Equations of Change for Isothermal Systems

- In the previous lecture, we showed how to derive the velocity distribution for simple flows by the application of the shell momentum balance or the force balance.
- It is however more reliable to start with general equations for
  - the conservation of mass (continuity equation)
  - the conservation of momentum (equation of motion, N2L)

  to describe any flow problem and then simplify these equations for the case at hand.
- For non-isothermal fluids (heat transfer + flow problems), the same technique can be applied combined with the use of the equation for the conservation of energy

Time Derivatives

Any quantity c which depends on time as well as position can be written as c=f(t, x, y, z) then it differential dc is:

$$dc = \left(\frac{\partial c}{\partial t}\right) dt + \left(\frac{\partial c}{\partial x}\right) dx + \left(\frac{\partial c}{\partial y}\right) dy + \left(\frac{\partial c}{\partial z}\right) dz$$

- Partial Time Derivative $\left(\frac{\partial c}{\partial t}\right)$

  The partial time derivative is the derivative of the function c with time holding x, y, z constant. (fixed observer)

- Total Time Derivative $\left(\frac{dc}{dt}\right)$

  The Total Time Derivative accounts for the fact that the observer is moving (how c varies with t because of changing location).

$$\frac{dc}{dt} = \left(\frac{\partial c}{\partial t}\right) + \left(\frac{\partial c}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial c}{\partial y}\right)\frac{dy}{dt} + \left(\frac{\partial c}{\partial z}\right)\frac{dz}{dt}$$

- Substantial Time Derivative $\left(\frac{Dc}{Dt}\right)$

  The substantial Time Derivative is a particular case of the Total Time Derivative for which the velocity **v** of the observer is the same as the velocity of the flow. It is also called the Derivative Following the Motion

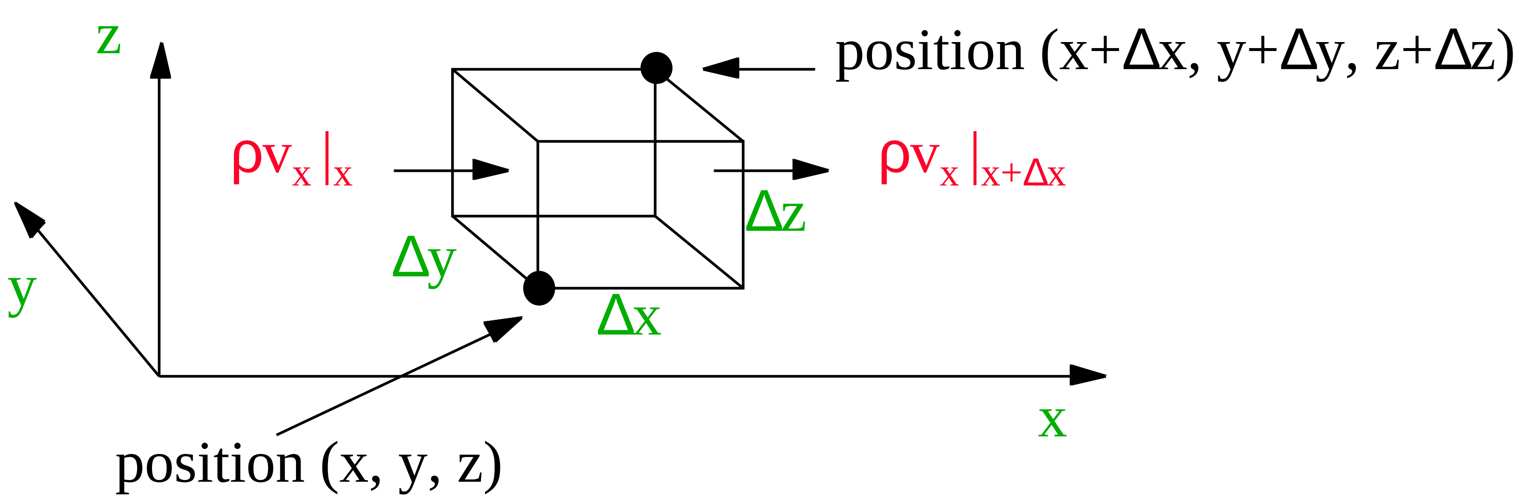
$$\frac{Dc}{Dt} = \left(\frac{\partial c}{\partial t}\right) + \left(\frac{\partial c}{\partial x}\right)v_x + \left(\frac{\partial c}{\partial y}\right)v_y + \left(\frac{\partial c}{\partial z}\right)v_z$$

  where the local fluyid velocity is defined by: $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

- Analogy of the river, the fisherman and the boat...

Equation of Continuity

- Write the mass balance over a stationary elementary volume $\Delta x \Delta y \Delta z$ through which the fluid is flowing



- Rate of Mass accumulation = (Rate of Mass In) - (Rate of Mass Out)
  Rate of Mass In through face at x is $\rho v_x|_x \, \Delta y \, \Delta z$
  Rate of Mass Out through face at x+Δx is $\rho v_x|_{x+\Delta x} \, \Delta y \, \Delta z$
  Same Thing as Above for other faces
  Rate of Mass Accumulation is $\Delta x \, \Delta y \, \Delta z \left(\frac{\partial \rho}{\partial t}\right)$

$$\Delta x\Delta y\Delta z\left(\frac{\partial \rho}{\partial t}\right) = \Delta y\ \Delta z\ (\rho v_x|_x - \rho v_x|_{x+\Delta x}) + \Delta x\ \Delta z\ (\rho v_y|_y - \rho v_y|_{y+\Delta y}) + \Delta y\ \Delta x\ (\rho v_z|_z - \rho v_z|_{z+\Delta z})$$

- Dividing on both sides by $\Delta x\Delta y\Delta z$ and taking the limit as $\Delta x\Delta y\Delta z$ goes to zero yields:

$$\boxed{\frac{\partial \rho}{\partial t} = -\left(\left(\frac{\partial(\rho v_x)}{\partial x}\right) + \left(\frac{\partial(\rho v_y)}{\partial y}\right) + \left(\frac{\partial(\rho v_z)}{\partial z}\right)\right) = -\left(\vec{\nabla}\cdot(\rho\vec{v})\right)}$$

which is known as the <u>Continuity Equation</u> (mass balance for fixed observer). The above equation can be rewritten as:

$$\frac{\partial \rho}{\partial t} + v_x\left(\frac{\partial \rho}{\partial x}\right) + v_y\left(\frac{\partial \rho}{\partial y}\right) + v_z\left(\frac{\partial \rho}{\partial z}\right) = -\rho\left(\left(\frac{\partial v_x}{\partial x}\right) + \left(\frac{\partial v_y}{\partial y}\right) + \left(\frac{\partial v_z}{\partial z}\right)\right)$$

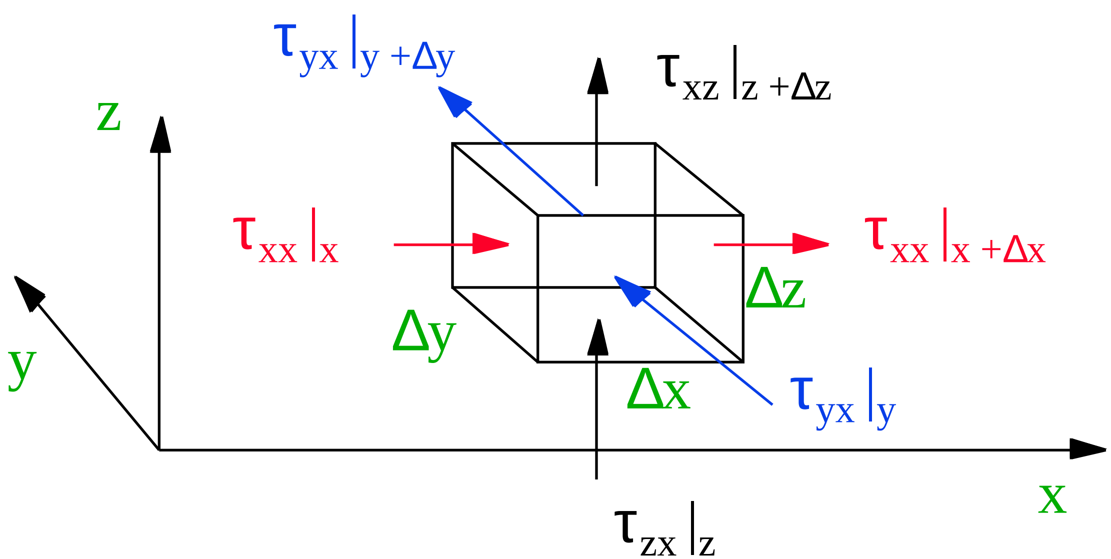
which is equivalent for an observer moving along the flow to:

$$\boxed{\frac{D\rho}{Dt} = -\rho\left(\vec{\nabla}\cdot\vec{v}\right)}$$

Note that for a fluid of constant density $D\rho/Dt = 0$ and Div(**v**) = 0 (incompressibility)

The Equation of Motion

- We will again consider a small volume element $\Delta x \Delta y \Delta z$ at position x, y, z and write the momentum balnce on the fluid element (similar approach to mass balance)
- Rate of Momentum Accumulation = (Rate of Momentum In) - (Rate of Momentum Out) + (Sum of Forces Acting on System) (Note that this is a generalization of what we did in the case of the Shell Momentum Balance for unsteady state conditions)



for transport of x-momentum through each of the 6 faces (the transport of y- and z-momentum through each of the faces is handled similarly)

- Note that momentum flows into and out of the volume element by two different mechanisms: 1- Convection (bulk fluid flow) and 2- Molecular Transfer (velocity gradient in Newton's or related laws)
- Molecular Transfer:
  Remembering that τ_{yx} is the flux of x-momentum through a face perpendicular to the y-axis, the rate at which x-momentum enters the face at y (perpendicular to the y-axis) is $\Delta x \Delta z\ \tau_{yx}|_y$ and the rate at which x-momentum leaves the face at y+Δy is $\Delta x \Delta z\ \tau_{yx}|_{y+\Delta y}$. Similarly, the rate at which x-momentum enters the face at x by molecular transfer is $\Delta y \Delta z\ \tau_{xx}|_x$ and the rate at which x-momentum leaves the face at x+Δx is $\Delta y \Delta z\ \tau_{xx}|_{x+\Delta x}$. Similarly, there is another term for the x-momentum entering by the face at z and leaving by face at z+ Δz

$$\Delta y \Delta z\left(\tau_{xx|x} - \tau_{xx|x+\Delta x}\right) + \Delta x \Delta z\left(\tau_{yx|y} - \tau_{yx|y+\Delta y}\right) + \Delta y \Delta x\left(\tau_{zx|z} - \tau_{zx|z+\Delta z}\right)$$

The same should be done with y-momentum and with z-momentum.

- Note that τ_{xx} is the normal stress on the x-face
  Note that τ_{yx} is the x-directed tangential (shear) stress on the y-face resulting from viscous forces
  Note that τ_{zx} is the x-directed tangential (shear) stress on the z-face resulting from viscous forces
- Convection:

Rate at which x-momentum enters face at x by convection	$\Delta y\ \Delta z\ (\rho v_x v_x\|_x)$
Rate at which x-momentum leaves face at x+Δx by convection	$\Delta y\ \Delta z\ (\rho v_x v_x\|_{x+\Delta x})$
Rate at which x-momentum enters face at y by convection	$\Delta x\ \Delta z\ (\rho v_y v_x\|_y)$
Rate at which x-momentum leaves face at y+Δy by convection	$\Delta x\ \Delta z\ (\rho v_y v_x\|_{y+\Delta y})$
Rate at which x-momentum enters face at z by convection	$\Delta y\ \Delta x\ (\rho v_z v_x\|_z)$
Rate at whic x-momentum leaves face at z+Δz by convection	$\Delta y\ \Delta x\ (\rho v_z v_x\|_{z+\Delta z})$

The same can be done for y- and z- momenta

- The other important contributions arise from forces acting on the fluid (fluid pressure and gravity)
  in the x-direction these forces contribute:

$$\Delta y \Delta z \left(p_{x|x} - p_{x|x+\Delta x}\right) + \Delta x \Delta y \Delta z \left(\rho g_x\right)$$

  p is a scalar quantity, which is a function of ρ and T
  g is a vectorial quantity with components (g_x, g_y, g_z)

- Summing all contributions and dividing by ΔxΔyΔz and taking the limit of ΔxΔyΔz to zero yields

$$\frac{\partial\left(\rho v_x\right)}{\partial t} = -\left(\frac{\partial\left(\rho v_x v_x\right)}{\partial x} + \frac{\partial\left(\rho v_y v_x\right)}{\partial y} + \frac{\partial\left(\rho v_z v_x\right)}{\partial z}\right)$$

$$-\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) - \frac{\partial p}{\partial x} + \rho g_x$$

Similarly for the contributions in the yu- and z-directions:

- Note that ρv_x, ρv_y, ρv_z are the components of the vector $\rho\mathbf{v}$
  Note that g_x, g_y, g_z are the components of $\mathbf{g}$

  Note that $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}$ are the components of **grad**(p)

  Note that $\rho v_x v_x$, $\rho v_y v_x$, $\rho v_z v_x$, $\rho v_x v_y$, ... are the nine components of the convective momentum flux (a dyadic product of $\rho\mathbf{v}$ and $\mathbf{v}$ (not the dot product).
  Note that τ_{xx} , τ_{xy} , τ_{xz} , τ_{yx} , τ_{yy} , ...are the nine components of the stress tensor τ.

$$\boxed{\frac{\partial(\rho\vec{v})}{\partial t} = -\left[\vec{\ }\cdot(\rho\vec{v}\vec{v})\right] - \vec{\ }p - \left[\vec{\ }\cdot\tau\right] + \rho\vec{g}}$$

Gravitational force per unit volume

Rate of increase of momentum per unit volume

Convection contribution

Pressure force per unit volume

Viscous Transfer Contribution

- The previous equation is Newton's second law (equation of motion) expressed for a stationary volume element
- This equation can be rewritten for a small volume element of fluid moving along the flow.

$$\boxed{\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} p - \left[\vec{\nabla} \cdot \tau\right] + \rho \vec{g}}$$

- To determine the velocity distribution, one now needs to insert expressions for the various stresses in terms of velocity gradients and fluid properties.
  For Newtonian Fluids one has:

$$\tau_{xx} = -2\mu \frac{\partial v_x}{\partial x} + \frac{2}{3}\mu\left(\vec{\nabla} \cdot \vec{v}\right) \qquad \tau_{yx} = \tau_{xy} = -\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$

$$\tau_{yy} = -2\mu \frac{\partial v_y}{\partial y} + \frac{2}{3}\mu\left(\vec{\nabla} \cdot \vec{v}\right) \qquad \tau_{yz} = \tau_{zy} = -\mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)$$

$$\tau_{zz} = -2\mu \frac{\partial v_z}{\partial z} + \frac{2}{3}\mu\left(\vec{\nabla} \cdot \vec{v}\right) \qquad \tau_{zx} = \tau_{xz} = -\mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$$

- Combining Newton's second law with the equations describing the relationships between stresses and viscosity allows to derive the velocity profile for the flow system. One may also need the fluid equation of state ($P = f(\rho,T)$) and the density dependence of viscosity ($\mu = f(\rho)$)along Boundary and Initial conditions.
- To solve a flow problem, write the Continuity equation and the Equation of Motion in the appropriate coordinate system and for the appropriate symmetry (cartesian, cylindrical, spherical), then discard all terms that are zero. Use your intuition, while keeping track of the terms you are ignoring (check your assumptions at the end). Use the Newtonian or Non-Newtonian relationship between velocity gradient and shear stresses. Integrate differential equation using appropriate boundary conditions

General Equation of Motion in Cartesian Coordinate System

x-component
$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \rho g_x$$

y-component
$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \rho g_y$$

z-component
$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z$$

Equation of Motion In Cartesian Coordinate System For Newtonian Incompressible Fluids

x-component
$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x$$

y-component
$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y$$

z-component
$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z$$

General Equation of Motion in the Cylindrical Coordinate System

r-component[a]
$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial p}{\partial r}$$
$$-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}-\frac{\tau_{\theta\theta}}{r}+\frac{\partial \tau_{rz}}{\partial z}\right)+\rho g_r$$

θ-component[b]
$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta})+\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta}+\frac{\partial \tau_{\theta z}}{\partial z}\right)+\rho g_\theta$$

z-component
$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}$$
$$-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \tau_{zz}}{\partial z}\right)+\rho g_z$$

Equation of Motion In the Cylindrical Coordinate System For Newtonian Incompressible Fluids

r-component[a]
$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial p}{\partial r}$$
$$+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right]+\rho g_r$$

θ-component[b]
$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]+\rho g_\theta$$

z-component
$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}$$
$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]+\rho g_z$$

General Equation of Motion in the Spherical Coordinate System

r-component
$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$
$$= -\frac{\partial p}{\partial r} - \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{r\theta}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial\tau_{r\phi}}{\partial\phi} - \frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r}\right) + \rho g_r$$

θ-component
$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} - \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{\theta\theta}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{\tau_{r\theta}}{r} - \frac{\cot\theta}{r}\tau_{\phi\phi}\right) + \rho g_\theta$$

ϕ-component
$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r}\cot\theta\right)$$
$$= -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} - \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\phi}) + \frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi\phi}}{\partial\phi} + \frac{\tau_{r\phi}}{r} + \frac{2\cot\theta}{r}\tau_{\theta\phi}\right) + \rho g_\phi$$

Equation of Motion In the Cylindrical Coordinate System For Newtonian Incompressible Fluids

r-component
$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$
$$= -\frac{\partial p}{\partial r} + \mu\left(\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}\right)$$
$$+\rho g_r$$

θ-component
$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2}\right.$$
$$\left.+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial\phi}\right) + \rho g_\theta$$

ϕ-component
$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r}\cot\theta\right)$$
$$= -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\phi\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2}\right.$$
$$\left.+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\phi}\right) + \rho g_\phi$$

Stress Tensor for Newtonian Fluids in the Cylindrical Coordinate System

$$\tau_{rr} = -\mu\left[2\frac{\partial v_r}{\partial r} - \tfrac{2}{3}(\nabla \cdot v)\right]$$

$$\tau_{\theta\theta} = -\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) - \tfrac{2}{3}(\nabla \cdot v)\right]$$

$$\tau_{zz} = -\mu\left[2\frac{\partial v_z}{\partial z} - \tfrac{2}{3}(\nabla \cdot v)\right]$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta z} = \tau_{z\theta} = -\mu\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]$$

$$\tau_{zr} = \tau_{rz} = -\mu\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]$$

$$(\nabla \cdot v) = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

Stress Tensor For Newtonian Fluids In Spherical Coordinates

$$\tau_{rr} = -\mu\left[2\frac{\partial v_r}{\partial r} - \tfrac{2}{3}(\nabla \cdot v)\right]$$

$$\tau_{\theta\theta} = -\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) - \tfrac{2}{3}(\nabla \cdot v)\right]$$

$$\tau_{\phi\phi} = -\mu\left[2\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}\right) - \tfrac{2}{3}(\nabla \cdot v)\right]$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = -\mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]$$

$$\tau_{\phi r} = \tau_{r\phi} = -\mu\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]$$

$$(\nabla \cdot v) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

The function $\tau : \ v = \mu\Phi_v$
for Newtonian Fluids

Rectangular

$$\Phi_v = 2\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right] + \left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]^2 + \left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right]^2 + \left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right]^2 - \frac{2}{3}\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right]^2$$

Cylindrical

$$\Phi_v = 2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right] + \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2 + \left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right]^2 + \left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right]^2 - \frac{2}{3}\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right]^2$$

Spherical

$$\Phi_v = 2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}\right)^2\right] + \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2 + \left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]^2 + \left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]^2 - \frac{2}{3}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right]^2$$

Axial Flow of an Incompressible Fluid in a Circular Tube of Length L and Radius R

- Use Cylindrical Coordinates
- Set $v_\theta = v_r = 0$ (flow along the z-axis)
- v_z is not a function of θ because of cylindrical symmetry
- Worry only about the z-component of the equation of motion

$$\rho v_z \frac{\partial v_z}{\partial z} = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{\partial^2 v_z}{\partial z^2}\right]$$

Continuity equation: $\frac{\partial v_z}{\partial z} = 0 \qquad \frac{\partial^2 v_z}{\partial z^2} = 0$

$$0 = -\frac{\partial p}{\partial z} + \rho g_z + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$$

Integrate twice w/respect to r using $v_z=0$ at r=R and v_z finite at r=0

$$\Rightarrow \quad v_z = \left(\frac{[P_0 - P_L]R^2}{4L\mu}\right)\left(1 - \frac{r^2}{R^2}\right)$$