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# Equation Solving in Indian Mathematics

Rania Al Homsy

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Handledare: Veronica Crispin Quinonez  
Examinator: Martin Herschend  
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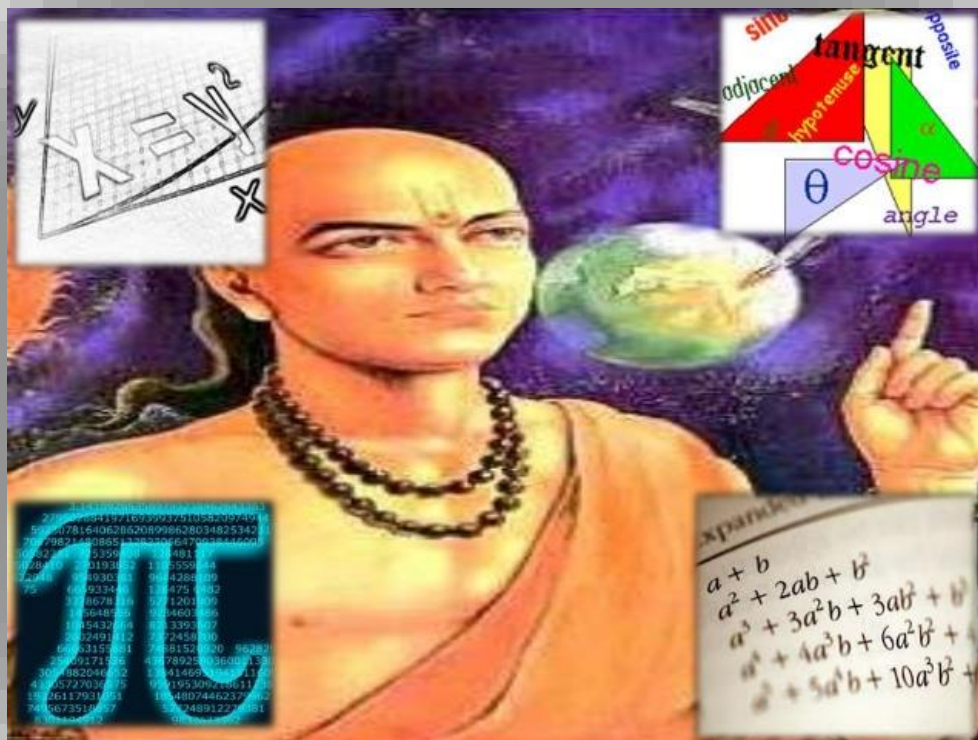
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Department of Mathematics  
Uppsala University



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*“We owe a lot to the ancient Indians teaching us how to count.*

*Without which most modern scientific discoveries would have been impossible”*

*Albert Einstein*

## Sammanfattning

Matematik i antika och medeltida Indien har påverkat utvecklingen av modern matematik signifikant. Vissa människor vet de matematiska prestationer som har sitt ursprung i Indien och har haft djupgående inverkan på matematiska världen, medan andra gör det inte. Ekvationer var ett av de områden som indiska lärda var mycket intresserade av. Vad är de viktigaste indiska bidrag i matematik? Hur kunde de indiska matematikerna lösa matematiska problem samt ekvationer? Indiska matematiker uppfann geniala metoder för att hitta lösningar för ekvationer av första graden med en eller flera okända. De studerade också ekvationer av andra graden och hittade heltalslösningar för dem.

Denna uppsats presenterar en litteraturstudie om indisk matematik. Den ger en kort översyn om matematikens historia i Indien under många hundra år och handlar om de olika indiska metoderna för att lösa olika typer av ekvationer. Uppsatsen kommer att delas in i fyra avsnitt:

- 1) Kvadratisk och kubisk extraktion av Aryabhata
- 2) Kuttaka av Aryabhata för att lösa den linjära ekvationen på formen  $c = ax + by$
- 3) Bhavana-metoden av Brahmagupta för att lösa kvadratisk ekvation på formen  $Dx^2 + 1 = y^2$
- 4) Chakravala-metoden som är en annan metod av Bhaskara och Jayadeva för att lösa kvadratisk ekvation  $Dx^2 + 1 = y^2$ .

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## 1 - Introduction

Indian civilization is one of the ancient human civilizations in the world. This civilization, which was unknown to many people due to a reluctance to recognize it over many hundreds of years, has a big and direct role in the development of mathematics of our current era. Through this essay I am going to shed light on the history of this civilization, specifically the history of mathematics in ancient and medieval India, and I will also highlight some mathematical knowledge and skills in India that produced some of the most remarkable results in mathematics. At first, I would like to point out that the use of the term Indian mathematics is not very accurate, but by using this term I mean the mathematics that originated in East Asia (including modern India, Pakistan, Nepal, Bangladesh and Sri Lanka).

As is known generally, significant contributions have been made by many Indian scholars in different fields over an extended period of time. Ancient India succeeded in providing the mathematics with many interesting ideas well ahead of their appearance elsewhere in the world. The earliest Indian mathematical works go back at least to the Vedas (ca. 2500 - 1700 BCE). The numbers were the first mathematical concept which was mentioned considerably in the Vedic literature. They spread through oral communication in combinations of powers of 10 from a hundred up to a trillion, which helped them to handle large numbers easily.<sup>1</sup> Geometrical constructions and numerical computations were recorded in the Baudhayana Sulbasutra (around 800 - 600 BCE), due to religious purposes especially the need to construct and design fire altars of specific shapes and sizes.<sup>2</sup>

Sulbasutra also discussed the transformations of one geometric figure to another with respect to preserving the area, for example how to transform a square into a circle and vice versa. The Indian numerals which were found on pillars from the Ashokas rule, were indeed one of the most significant mathematical achievements that made significant contributions to mathematics. Two mathematical concepts were known and established in India under the early centuries of the first millennium CE: the decimal place value system in writing the numbers and the using of the notation zero as a digit.

*Aryabhatiya* written by Aryabhata (476 - 550 CE), is one of the most influential works of Indian mathematics. Though the *Aryabhatiya* was very brief and difficult to understand, it contained many significant subjects like the arithmetic and geometric methods to compute areas, to extract the square

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<sup>1</sup> K. Plofker, *Mathematics in India*, p. 386.

<sup>2</sup> *Ibid*, p.387.

and cube roots (the cube root of 1444), arithmetic progressions, and finding solutions of equations of the first degree, such as finding a solution of equations of the form  $ax + c = by$  where  $a, b, c$  are unknown integers. Aryabhata found such solution by using the kuttaka method. This method will be useful in finding the roots of equations of the second degree, as we will see later.

The various arithmetical rules for calculating the four operations: addition, subtraction, multiplication and division on both positive and negative numbers, were mentioned in the major Indian work the *Brahmasphutasiddhanta*, by Brahmagupta (598 - 668 CE). Moreover, the roots of the general quadratic equations  $ax^2 + bx = c$  where  $a, b, c$  are integers and  $x$  is unknown. Brahmagupta went on to solve equations with multiple unknowns of the form  $Dx^2 + 1 = y^2$  (called Pell's equation) by using the pulveriser method. The Kuttaka and Chakravala methods for solving equations, were illustrated clearly with help of examples in the *Brahmasphutasiddhanta*.

*Siddhantasiromani* written by Bhāskara II (1114 - 1185 CE), was the work that dealt with algebra, geometry, permutations and combinations, and even quadratic equations with multiple roots. It also handled linear equation system in several variables and there is a proof that these equations have multiple solutions. He solved Pell's equation by using the chakravala method. Mahavira, the ninth century mathematician, composed the first Sanskrit textbook devoted completely to mathematics, rather than being an adjunct to astronomy.

The Kerala is the Indian mathematical school that succeeded in giving advanced contributions to mathematics, such as the developments of infinite series especially those associated to trigonometric functions.

## 2 - Purpose

This study focuses on Indian mathematical contributions, specifically solving equations. The Indian mathematicians contributed to very important achievements in mathematics, over a period of some thousands of years. Though the Indian mathematical development is very significant, the knowledge and awareness of it, is not spread, and even to that extent that some mathematical contributions are attributed to other mathematicians. We will write about the Indian mathematics mostly to highlight it, and this thesis concentrates on the ancient Indian discoveries in solving equations, especially Pell's equation which has significantly influenced the mathematics of current ages.



## 3 - Historical background

### 3.1 The history of mathematics in Ancient India

One can say that the Indus Valley civilization (The Harappan civilization) was the earliest Indian civilization that originated around 3000 BCE, near the Indus River. Unfortunately, the historians could not arrive to a certain evidence of mathematical development at that period, but some indirect evidence came to light later.

The Aryans people who moved from the Asian Steppes late in the second millennium BCE to the north of India, succeeded in establishing a mathematical civilization, based upon evidence on the Ganges River. These Aryan tribes spread over the region and found monarchical states headed by the king and brahmins (priests). The brahmins transmitted their Sanskrit literature (hymns, spells, and religious ritual), in similarity with other Vedic knowledge, in lengthy verses called "Vedas" which were handed down later in manuscripts by the sixth century BCE.

The *Sulbasutras* "Cord-Rules" were the first Indian mathematical knowledge survived from the Vedic ages to our current times with direct evidence. These *Sulbasutras* texts contain description of how to form certain shapes using stakes and cords, to help the priests to construct and design the fire-altars. We know nothing about the scribes of these *Sulbasutras* other than two things, their names and their writings.<sup>3</sup> The *Sulbasutras* were associated with religious purposes, significantly. The brahmins religion grew up to Hinduism and the content of *Sulbasutras* included description for the sacrificial system of the brahmins, in addition to very brief arithmetic rules for construction of the bricks altars called Sutas. As far as the technology of using the bricks was just used in the Harappan culture, this gave the historians a clue to believe that the *Sulbasutras* were works belonging to the Harappan era. The process of translating these concise Sutas needed undoubtedly more explaining and additional commentaries in aim to reach clear and understandable texts.<sup>4</sup>

Greek influence began with the conquest of the northeastern India by Alexander the Great (356 - 323 BCE). Ashoka rose to power, after the death of Chandragupta Maurya (340 - 297 BCE) the king of the major north Indian kingdom of the time, and tried to convert the other kingdoms around his. The earliest evidence of the Indian numerals which we know, was found in edicts carved on pillars from Ashoka's throne along with another subject about his reign.

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<sup>3</sup> V. J. Katz, A history of mathematics, p. 232.

<sup>4</sup> K. Plofker, Mathematics in India, p. 387.

### 3.2 The history of mathematics in Medieval India

One of the most noteworthy features of the first century CE was the commercial prosperity in the period of the Kushan empire in the north of India, while it was the culture which took the first place of importance under the rule of the Guptas in the early fourth century CE.<sup>5</sup>

Aryabhatya, the important work written by Aryabhata (b. 476 CE), consists of 123 verses divided into four chapters, and it was centered a lot of astronomy and many mathematical themes without any proofs.

Bhaskara I and Brahmagupta were the famous mathematicians in the early seventh century, who were known under the rule of Harsha. As Sanskrit was the common language in the Indian subcontinent that enabled the scholars to discuss both astronomy and mathematics. Along all these thousands of years the mathematics had been associated with astronomy so much that most the mathematicians were astronomers at the same time.

The first mathematical work that separated astronomy from mathematics was written in Sanskrit by Mahavira in the ninth century and it was quite dedicated to the mathematics.

In the twelfth century, Bhāskara II (1114 - 1185 CE) made a number of contributions to mathematics by writing his two-intelligent works, the *Lilavati* (it is believed without definite evidence, that Lilavati was the daughter of Bhaskara), and the *Bijaganit* (1150 CE), on arithmetic and algebra, respectively. Ancient and medieval mathematical works consist generally of rules or problems without any proofs. They composed in verse in order to aid memorisation by the students and were explained later with more details by other mathematicians.

Under two centuries, the fourteenth and the sixteenth centuries, the Kerala school of astronomy and mathematics in southern India which was founded by the Indian mathematician Madhava (1350 - 1425 CE), managed through oral transmission from teachers to students to prove many of the results that have been handed down in India for long times.<sup>6</sup>

## 4 - Arithmetic algorithms

The discovery of the decimal place value system, was the basis for most of the excellence and skills achieved by Indian mathematicians in arithmetic. For instance, the two ingenious algorithms for calculating square and cube roots were mainly based on using the decimal value system in a technique

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<sup>5</sup> V. J. Katz, A history of mathematics, p. 232.

<sup>6</sup> Ibid, p. 233.

similar to the long division method. These two methods were described digit by digit by Aryabhata, in his work *Aryabhatiya*.

#### 4.1 The square root extraction method

Aryabhata represents the rule of square root extraction method as follows:

*“2.4. One should divide, constantly, the none-square [place] by twice the square-root. When the square has been subtracted from the square [place], the quotient is the root in a different place.”*

Plofker interprets this algorithm by six steps:

- 1) Consider that the square places refer to the even power of 10, while the non-square places refer to the odd powers of 10.
- 2) Determine the greatest possible perfect square. The root of this perfect square, gives the first and the highest digit of the desired square root.
- 3) Subtract the perfect square from the digit(s) in the highest square place and obtain a remainder
- 4) Divide the digit in the non-square place by twice the preceding root.
- 5) The remainder from the division is substitute into the current non-square place.
- 6) Repeat the process as long as there are still digits on the right.<sup>7</sup>

Example: This rule for extracting the square root of a number was illustrated with examples in order to be understood. K. Plofker presents the technique by computing the square root of 1444 by using Aryabhata’s method of square root extraction: The first square place, that is in the hundred places, containing 14, is the highest square place.

$$\begin{array}{r} \text{---} \\ 1\ 4\ 4\ 4 \end{array}$$

As  $3 < \sqrt{14} < 4$  the square of 4 is too big and not possible to take in this calculation, and then the greatest possible perfect square is  $9 = 3^2$ . According to Plofker’s interpretation, 3 is the significant digit in the square root that is determined by trial and error. Subtract the current square from 14 ( $14 - 9 = 5$ ) and then moving the next digit 4 down next to the difference gives 54 in the non-square root.

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<sup>7</sup> K. Plofker, *Mathematics in India*, p. 403.

$$\begin{array}{r}
 1\ 4\ 4\ 4 \\
 -\ 9 \\
 \hline
 5
 \end{array}
 \quad \text{Root result} = 3$$

In the fourth step 54 should be divided by twice the current root that already obtained in the last step  $\frac{54}{2 \cdot 3} = 9$ . Requiring a non-zero remainder, we take 8 instead of 9 to be the second digit in the square root. Subtract the last quotient times twice the current root ( $8 \cdot 2 \cdot 3 = 48$ ) from 54 gives the new remainder 6.

$$\begin{array}{r}
 1\ 4\ 4\ 4 \\
 \dots\dots\dots \\
 5\ 4 \\
 -\ 4\ 8 \\
 \hline
 6
 \end{array}
 \quad \text{Root result} = 38$$

Moving the next digit down next to the result leaving the remainder 64 in the next square root place. It is supposed to repeat the same process again. The last place has been achieved, and now it is required to subtract the square of the last digit in the current root from 64 then null remainder is obtained.

$$\begin{array}{r}
 1\ 4\ 4\ 4 \\
 \dots\dots\dots \\
 6\ 4 \\
 -\ 6\ 4 \\
 0
 \end{array}
 \quad \text{Root result} = 38$$

Thus, the resulting calculation is as follows:

$$\begin{array}{r}
 1\ 4\ 4\ 4 \\
 -\ 9 \\
 \hline
 5\ 4 \\
 -\ 4\ 8 \\
 \hline
 6\ 4 \\
 -\ 6\ 4 \\
 \hline
 0
 \end{array}
 \quad \text{Root Result} = 38$$

$$\begin{array}{r} \hline 00 \end{array}$$

Thus 38 is the square root of the number 1444. Similarly, Aryabhata presents cube root extraction procedures as follows:

*“2.5. One should divide the second non-cube [place] by three times the square of the root of the cube. The square [of the quotient] multiplied by three and the former [quantity] should be subtracted from the first [non-cube place] and the cube from the cube [place]”.*<sup>8</sup>

Example: Find the cube root to number 12977875.

Solution: By Aryabhata’s procedure the unit is always a cube place. Let define the first, the fourth and the seventh places are names as the cube places. The second, the fifth and so on are named as the first non-cube place. The third, the eighth and so on are named as the second-non-cube place. The first step is to determine the last cube place and find the number that holds this position which is 12. Now find the nearest lesser or equal cube to 12, that is 2. The significant digit in the cube root is 2.

We subtract the current cube 8 from 12

$$\begin{array}{r} 12977875 \\ - 8 \\ \hline 49 \end{array} \quad \text{Root result} = 2$$

Dividing 49 by 3 times the square of the current root 2, we approximate  $\frac{49}{3 \cdot 2^2}$  to 3 (4 it is too large).

Thus 3 is the second digit in the cube root. We subtract 3 times the square of the current root times the last quotient, that is,  $3 \cdot 2^2 \cdot 3$ , from 49. Then we take the next digit of the number to the right down to get 137.

$$\begin{array}{r} 12977875 \\ \dots\dots\dots \\ 49 \\ - 36 \\ \hline 137 \end{array} \quad \text{Root result} = 23$$

<sup>8</sup> K. Plofker, Mathematics in India, p. 403.

Now it is required to subtract 3 times the first digit in the current root times the square of the last quotient (which is the second digit in the root)  $3 \cdot 2 \cdot 3^2 = 54$ . Then we take the next digit 7 of the number down

$$\begin{array}{r}
 12977875 \\
 \dots\dots\dots \\
 137 \\
 - 54 \\
 \hline
 837
 \end{array}$$

We subtract the cube of the second root which is 3 from the previous difference and then take the next digit down

$$\begin{array}{r}
 12977875 \\
 \dots\dots\dots \\
 837 \\
 - 27 \\
 \hline
 8108
 \end{array}$$

We divide the last result by 3 times the square of the current root 23 and get 5 as an approximation

$$\frac{8108}{3 \cdot 23^2} = \frac{8108}{1587} = 5$$

Now 5 is the next digit in the cube root. Subtracting 3 times the square of the current square times the last quotient  $3 \cdot 23^2 \cdot 5 = 7935$ , from 8108 and then taking the next digit 7 down gives

$$\begin{array}{r}
 12977875 \\
 \dots\dots\dots \\
 8108 \qquad \text{Root result} = 235 \\
 - 7935 \\
 \hline
 1737
 \end{array}$$

We subtract 3 times the current root times the square of the last quotient (the last digit in the root)

$3 \cdot 23 \cdot 5^2 = 1725$  and then take the last digit in the number down, which gives

$$\begin{array}{r}
 12977875 \\
 \dots\dots\dots \\
 1737 \\
 - 1725
 \end{array}$$

$$\begin{array}{r} \hline 125 \end{array}$$

We subtract the cube of the last digit in the root from 125 to obtain the remainder as null.

$$12977875$$

.....

$$125$$

$$-125$$

$$\begin{array}{r} \hline 000 \end{array}$$

Thus, the cube root of the number 12977875 is 532.

Although, there was no justification for the correctness of the given method, the procedure of computing square and cube roots by Aryabhatya enabled one to get the square and cube roots to numbers having many digits easily. Aryabhata did not give any proof or explanation to how he reached these rules, but he illustrated his algorithms by examples. From those examples it is obvious that Aryabhata's procedure in calculating the square and cube roots, respectively, resembles the binomial expansion  $(a + b)^2$  and  $(a + b)^3$ .

## 5 - Equation Solving

### 5.1 Quadratic equations

Katz (2009) writes that Indian mathematicians were aware of solving quadratic equations, from almost the end of the fifth century.<sup>9</sup> Aryabhata introduced a mathematical rule for the sum of an arithmetic progression  $S_n$  and another one for the number of terms  $n$  given that the sum is known. Aryabhata's procedure for computing the sum of an arithmetic progression is based on the mathematical relationship between the initial term  $a$  and the common difference  $d$  of alternate terms. The formula for  $S_n$ , presented by Aryabhata, is given as follows

$$S_n = \left[\left(\frac{n-1}{2}\right)d + a\right] = \frac{n}{2}[a + (a + (n - a)d)]$$

and the formula for finding  $n$  (when  $S_n$  is known) is given by

$$n = \frac{1}{2} \left[ \frac{\sqrt{8S_n d + (2a - d)^2} - 2a}{d} + 1 \right]$$

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<sup>9</sup> V. J. Katz, A history of mathematics, p. 243.

The formula for  $S_n$  leads us to the quadratic equation  $dn^2 + (2a - d) \cdot n - 2Sn = 0 \Leftrightarrow n^2 + \left(\frac{2a-d}{d}\right)n - \frac{2S_n}{d} = 0$ . Finding the roots of this equation, namely the value for  $n$ , easily gives

$$n = -\frac{2a-d}{2d} \pm \sqrt{\left(\frac{2a-d}{2d}\right)^2 + \frac{2S_n}{d}} \Leftrightarrow n = \frac{d-2a}{2d} + \pm \frac{\sqrt{4a^2-4ad+d^2+8dS_n}}{2d}. \text{ Comparing Aryabhata's formula}$$

for finding the number of terms  $n$  with our result using the formula for general quadratic equations applied on  $S_n$  to find  $n$ , we see that they are equivalent in the positive case. Thus, Aryabhata basically succeeded in solving the quadratic equations but he did not give a general quadrant formula.

Brahmagupta was one of the early Indian mathematicians who gave a clear and general formula for solving quadratic equations in one unknown. He gave examples and presented solutions, depending on his formula. Brahmagupta's quadratic formula has the same form as we know it except that he did not refer to the part of solutions which results from the negative square roots. Brahmagupta came to his ingenious result in solving quadratic equations a century and a quarter after Aryabhata did.<sup>10</sup>

Bhaskara II was among the first Indian mathematician to deal with two positive roots of quadratic equations. According to Katz (2009) the method used by Bhaskara II is different from the method used by Aryabhata or Brahmagupta. Bhaskara II used the technique of completing the square, by combining the square of half the coefficient of  $x$  on both sides of the equation  $ax^2 + bx = c$  in order to get a complete square in the form of  $(rx + s)^2 = d$  and to get then the value of  $x$  in the form  $x = \frac{s+\sqrt{d}}{r}$ . He continued that if  $\sqrt{d} < s$  then he took the other solution  $x = \frac{s-\sqrt{d}}{r}$ . Bhaskara II mentioned also, that some roots are acceptable and others incongruous, and he presented different examples with a view to clarify that. However, all the quadratic equations examples given by him never handled negative or irrational solutions.<sup>11</sup>

## 5.2 Bhaskara's procedure for a linear equation

Linear equations in several variables were one of the equations that Bhaskara II was interested in, and he presented the next example to illustrate it.

*“Doves are sold at the rate of 5 for 3 coins, cranes at the rate of 7 for 5, swans at the rate of 9 for 7, and peacocks at the rate of 3 for 9. A certain man was told to bring at these rates 100 birds for 100 coins for the amusement of the king's son and was sent to do so. What amount does he give for each?”*

<sup>10</sup> V. J. Katz, A history of mathematics, p. 243.

<sup>11</sup> Ibid, p. 244.



To solve this example, Bhaskara II began with formulating of the problem in mathematical models, and reduced his problem to a system of linear equations in four unknowns. He named the sets of doves, cranes, swans and peacocks as  $d$ ,  $c$ ,  $s$  and  $p$  respectively. Looking at the part of the problem, describing the total cost of the birds as 100 coins (a set of dove's costs 3coins, etc). The first equation was formed as  $3d + 5c + 7s + 9p = 100$  Then the total of 100 birds, (where every dove set consists of 5 doves, etc) where the second equation as  $5d + 7c + 9s + 3p = 100$ . After getting the two equations Bhaskara did some algebraical operations to determinate the unknowns. He solved the first equation for  $d$ :

$$(1) \quad d = (100 - (5c + 7s + 9p))/3$$

and then he solved the second equation in the same way:

$$(2) \quad d = \frac{100 - (7c + 9s + 3p)}{5}.$$

By equaling both equations  $\frac{100 - (5c + 7s + 9p)}{3} = \frac{100 - (7c + 9s + 3p)}{5} \Leftrightarrow$

$$500 - 25c - 35s - 45p = 300 - 21c - 27s - 9p \Leftrightarrow 50 - c - 2s - 9p = 0$$

Bhaskara II solved the last equation easily by assuming  $p = 4$  and then substituting  $50 - c - 2s - 36 = 0 \Leftrightarrow c = 14 - 2t$ . The next step was to let  $s = t$  (where  $t$  is an arbitrary number) and insert it in (1):  $d = \frac{-6+3t}{3} = -2t + t$ . Since  $t$  is an arbitrary number, Bhaskara II considered  $t = 3$ . Depending on the value of  $t$ , he found the corresponding values of the other unknowns.

- 1)  $d = 1$  One set of doves means  $1 \cdot 5 = 5$  doves.
- 2)  $c = 8$  8 sets of cranes mean  $8 \cdot 7 = 56$  cranes.
- 3)  $s = 3$  3 sets of swans give  $3 \cdot 9 = 27$  swans.
- 4)  $p = 4$  and 4 sets of peacocks mean  $4 \cdot 3 = 12$  peacocks.

Through the past procedure, it is remarkable that Bhaskara II found the solutions for the starting linear equation so simply, by assuming arbitrary numbers at some steps. It can be said that he found actually more than one solution, because one set of a possible roots is obtained and others can be obtained by changing the value of those arbitrary numbers.

## 6 - Linear congruences

### 6.1 Kuttaka

The Indian interest in solving linear equations with integer coefficients, when the solution is to be found in integers, can be traced back to Aryabhata. Aryabhata was the first Indian mathematician who drew attention to the method for solving linear equation  $ax - by = c$ , where  $a, b, c$  are integers and  $x, y$  are integer unknowns. Furthermore, he gave a systematic method for finding integer solution that was termed by the later mathematicians as “Kuttaka” or “pulverizer” because of the procedure of the continued division carried out to obtain smaller remainders.<sup>12</sup> Brahmagupta was the first who subsequently succeeded in clarifying the solution fully, and he gave an explicit description for the solution of this equation.

The problem was to solve linear congruences of the form  $N \equiv a \pmod{r}$  and  $N \equiv b \pmod{s}$ , in other words, it was asked to find the integer  $N$  satisfying  $N = a + rx = b + sy$ . By letting  $c = a - b$  the problem became to find  $x$  and  $y$  in  $rx + c = sy$ , where  $r, c, s$  are integers. The performed procedure, was to reduce the first equation  $ax - by = c$  to an equivalence equation with smaller coefficients which is easier to solve. Brahmagupta gave several examples and described the method step by step, clearly.<sup>13</sup>

### 6.2 Brahmagupta algorithm

Example: One was to find  $N$  such that  $N \equiv 10 \pmod{137}$  and  $N \equiv 0 \pmod{60}$ .

The problem can be rewritten as the linear equation  $137x + 10 = 60y$ . Brahmagupta’s method was presented as follows

*“Divide the divisor having the greatest remainder(agra)by the divisor having the least remainder; whatever the remainder is mutually divided; the quotients are to be placed separately one below the other.”*

Brahmagupta began determining the quotient by applying the previous technique to the new equation, and he got (137, 60, 17, 9) as dividends, (2, 3, 1, 1) as quotients, (60, 17, 9, 8) as divisors and (17, 9, 8, 1) as the remainders. He did that in an analogous way as the Euclidean algorithm

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<sup>12</sup> V. J. Katz, A history of mathematics, p.245.

<sup>13</sup> Ibid, p. 245.

$$137 = 2 \cdot 60 + 17$$

$$60 = 3 \cdot 17 + 9$$

$$17 = 1 \cdot 9 + 8$$

$$9 = 1 \cdot 8 + 1$$

Then he considered  $60 = 0 \cdot 137 + 60$  the first division, contrary to the method, and he arranged the quotients above each other by considering 0 as the first quotient. The series which results is:

0  
2  
3  
1  
1

*“Multiply the remainder by an arbitrary number such that, when increased by the difference between the two remainders (agras), it is eliminated. The multiplier is to be set down as the quotient.”*

Continuing the procedure, the last non-vanishing remainder is the greatest common divisor, in this case it is equal to 1. Hence by choosing some multiplier  $v$ , such that  $1 \cdot v - 10$  is divisible by the last divisor 8 ( $1 \cdot v - 10$  is a multiple of 8), an equivalent equation with smaller coefficients and an obvious solution  $1 \cdot v - 10 = 8w$  is formed. Brahmagupta required increasing the difference between the two remainders  $10 - 0 = 10$ , instead of decreasing, depending on whether the number of the quotient is even or not. He chose later  $v = 18$  and  $w = 1$ . The new resulting series is:

0  
2  
3  
1  
1  
18  
1

*“Beginning from the last, multiply the next to the last by the one above it; the product, increased by the last, is the end of the remainders (agranta). [ continue to the top of the column.]”*

From this obvious solution Brahmagupta began determining the desired solution for the initial equation. By applying the last algorithm, he got the term  $(18 \cdot 1 + 1 = 19)$  and then  $(19 \cdot 1 + 18 = 37)$ . Repeating this procedure gave  $(37 \cdot 3 + 19 = 130)$  then  $(130 \cdot 2 + 37 = 297)$  and at last  $(290 \cdot 0 + 130 = 130)$ . The process is over and the value of both the unknowns  $x, y$  in the initial equation are  $x = 130$  (the top term that has the greatest remainder) and  $y = 297$ .<sup>14</sup> To determine  $N$  in order to get a smaller solution Brahmagupta said:

*“Divide it (the agranta), by the divisor having the least remainder. Increase the product by the greatest remainder. The result is the remainder of the product of the divisors.”*

The last procedure to be applied is as follows

$$\frac{130}{60} = 2 \quad \text{and the remainder is equal to } 10$$

$$10 \cdot 137 = 1370$$

$$1370 + 10 = 1380$$

The underlying idea of this procedure can be summed up in three steps.

- 1) Assume the existence of the integer solution to this linear equation
- 2) Work out to get an equivalent equation with smaller coefficients and an obvious solution
- 3) Work backwards from this solution to get the required solution to the original solution.

Brahmagupta achieved the value for the required number  $N$  modulo the product of  $x$  and  $y$ ,  $x \cdot y = 137 \cdot 60 = 8220$ . Then  $N \equiv 1380 \pmod{8220}$  and  $y$  can be calculated simply to get the new solution  $x = 10$  and  $y = 23$ .

Verifying by the solution  $N = 137x + 10 = 60y$

gives  $1380 = 137 \cdot 10 + 10 = 60 \cdot 23$

Kuttaka or “pulverizer” is considered to be the most significant topical mathematical contribution of geniuses Indians. The kuttaka method of solving linear equations is a method to obtain the greatest common divisor. Furthermore, it plays an important role in the solution of the much more difficult second-degree equations by Brahmagupta, as it will be discussed later in the next chapter.<sup>15</sup>

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<sup>14</sup> V. J. Katz, A history of mathematics, p. 246.

<sup>15</sup> Ibid, p.247.

## 7 - Pell's equation

In this chapter we are going to discuss the solution of another form of equation, the quadratic equation of the form

$$Dx^2 + k = y^2, \quad (1)$$

and the special case where  $k = 1$

$$Dx^2 + 1 = y^2. \quad (2)$$

This equation is commonly known as Pell's equation in modern mathematics. Pell's equation with two unknowns, is supposed to be solved in positive integers  $x, y$  for a given non-square integer  $D$ . The importance of this equation goes back essentially, to that just integer solutions are allowed, as well as, finding rational solutions to  $\sqrt{D}$ , where  $D$  is not a perfect square.

### 7.1 Historical background

Pell's equation took this name by mistake. John Pell the seventeenth century English algebraist was not the one who studied and discussed the equation. The only connection between them is just the name.<sup>16</sup> In fact, Brahmagupta was the first mathematician who studied this equation and discovered a special procedure to solve it successfully. His method bhavana, was explained in *Brahmasputasiddhanta* in 628 CE, and it based on the principle of composition. In 1954, Kripa Shankar Shukla mentioned that Acarya Jayadeva, who lived at least one century before Bhaskara II, was the second mathematician to contribute to Pell's equation. Shukla explained that Jayadeva obtained another method to solve it completely, and his method was described in 1073 by Srimad Udayadivakara, as a commentary on Laghu-Bhaskariya of Bhaskara I. Jayadeva's process was basically the same that Bhaskara II expounded in his Lilavati in 1150 CE. Bhaskara II has named this method chakravala, otherwise it is known as the cyclic method. Bhaskara II mentioned earlier mathematicians who had contact with Pell's equation but not Jayadeva.<sup>17</sup>

### 7.2 Solution methods

#### 7.2.1 Method of composition (Bhavana)

Brahmagupta was the first one who discovered the method to find the solution of the special case (2) of the equation  $Dx^2 + 1 = y^2$  called varga-prakrti. This method, known as bhavana or the method

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<sup>16</sup> V. J. Katz, A history of mathematics, p. 248.

<sup>17</sup> C.-O. Selenius, Rationale of the chakravala process, p.168.

of composition, enabled one to produce many solutions for certain difficult cases in a few simple steps, starting off from one solution. He saw that if one could find one solution to this equation, then he could find others. Brahmagupta gave the rule of this method and illustrated it with examples for some special values for  $D$ , like  $D = 83$  or  $D = 92$ . The rule of bhavana is as follows

*“Put down twice the square root of a given square multiplied by a multiplier and increased or diminished by an arbitrary number. The product of the first pair, multiplied by the multiplier, with the product of the last pair is the last computed. The sum of the thunderbolt product is the first. The additive is equal to the product of the additive. The two square roots, divided by the additive or the subtractive, are the additive unity.”*

**Example:** Solve the equation  $83x^2 + 1 = y^2$

**Solution:** Brahmagupta starts his procedure by trying to find a solution  $(a, b)$  for an auxiliary equation  $Da^2 + k = b^2$  where  $D = 83$ , and he also defines the value of the arbitrary number  $k$  by trial and error. Thereafter, he works with the solution  $(a, b)$  to get a solution of the initial equation.

Brahmagupta began the solution of the given equation by finding an arbitrary value of  $a$  (he considered it to be 1), then increasing or decreasing the result by an arbitrary number such that a complete square value appears in the right-hand side of the equation. He noted that taking  $k = 2$  leads to the solution  $(a, b) = (1, 9)$  for the new equation  $83a^2 - 2 = b^2$ .

He put down the two square roots to get

$$\begin{array}{l} a \quad b \qquad \qquad \qquad 1 \quad 9 \\ a \quad b \quad \text{and then the corresponding values are} \quad 1 \quad 9 \end{array}$$

Combining this solution with itself gives a new value for the first and the last roots. Namely, the sum of the thunderbolt-products,  $a_1 = 1 \cdot 9 + 1 \cdot 9 = 18$ , which is the first root. The second root can be determined by adding the multiplier  $D = 83$ , multiplied by the product of the first pair  $a \cdot a$ , to the product of the last pair  $b \cdot b$ , which gives  $b_1 = 83(1 \cdot 1) + 9 \cdot 9 = 83 + 81 = 164$ .

Thus, the solution of the last equation is defined by the pair  $(a_1, b_1) = (18, 164)$  for the new additive  $2 \cdot 2 = 4$ . Then this pair satisfies the equation  $83 \cdot 18^2 + 4 = 164^2$ . Brahmagupta said that to achieve the desired roots  $x$  and  $y$ , one should divide the first and the last roots by  $k = 2$  (from the auxiliary equation). Therefore, the desired solution is  $x = \frac{18}{2} = 9$  and  $y = \frac{164}{2} = 82$ . Which solves

Pell’s equation

$$83x^2 + 1 = 83 \cdot 9^2 + 1 = 83 \cdot 81 + 1 = 6723 + 1 = 6724 = 82^2 .^{18}$$

Brahmagupta explained in his method of composition, that if  $(a, b)$  and  $(c, d)$  are two solutions of the Pell-type equation in the form  $Da^2 + p = b^2$  and  $Dc^2 + q = d^2$  respectively, then the combination of the two solutions,  $(ad + cb, Dac + bd)$  is a solution to the equation  $D(ad + cb)^2 + pq = (Dac + bd)^2$ . Namely, if it is possible to find two solutions to Pell's equation for two distinct additives, then by combining them, one can find a solution to the product of their additives.

He showed also, using the bhavana method, that if there is a solution  $(a, b)$ , satisfying the equation  $Dx^2 + 1 = y^2$ , then by applying the method of composition, it is found that  $(2ab, Da^2 + b^2)$  is a solution for the same equation. Then by repeating the same method, one can obtain infinitely many solutions.

Brahmagupta discovered also that if there is a solution  $(a, b)$  to the equation  $Dx^2 + k = y^2$  where  $k = \pm 1, \pm 2, \pm 4$  then an integer solution to Pell's equation  $Dx^2 + 1 = y^2$  can be found.

Although Brahmagupta gave correct solutions to Pell's equation always, he could never achieve the general solution and therefore his method is incomplete. There are no proofs in his texts and no one can know how he could achieve his method.<sup>19</sup>

### 7.2.2 Method of chakravala

Both Acarya Jayadeva (c.1000) and Bhaskara II, studied Pell's equation and discovered the complete solution of it. Bhaskara's method is easier to follow, and it determines the smallest solution of the equation, as well as all the solutions. It is known as the cyclic method, referring to the cyclic nature of the steps, where the same procedure can be repeated again and again in a circle, otherwise called as chakravala. According to Selenius (1975), there is a remarkable interaction between the Pell's equation and the Kuttaka, that Bhaskara II in his *Bijaganita* made use of the Kuttaka in his procedure to get the desired solution of the equation.<sup>20</sup> The main idea of this method is to use the principle of composition, given by Brahmagupta, to repeatedly obtain solutions to  $Dx^2 + k = y^2$  (for different values of  $k$ ). Eventually, this method will introduce a solution to  $Dx^2 + 1 = y^2$  and furthermore, the smallest solution of the given equation will be obtained. He writes that to deal with the quantities in the equation  $Dx^2 + 1 = y^2$ , one uses the following terms:  $x$  is the lesser root,  $y$  is the greater root,  $D$  is the multiplier or prakrti, and  $k$  is the interpolator or additive.<sup>21</sup> For the kuttaka equation  $(ax +$

<sup>18</sup> K. Plofker, Mathematics in India, p. 432.

<sup>19</sup> V. J. Katz, A history of mathematics, p. 249.

<sup>20</sup> C.-O. Selenius, Rationale of the chakravala process, p. 169.

<sup>21</sup> Ibid, p. 173.

$c)/b = y$  the considered terms are:  $x$  is the multiplier,  $y$  is the quotient,  $a$  is the dividend,  $d$  is the divisor and  $c$  is the additive (interpolator).

*“Making the smaller and larger roots and the additive into the dividend, the additive and the divisor, the multiplier is to be imagined. When the square of the multiplier is subtracted from the “nature” or is diminished by the “nature” so that the remainder is small, that divided by the additive is the new additive. It is reversed if the square of the multiplier is subtracted from the “nature”. The quotient of the multiplier is the smaller square root; from that is found the greatest root. Then it is done repeatedly, leaving aside the previous square roots and additive. They call it the chakravala (circle). Thus, there are two integer square roots increased by four, two or one. The supposition for the sake of an additive one is from the roots with four and two as additives.”*

According to Bhaskara II, every cycle in chakravala procedure consists of four steps:

- 1) The first step begins with finding  $(u, v)$ , for some additive  $k$ , satisfying the auxiliary equation  $Dx^2 + k = y^2$  (the same procedure as in the composition method), then forming the kuttaka’s equation by taking the lesser root  $u$  to be the dividend, the greater root  $v$  to be the additive, and the additive  $k$  to be the divisor.
- 2) The kuttaka equation  $\frac{um+v}{k} = n$  (that was formed in the first step), has more than one solution. In the second step,  $(m, n)$  should be chosen such that the square of  $m$  (the multiplier in the kuttaka equation), is as close to the multiplier  $D$  (the nature) as possible. In other words, we want  $|D - m^2|$  to be minimal. The value of  $n$  that obtained from the kuttaka equation (corresponding to that value of  $m$ ), is the new lesser root  $u_1$ . The new first root is  $u_1 = \frac{um+v}{k}$ .
- 3) The third step is to find the new additive or interpolator  $k_1$ . It is the result obtained by dividing the residue of  $(D - m^2)$  or  $(m^2 - D)$  by the original divisor in the kuttaka equation  $k$  (if  $D > m^2$  one takes  $-(D - m^2)$  but if  $D < m^2$  then one takes  $m^2 - D$ ). The new additive is  $k_1 = \pm \frac{D-m^2}{k}$ .
- 4) The last step is to determine the new greater root  $v_1$ . Selenius (1975), presents the three paths that the Indian mathematicians used to find  $v_1$ .

4.1 The value of  $v_1$  can be naturally obtained, from the new auxiliary equation  $Du_1^2 + k_1 = v_1^2$  where  $D$  is the original multiplier and both  $u_1, k_1$  have been obtained in the second and the third steps, respectively.

4.2 According to the method given by the Indian mathematician Narayana, the value of the greater root  $v_1$ , is determined in a very different rule “...and that [the new lesser root] multiplied by the



multiplier  $[m]$  and diminished by the product of the previous lesser root  $[u]$  and (new) interpolator  $[k_1]$  will be its greater root.” Then the greatest root is  $v_1 = u_1m - uk_1$

4.3 The last path to find  $v_1$  is by performing Bhaskara’s II rule.” *The original greatest root  $v$  multiplied by the multiplier  $m$ , is added to the least root  $u$  multiplied by the given coefficient  $D$ ; and the sum is divided by the additive  $k$ ”.* Then the greatest root is  $v_1 = \frac{vm+Du}{k}$ .

As soon as the four steps in chakravala procedure are done, then the first cycle is complete, and from initial value  $(u, v, k)$  we obtain new quantities  $(u_1, v_1, k_1)$ .<sup>22</sup> The same procedure is repeated again and again until one achieves the integral solution for an equation with the additive  $k = \pm 1, \pm 2, \pm 4$ . Applying the principle of composition, the same integral roots corresponding to the additive 1 will be derived with much fewer steps than continuing with the method of chakravala.<sup>23</sup>

### 7.2.3 Interpretation of the chakravala rules

The clear explanation of the previous procedure depends basically on the principle of composition given by Brahmagupta. That is, if  $(u, v)$  and  $(1, m)$  are solutions to two auxiliary equations with additive  $k$  and  $D - m^2$  respectively, such that

$$Du^2 + k = v^2 \quad (1)$$

$$D \cdot 1^2 - m^2 = D - m^2 \quad (2)$$

Applying the method of composition on the two solutions, the new equation is obtained  $D(um + v)^2 + k(D - m^2) = (Du + vm)^2$ . Dividing by  $k^2$  the new equation  $D \frac{(um+v)^2}{k^2} + \frac{k(D-m^2)}{k^2} = \frac{(Du+vm)^2}{k^2}$  and thus  $D(\frac{um+v}{k})^2 + \frac{(D-m^2)}{k} = (\frac{Du+vm}{k})^2$ . It is required to obtain integral solutions, so Bhaskara’s method now is to choose  $m$  such that the greater root  $\frac{Du+vm}{k}$  is an integer (in other words  $(Du + vm)$  is divisible by  $k$ ). Thus, the two other terms  $\frac{D-m^2}{k}$  and  $\frac{um+v}{k}$  will be both integers. It should also pick up  $m$  such that  $|m^2 - D|$  is minimal, thus  $\frac{m^2-D}{k}$  will be minimal. This process ensures the access to a smaller additive in every cycle. The procedure can be applied again and again until getting the additive  $k = \pm 1, \pm 2, \pm 4$  then the Pell’s equation is solved by the method of composition.

Example: Solve Pell’s equation  $67x^2 + 1 = y^2$

<sup>22</sup> C.-O. Selenius, Rationale of the chakravala process, p. 174.

<sup>23</sup> Ibid, p. 169.

- 1) We start with choosing a solution for the auxiliary equation  $67u^2 + k = v^2$ . We begin with considering  $u = 1$ , then it is easy to see that the other quantities are  $v = 8$  and the additive  $k = -3$ . Thus the auxiliary equation is  $67 \cdot 1^2 - 3 = 8^2$ . We form the kuttaka equation  $1 \cdot m + 8 = -3n$ .
- 2) The previous equation gives  $m = 1 + 3t$  and  $n = -3 - t$  for any integer  $t$ , ( $1m + 8$  must be divisible by 3). The values of  $m$  can be (1, 4, 7, 10, 13...) and we choose the value that minimises the value of  $|67 - D^2|$ . The desired value of  $m = 7$ , and then  $u_1 = \frac{1 \cdot 7 + 8}{-3} = -5$  is an integer. Because the roots are always squared then we can take  $u_1 = 5$ . At the same time  $|67 - 7^2|$  is minimal.
- 3) The third step by Bhaskara is to determine the new additive  $k_1$  and the solution of the new auxiliary equation. The new additive is  $\frac{D-m^2}{k} = \frac{67-49}{-3} = -6$  and because  $D > m^2$  we take  $k_1 = 6$ .
- 4) To compute the new greater root, we use the first rule in the fourth step to get  $v_1 = \sqrt{67 \cdot 25 + 6} = \sqrt{1681} = 41$ . From the four previous steps we get, (5,41,6) is solution for the auxiliary equation, thus we can form the new auxiliary equation now:  $67 \cdot 5^2 + 6 = 41^2$ .<sup>24</sup>
- The cycle is complete now, and we repeat the same procedure. We should find the solution for the new equation. We form the kuttaka equation  $5 \cdot m + 41 = 6 \cdot n$  and  $m = 6t + 5$ . The set of the values of  $m$  is (5, 11, 17...) and we will choose  $m = 11$ .
- The new additive is  $k_1 = \frac{67-5^2}{6} = 7$  and because  $D > m^2$  we take  $k_1 = -7$ . The new lesser root is  $u_2 = \frac{5 \cdot 5 + 41}{6} = 11$ , and the new greater root is  $v_2 = \sqrt{67 \cdot 11^2 - 7} = \sqrt{8100} = 90$ . The new derived auxiliary equation is  $67 \cdot 11^2 - 7 = 90^2$ . Then the next triple is (11, 90, -7).
- By the same procedure we obtain the next triple (27, 221, -2). Since the value of the additive is equal to -2 then either the method of Brahmagupta can be applied or the chakravala method. Combining the last solution with itself we get (11934, 97684, 4). Finally, the solution for the additive one can be obtained by dividing the last solution by 2, and then the desired solution to the original equation  $x^2 - 67y^2 = 1$  is  $x = 5967$  and  $y = 48842$ .<sup>25</sup>

#### 7.2.4 The rationale of chakravala

The problem of solving Pell's equation later drew the attention from the European mathematicians. Some European names were closely associated with solving this type of equations, such as Euler, Lagrange and Fermat. Lagrange's solution for Pell's equation was based on the regular continued

<sup>24</sup> V. J. Katz, A history of mathematics, p. 250.

<sup>25</sup> Ibid, p. 251.

fraction expansion which has nothing to do with the chakravala method. According to Selenius (1975), Euler rediscovered the property of Brahmagupta's principle of composition, that is if  $Dx_1^2 + k_1 = y_1^2$  and  $Dx_2^2 + k_2 = y_2^2$  then  $D(x_1y_2 + x_2y_1)^2 + k_1k_2 = (Dx_1x_2 + y_1y_2)^2$ .<sup>26</sup>

For more than 150 years, there was a common misconception, that both of the two methods, the chakravala and the regular continued fraction, which were used to determine the solution of Pell's equation, were the same. The periodic nature of the chakravala method is the cause of the similarity between it and the regular continued fraction, and it is the reason of the misunderstanding of the method.

Selenius (1975) emphasises, that the regular continued fraction method of Euler and Lagrange is not a rediscovered Hindu method, and the prevailing belief at the time, that the cyclic method of Jayadeva and Bhaskara II, corresponds to the method of Euler and Lagrange in solving Pell's equation, is a serious mistake.<sup>27</sup>

Krishnaswami Ayyangar was the first one who proved that the regular continued fraction expansion has nothing to do with chakravala method. Ayyangar tried to imitate the chakravala process during 1929 and 1941, in the form of a continued fraction process. He studied only the state of the square root  $\sqrt{2}$  and not other real numbers. He recognized that the chakravala in fact, is its own method, which differs from that of European mathematicians, and that the continued fractions, are used in the European methods, but there is indeed, no exact equivalent to that in chakravala. It was shown later also, that the cyclic method could be represented as the expansion of  $\sqrt{D}$  into a type of semi regular continued fraction, which is known as the ideal continued fraction expansion of  $\sqrt{D}$ .<sup>28</sup> When expressing  $\sqrt{D}$  as an ideal continued fraction, then the developed  $\frac{x}{y}$  from this continued fraction, represents the best a proximation to  $\sqrt{D}$ .

The mathematicians who studied the relation between the chakravala algorithm and the regular continued fraction, and considered them as the same, actually did not observe that not all the quantities in chakravala belong to the regular continued fraction, while these quantities do belong to the semi-regular algorithm. The procedure of chakravala could be applied again and again until getting  $k = 1$  and getting the desired solution, but most often, if interpolator (additive)  $k = \pm 1, \pm 2, \pm 4$  appears, Brahmagupta's short cut method (bhavana) is to be used, to skip some steps and get directly the same solution.<sup>29</sup> When expressing  $\sqrt{58}$  in its ideal continued fractions we get  $\sqrt{58} =$

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<sup>26</sup> C.-O. Selenius, Rationale of the chakravala process, p.168.

<sup>27</sup> Ibid, p.170.

<sup>28</sup> Ibid, p. 172.

<sup>29</sup> Ibid, p.175.

[8,3,2,2,16,8,3,2,2,16...]. The smallest solution is reached after the eighth step, if chakravala algorithm is used. One should continue applying the process until getting  $k = 1$ . However, on the other side, when getting:  $x_4^2 - 58y_4^2 = 99^2 - 58 \cdot 13^2 = q_4 = -1$  and then applying the short cut method (bhavana), one can jump over some steps to get the desired solution immediately  $x_8^2 - 58y_8^2 = 19603^2 - 58 \cdot 2574^2 = q_8 = +1$ .

## 8 - Discussion

Mathematics in India has a very long history. Ancient and medieval Indian mathematicians offered a lot of significant contributions to the evolution of mathematics. The first using of mathematics in ancient India goes back to the Vedas. In the ancient time, mathematics was associated with astronomy so much, and it was used in the purpose of construction the religious fire altars. The rules of the construction of theses altars are found in the so called Sulbasutras but without any proofs. Mathematical knowledge transmitted through oral communication until the sixth century, then it transmitted both orally and in Sanskrit manuscript form. Indian mathematicians in ancient times were aware of most of the modern mathematics that taught in our schools. In extracting the square and cube roots, Aryabhata provides a good method to find roots of numbers with many digits. His method is a straightforward method, but not the easiest one comparing with other several different methods. In solving linear equation of the form  $c = ax + by$  in integer, the Kuttaka method is one of the most significant contributions of Indian scholars. It was originally invented by Aryabhata, and then completely described by Bhaskara II. This method plays a significant role in solving equations from the second degree with two unknowns. Pell's equation is a quadratic equation with two unknowns of the form  $Dx^2 + 1 = y^2$ . The bhavana method was discovered by Brahmagupta in order to find integer solution for Pell's equation. This method enabled Brahmagupta to obtain infinitely many solutions to the equation beginning from one solution, but he did not give the general solution. The chakravala method that was described by Jayadeva first, then by Bhaskara, based basically on the bhavana method. It offers the smallest solution to Pell's equation; thus, the whole set of solution can be obtained. Applying Brahmagupta's method in a specific step allows one to get the desired solution with much fewer steps compared with the Chakravala method. The chakravala method is one of the most essential contributions to mathematics and shows the ingenuity of Indian mathematicians. It has a minimization property and arrive to the value of  $x$  and  $y$  much faster than Euler's and Lagrange's algorithms did. The importance of Pell's equation comes from that if  $x, y$  is a solution, then  $\frac{x}{y}$  is a good approximation to  $\sqrt{D}$ . The Kerala works contain an advanced mathematical level, more than

the other Indian mathematics. Madhava introduced infinite series for sine and cosine functions. He improved also the value of  $\pi$  given by Aryabhata. Forces of Islam conquered parts of India in the nineteenth century, and the mathematical contribution of this period was just a few mathematical translations from Sanskrit to Persian and vice versa. Srinivasa Ramanujan was the most genius mathematicians in the twentieth century. Ramanujan and other mathematician's achievements became a part of the global mathematics.

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