Algebra 2/Pre-Calculus

Name__________________

Equation Solving and More Factoring (Day 2 and 3, Polynomial Unit)

In this problem set, we will learn how to use our knowledge of factoring to solve polynomial equations. We will also learn some methods for factoring polynomials.

1. Suppose $ab = 0$. What conclusions can we make about the numbers a and b? Why?

2. Suppose $abc = 0$. What conclusions can we make about the numbers a, b, and c? Why?

3. One property of the real number system is that if $ab = 0$ then either $a = 0$ or $b = 0$. Similarly, if $abc = 0$ then $a = 0$ or $b = 0$ or $c = 0$. (These were the conclusions you should have made in problems **1** and **2**.) Explain how you can use this idea to solve the equation $x^2 - 3x = 0$.

4. Here's the solution to the last problem:

$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

$$x = 0 \text{ or } x - 3 = 0$$

$$x = 0 \text{ or } x = 3$$

Notice that factoring is the key to solving the equation. Solve each of the following polynomial equations by factoring. ***Note:*** Some of these equations are quite easy, others are harder and will require more steps. Answers are provided at the end of the problem, so don't hesitate to check your answers as you work.

a. $x^2 + 56 = 15x$

b. $x^2 = 11x$

c. $x^3 = 2x^2 + 63x$

d. $x^4 = 10x^3$

e. $x^2 - 16 = 0$

f. $3x^3 = 6x^2 + 24x$

g. $(x-4)(x-5) = 0$

h. $(x-1)(x-4)(x+9) = 0$

i. $5x^2(2x-1)(3x-4)=0$

j. $3x^3+7x=22x^2$

k. $16x=x^2+64$

l. $4x^2-20x+25=0$

m. $(x^2+17x+72)(x^2-13x+12)=0$

n. $x^2(4x^2-1)(3x-2)=0$

Answers a. 7, 8 b. 0, 11 c. 0, 9, -7 d. 0, 10 e. 4, -4 f. 0, 4, -2 g. 4, 5 h. 1, 4, -9 i. 0, $\frac{1}{2}$, $\frac{4}{3}$ j. 0, $\frac{1}{3}$, 7 k. 8 only l. $\frac{5}{2}$ only m. -9, -8, 1, 12 n. 0, $\pm\frac{1}{2}$, $\frac{2}{3}$

5. Clearly, factoring is really important for solving polynomial equations. In the next few problems, we will explore more approaches to factoring.

a. Factor $x^2 - 9$.

b. Factor $x^2 - 3$. ***Hint:*** Use the same method as part **a** and don't be afraid to write something that looks strange.

c. You should have found that $x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$. Now factor $7x^2 - 1$.

d. You should have found that $7x^2 - 1 = (\sqrt{7}x - 1)(\sqrt{7}x + 1)$. Now factor $3x^2 - 5$.

e. Factor $a^2 - b^2$.

f. What about $a^2 + b^2$? Does this factor? Explain why or why not. ***Hint:*** Start with a concrete example such as $x^2 + 9$.

g. **Optional Challenge Problem** We've found a formula for the difference of two squares. What about the difference of two cubes? For example, can you factor $x^3 - 27$? ***Hint:*** Start by seeing if you can find one of the factors.

6. In this problem, we will explore another factoring method, called *factoring by grouping*.

a. Is $x^2(x+5)+4(x+5)$ the same as $(x^2+4)(x+5)$. Explain why or why not.

b. You should have found that $x^2(x+5)+4(x+5)=x^3+5x^2+4x+20$ and $(x^2+4)(x+5)=x^3+5x^2+4x+20$. Therefore, they are equal. Notice that we now have three different forms for this polynomial

Standard Form: $x^3+5x^2+4x+20$

Factored Form: $(x^2+4)(x+5)$

"In Between" Form: $x^2(x+5)+4(x+5)$

Here's another polynomial in "in between" form: $x^2(x+8)+3(x+8)$. Rewrite $x^2(x+8)+3(x+8)$ in standard form and in factored form.

c. Factor $x^3+7x^2+5x+35$. ***Note:*** The later parts of this problem will explain how to do this, but try it on your own first.

d. Sometimes it is helpful to break a large problem into smaller parts. Factor $x^3 + 7x^2$. Then factor $5x + 35$. What special thing happens? Explain in words.

e. Rewrite the polynomial $x^3 + 7x^2 + 5x + 35$ in "in between" form, then see if you can factor it. (You can skip this if you already factored successfully in part **c**.)

f. In part **e**, you should have found that

$$x^3 + 7x^2 + 5x + 35 = x^2(x + 7) + 5(x + 7) = (x^2 + 5)(x + 7).$$

Now use the same approach to factor $x^3 - 7x^2 - 6x + 42$.

g. Here's the factoring for the last problem:

$$x^3 - 7x^2 - 6x + 42 = x^2(x - 7) - 6(x - 7) = (x^2 - 6)(x - 7).$$

Now factor $x^3 - 8x^2 + 2x - 16$. ***Note:*** Answers are provided at the end of this problem.

h. Now factor $x^3 + 5x^2 - 9x - 45$ as much as possible. Your final answer should be three linear factors.

i. Can we use this method to factor $x^3 + 5x^2 + 7x + 28$? Explain.

Some answers g. $(x^2 + 2)(x - 8)$ h. $(x + 3)(x - 3)(x + 5)$ i. No (but why not?)

7. In this problem, we will explore one more method of factoring, called *factoring by quadratic forms*.

a. Do each of the following multiplications. How are they similar? How are they different?

i. $(x + 2)(x + 5)$ **ii.** $(x^2 + 2)(x^2 + 5)$

b. Factor $x^2 + 7x + 12$.

c. Factor $x^4 + 7x^2 + 12$. ***Hint:*** Obviously, this is related to the last problem. If you get stuck, look ahead to the next part, but try it on your own first.

d. Use the substitution $u = x^2$ to factor $x^4 + 7x^2 + 12$.

e. Here are two ways of approaching the factoring of $x^4 + 7x^2 + 12$. The method on the left uses the substitution, whereas the method on the right does it all in one step. Either way is fine, but make sure you understand the steps that you are doing.

$$\begin{aligned} & x^4 + 7x^2 + 12 \\ &= u^2 + 7u + 12 \\ &= (u+3)(u+4) \\ &= (x^2+3)(x^2+4) \end{aligned}$$

$$\begin{aligned} & x^4 + 7x^2 + 12 \\ &= (x^2+3)(x^2+4) \end{aligned}$$

Now factor $x^4 + 17x^2 + 70$.

f. Factor $x^4 + x^2 - 30$.

g. Completely factor $x^4 - 10x^2 + 9$. Your final answer should be four linear factors.

h. Factor $x^4 - 49$. ***Hint:*** How would you factor $x^2 - 49$?

i. Completely factor $x^4 - 16$. How far can you go?

j. **Optional Challenge** Completely factor $x^8 - 256$.

Some answers e. $(x^2+7)(x^2+10)$ f. $(x^2+6)(x^2-5) = (x^2+6)(x+\sqrt{5})(x-\sqrt{5})$

g. $(x+3)(x-3)(x+1)(x-1)$ h. $(x^2+7)(x^2-7) = (x^2+7)(x+\sqrt{7})(x-\sqrt{7})$

i. $(x^2+4)(x+2)(x-2)$ j. $(x^4+16)(x^2+4)(x+2)(x-2)$

8. Factor each of the following polynomials. You will need to use all of the factoring methods we have seen thus far. ***Note:*** Answers are provided at the end of this problem.

a. $x^2 + x - 56$

b. $x^2 + 9x$

c. $3x^2 - 22x + 7$

d. $2x^2 - x - 6$

e. $3x^3 - 15x^2 - 18x$

f. $-x^3 + 9x^2 + 22x$

g. $x^2 - 100$

h. $x^5 - x^3$

i. $x^3 - 6x^2 + 8x - 48$

j. $x^4 - 6x^2 - 7$

k. $x^4 + 8x^3 - 4x^2 - 32x$

l. $x^4 - 1$

Answers a. $(x+8)(x-7)$ b. $x(x+9)$ c. $(3x-1)(x-7)$ d. $(2x+3)(x-2)$
e. $3x(x+1)(x-6)$ f. $-x(x+2)(x-11)$ g. $(x-10)(x+10)$ h. $x^3(x-1)(x+1)$
i. $(x^2+8)(x-6)$ j. $(x^2-7)(x^2+1)$ k. $x(x-2)(x+2)(x+8)$ l. $(x^2+1)(x+1)(x-1)$

9. In the next few problems, we will explore a few more things that can happen when we are solving quadratic equations.

a. Solve each of the following equations. Find all real solutions. ***Hint:*** Use square roots.

i. $3x^2 + 2 = 23$

ii. $2(x-5)^2 - 1 = 19$

b. Here are the solutions to the last two problems:

$$3x^2 + 2 = 23$$
$$3x^2 = 21$$
$$x^2 = 7$$
$$x = \pm\sqrt{7}$$

$$2(x-5)^2 - 1 = 19$$
$$2(x-5)^2 = 20$$
$$(x-5)^2 = 10$$
$$x - 5 = \pm\sqrt{10}$$
$$x = 5 \pm \sqrt{10}$$

Now use the same approach to solve each of the following two equations. Find all real solutions. ***Note:*** What happens differently in these equations?

i. $x^2 + 6 = 1$

ii. $5x^2 + 4 = 4$

Answers You should have found that the first equation had no real solutions and the second had only one solution: $x = 0$. ***Note:*** Quadratic equations can either have two, one, or zero real solutions.

10. When solving polynomial equations, we will sometimes need to use the quadratic formula, which you studied last year and ***should have memorized already.***

a. Write the quadratic formula from memory.

b. There are two ways that we can write the quadratic formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ or } -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Explain why these are equivalent. ***Hint:*** Think about common denominators.

c. Solve the equation $2x^3 - 3x^2 - 7x = 0$. Find all real solutions.

d. Solve the equation $(x^2 + x - 4)(x^2 + 2x + 5) = 0$. Find all real solutions.

Some answers: c. 0, $\frac{3}{4} \pm \frac{\sqrt{65}}{4}$ d. $-\frac{1}{2} \pm \frac{\sqrt{17}}{2}$ (no other real solutions)

11. More equations. Find all real solutions. ***Note:*** Some problems will require you to use the factoring methods we've been practicing in the problem set.

a. $x^4 + 20 = 9x^2$

b. $x^3 + 2x^2 = 7x + 14$

c. $x^4 + 9x^2 + 8 = 0$

d. $x^3 + 5x = x^4 + 5x^2$

e. $(3x - 5)(2x^2 + 3x - 3) = 0$

f. $0 = (x^2 - 2)(3x^2 + 2x - 5)$

g. $2x^5 + 20x^2 = 4x^4 + 10x^3$

h. $2x^4 + 3 = 7x^2$

Answers: a. ± 2, $\pm\sqrt{5}$ b. -2, $\pm\sqrt{7}$ c. No real solutions d. 0, 1 (only real solutions) e. $\frac{5}{3}$, $-\frac{3}{4} \pm \frac{\sqrt{33}}{4}$ f. $\pm\sqrt{2}$, $-\frac{5}{3}$, 1 g. 0, 2, $\pm\sqrt{5}$ h. $\pm\sqrt{3}$, $\pm\sqrt{\frac{1}{2}}$