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$

$$\Rightarrow (D^2 - \ 4DD' + 4D'^2)z = e^{2x+y}$$

Auxiliary equation is:$m^2 - 4m + 4 = 0$

$\Rightarrow\ (m-2)^2 = 0$

$\Rightarrow\ m = 2,2$

C.F. = $f_1(y+\ 2x) + \ xf_2(y+2x)$

P.I = $\frac{e^{2x+y}}{(D-2D')^2}$

Putting $D\ = 2,\ D' = 1$, denominator = 0

P.I = $\frac{xe^{2x+y}}{\frac{d}{dD}(D-2D')^2} = \frac{xe^{2x+y}}{2(D-2D')}$

Putting $D\ = 2,\ D' = 1$, again denominator = 0

P.I. = $\frac{x^2e^{2x+y}}{\frac{d}{dD}2(D-2D')}$

$\Rightarrow$ P.I. $= \dfrac{x^2 e^{2x+y}}{2}$

Complete solution is Z= C.F. + P.I.

$\Rightarrow$ Z = $f_1(y+2x) + xf_2(y+2x) + \dfrac{x^2 e^{2x+y}}{2}$

Case II : When $F(x,y) = e^{ax+by}\phi(x,y)$, $\phi(x,y)$ is a trigonometric function of sine or cosine.

$$\text{P.I} = e^{ax+by}\frac{1}{f(D+a,\ D'+b)}\phi(x,y)$$

**Example 1.6** Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \sin 2y$

Solution: Auxiliary equation is: $m^3 + m^2 - m - 1 = 0$

$\Rightarrow m = 1, -1, -1$

C.F. = $f_1(y+x) + f_2(y-x) + xf_3(y-x)$

$$\text{P.I} = \frac{e^x \sin 2y}{D^3 + D^2D' - DD'^2 - D'^3}$$

$$= \frac{e^x \sin 2y}{D^2(D+D') - D'^2(D+D')} = \frac{e^x \sin 2y}{(D+D')^2(D-D')}$$

$$= e^x \frac{1}{f(D+1,\ D')}\sin 2y \quad (\because\ a = 1, b = 0)$$

$$= e^x \frac{1}{(D+1+D')^2(D+1-D')}\sin 2y$$

$$= e^x \frac{1}{(D^2+1+D'^2+2D+2D'+2DD')(D+1-D')}\sin 2y$$

Put $D^2 = 0,\ DD' = 0\ ,\ D'^2 = -4$

P.I. = $e^x \frac{1}{(0+1-4+2D+2D'+0)(D+1-D')} \sin 2y$

$= e^x \frac{1}{(-3+2D+2D')(D+1-D')} \sin 2y$

$= e^x \frac{1}{(-3D+2D^2+2DD'-3+2D+2D'+3D'-2DD'-2D'^2)} \sin 2y$

$= e^x \frac{1}{(-D+0+0-3+5D'-0+8)} \sin 2y$

$= e^x \frac{1}{(5D'-D)+5} \sin 2y$

$= e^x \frac{(5D'-D)-5}{(5D'-D)^2-25} \sin 2y$

$= e^x \frac{(5D'-D-5)\sin 2y}{25D'^2+D^2-10DD'-25}$

$= \ e^x \frac{(10\cos 2y-0-5\sin 2y)}{25(-4)+0+0-25}$

$= \frac{e^x}{-125}(10\cos 2y - 5\sin 2y)$

$= \frac{e^x}{25}(\sin 2y - 2\cos 2y)$

$\Rightarrow$ Z = $f_1(y+x) + \ f_2(y-x) + xf_3(y-x) + \frac{e^x}{25}(\sin 2y - 2\cos 2y)$

Case III : When $F(x,y) = Sin\ (ax+by)$ or $Cos\ (ax+by)$

P.I = $\frac{Sin(ax+by) \text{ or } Cos\,(ax+by)}{f(D^2, DD', D'^2)}$

Put $D^2 = -a^2$, $D' = -ab$ ,$D'^2 = -b^2$

Hence P.I = $\frac{Sin(ax+by) or Cos\,(ax+by)}{f(-a^2, -ab, -b^2)}$

**Example 1.7** Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = Sinx\,Cos2y$

Solution:⇒ $(D^2 - DD')z = Sinx\,Cos2y$

Auxiliary equation is:$m^2 - m = 0$

$\Rightarrow m(m-1) = 0$

$\Rightarrow m = 0, 1$

C.F. = $f_1(y) + f_2(y+x)$

P.I = $\frac{Sinx\,Cos2y}{D^2 - DD'} = \frac{1}{D^2 - DD'} \frac{1}{2}[Sin\,(x+2y) + Sin(x-2y)]$

$$= \frac{1}{2} \frac{1}{D^2 - DD'} Sin(x+2y) + \frac{1}{2} \frac{1}{D^2 - DD'} Sin(x-2y)$$

Putting $D^2 = -1$, $DD' = -2$ in the 1st term, $D^2 = -1$, $DD' = 2$ in the 2nd term

$$= \frac{1}{2} \frac{Sin\,(x+2y)}{-1-(-2)} + \frac{1}{2} \frac{Sin\,(x-2y)}{-1-(2)}$$

$\Rightarrow$ P.I. $= \frac{1}{2} Sin(x+2y) - \frac{1}{6} Sin\,(x-2y)$

Complete solution is Z= C.F +P.I

$\Rightarrow$ Z= $f_1(y) + f_2(y+x) + \frac{1}{2} Sin(x+2y) - \frac{1}{6} Sin\,(x-2y)$

Case IV : When $F(x,y) = x^m y^n$

P.I = $\frac{x^m y^n}{f(D,D')}$

$= [f(D,D')]^{-1}.x^m y^n$

Expand $[f(D,D')]^{-1}$ in ascending powers of $D$ or $D'$ and operate on $x^m y^n$ term by term.

**Example 1.8** Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = x + y$

Solution:⇒ $\left(D^2 + DD' - 6D'^2\right)z = x + y$

Auxiliary equation is:$m^2 + m - 6 = 0$

$\Rightarrow (m+3)(m-2) = 0$

$\Rightarrow m = -3, 2$

C.F. = $f_1(y - 3x) + f_2(y + 2x)$

P.I = $\frac{x+y}{D^2 + DD' - 6D'^2}$

$= \frac{1}{D^2}\left[1 + \frac{D'}{D} - 6\frac{D'^2}{D^2}\right]^{-1}(x+y)$

$= \frac{1}{D^2}\left[1 - \frac{D'}{D} + - - - - -\right](x+y)$

$\because (1+t)^{-1} = 1 - t + t^2 - t^3 + - - -$

$= \frac{1}{D^2}\left[(x+y) - \frac{1}{D}(0+1)\right]$

$= \frac{1}{D^2}[x + y - x]$

P.I. $= \frac{1}{D^2}[y] = \frac{yx^2}{2}$

Complete solution is Z= C.F +P.I

$\Rightarrow$ Z $= f_1(y - 3x) + f_2(y + 2x) + \frac{yx^2}{2}$

**Example 1.9** Solve$(D^3 - 3\,D^2D')z = x^2y$

Solution: Auxiliary equation is: $m^3 - 3\,m^2 = 0$

$\Rightarrow m^2(m - 3) = 0$

$\Rightarrow\ m = 0,0,3$

C.F. $= f_1(y) + x f_2(y) + f_3(y + 3x)$

P.I = $\frac{x^2y}{D^3 - 3\,D^2D'}$

$= \frac{1}{D^3}\left[1 - \frac{3D'}{D}\right]^{-1}(x^2y)$

$= \frac{1}{D^3}\left[1 + \frac{3D'}{D}\right](x^2y)$

$\because (1 - t)^{-1} = 1 + t + t^2 + t^3 + - - -$

$= \frac{1}{D^3}\left[(x^2y) + \frac{3}{D}(x^2)\right]$

$= \frac{1}{D^3}[x^2y + x^3]$

$$=\frac{1}{D^2}\left[\frac{x^3y}{3}+\frac{x^4}{4}\right]$$

$$=\frac{1}{D}\left[\frac{x^4y}{12}+\frac{x^5}{20}\right]$$

$$\Rightarrow \text{P.I.} = \left[\frac{x^5y}{60}+\frac{x^6}{120}\right]$$

Complete solution is Z= C.F +P.I

$\Rightarrow$ Z $= f_1(y) + \; xf_2(y) + f_3(y+3x) + \left[\frac{x^5y}{60}+\frac{x^6}{120}\right]$

Case V: In case of any function of $F(x,y)$ or when solution fails for any case by above given methods

$$\text{P.I} = \frac{F(x,y)}{f(D,D')}$$

Resolve $\frac{1}{f(D,D')}$ into partial fractions considering $f(D,D')$ as function of $D$ alone.

$$\text{P.I} = \frac{F(x,y)}{D-m\,D'} = \int F(x, c-mx)dx$$

where $C$ is replaced by $y+mx$ after integration.

**Example 1.10** Solve $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

Solution: Auxiliary equation is: $m^2 - m - 2 = 0$

$\Rightarrow (m-2)(m+1) = 0$

$\Rightarrow m = 2, -1$

C.F. $= f_1(y+2x) + \; f_2(y-x)$

$$\text{P.I} = \frac{(y-1)e^x}{D^2 - DD' - 2D'^2} = \frac{(y-1)e^x}{D^2 - 2DD' + DD' - 2D'^2} = \frac{(y-1)e^x}{(D-2D')(D+D')}$$

$$= \frac{1}{(D-2D')} \int (c + x - 1)\, e^x dx$$

Putting $y = c + x \;\; as \;\; m = -1$

$$= \frac{1}{(D-2D')} [(c + x - 1)e^x - e^x]$$

$$= \frac{1}{(D-2D')} [(c + x)e^x - 2e^x]$$

$$= \frac{1}{(D-2D')} [(y - x + x)e^x - 2e^x]$$

Putting $c = y - x$

$$= \frac{1}{(D-2D')} [ye^x - 2e^x]$$

$$= \frac{1}{(D-2D')} [(y - 2)e^x]$$

$$= \int (c - 2x - 2)e^x\, dx$$

Putting $y = c - 2x$

$$= (c - 2x - 2)e^x - (-2)e^x$$

$$= (c - 2x)e^x$$

$$= (y + 2x - 2x)e^x$$

Putting $c = y + 2x$

$\Rightarrow$P.I. $= ye^x$

Complete solution is Z= C.F +P.I

$\Rightarrow$ Z $= f_1(y + 2x) + f_2(y - x) + ye^x$

**Example 1.11** Solve $2\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 5sin(2x + y)$

Solution: $\Rightarrow \left(2D^2 - 5DD' + 2D'^2\right)z = 5sin(2x + y)$

Auxiliary equation is: $2m^2 - 5m + 2 = 0$

$\Rightarrow (2m - 1)(m - 2) = 0$

$\Rightarrow m = \frac{1}{2}, 2$

C.F. $= f_1\left(y + \frac{x}{2}\right) + f_2(y + 2x)$

P.I = $\frac{5sin(2x+y)}{2D^2 - 5DD' + 2D'^2}$

Putting $D^2 = -4,\ DD' = -2, D'^2 = -1$, denominator =0

$\therefore$ solution fails as per case II, resolving denominator into partial fractions

P.I = $\frac{5sin(2x+y)}{\left(2D-D'\right)\left(D-2D'\right)}$

$= \frac{5}{(2D-D')} \int sin(2x + (c - 2x))\, dx$

Putting $y = c - 2x$

$$=\frac{5}{2}\frac{1}{\left(D-\frac{D'}{2}\right)}\int sin\ cdx$$

$$=\frac{5}{2}\frac{x\ sinc}{\left(D-\frac{D'}{2}\right)}$$

$$=\frac{5}{2}\frac{x\ sin(y+2x)}{\left(D-\frac{D'}{2}\right)}$$

$$=\frac{5}{2}\int x\ sin\left[\left(c-\frac{x}{2}\right)+2x\right]dx$$

Putting $y=c-\frac{x}{2}$

$$=\frac{5}{2}\int x\ sin(c+\frac{3}{2}x)\ dx$$

$$=\frac{5}{2}\left[(x)\frac{(-cos(c+\frac{3}{2}x)}{\frac{3}{2}}-(1)\frac{(-sin(c+\frac{3}{2}x)}{\frac{9}{4}}\right]$$

$$=-\frac{5}{3}x\ cos\left(c+\frac{3}{2}x\right)+\frac{10}{9}sin(c+\frac{3}{2}x)$$

$$=-\frac{5}{3}x\ cos\left(y+\frac{x}{2}+\frac{3}{2}x\right)+\frac{10}{9}sin(y+\frac{x}{2}+\frac{3}{2}x)$$

Putting $c=y+\frac{x}{2}$

$\Rightarrow$P.I. $=-\frac{5}{3}x\ cos(y+2x)+\frac{10}{9}sin(y+2x)$

Complete solution is Z= C.F +P.I

$\Rightarrow$ Z $=f_1\left(y+\frac{x}{2}\right)+\ f_2(y+2x)-\frac{5}{3}x\ cos(y+2x)+\frac{10}{9}sin(y+2x)$

### 1.3.2 Non Homogeneous Linear Equations

If in the equation $(D, D')z = F(x, y)$ , the polynomial $f(D, D')$ in $D$, $D'$ is not homogeneous, then it is called a non-homogeneous partial differential equation.

Working Rule to Solve a Non Homogeneous Linear Equation

Step1: Resolve $f(D, D')$ into linear factors of the form

$(D - m_1 D' - a_1)(D - m_2 D' - a_2)\ldots\ldots\ldots(D - m_n D' - a_n)$

Step2: Auxiliary equation is

$(D - m_1 D' - a_1)(D - m_2 D' - a_2)\ldots\ldots\ldots(D - m_n D' - a_n) = 0$

Step3: C.F.= $e^{a_1 x} f_1(y + m_1 x) + e^{a_2 x} f_2(y + m_2 x) + \cdots \ldots \ldots \ldots \ldots + e^{a_n x} f_n(y + m_n x)$

In case of two repeated factors

C.F.= $e^{ax} f_1(y + mx) + x e^{ax} f_2(y + mx)$

Step4: Find P.I. by using usual methods of homogeneous PDE.

Step5: Complete solution is Z = C.F. + P.I.

**Note:** If the Auxiliary equation is of the form

$(D' - m_1 D - a_1)(D' - m_2 D - a_2)\ldots\ldots\ldots(D' - m_n D - a_n) = 0$

Then C.F.= $e^{a_1 y} f_1(x + m_1 y) + e^{a_2 y} f_2(x + m_2 y) + \cdots \ldots \ldots \ldots \ldots + e^{a_n y} f_n(x + m_n y)$

**Example 1.12** Solve $(D^2 - D'^2 - 3D + 3D')z = e^{x+2y}$

Solution: Auxiliary equation is: $(D^2 - D'^2 - 3D + 3D') = 0$

Clearly $D = D'$ is satisfying the equation, $\therefore (D - D')$ is a factor.

Dividing by $(D - D')$ , we get

$(D - D')(D + D' - 3) = 0$

$\Rightarrow (D - D' - 0)(D + D' - 3) = 0$

$\Rightarrow$ C.F. = $f_1(y + x) + e^{3x} f_2(y - x)$

$$\text{P.I} = \frac{e^{x+2y}}{(D^2 - D'^2 - 3D + 3D')}$$

Putting $D = 1,\ D' = 2, f(a,b) = 0$

$$\Rightarrow \text{P.I} = \frac{xe^{x+2y}}{\frac{d}{dD}(D^2 - D'^2 - 3D + 3D')} = \frac{xe^{x+2y}}{2D - 3}$$

Putting D = 1

P.I. $= -xe^{x+2y}$

Complete solution is Z= C.F. + P.I.

$\Rightarrow$ Z = $f_1(y + x) + e^{3x} f_2(y - x) - xe^{x+2y}$

**Example 1.13** Solve $(2DD' + D'^2 - 3D')z = 3\cos(3x - 2y)$

Solution: Auxiliary equation is: $(2DD' + D'^2 - 3D') = 0$

$$.\Rightarrow D'(D' + 2D - 3) = 0$$

$\Rightarrow$ C.F. = $f_1(x) + e^{3y} f_2(x - 2y)$

$$\text{P.I} = \frac{3\cos(3x - 2y)}{(2DD' + D'^2 - 3D')}$$

Putting $DD' = 6\ ,\ D'^2 = -4$

$$\Rightarrow \text{P.I} = \frac{3\cos(3x - 2y)}{(12 - 4 - 3D')} = \frac{3\cos(3x - 2y)}{(8 - 3D')}$$

$$= \frac{3\,(8+3D')\cos(3x-2y)}{(8-3D')(8+3D')}$$

$$= \frac{3\,(8+3D')\cos(3x-2y)}{(64-9D'^2)}$$

$$= \frac{3\,(8+3D')\cos(3x-2y)}{(64+36)}$$

$$= \frac{3}{100}(8+3D')\cos(3x-2y)$$

$$= \frac{3}{100}[8\cos(3x-2y)+6\sin(3x-2y)]$$

$$= \frac{3}{50}[4\cos(3x-2y)+3\sin(3x-2y)]$$

Complete solution is Z= C.F. + P.I.

$\Rightarrow$ Z = $f_1(x) + e^{3y} f_2(x-2y) + \frac{3}{50}[4\cos(3x-2y)+3\sin(3x-2y)]$

## 1.4 Applications of PDEs (Partial Differential Equations)

In this Section we shall discuss some of the most important PDEs that arise in various branches of science and engineering. Method of separation of variables is the most important tool, we will be using to solve basic PDEs that involve wave equation, heat flow equation and laplace equation.

Wave equation (vibrating string) : $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

One- dimensional heat flow (in a rod) : $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Two- dimensional heat flow in steady state (in a rectangular plate ) : $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Note: Two dimension heat flow equation in steady state is also known as laplace equation.

## Working Rule for Method of Separation of Variables

Let $u$ be a function of two independent variables $x$ and $t$.

Step1. Assume the solution to be the product of two functions each of which involves only one variable.

Step 2. Calculate the respective partial derivative and substitute in the given PDE.

Step 3. Arrange the equation in the variable separable form and put LHS = RHS = K

(as both $x$ and $t$ are independent variables)

Step 4. Solve these two ordinary differential equations to find the two functions of $x$ and $t$ alone.

**Example 1.14** Solve the equation $\frac{\partial u}{\partial x} = 4\ \frac{\partial u}{\partial t} + u$ given that $u(x,0) = 5e^{-2x}$

Solution: Step1.

Let $u = XT$ ................ ①

where $X$ is a function of $x$ alone and $T$ be a function of $t$ alone.

Step 2.

$\frac{\partial u}{\partial x} = X'T\ ,\ \frac{\partial u}{\partial t} = XT'$

Substituting these values of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}$ in the given equation

$X'T = 4XT' + XT \qquad \Rightarrow X'T = X(4T' + T)$

$Step\ 3.$

$$\Rightarrow \frac{X'}{X} = \frac{4T'}{T} + 1$$

Putting LHS = RHS = K

Step 4.

i.e. $\frac{X'}{X} = K$

$$\Rightarrow \log X = Kx + \log C_1$$

$$\Rightarrow \log \frac{X}{C_1} = Kx$$

$$\Rightarrow X = C_1 e^{Kx} \ldots\ldots.. ②$$

$$\frac{4T'}{T} + 1 = K$$

$$\frac{T'}{T} = \frac{1}{4}(K-1)$$

$$\Rightarrow \log T = \frac{1}{4}(K-1)t + \log C_2$$

$$\Rightarrow T = C_2 e^{\frac{1}{4}(K-1)t} \ldots\ldots\ldots ③$$

Using ② and ③ in ①

$$u(x,t) = C_1 C_2 e^{Kx} e^{\frac{1}{4}(K-1)t} \ldots\ldots\ldots..④$$

$$\Rightarrow u(x,0) = C_1 C_2 e^{Kx} \ldots\ldots\ldots.. ⑤$$

Given that $u(x,0) = 5e^{-2x}$, using in ⑤

$$\Rightarrow 5e^{-2x} = C_1 C_2 e^{Kx}$$

$$\Rightarrow C_1 C_2 = 5, K = -2 \ldots\ldots\ldots\ldots.⑥$$

Using ⑥in ④

$$u(x,t) = 5e^{-\left(2x + \frac{3t}{4}\right)}$$

1.4.1 Solution of wave equation using method of separation of variables

Wave equation is given by $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \ldots\ldots\ldots\ldots\ldots..$ ①

where $u$ gives displacement at distance $x$ from origin at any time $t$. To solve wave equation using method of separation of variables,

Let $u = XT \ldots\ldots\ldots\ldots\ldots\ldots..$ ②

where $X$ is a function of $x$ alone and $T$ be a function of $t$ alone.

$$\therefore \frac{\partial^2 u}{\partial t^2} = XT'', \frac{\partial^2 u}{\partial x^2} = X''T$$

Substituting these values of $\frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x^2}$ in the wave equation given by ①

$XT'' = c^2 X'' T$

Arranging in variable separable form$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$

Equating LHS = RHS = K ($\because X$ *and* $T$ *are independent*)

$\Rightarrow \frac{X''}{X} = K$ and $\frac{1}{c^2}\frac{T''}{T} = K$

$\Rightarrow X'' - KX = 0$ and $T'' - KC^2T = 0$ ................ ③

Solving ordinary differential equations given in ③, three cases arise

(i) When $K$ is +ve and = $p^2$say

$$X = c_1e^{px} + c_2e^{-px},\ T = c_3e^{cpt} + c_4e^{-cpt}$$

(ii) When $K$ is -ve and = $-p^2$say

$$X = c_1 cospx + c_2 sinpx,\ T = c_3 coscpt + c_4 sincpt$$

(iii)When $K$ = 0

$$X = c_1x + c_2,\ T = c_3t + c_4$$

Again since we are dealing with wave equation, $u$ must be a periodic function of $x$and $t$, ∴ the solution must involve trigonometric terms. Hence the solution given by (ii) i.e. corresponding to $K = -p^2$, is the most plausible solution, substituting (ii) in equation ②

$u(x,t) = (c_1 cospx + c_2 sinpx)\,(c_3 coscpt + c_4 sincpt)$ ........ ④

Which is the required solution of wave equation.

Again if we consider a string of length $l$ tied at both ends at $x = 0$ and $x = l$,

then displacement of the string at end points at any time $t$ is zero .

$\Rightarrow u(0,t) = 0$ ........... ⑤

and$u(l,t) = 0$ ...........⑥

using ⑤in ④$\Rightarrow 0 = C_1(c_3 coscpt + c_4 sincpt)$

$\Rightarrow C_1 = 0$ ................ ⑦

Using ⑦ in ④ , wave equation reduces to

$u(x,t) = c_2 sinpx\,(c_3 coscpt + c_4 sincpt)$ ........ ⑧

Now using ⑥ in ⑧⇒ $0 = c_2 sinpl\,(c_3 coscpt + c_4 sincpt)$

$\Rightarrow \sin pl = 0 \quad \because c_2 \neq 0$ and $(c_3 coscpt + c_4 sincpt) \neq 0$

$\Rightarrow pl = n\pi \Rightarrow p = \frac{n\pi}{l}$ , $n = 1,2,3 \ldots \ldots \ldots$--------⑨

using ⑨ in ⑧⇒ $u(x,t) = c_2 sin\frac{n\pi x}{l}\,(c_3 \cos\frac{n\pi ct}{l} + c_4 sin\frac{n\pi ct}{l})$

Adding up the solutions for different values of $n$, we get

$u(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi ct}{l} + b_n sin\frac{n\pi ct}{l}\right) sin\frac{n\pi x}{l}$.........⑩

⑩ is also a solution of wave equation

**Example 1.15 :** A string is stretched and fastened to 2 points $l$ apart. Motion is started by displacing the string in the form $y = a\sin\frac{\pi x}{l}$ from which , it is released at time $t = 0$. Show that the displacement at any point at a distance $x$ from one end at time $t$ is given by

$$y(x,t) = a\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}$$

Solution: Let the equation of vibrating string be given by $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ ...... ①

Boundary value conditions are given by :

$y(0,t) = 0$ ............②

$y(l,t) = 0$ ............③

$\frac{\partial y}{\partial t}_{t=0} = 0$............④

$y(x,0) = a\sin\frac{\pi x}{l}$......⑤

Let the solution of ①be given by

$y(x,t) = (c_1 cospx + c_2 sinpx)\,(c_3 coscpt + c_4 sincpt)$......⑥

Using ② in ⑥

$\Rightarrow 0 = C_1(c_3 coscpt + c_4 sincpt)$

$\Rightarrow C_1 = 0$ ................ ⑦

Using ⑦ in ⑥ , wave equation reduces to

$y(x,t) = c_2 sinpx\,(c_3 coscpt + c_4 sincpt)$ …….. ⑧

Now using ③ in⑧⇒ $0 = c_2 sinpl\,(c_3 coscpt + c_4 sincpt)$

$\Rightarrow \sin pl = 0 \quad \because c_2 \neq 0$ and $(c_3 coscpt + c_4 sincpt) \neq 0$

$\Rightarrow pl = n\pi \Rightarrow p = \frac{n\pi}{l}$ -------- ⑨

using ⑨ in ⑧⇒ $y(x,t) = c_2 sin\frac{n\pi x}{l}\,(c_3 \cos\frac{n\pi ct}{l} + c_4 sin\frac{n\pi ct}{l})$……..⑩

Now to use ④, differentiating ⑩ partially w.r.t. $t$

$\Rightarrow \frac{\partial y}{\partial t} = c_2 sin\frac{n\pi x}{l}\,(-c_3\frac{n\pi c}{l}\sin\frac{n\pi ct}{l} + c_4\frac{n\pi c}{l}cos\frac{n\pi ct}{l})$

Putting $t = 0$

$\frac{\partial y}{\partial t}_{t=0} = c_2 sin\frac{n\pi x}{l}\left(c_4\frac{n\pi c}{l}\right) = 0$ using④

$\Rightarrow c_4 = 0 \quad \because c_2 \neq 0$ ·········⑪

Using ⑪ in ⑩

$\Rightarrow y(x,t) = c_2 c_3 sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$……..⑫

Using ⑤ in ⑫

$\Rightarrow a\sin\frac{\pi x}{l} = c_2 c_3 sin\frac{n\pi x}{l}$

$\Rightarrow c_2 c_3 = a\,, n = 1$

using in ⑫

$\Rightarrow y(x,t) = a sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}$

Note : Above example can also be solved using solution of wave equation given by equation ⑩ in section 1.4.1

$$y(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi ct}{l} + b_n sin\frac{n\pi ct}{l}\right)sin\frac{n\pi x}{l}$$

It is to be noted that boundary value conditions ②and ③have been already used in this solution.

**Example 1.16** A tightly stretched string with fixed end points at $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a velocity $\lambda x\,(l - x)$, find the displacement of the string at any distance $x$ from one end at any time $t$.

Solution: Let the equation of vibrating string be given by $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ …… ①

Boundary value conditions are given by :

$u(0,t) = 0$ …………②

$u(l,t) = 0$ ……..…..③

$u(x,0) = 0$ ……..…..④

$\frac{\partial u}{\partial t}_{t=0} = \lambda x\,(l - x)$………⑤

solution of wave equation as given by equation ⑩in section 1.4.1 is

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \text{ …..⑥}$$

It is to be noted that boundary value conditions ②and ③ have been already used in this solution.

Now using ④ in ⑥

$$0 = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

$$\Rightarrow a_n = 0 \quad \cdots\cdots\cdots \quad ⑦$$

Using ⑦ in ⑥

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi ct}{l} \sin\frac{n\pi x}{l} \quad \cdots\cdots\cdots\cdots \quad ⑧$$

Now to use ⑤, differentiating ⑧ partially w.r.t.$t$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{l} \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}$$

Putting $t = 0$,

$$\Rightarrow \frac{\partial u}{\partial t}_{t=0} = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \sin \frac{n\pi x}{l} \text{ ..........⑨}$$

Using ⑤ in ⑨

$$\lambda x\,(l-x) = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \sin \frac{n\pi x}{l}$$

Multiplying both sides by $\sin \frac{n\pi x}{l}$ and integrating w.r.t. $x$ within the limits $0$ to $l$

$$\Rightarrow \int_0^l \lambda x\,(l-x) \sin \frac{n\pi x}{l} dx = \frac{\pi c}{l} n b_n \int_0^l \sin^2 \frac{n\pi x}{l} dx$$

$$= \frac{\pi c}{2l} n b_n \int_0^l (1 - \cos \frac{2n\pi x}{l}) dx$$

$$= \frac{\pi c}{2} n b_n$$

$$\Rightarrow \pi c n b_n = 2 \int_0^l \lambda x\,(l-x) \sin \frac{n\pi x}{l} dx$$

$$= 2\lambda \int_0^l (lx - x^2) \sin \frac{n\pi x}{l} dx$$

$$\Rightarrow \pi c n b_n = 2\lambda \left[ (lx - x^2).\left(\frac{-l}{n\pi} \cos \frac{n\pi x}{l}\right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3} \cos \frac{n\pi x}{l}\right)\right]_0^l$$

$$\Rightarrow \pi c n b_n = 2\lambda \left[ \frac{-2l^3}{n^3\pi^3} \cos n\pi + \frac{2l^3}{n^3\pi^3} \right]$$

$$\Rightarrow \pi c n b_n = \frac{4\lambda l^3}{n^3\pi^3} [1 - (-1)^n]$$

$$\Rightarrow \pi c n b_n = \begin{bmatrix} \frac{8\lambda l^3}{n^3\pi^3} & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{bmatrix}$$

taking $n = 2m - 1$

$$b_n = \frac{8\lambda l^3}{C\pi^4(2m-1)^4} \ldots \ldots \ldots ⑩$$

using⑩ in ⑧, required solution is

$$u(x,t) = \frac{8\lambda l^3}{C\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^4} sin\frac{(2m-1)\pi ct}{l} sin\frac{(2m-1)\pi x}{l}$$

## 1.4.2 Solution of heat equation using method of separation of variables

One dimensional heat flow equation is given by $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ ................. ①

where $u(x,t)$ is temperature function at distance $x$ from origin at any time $t$. To solve heat equation using method of separation of variables,

Let $u = XT$ ...................... ②

where $X$ is a function of $x$ alone and $T$ be a function of $t$ alone.

$$\therefore \frac{\partial u}{\partial t} = XT', \frac{\partial^2 u}{\partial x^2} = X''T$$

Substituting these values of $\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}$ in the heat equation given by ①

$$XT' = c^2 X''\mathrm{T}$$

Arranging in variable separable form $\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T}$

Equating LHS = RHS = K $\quad (\because X\ and\ T\ are\ independent)$

$$\Rightarrow \frac{X''}{X} = K \text{ and } \frac{1}{c^2}\frac{T'}{T} = K$$

$$\Rightarrow X'' - KX = 0 \text{ and } T' - Kc^2T = 0 \ldots\ldots\ldots\ldots\ldots ③$$

Solving ordinary differential equations given in ③, three cases arise

(i) When $K$ is +ve and = $p^2$say

$X'' - p^2X = 0$and$T' = p^2c^2T$

$$X = Ae^{px} + Be^{-px}, T = Ce^{p^2c^2t}$$

$$\Rightarrow u(x,t) = (Ae^{px} + Be^{-px})Ce^{p^2c^2t}$$

$$\Rightarrow u(x,t) = ((c_1e^{px} + c_2e^{-px})e^{p^2c^2t}$$

(ii) When $K$ is −ve and = $-p^2$say

$$X = Acospx + Bsinpx, T = Ce^{-c^2p^2t}$$

$$\Rightarrow u(x,t) = (Acospx + Bsinpx)\ Ce^{-c^2p^2t}$$

$$\Rightarrow u(x,t) = (c_1cospx + c_2sinpx)\ e^{-c^2p^2t}$$

(iii)When $K$ = 0

$$X'' = 0\text{and}T' = 0$$

$$\Rightarrow X = Ax + B, T = C$$

$$\Rightarrow u(x,t) = (Ax + B)C$$

$$\Rightarrow u(x,t) = c_1x + c_2$$

The solution given by (ii) i.e. corresponding to $K = -p^2$, is the most plausible solution for steady state.

**<u>Special case</u>:** When the ends of a rod are kept at $0^\circ$C

One dimensional heat equation in steady state is given by :

$u(x,t) = (c_1cospx + c_2sinpx)\ e^{-c^2p^2t}$ …………..④

Also since ends of a rod are kept at $0^\circ$C

$$u(0,t) = 0 \text{ …………⑤}$$

$$u(l,t) = 0 \text{ ………..…..⑥}$$

Using ⑤in ④⇒ 0 = $C_1e^{-c^2p^2t}$

⇒ $C_1 = 0$ ……………. ⑦

Using ⑦ in ④, wave equation reduces to

$u(x,t) = c_2 sinpxe^{-c^2p^2t}$........ ⑧

Now using ⑥ in ⑧⇒ $0 = c_2 sinple^{-c^2p^2t}$⇒$\sin pl = 0$

$\because c_2 \neq 0$ and $e^{-c^2p^2t} \neq 0$

⇒$pl = n\pi$⇒$p = \frac{n\pi}{l}$ , $n = 1,2,3\ldots\ldots\ldots$⑨

using ⑨ in⑧⇒ $u(x,t) = c_2 sin\frac{n\pi x}{l} e^{\frac{-c^2n^2\pi^2t}{l^2}}$

Adding up the solutions for different values of $n$, the most general solution is given by

$u(x,t) = \sum_{n=1}^{\infty} b_n sin\frac{n\pi x}{l} e^{\frac{-c^2n^2\pi^2t}{l^2}}$.........⑩

**Example 1.17 :**A rod of length $l$ with insulated sides is initially at a uniform temperature $u_0$. Its ends are suddenly cooled to $0^\circ$C and are kept at that temperature, find the temperature formula $u(x,t)$.

Solution: One dimensional heat flow equation is given by $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Solution of one dimensional heat equation in steady state is given by :

$u(x,t) = (c_1 cospx + c_2 sinpx)\, e^{-c^2p^2t}$ ..............①

$u(0,t) = 0$ ............②

$u(l,t) = 0$ ..............③

Also initial condition is $u(x,0) = u_0$ ..........④

The most general solution of heat equation① using ②and③ is given by

$u(x,t) = \sum_{n=1}^{\infty} b_n sin\frac{n\pi x}{l} e^{\frac{-c^2n^2\pi^2t}{l^2}}$......... ⑤

Using ④in ⑤⇒ $u_0 = \sum_{n=1}^{\infty} b_n sin\frac{n\pi x}{l}$ .........⑥

Multiplying both sides of ⑥ by $sin\frac{n\pi x}{l}$, and integrating w.r.t. $x$ within the limits $0$ to $l$

$$\int_0^l u_0 \, sin\frac{n\pi x}{l}dx = b_n \int_0^l sin^2\frac{n\pi x}{l}dx$$

$$= b_n \frac{l}{2}$$

$$\Rightarrow b_n = \frac{2}{l}\int_0^l u_0 \; sin\frac{n\pi x}{l}dx$$

$$= \frac{-2\,u_0}{l}\left[\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_0^l$$

$$\Rightarrow b_n = \frac{2\,u_0}{n\pi}[1-(-1)^n] = \begin{bmatrix} \frac{4\,u_0}{n\pi}, & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{bmatrix}$$

Putting $n = 2m-1, \Rightarrow\ b_n = \frac{4\,u_0}{(2m-1)\pi}$ …………⑦

Using ⑦ in ⑤, the required temperature formula is

$$u(x,t) = \frac{4\,u_0}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1} sin\frac{(2m-1)\pi x}{l} e^{\frac{-c^2(2m-1)^2\pi^2 t}{l^2}}$$

### 1.4.3 Solution of laplace equation (two dimensional heat flow) using method of separation of variables

Consider the heat flow in a uniform rectangular metal plate at any time $t$; if $u(x,y)$ be the temperature at time $t$, two dimensional heat flow equation is given by

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

In steady state, $u$ doesn't change with $t$ and hence $\frac{\partial u}{\partial t} = 0$

∴ Two dimensional heat flow equation in steady state is given by $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$………①

where $u(x,y)$is temperature function at any point $(x,y)$ of the rectangular metal plate. This is called Laplace equation in two dimensions. To solve Laplace equation using method of separation of variables,

Let $u = XY$ ………………….. ②

where $X$ is a function of $x$ alone and $Y$ be a function of $y$ alone.

$$\therefore \frac{\partial^2 u}{\partial x^2} = X''Y, \quad \frac{\partial^2 u}{\partial y^2} = XY''$$

Substituting these values of $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ in the Laplace equation given by ①

$X''Y + XY'' = 0$

Arranging in variable separable form$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y}$

Equating LHS = RHS = K $\quad (\because x \text{ and } y \text{ are independent})$

$\Rightarrow \frac{X''}{X} = K$ and $-\frac{Y''}{Y} = K$

$\Rightarrow X'' - KX = 0$ and $Y'' + KY = 0$ ................ ③

Solving ordinary differential equations given in ③, three cases arise

(i) When $K$ is +ve and = $p^2$ say

$$X = c_1e^{px} + c_2e^{-px}, Y = c_3\cos py + c_4\sin py$$

$$\Rightarrow u(x,y) = (c_1e^{px} + c_2e^{-px})(c_3\cos py + c_4\sin py)$$

(ii) When $K$ is $-$ve and = $-p^2$say

$$X = (c_1\cos px + c_2\sin px, Y = (c_3e^{py} + c_4e^{-py})$$

$$\Rightarrow u(x,y) = (c_1\cos px + c_2\sin px)(c_3e^{py} + c_4e^{-py})$$

(iii)When $K$ = 0

$$X'' = 0 \text{ and } Y'' = 0$$

$$\Rightarrow X = c_1x + c_2, Y = c_3y + c_4$$

$$\Rightarrow u(x,t) = (c_1x + c_2)(c_3y + c_4)$$

The solution given by (ii) i.e. corresponding to $K = -p^2$, is the most plausible solution for steady state.

**Example 1.18** An infinitely long rectangular uniform plate with breadth $\pi$ is bounded by two parallel edges maintained at $0^\circ$C. Base of the plate is kept at a temperature $u_0$ at all points. Determine the temperature at any point of the plate in the steady state.

Solution : In steady state, two dimensional heat flow equation is given by$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$....①

Boundary value conditions are

$u(0, y) = 0$…………②

$u(\pi, y) = 0$…………③

$\lim_{y\to\infty} u(x, y) = 0 \;, \quad 0 < x < \pi$……… ④

$u(x, 0) = u_0 \;, \quad 0 < x < \pi$……………⑤

Solution of ① is given by :

$u(x, y) = (c_1 cospx + c_2 sinpx)(c_3 e^{py} + c_4 e^{-py})$……….⑥

Using ② in ⑥

$\Rightarrow u(0, y) = 0 = c_1(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow c_1 = 0$………. ⑦

Using ⑦ in ⑥

$u(x, y) = c_2 sinpx(c_3 e^{py} + c_4 e^{-py})$………….⑧

Using ③ in ⑧

$\Rightarrow u(\pi, y) = 0 = c_2 sinp\pi(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow sinp\pi = 0$

$\Rightarrow p\pi = n\pi$

$\Rightarrow p = n$ …………⑨

Using ⑨ in ⑧

$\Rightarrow u(x, y) = c_2 sinnx(c_3 e^{ny} + c_4 e^{-ny})$ ……….⑩

Using ④ in ⑩

$\lim_{y\to\infty} u(x, y) = 0 = c_2 sinnx \lim_{y\to\infty}(c_3 e^{ny} + c_4 e^{-ny})$

$\Rightarrow c_3 = 0$ ……… ⑪

Using ⑪ in ⑩

$\Rightarrow u(x, y) = c_2 c_4 sinnxe^{-ny}$

The most general solution of heat equation is given by

$u(x, t) = \sum_{n=1}^{\infty} b_n sin\, nx.\, e^{-ny}$……… ⑫where $c_2 c_4 = b_n$

Using ⑤ in ⑩

$\Rightarrow u(x,0) = u_0 = \sum_{n=1}^{\infty} b_n \sin nx$

Multiplying both sides by $\sin nx$, and integrating w.r.t. $x$ within the limits $0$ to $\pi$

$$b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx \, dx$$

$$\Rightarrow b_n = \frac{2u_0}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \begin{bmatrix} \frac{4\,u_0}{n\pi}, & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{bmatrix} \ldots\ldots ⑬$$

Let $n = 2m - 1$ as $n$ is odd

Using ⑬ in ⑫ the required temperature formula is:

$$u(x,t) = \frac{4\,u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2m-1)}\sin(2m-1)x.e^{-(2m-1)y}$$