English/Latin Scroll Mathematical Olympiad

Annus: **MMXVIII**

20$^{\text{th}}$ ELSMO
PITTSBURGH, PA

Dies: **I**

V. Ide. Iun. A.D. MMXVIII
hora VI — hora IXS

Quaesitum Primum. Numerus integer positivus n sit. Urbes MMXVIIIn+I in regnum Arabiae Sellkei sunt. Rex Marcus vias quae commercium ab utro cursu accipiunt et quae quosdam duo urbes committent sic statuit ut quisque urbs quam "C" appellatur et quisque numerus integer i quae plus aequalisve I et minus aequalisve MMXVIII est subtiliter n urbes quae ab spatio i ex C dividetur habeat. Quis n Marcum sinet ut hoc opus attingat?

N.B. *Spatium* inter duo urbes est minimum numerum viae in quodquam iter quod inter idem duo urbes est.

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Problem 1. Let n be a positive integer. There are $2018n + 1$ cities in the Kingdom of Sellke Arabia. King Mark wants to build two-way roads that connect certain pairs of cities such that for each city C and integer $1 \le i \le 2018$, there are exactly n cities that are a distance i away from C. (The *distance* between two cities is the least number of roads on any path between the two cities.)

For which n is it possible for Mark to achieve this?

Horas IIIS habes. (You have 4 hours and 30 minutes.)
Quodque quaesitum septem denarii precium est. (Each problem is worth 7 points.)
Explicationes Latine Angliceve scribi liceat. (You may write solutions in Latin or English.)

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Quaesitum Secundum. $a_1, a_2, \cdots$ sequentia infinitiva positivorum numerorum integrorum sic sit ut $a_1 = 1$ et

$$a_n \mid a_k + a_{k+1} + \cdots + a_{k+n-1}$$

quique positivi integri k, n. Decerne quem maximum numerum esse a_{2m} possit.

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Problem 2. Consider infinite sequences $a_1, a_2, \cdots$ of positive integers satisfying $a_1 = 1$ and

$$a_n \mid a_k + a_{k+1} + \cdots + a_{k+n-1}$$

for all positive integers k and n. For a given positive integer m, find the maximum possible value of a_{2m}.

Horas IIIS habes. (You have 4 hours and 30 minutes.)
Quodque quaesitum septem denarii precium est. (Each problem is worth 7 points.)
Explicationes Latine Angliceve scribi liceat. (You may write solutions in Latin or English.)

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Quaesitum Tertium. Punctum A in planitie sit, et linea ℓ, in qua A non inest, in eiusdem planitie sit. Evanus regulam non habet, sed autem circinum peculiarem habet, qui circulum per tria puncta pingere possit, si hic circulus exstet. (Circinus centrum huius circuli non instruit.) Quoque, Evanus decussationes inter duo obiectes instruere potest, et Evanus punctum arbitrarium in obiecto certo instruere potest.

- Evanusne reflexionem de A trans ℓ instruere* potest?
- Evanusne instruere punctum in linea ℓ, linea per quem lineae ℓ perpendicularis sit, potest?

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Problem 3. Let A be a point in the plane, and ℓ a line not passing through A. Evan does not have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is *not* marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.

(i) Can Evan construct† the reflection of A over ℓ?

(ii) Can Evan construct the foot of the altitude from A to ℓ?

*Evano algorithmus determinabilis, qui punctum pingit, habendus est ut punctum instruat.

†To construct a point, Evan must have an algorithm which marks the point in finitely many steps.

Horas IIIS habes. (You have 4 hours and 30 minutes.)
Quodque quaesitum septem denarii precium est. (Each problem is worth 7 points.)
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XVI. Kal. Iul. A.D. MMXVIII
hora VI — hora IXS

Quaesitum Quaternum. ABC triangulum est, et orthocentrum circumcentrumque huius trianguli H, O in illo ordine sunt. Punctum P medium punctum segmenti lineae $\overline{AH}$ est, atque punctum T in linea BC sic est ut $\angle TAO$ angulum rectum sit. Punctum X in linea PT sic est ut linea OX lineae PT perpendicularis sit. Proba ut medium punctum segmenti lineae PX circulo novem punctorum ABC insit. *

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Problem 4. Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90^\circ$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle of $\triangle ABC$. †

*Circulum novem punctorum ΔABC circulum per media puncta segmentorum linearum AB, BC, CA, AH, BH, CH est.

†The nine-point circle of ΔABC is the unique circle passing through the following nine points: the midpoint of the sides, the feet of the altitudes, and the midpoints of $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$.

Horas IIIS habes. (You have 4 hours and 30 minutes.)
Quodque quaesitum septem denarii precium est. (Each problem is worth 7 points.)
Explicationes Latine Angliceve scribi liceat. (You may write solutions in Latin or English.)

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Quaesitum Quintum. $a_1, a_2, \cdots, a_m$ sequentia definitiva positivorum numerorum integrorum sit. Proba quod nonnegativi integri b, c, N sic exstent ut, quique positivus integrus $n > N$,

$$\left\lfloor \sum_{i=1}^{m} \sqrt{n + a_i} \right\rfloor = \left\lfloor \sqrt{bn + c} \right\rfloor.$$

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Problem 5. Let $a_1, a_2, \cdots, a_m$ be a finite sequence of positive integers. Prove that there exist nonnegative integers b, c, and N such that

$$\left\lfloor \sum_{i=1}^{m} \sqrt{n + a_i} \right\rfloor = \left\lfloor \sqrt{bn + c} \right\rfloor.$$

holds for all integers $n > N$.

Horas IIIS habes. (You have 4 hours and 30 minutes.)
Quodque quaesitum septem denarii precium est. (Each problem is worth 7 points.)
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Quaesitum Sextum. *Ventimolina* segmentum clausum lineae longitudine assis ex quo unus terminus, qui *cnodax* appellatur, cernitur est. S copia definitiva sic sit ut spatium inter quaeque duo puncta in S amplium quam c sit. Deformatio n ventimolinis *patricia* appelletur si quaeque duae ventimolinae non decussent et quodque punctum in S cnodax ventimolinae semel subiliter fiat.

Deformatio patricia ventimolinarum in planitie Evano in initio datur. Per unam operationem, Evanus ullam ventimolinam circum eius cnodacem rotare possit, ad utram partem, si quaeque duae ventimolinae per actionem non decussent. Demonstra ut, si

(i) $c = \sqrt{3}$,

(ii) $c = \sqrt{2}$.

Evanus, per operationes numerorum finitorum, ullam aliam patriciam deformationem instruere possit.

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Problem 6. A *windmill* is a closed line segment of unit length with a distinguished endpoint, the *pivot*. Let S be a finite set of n points such that the distance between any two points of S is greater than c. A configuration of n windmills is *admissible* if no two windmills intersect and each point of S is used exactly once as a pivot.

An admissible configuration of windmills is initially given to Evan in the plane. In one operation Evan can rotate any windmill around its pivot, either clockwise or counterclockwise and by any amount, as long as no two windmills intersect during the process. Show that Evan can reach any other admissible configuration in finitely many operations, where

(i) $c = \sqrt{3}$,

(ii) $c = \sqrt{2}$.

Horas IIIS habes. (You have 4 hours and 30 minutes.)
Quodque quaesitum septem denarii precium est. (Each problem is worth 7 points.)
Explicationes Latine Angliceve scribi liceat. (You may write solutions in Latin or English.)