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7 Analytic Geometry

7.1 Coordinates of points

Suppose (x_1, y_1) and (x_2, y_2) are point in a plane.

The distance between (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The midpoint of (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right)$$

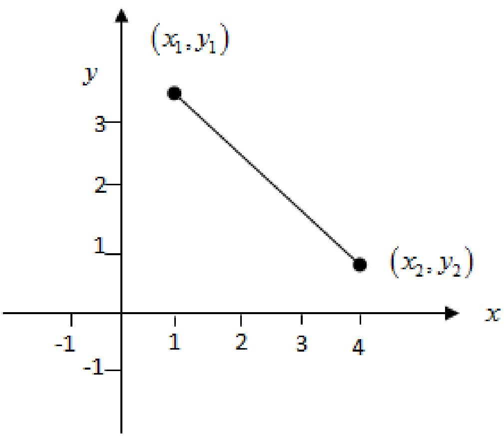


Fig7.1

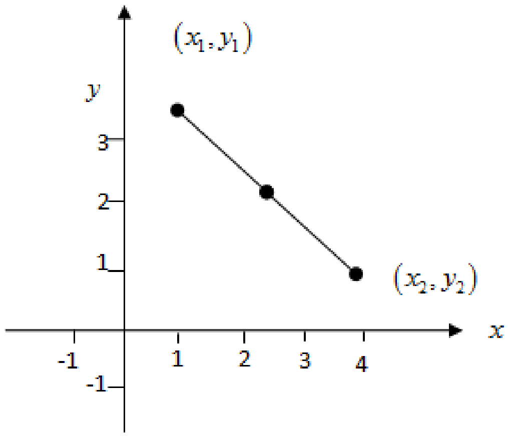


Fig7.2

7.1.1 Examples

1. Find the distance between $(3,4)$ and $(-1,8)$.

Solution

Use the formula above , being careful with minus signs.

$$\text{Distance} = \sqrt{(3-(-1))^2 + (4-8)^2}$$
$$= \sqrt{32}$$
$$= 5.66$$

2. Find the midpoint of the line joining $(3,4)$ and $(-1,8)$.

Solution

Use the formula above

$$\left(\frac{1}{2}(3+-1),\frac{1}{2}(4+8)\right)=(1,6)$$

7.1.2 Exercise

1. Find the distance between the following pairs of points:

   (a) $(2,1)$ and $(4,7)$ (b) $(0,0)$ and $(2,5)$

   (c) $(4,6)$ and $(2,9)$ (d) $(2,-3)$ and $(-2,8)$

   (e) $(-3,-2)$ and $(1,5)$ (f) $(2,-6)$ and $(-5,-3)$

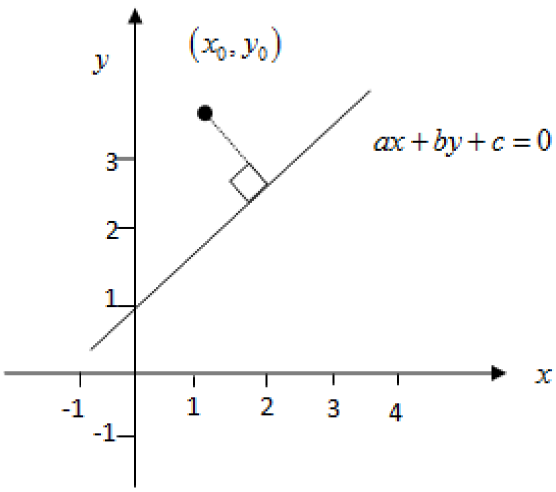
2. By finding the lengths of its sides show that the quadrilateral formed by the points $(1,1),(4,5),(4,10)$ and $(1,6)$ is a rhombus.
3. Show that the points $(1,1),(4,3)$ and $(-1,4)$ form a right-angled triangle. What is its area?
4. Find the midpoints of the pairs of points in Question 1.
5. By showing that its diagonals bisect each other, show that the quadrilateral formed by the points $(1,1),(3,5),(9,7)$ and $(7,3)$ is a parallelogram.
6. Is the quadrilateral of Question 6 a rhombus?
7. Show that points $(0,0),(m,1),(-1,m)$ form a right-angled triangle.
8. Show that the points $(x,y),(x+\lambda,y+\mu),(x+\mu,y-\lambda)$ form a right –angled triangle.

7.2 Equations of straight lines

The gradient of a straight line is the ratio of its y-change to its x-change.

The equation of a straight line can be written in the following forms :

- $ax+by+c=0$, where a,b,c are constants.
- $y=mx+c$, where m is the gradient and $(0,c)$ the intercept on the y-axis.



- $y-y_0=m(x-x_0)$, where the gradient is m and the line goes through (x_0,y_0).
- $\frac{x}{a}+\frac{y}{b}=1$, where $(a,0)$ and $(0,b)$ are the intercepts on the x and y axes respectively.

Fig 7.3

$y = mx + c$ and $y = nx + d$ are parallel if $m = n$.

$y = mx + c$ and $y = nx + d$ are perpendicular if $m \times n = -1$.

7.2.1 Examples

1. Find the equation of the line through $(1,4)$ and $(2,6)$.

   Solution

   The ratio of the y-change to the x-change is $6-4 : 2-1$.Use the third form of the equation of a line.

   $$y - 4 = 2 \times (x - 1)$$
   $$y = 2x + 2$$

2. A triangle has its vertices at $A(6,1), B(1,6)$ and $(-3,-2)$.Find the equations of the perpendiculars of BC and AB.Find the coordinates of the point D where these lines meets, and verify that D is the circumcentre of the triangle, i.e. is the same distance from A, B and C.

   Solution

   The midpoint of BC is $(-1,2)$.The gradient of BC is 2, hence the gradient of the perpendicular is $-\frac{1}{2}$.

   The equation of the perpendicular is therefore :

   $$y = -\frac{1}{2}x + \frac{3}{2}.$$

   Similarly, the equations of the perpendicular bisector of AB is :

   $$y = x$$

   Solve these equations simultaneously :

   D is at $(1,1)$

   Use the definition of distance in 7.1:

   $$AD = BD = CD = 5$$

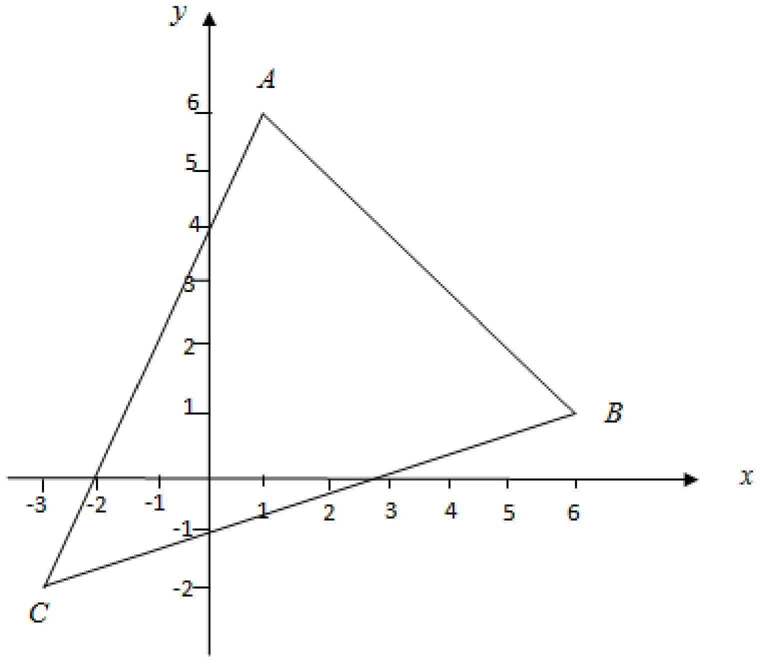


Fig 7.4

7.2.2 Exercise

1. Find the equations of the following straight lines.

(a) Through $((1,1)$ and $(2,5)$

(b) Through $(2,4)$ and $(5,5)$

(c) Through $(2,3)$ and $(3,1)$

(d) Through $(6,3)$ and $(2,6)$

(e) With gradient and 2 through $(3,4)$

(f) With gradient and $\frac{1}{2}$ through $(2,1)$

(g) With gradient and -3 through $(1,5)$

(h) With gradient and $-\frac{3}{4}$ through $(1,-3)$

(i)Through $(2,3)$, parallel to $y=4$

(j) Through $(2,1)$, parallel to $y=2-x$

(k) Through $(5,7)$, perpendicular to $y=x-3$

(l) Through $(2,-3)$, perpendicular to $y=2x+1$

2. Find the perpendicular distances between the following points and lines.

(a) $(1,2)$ and $2y+3X-1=0$

(b) $(3,4)$ and $x+y+3=0$

(c) $(2,3)$ and $y=2x-4$

(d) $(0,0)$ ans $2y=3x-1$

3. Show that opposite sides of the quadrilaterals in Questions 2 and 6 of $7.1.2$ are parallel.

4. Show that the diagonals of the quadrilaterals in Questions 2 of $7.1.2$ are perpendicular.

5. Show that the quadrilateral formed by the points $(1,1),(5,5),(8,2)$ and $(4,-2)$ is a rectangle. Is it a square?
6. By the method of Question 2 of 7.2.1 find the circumcentre of the triangle formed by $(1,4),(0,1)$ and $(4,1)$. Find the radius of the circumcircle.
7. ABC is the triangle .Whose vertices are $(2,3),(1,-1),(4,3)$. Find the equations of the sides. Find the equation lines through A,B,C perpendicular to BC,AC,AB respectively. Find the point where these three lines intersect.(The orthocenter of ΔABC).

7.3 Circles

If a circle has radius r and centre (h,k), then its equation is :

$$(x-h)^2+(y-k)^2=r^2$$

7.3.1 Examples

1. Find the centre and radius of the circle with equation:
   $x^2+y^2+2x-4y-15=0$

   Find the equation of the tangent at $(1,-2)$.

   Solution

   Complete the square for x and y.

   $(x+1)^2+(y-2)^2=1+4+15=20$

The centre is at $(-1,2)$ and the radius is $\sqrt{20}$

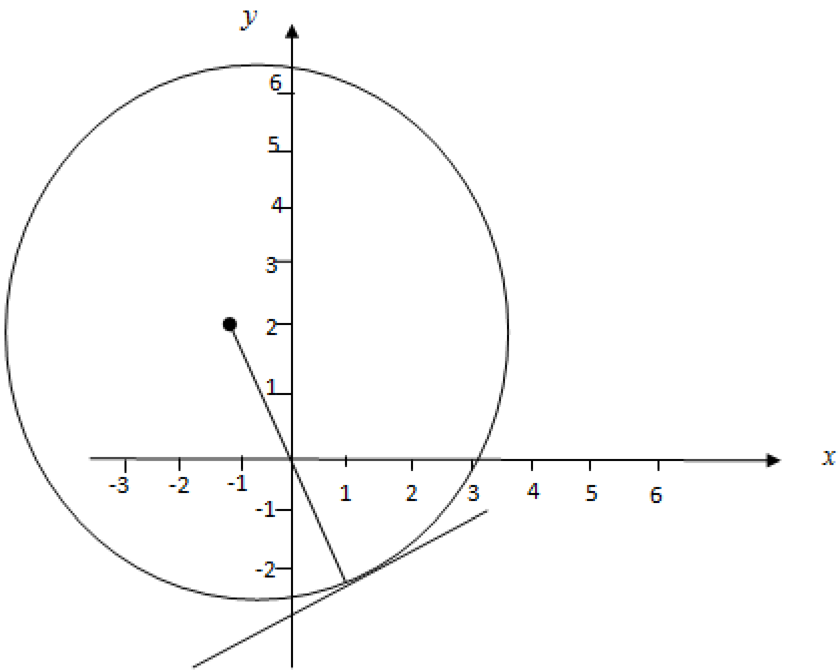


Fig7.5

The radius form the centre to $(1,-2)$ has gradient -2. Hence the gradient of the tangent is $\frac{1}{2}$.

The tangent has equation $y+2=\frac{1}{2}(x-1)$.

The equation is $y=\frac{1}{2}x-\frac{5}{2}$

2. A is at $(0,3)$ and B is at $(6,0)$. P is such that $PA=2PB$.Show that the locus of P is a circle, and find its centre and radius .

Solution

Let P be at (x,y).Use the formula for distance:

$PA=\sqrt{(x^2+(y-3)^2)}$ $PB=\sqrt{((x-6)^2+y^2)}$

Use the fact that $PA=2PB$, and hence $PA^2=4PB^2$:

$$x^2+(y-3)^2=4((x-6)^2+y^2)$$

Expand and collect like terms.

$$0=3x^2+3y^2-48x+6y+135$$

$$x^2+y^2-16x+2y+45=0$$

$$(x-8)^2+(y+1)^2=20$$

The locus is a circle , with centre $(8,-1)$ and radius $\sqrt{20}$

7.3.2 Exercises

1. Write down the equations of the following circles:

   (a) centre $(1,2)$ and radius 5

   (b) centre $(1,-3)$ and radius 8

2. Find the centre and radii of the following circles:

   (a) $x^2+y^2+4x-2y-3=0$

   (b) $x^2+y^2-3x-y+1=0$

   (c) $x^2+y^2-6y+2=0$

   (d) $x^2+y^2+3x=0$

3. Find the equation of the circle with centre $(1,3)$ which goes through the point $(2,7)$.
4. Find the equation of the circle whose diameter is the between $(1,4)$ and $(-3,6)$.
5. Find the distance from $(-1,1)$ to the line $3x+4y=2$,Hence find the equation of the circle with centre $(-1,1)$ which touches $3x+2y=2$.
6. Find the equation of the circle which has centre $(2,3)$ and which touches the line $x-3y+5=0$.
7. Find the equation of the tangent to $x^2+y^2+2x-6y+2=0$ at the point $(1,1)$.

8. Find where the circle $x^2+y^2+3x+1=0$ cuts the x-axis .Find the equations of the tangents at these points.
9. Find the points where the circle $x^2+y^2+3x+1=0$ cuts the line $y=x$.Find the equation of the tangents at these points.
10. P moves so that its distance from the origin is twice its distance from $(3,-9)$.Prove that P moves in a circle , and find its centre and radius .
11. The centre of a circle is on the line $x=3$, and both the axes are tangents to the circle. Find the equation of the circle.
12. A is at $(2,-3)$ and B is at $(4,5)$. P moves so that PA is perpendicular to PB.. Show that P moves in a circle , and find its centre and radius .
13. With A and B as in Question 12, let P moves so that $PA^2+PB^2=100$. Show that the locus of P is a circle , and find its centre and radius .
14. Let a circle have centre at $C(1,1)$ and radius 3.Find the distance from $P(4,5)$ to C .Does P lie inside or outside the circle? Find the length of the tangent from P to the circle.
15. Find the values of m so that $y=mx$ is a tangent to the circle $x^2+y^2+3x-2y+2=0$.
16. Find the values of k so that the line $y+x=k$ is a tangent to the circle .

7.4 Parameters

If a curve is defined by an equation relating x and y then the equation is the Cartesian form of the curve.

If x and y are expressed in terms of a third variable t then t is a parameter for the curve . The equations which define x and y in terms of t are the parametric equations of the curve .

7.4.1 Examples

1. Let the parametric equations of a curve be $x=t^2+1$ and $y=t^3$.Obtain the Cartesian equation of the curve .

Solution

Re-write the equations as $t^2=x-1$ and $t = y$.Cube the first equation and square the second.

$t^6=(x-1)^3=y^2$

The equation is $(x-1)^3=y^2$

2. Verify that the point $(t^2,2t)$ lies on the curve $y^2=4x$.Find the equation of the chord from $(t^2,2t)$ to $(1,2)$.By letting t tend to 1 find the equation of the tangent at $(1,2)$.

Solution

If $y=2t$, then $y^2=4t^2=4x$.

$(t^2,2t)$ lies on the curve.

The gradient of the chord is

$$\frac{2t-2}{t^2-1}=\frac{2(t-1)}{(t-1)(t+1)}=\frac{2}{t+1}$$

The equation is of the chord is :

$$y-2=\frac{2}{t+1}(x-1)$$
$$(y-2)(t+1)=2(x-1)$$

The equation is : $ty+y-2x=2t$

Now put $t=1$.

The equation of the tangent is $y-x=1$

7.4.2Exercises

1. The parametric equations of curves are given below. Obtain the equations of the curves in a form which does not mention the parameter.

(a) $x=t^2, y=t^3$ (b) $x=t, y=\frac{5}{t}$

(c) $x=t+1, y=3t-2$ (d) $x=2t, y=t^2-1$

(e) $x=t^3, y=\frac{1}{t}$ (f) $x=t^2, y=\frac{1}{t+1}$

(g) $x=\frac{1}{2}\left(t+\frac{1}{t}\right), \frac{1}{2}\left(t-\frac{1}{t}\right)$ (h) $x=\frac{1}{1+2t}, y=\frac{t}{1+2t}$

2. Verify that the point $\left(t,\frac{3}{t}\right)$ lies on the curve $xy=3$.

   Find the equation of the chord joining the point with parameters p and q.

3. Verify that the $\left(4t,t^2\right)$ lies on the curve $16y=x^2$ .Find the equation of the chord joining the points with parameters 2 and p. By letting p tend to 2 find the equation of the tangent to the curve at $t=2$.

4. Show that the point $\left(\frac{1}{1+t^2},\frac{t}{1+t^2}\right)$ lies on a circle , and find its centre and radius .

7.5Angles and lines

If a line has equation $y=mx+c$ m, then the angle θ it makes with the x-axis is given by $\tan\theta=m$.

The angle between two lines is the difference between the angles they make with the x-axis .

7.5.1 Example

Find the angle between this line and line $2y=x-4$,

Solution

If angle is θ then $\tan\theta=2$.

The angle is $63.4°$

Rewrite the equation of the second line as $y=\frac{1}{2}x-2$. The angle this makes with the x-axis is $\tan^{-1}\frac{1}{2}=26.6°$ .Now subtract.

The angle between the lines is $36.9°$

7.5.2 Exercise

1. Find the angles the following lines make with the x-axis .

   (a) $y=x+1$ (b) $y=3x-2$

   (c) $y=-2x-1$ (d) $2y=3x-1$

   (e) $4y+3x=7$ (f) $\frac{1}{2}y=\frac{1}{3}x+1$

2. Find the angles between the following pairs of lines from Question 1:

   (i) (a) and (b) (ii) (a)and (d)

   (iii) (b)and (c) (iv) (c)and(e)

7.6 Examination questions

1. Find the gradient of the line through $A(-4,-1)$ and $C(8,7)$ and write down the coordinates of E, the middle point of AC.

   Find the equation of the line through C parallel to AB and calculate the coordinates of B where this line meets BE produced.

2. (a) Find an equation of the line l which passes through the points $A(1,0)$ and $B(5,6)$.

   The line m with equation $2x+3y=15$ meets l at the point C.

   (b) Determine the coordinates of C.

   (c) Show, by calculation, that $PA=PB$.

3. The line L passes through the points $A(1,3)$ and $B(-19,-19)$.

   (a) Calculate the distance between A and B.

   (b) Find an equation of L in the form $ax+by+c=0$, where a,b and c are integers.

4. Find the point of intersection of the circle $x^2+y^2-6x+2y-17=0$ and the line $x-y+2=0$.Show that an equation of the circle which has these points as the ends of a diameter is $x^2+y^2-4y-5=0$. Show also that this circle and circle $x^2+y^2-8x+2y+13=0$ touch externally.

5. (a) Calculate the coordinates of the centres C_1,C_2 and the radii of the two circle whose equations are $x^2+y^2+2x-4y-5=0$ and $x^2+y^2-2x-6y+5=0$.

   Find the coordinates of the point of intersection A,B of the two circles , and show that AB is perpendicular to C_1C_2.

   (b) Show that the point $P(aq^2,2apq)$ lies on the having equation $y^2=4ax$ for all values of p.

The point $Q(aq^2, 2aq)$ is another point on the curve. Show that the equation of PQ is $(p+q)y-2x=2apq$.

If PQ passes through the point $S(a,0)$, show that $q=-\frac{1}{p}$, and in this case deduce the coordinates of R, the midpoint of PQ, in terms of p. Hence show that the equation of the locus of R as p varies is $y^2=2a(x-a)$.

Common errors

1. **Suffices**

   Do not confuse x_3 (the third variable in the series x_1, x_2, x_3) with x^3 (x-cubed).

2. **Arithmetic**

   Throughout coordinate geometry, be very careful of negative signs. The difference between 5 and -3 is 8, not 2.

   When finding distances , d o not oversimplify the expression.

   $$\sqrt{x^2+y^2} \neq x+y$$

3. **Gradients**

   The gradient of the line between two points is the $y-change$ over the $x-change$. It is not $y-value$ over the $x-value$

   When finding the gradient of a line from its equation, it must first be written in the form $y=mx+c$. The gradient of $2y=3x+4$ is $\frac{3}{2}$. It is not 3.

Solution(to exercise)

7.1.2

1.

(a) 6.32

(b) 5.39

(c) 3.61

(d) 11.7

(e) 5 7.62

3. 6.5

4.

(a) $(3,4)$

(b) $\left(1, 2\frac{1}{2}\right)$

(c) $\left(3, 7\frac{1}{2}\right)$

(d) $\left(0, 2\frac{1}{2}\right)$

(e) $\left(-1, -3\frac{1}{2}\right)$

(f) $\left(-1\frac{1}{2}, -4\frac{1}{2}\right)$

6.No.

7.2.2

1.

(a) $y = 4x - 3$

(b) $3y = x + 10$

(c) $y = -2x + 7$

(d) $4\mathrm{y} + 3\mathrm{c} = 30$

(e) $y = 2x - 2$

(f) $2y = x$

(g) $y = 3x + 8$

(h) $4y + 3x = -9$

(i) $y = 4x - 5$

(j) $y = 3 - x$

(k) $y = 12 - x$

(l) $2y + x = -4$

2.

(a) 1.66

(b) 7.07
(c) 1.34
(d) 0.277

5.No.

6. $(2,2), \sqrt{5}$

7. $y=4x-5, 3y=4x-7, y=3 : 4y+x=16, 4y+3x=18, x=1$. Meet at $\left(1, 3\frac{3}{4}\right)$

7.3.2

1.

(a) $(x-1)^2+(y-2)^2=25$

(b) $(x-1)^2+(y+3)^2=64$

2.

(a) $(2,1), \sqrt{8}$

(b) $\left(1\frac{1}{2}, \frac{1}{2}\right), \sqrt{\frac{3}{2}}$

(c) $(0,3), \sqrt{7}$

(d) $\left(-1\frac{1}{2}, 0\right), 1\frac{1}{2}$

3. $(x-1)^2+(y-2)^2=25$

4. $(x+1)^2+(y-5)^2=5$

5. $0.6, (x-1)^2+(y+1)^2=0.6^2$

6. $(x-2)^2+(y-3)^2=0.4$

7. $y=x$

8. $(1,0) \& (2,0).2y=x-1, 2y=2-x$

9. $(-1.-1) \& \left(-\frac{1}{2}, -\frac{1}{2}\right).2y=x-1, y=2x+\frac{1}{2}$

10. $(4,-12), \sqrt{40}$

11. $(x-3)^2+(y-3)^2=9$

12. $(3,1), \sqrt{17}$

13. $(3,1), \sqrt{33}$

14. 5 outside, 4.

15. $m=0.0811$ or -3.0811

16. $k=-2\pm\sqrt{14}$

7.4.2

1.

(a) $x^3 = y^2$

(b) $xy = 5$

(c) $y = 3x - 5$

(d) $y + 1 = \left(\frac{1}{2}x\right)^2$

(e) $xy^3 = 1$

(f) $x = \left(\frac{1}{y} - 1\right)^2$

(g) $x^2 - y^2 = 1$

(h) $x + 2y = 1$

2. $pqy + 3x = 3(p + q)$

3. $4y = (p+2)x - 8p, y = x - 4$

4. $\left(\frac{1}{2}, 0\right), \frac{1}{2}$

7.5.2

1.

(a) 45°

(b) 71.6°

(c) –63.4°

(d) 56.6°

2. 26.6° 11.3° 135° 26.6°

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References:

Solomon, R.C. (1997), *A Level: Mathematics* (4^{th} Edition) , Great Britain, Hillman Printers(Frome) Ltd.

More: (in Thai)

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