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EQUIVARIANT DIFFERENTIAL TOPOLOGY†

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INTRODUCTION

THE AIM of this paper is to establish the basic propositions of differential topology (as presented in Milnor [9], for example) for G-manifolds where G is a compact Lie group.

Mostow [11] and Palais [12] proved that any compact G-manifold can be imbedded in a Euclidean G-space. In §1 a technique of de Rham's [5] is used to prove an analogue of the Whitney Imbedding Theorem, namely that any G-manifold M^n, "subordinate" to the representation V, can be imbedded in V^{2n+1}.

Section 2 concerns the classification of G-vector bundles. The precise statement is: The equivalence classes of k-dimensional G-vector bundles over M^n "subordinate" to V are in a natural one-to-one correspondence with the equivariant homotopy classes of maps of M into $G_k(V^t)$, the grassmannian of k-planes in V^t, if $t > n + k$. The existence of a classifying map is proved via a transversality argument. The equivalence of bundles induced by homotopic maps can be shown to follow from the existence and uniqueness of solution curves of vector fields. Atiyah [1] has proved a similar theorem for compact topological spaces.

Section 3 develops a cobordism theory for G-manifolds. Equivariant homotopy groups are defined and it is shown that the unoriented cobordism group of G-manifolds of dimension n, subordinate to V are isomorphic to the equivariant homotopy classes of maps of the sphere in $V^{2n+3} \oplus \mathbf{R}$ into the Thom space of the universal bundle over $G_k(V^{2n+3} \oplus \mathbf{R})$ where $k + n = (2n + 3)$ dimension of V, if G is abelian or finite. There is a severe technical difficulty in establishing even a weak transversality theorem for G-manifolds; hence, the existence of the isomorphism for arbitrary compact Lie groups is still an open question.

Section 4 generalizes the results of R. Palais [14] on Morse Theory on Hilbert Manifolds to the case of G-manifolds. It is shown that "Morse functions" are dense in the set of *invariant* real valued functions on M if M is finite dimensional. Also it is shown that passing a critical value of a Morse function corresponds to adding on "handle-bundles" over orbits or more generally over non-degenerate critical submanifolds. Morse inequalities are then deduced for the case of critical submanifolds. The results in this section were announced in [15]. Some of the results in this section have been obtained independently by Meyer [6].

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§0. NOTATIONS AND DEFINITIONS

Let G be a compact Lie group and X a completely regular topological space. An action of G on X is a continuous map $\psi: G \times X \to X$ such that $\psi(e, x) = x$ and $\psi(g_1 g_2, x) = \psi(g_1, \psi(g_2, x))$ for all $x \in X$ and $g_1, g_2 \in G$. The pair (X, ψ) will be called a G-space. We will denote by $\bar{g}: X \to X$ the map given by $\bar{g}(x) = \psi(g, x)$ and $\psi(g, x)$ will be shortened to gx. X_G will denote $\{x \in X | gx = x \text{ for all } g \in G\}$; G_x is the isotropy group, $\{g \in G | gx = x\}$. If Y is another G-space and $f: X \to Y$ then f is equivariant if, for all $g \in G$, $f \circ \bar{g} = \bar{g} \circ f$ and invariant if $f \circ \bar{g} = f$. If $\mathcal{M}(X, Y)$ is some set of maps of X into Y (differentiable, linear, etc.) then G acts on $\mathcal{M}(X, Y)$ by $gf = \bar{g} \cdot f \cdot \bar{g}^{-1}$. Clearly $\mathcal{M}(X, Y)_G$ is the set of equivariant maps in $\mathcal{M}(X, Y)$. If $H \subset G$ is a closed subgroup $X|H$ will denote the pair $(X, \psi | X \times H)$.

Let M be a C^∞ Hilbert manifold [7] with or without boundary. M will be called a G-manifold if the action $\psi: G \times M \to M$ is a differentiable map. The tangent bundle $T(M)$ of a G-manifold M is also a G-manifold with the action $gX = d\bar{g}_p(X)$ for $X \in T(M)_p$. More generally, if $\pi: E \to B$ is a fibre bundle and each $\bar{g}: E \to E$ is a bundle map then π will be called a G-bundle; if, in addition, π is a differentiable fibre bundle and E and B are G-manifolds then π is a differentiable G-bundle. If the G-vector bundle $\pi: E \to B$ has a Riemannian metric, $\langle\ ,\ \rangle$, and $\bar{g}$ is an isometry for each $g \in G$ then π is called a Riemannian G-vector bundle. If E is a Riemannian G-vector bundle then $\|e\| = \langle e, e\rangle^{1/2}$, $E(r) = \{e \in E | \, \|e\| \le r\}$, $\mathring{E}(r) = \{e \in E | \, \|e\| < r\}$ and $\dot{E}(r) = \{e \in E | \, \|e\| = r\}$. We write $\dot{E} = \dot{E}(1)$ and $\mathring{E} = \mathring{E}(1)$. Note that $T(M) \to M$ is a differentiable G-vector bundle; if $T(M) \to M$ is a Riemannian G-vector bundle then M is a Riemannian G-space. A Riemannian G-vector bundle V over a point is an (orthogonal) representation. V^t will denote the t-fold direct sum of V with itself.

If M is a G-manifold and $\Sigma \subset M$ is a compact invariant submanifold then $\pi: \nu(\Sigma) \to \Sigma$ the normal bundle of Σ is a differentiable G-vector bundle; moreover, by a theorem of Koszul [6], there is an equivariant diffeomorphism $\nu(\Sigma) \to U$ where U is an open neighborhood of Σ in M. In particular, if $x \in M$, $B_x(r)$ will denote the image of $\nu(Gx)(r)$ under some such diffeomorphism, $S_x(r)$ will denote the image of $\pi^{-1}(x)(r)$. We write $B(x) = B_x(1)$, $S(x) = S_x(1)$. $B_x(r)$ is a tubular neighborhood of Gx and $S_x(r)$ is a slice at x.

If V is a representation of G then $G_k(V)$ will denote the grassmanian of k-planes in V. $G_k(V)$ may be thought of as orthogonal projections on V with nullity k; hence G acts on $G_k(V) \subset \mathcal{M}(V, V)$ and $G_k(V)$ is a G-manifold with this action. Denote by $\mu_k(V)$ the universal bundle over $G_k(V)$; the fibre at $P \in G_k(V)$ is the null space of P. The inner product on V induces a metric on $\mu_k(V)$ and with this metric $\mu_k(V) \to G_k(V)$ is a Riemannian G-vector bundle. Let $W \subset V$ be an invariant subspace of dimension k. For each $P \in G_k(V)$ we have a representation of G_P on the null space of P; in particular, for $P \in G_k(V)_G$ we have a representation of G and if Q and P are in the same component of $G_k(V)_G$ the representations at P and Q are equivalent. Hence, we denote by $G_W(V)$ the set of k-planes $G_k(V)_G$ which are equivalent to W. Clearly $G_W(V)$ is a component of $G_k(V)_G$. We write $\mu_W(V)$ for $\mu_k(V)|G_W(V)$.

If $f: X \to V$ is any map into a Euclidean G-space then averaging f over the group means an equivariant map f^* defined by $f^*(x) = \int_G g^{-1} f(gx)\,dg$ or the invariant map $\bar{f}$ defined by $\bar{f}(x) = \int_G f(gx)\,dg$ as the context dictates.

Let X be an equivariant vector field on M, i.e., $X_{gp} = gX_p$. If $\sigma_p(t)$ denotes the maximal solution curve to X with initial condition p then by the equivariance of X, $g\sigma_p(t)$ and $\sigma_{gp}(t)$ are both solution curves with initial condition gp and, hence, by uniqueness of solution curves $g\sigma_p(t) = \sigma_{gp}(t)$. Therefore the flow generated by X is equivariant. If $f: M \to \mathbf{R}$ is an invariant function on the Riemannian G-space M then f gives rise to the vector field gradient of f, ∇f, by $\langle \nabla f_p, X\rangle = df_p(X)$. Note that $\langle g\nabla f_p, X\rangle = \langle \nabla f_p, g^{-1}X\rangle = df_p(g^{-1}X) = d(f\cdot g^{-1})_{gp}(X) = df_{gp}(X) = \langle \nabla f_{gp}, X\rangle$ for all $X \in T(M)_{gp}$ so $g\nabla f_p = \nabla f_{gp}$ and hence ∇f is an equivariant vector field.

$C_G(M, N)$ will denote the equivariant C_∞ maps between the finite dimensional G-manifolds M and N with the C^k topology for some fixed k. If $f \in C_G(M, N)$, $\varepsilon > 0$ and $\psi: R^n \to M$, $\varphi: R^n \to N$ are coordinate charts for M and N respectively then a sub-base for the neighborhoods of f in the C^k topology is given by $\{h \in C_G(M, N) | N_k(\varphi^{-1}\cdot f\cdot\psi - \varphi^{-1}\cdot h\cdot\psi)(x) < \varepsilon$ for $\|x\| \le 1\}$ where $N_k(w)(x) = \sum_{j=0}^{k} \|d^j w_x\|$, $w: R^m \to R^n$ and $\| \;\|$ denotes the usual norm on multilinear transformations. $C_G(M, N)$ is a space of the second category.

§1. GENERALIZED WHITNEY THEOREM

In this section we prove an analogue of the Whitney imbedding theorem for G-manifolds. Let V be a finite dimensional orthogonal representation of G.

PROPOSITION 1.1. *If M^n can be immersed in V^t then M can be immersed in V^{2n}.*

Proof. Let $f: M \to V^t$ be an immersion and let W be a k-dimensional irreducible representation of G contained in V. It will be sufficient to show that if W occurs s times in V^t and $s > 2n$, then there is an equivariant projection $P: V^t \to V^t$ with null space isomorphic to W such that $P\cdot f$ is an immersion.

To that end consider the diagram $\dot{T}(M) \xrightarrow{\widetilde{df}} \dot{V}^t \xleftarrow{i} \dot{\mu}_W(V^t) \xrightarrow{\pi} G_W(V^t)$ where $\widetilde{df}(X) = df(X)/\|df(X)\|$, $i(P, w) = w$ and $\pi(P, w) = P$. The pair (P, w) represents a point in $\dot{\mu}_k(V^t)$ as a projection with null space isomorphic to W and a unit vector in that null space. Since W is irreducible, i is a differentiable homeomorphism into. To show that i is an imbedding we let $X_{(P,w)}$ be any tangent vector at (P, w) and let $\lambda(t) \in \dot{V}^t$, $\gamma(t) \in G_W(V^t)$ be curves such that $(\gamma'(0), \lambda'(0)) = X_{(P,w)}$. Then $di_{(P,w)}X = \lambda'(0)$; but if $\lambda'(0) = 0$, $\gamma'(0) = d\pi\lambda'(0) = 0$ since $\gamma(t) = \pi \circ \lambda(t)$. Hence $di(X) = 0$ implies $X = 0$ and so i is an imbedding.

Since the dimension of $\dot{T}(M) = 2n - 1$, dim $\widetilde{df}(\dot{T}(M)) \cap i(\dot{\mu}_W(V^t)) \le 2n - 1$ and since i is an imbedding and π is differentiable the dim $\pi \circ i^{-1}(\widetilde{df}(\dot{T}(M)) \cap i(\dot{\mu}_W(V^t)) \le 2n - 1$. But the dim of $G_W(V^t)$ is $(s - 1)l$ where l is the dimension of the division algebra $\mathrm{Hom}(W, W)_G$. Hence, if $(s - 1)l > 2n - 1$, and in particular if $s > 2n$ there is a projection P such that $P \circ \widetilde{df}(w) = 0$ if and only if $w = 0$, i.e., $P \circ f$ is an immersion. Moreover, if $P_0 \in G_W(V^t)$, P can be chosen arbitrarily close to P_0.

Continuing in this fashion, we eventually find a projection T, the composition $\cdots P_4 \cdot P_3 \circ P_2 \circ P_1$, such that $T \circ f$ is an immersion and the range of T is isomorphic to V^{2n}.

PROPOSITION 1.2. *If M^n admits a* 1–1 *immersion in V^t, then M can be* 1–1 *immersed in V^{2n+1}.*

Proof. Let $f: M \to V^t$ be a 1–1 immersion and consider the diagram $M \times M - \Delta \xrightarrow{\alpha} \dot{V}^t \xleftarrow{i} \dot{\mu}_W(V^t) \xrightarrow{\pi} G_W(V^t)$ where $\alpha(x, y) = f(x) - f(y)/\|f(x) - f(y)\|$. Since $\dim(M \times M) = 2n$, $\dim \pi \circ i^{-1}[\alpha(M \times M - \Delta) \cap i(\dot{\mu}_W(V^t))] \leq 2n$ and hence if $t > 2n + 1$ we can find a projection $P: V^t \to V^t$ with null space isomorphic to W such that $i(\pi^{-1}(P))$ is disjoint from the image of df (so that $P \circ f$ is an immersion) and from the image of α. If $P(f(x)) = P(f(y))$ then $P \circ \alpha(x, y) = 0$ and hence $\alpha(x, y) \in i(\pi^{-1}(P))$; thus $P \circ f$ is 1–1.

COROLLARY 1.3. *Suppose that M admits an immersion, f, in V^t. Then any map $g: M \to V^{2n}$ can be C^k-approximated by an immersion. The approximation is also uniform.*

Proof. The approximation, $\bar{g}$, will be of the form $\bar{g}(x) = g(x) + Af(x)$ where A is a bounded linear map: $V^t \to V^{2n}$ and $\|A\| < \varepsilon$. By a diffeomorphism of V^t we may assume $\|f(x)\| < 1$ for all x and hence $\bar{g}$ will be a uniform approximation. To make $\bar{g}$ a C^k approximation on some compact set C, we need only replace $f(x)$ by $\delta f(x)$ where $\delta = \varepsilon/\sup_{x\in C} N_k(f(x))$ (see §0). Let i_1 (resp. i_2) denote the inclusion of V^t (resp. V^{2n}) in $V^t \times V^{2n}$ and let P_0 denote the internal projection of $V^t \times V^{2n}$ onto the second factor. Applying Prop. 1.1 to the map $f \times g: M \to V^t \times V^{2n}$ yields a projection P such that $P \circ (f \times g)$ is an immersion and $\|P - P_0\| < \varepsilon$. If $E = P(V^t \times V^{2n})$, then $P \circ i_2$ is an isomorphism onto E for ε sufficiently small and thus $(P \circ i_2)^{-1}: E \to V^{2n}$ is defined. Let $\bar{g} = (P \circ i_2)^{-1} \circ P \circ (f \times g)$. Note that $\bar{g}$ is an immersion and $\bar{g}(x) = g(x) + (P \circ i_2)^{-1} \circ P \circ (f(x), 0) = g(x) + (P \circ i_2)^{-1} \circ P \circ i_1(f(x)) = g(x) + Af(x)$.

COROLLARY 1.4. *Suppose that M admits a one-to-one immersion, f, in V^t. Then any map $g: M \to V^{2n+1}$ can be C^k-approximated by a one-to-one immersion. The approximation is also uniform.*

Proof. Essentially the same as above.

COROLLARY 1.5. *If M admits a one-to-one immersion in V^t then M can be imbedded as a closed subset of V^{2n+1}.*

Proof. Let $g: M \to V^{2n+1}$ be a proper map and apply the previous corollary. To get a proper map, let $\bar{\psi}_i$ be a locally finite partition of unity with compact support and average over the group to get ψ_i, an invariant partition of unity. Let $f: M \to V^{2n+1}$ be a one-to-one immersion (Cor. 1.4). If $f(y) = 0$ (there is at most one such point), let $\psi_1, \ldots, \psi_r$ denote those functions with $y \in$ support ψ_i and let $m_i = \inf_{\psi_i(x) > 0} \|f(x)\|$ $i > r$. Then define

$$g(x) = \sum_{i=r+1}^{\infty} i\psi_i(x) f(x)/m_i.$$

Since $g^{-1}([0, n]) \subset \bigcup_{i=1}^{n}$ support ψ_i = compact set for $n > r$, g is proper.

Remark. If the origin is not in the image of f in Props .1.1, 1.2, 1.3, 1.4, 1.5, then the new map can be chosen so as to avoid the origin also. If $\beta: M \to \dot{V}^t$ is defined by $\beta(x) = f(x)/\|f(x)\|$ then the dimension of the image of β is less than n, choose the projection, P, in Props. 1.1, 1.2 so as to avoid the n-dimensional set $\pi \circ i^{-1}(\beta(M) \cap i(\dot{\mu}_W(V^t)))$. With such a choice of P the conclusion follows in Cors. 1.3, 1.4, and 1.5.

Definition. Let V be a finite dimensional orthogonal representation of G. A G-manifold M is said to be subordinate to V is for each $x \in M$ there exists an invariant neighborhood U of x and an equivariant differentiable imbedding of U in $V^t - \{0\}$ for some t. $\mathcal{G}(V)$ is the category whose objects are G-manifolds subordinate to V and whose maps are continuous equivariant maps.

PROPOSITION 1.6. *There are only a finite number of orbit types in $\mathcal{G}(V)$.*

Proof. Let Ω be an orbit type in $\mathcal{G}(V)$ and $x \in \Omega$. By assumption there is a differentiable imbedding of an invariant neighborhood of x in V^t for some t. Hence there is a one-to-one equivariant immersion of Ω in V^{2n+1} where $n = \dim \Omega$; in particular since Ω is compact Ω can be imbedded in $V^{2 \dim G+1}$. But $V^{2 \dim G+1}$ contains only a finite number of orbit types [13]

PROPOSITION 1.7. *M^n if in $\mathcal{G}(V)$ if and only if* (i) *for each $m \in M$, G/G_m is one of the orbit types in $\mathcal{G}(V)$ and* (ii) *there is a G_m equivariant monomorphism*

$$T(M)_m/T(Gm)_m \to V^n.$$

Proof. Necessity is clear and sufficiency follows from 1.7.10 of [13].

COROLLARY 1.8. *M^n is in $\mathcal{G}(V)$ if and only if M is locally imbeddable in $V^{n+2 \dim G+1} - \{0\}$.*

PROPOSITION 1.9. *If M is in $\mathcal{G}(V)$ then M can be imbedded in V^t for some t.*

Proof. By Cor. 1.8 we may cover M by the interiors of compact invariant sets U_α such that each U_α admits an imbedding $f_\alpha: U_\alpha \to V^s - \{0\}$ where $s = 2 \dim G + \dim M + 1$. Since M is paracompact and has dimension n there is a countable refinement of U_α by compact invariant sets U_{ij} $i = 0, 1, \ldots, n; j \in Z^+$, such that $U_{ij} \cap U_{ik} = \varnothing$ if $j \neq k$ [8]. Let $f_{ij}: U_{ij} \to V^s - \{0\}$ be an imbedding; let r_j be a diffeomorphism of the positive reals onto $(j, j+i)$ and let

$$\bar{f}_{ij}(x) = r_j(\|f_{ij}(x)\|)\frac{f_{ij}(x)}{\|f_{ij}(x)\|}.$$

Then each $\bar{f}_{ij}$ is an imbedding and the images of f_{ij}, f_{ik} are disjoint if $j \neq k$; hence the map $f_i: U_i = \bigcup_{j=1}^{\infty} U_{ij} \to V^s - \{0\}$ given by $f_i(x) = \bar{f}_{ij}(x)$, $x \in U_{ij}$, is an imbedding. Let $\bar{f}_i$ imbed U_i in the unit sphere in V^{2s} by $\bar{f}_i(x) = (r_0(\|f_i(x)\|)f_i(x)/\|f_i(x)\|, \sqrt{1 - r_0(\|f_i(x)\|)^2}f_i(x)/\|f_i(x)\|)$. Finally, let h_i: $M \to I$ be differentiable invarient functions with support $h_i \subset U_i$ and such that $\bigcup_{i=0}^{n} \text{Int } h_i^{-1}(1)$ covers M and define $f: M \to V^{2(n+1)s}$ by $f(x) = (h_0(x)\bar{f}_0(x), h_1(x), \bar{f}_1(x), \ldots, h_n(x)\bar{f}_n(x))$. f is clearly equivariant and differentiable. If $x \in \text{Int } h_i^{-1}(1)$, $\pi_i \circ df = d\bar{f}_i$ and hence f is an immersion; if $f(x) = f(y)$ then $h_i(y) = 1$ and $\bar{f}_i(y) = \bar{f}_i(x)$ and so $x = y$ and $\bar{f}$ is 1-1. If $\{f(x_n)\} \to f(x)$ then $\{h_i(x_n)\} \to h_i(x) = 1$ and hence $x_n \in U_i$ for n large and since $\{h_i(x_n)\} \to 1$, $\{\bar{f}_i(x_n)\} \to \bar{f}_i(x)$ but since $\bar{f}_i$ is an imbedding $\{x_n\} \to x$ and hence f is an imbedding.

COROLLARY 1.10. (Generalized Whitney Theorem). *If M is in $\mathcal{G}(V)$ then any map $f: M \to V^t$ can be approximated C^k and uniformly by an equivariant immersion if $t \geq 2n$ and by an equivariant* 1-1 *immersion if $t \geq 2n + 1$. Moreover, if C is a closed subset of M and $f|C$ is an mmersion* (1-1 *immersion*), *the approximation $\bar{f}$ may be chosen to agree with f on C.*

Proof. The first statement follows from Prop. 1.9 and Cors. 1.3, 1.4. To prove the last statement let $g: M \to V^l$ be an imbedding with $\|g(x)\| = 1$. Let $h: M \to I$ be an invariant function such that $C \subset \operatorname{Int} h^{-1}(1)$ and $f|\text{support } h$ is an immersion (1–1 immersion). Then $x \to (f(x), (1 - h(x))g(x))$ is an immersion (1–1 immersion) of M in $V^t \times V^l$. The approximation of Cor. 1.3 (Cor. 1.4) has the desired properties.

Remark. If V is a representation of G in a Hilbert space then by the Peter–Weyl theorem, V can be decomposed $V = \bigoplus_{i=1}^{\infty} V_i^{r_i}$ where the V_i are finite dimensional irreducible representations of G, $0 \le r_i \le \infty$ and the direct sum is in the Hilbert sense. Let $V^* = \bigoplus_{i=1}^{\infty} V_i$. Then by Prop. 1.7 and the fact that closed subgroups of G obey the descending chain condition we see that M is in $\mathscr{G}(V)$ if and only if M is in $\mathscr{G}(V^*)$. In addition, all propositions of this section except 1.6 hold for $\mathscr{G}(V^*)$ and hence for $\mathscr{G}(V)$. As a consequence of this remark we have that any equivariant differentiable map $M^n \to L^2(G)^{2n+1}$ can be approximated by an equivariant 1–1 immersion since $L^2(G)$ contains at least one copy of each irreducible representation of G. In particular, if $f: M \to \mathbf{R} \subset L^2(G)^{2n+1}$ is proper, say $f(x) = \sum i\psi_i(x)$ where ψ_i is an equivariant partition of unity, the approximation will be an imbedding. Hence

COROLLARY 1.11. *Any G-manifold M^n can be imbedded as a closed subset of $L^2(G)^{2n+1}$ and hence has a complete invariant metric.*

COROLLARY 1.12. *If $f: M \to N^n$ is a continuous equivariant map then f can be approximated by a differentiable map.*

Proof. By Cor. 1.11, N may be considered as a retract of an open invariant neighborhood U of $N \subset L^2(G)^{2n+1}$ with retraction $r: U \to N$. Let $f_1: M \to U$ be a differentiable approximation to f ([9]) and average f_1 over the group to get f^*. The approximation is given by $r \circ f^*$.

§2. CLASSIFICATION OF G-VECTOR BUNDLES

Definition. Let $\pi: E \to M$ be a G-vector bundle of fibre dimension $k < \infty$ over the G-manifold M. π is said to be subordinate to the representation V of G if, for each $m \in M$, the representation of G_m on $\pi^{-1}(m)$ is equivalent to a subrepresentation of $V^k|G_m$. The category $\mathscr{B}(V)$ will have as objects G-vector bundles subordinate to V and bundle homomorphisms for maps.

Remark. $\mathscr{B}(V)$ and $\mathscr{B}(V^*)$ are the same category where V^* contains exactly one copy of each irreducible representation occurring in V.

If $\pi: E \to M$ is a G-vector bundle and $f: N \to M$ is equivariant then $f^*\pi \subset N \times E$ inherits a natural G-structure from the product which makes $f^*\pi \to N$ a G-vector bundle. Moreover, if π is in $\mathscr{B}(V)$ then so is $f^*\pi$. In particular, $\pi: \mu_k(V^t) \to G_k(V^t)$ is in $\mathscr{B}(V)$ and hence so is $f^*\pi$ for any equivariant map $f: N \to G_k(V^t)$. The next theorem due to R. Palais shows that "all" bundles over G-manifolds are obtained in this way.

THEOREM 2.1. *Let* $\pi: E^{n+k} \to M^n$ *be in* $\mathscr{B}(V)$ *and let* $f: E|C \to \mu_k(V^t)$ *be a bundle map where* $C \subset M$ *is a closed invariant subspace. If* $t \geq n + k$, *then* f *can be extended to a bundle map* $h: E \to \mu_k(V^t)$.

Proof. Consider the G-vector bundle $\mathrm{Hom}(E, V^t)$ over M with fibre $\mathrm{Hom}(\pi^{-1}(m), V^t)$ at m. The action of G is given by $gT = \bar{g}\cdot T\cdot \bar{g}^{-1}$ where $T \in \mathrm{Hom}(\pi^{-1}(m), V^t)$ and $gT \in \mathrm{Hom}\,(\pi^{-1}(gm), V^t)$. A section s of $(\mathrm{Hom}(E, V^t))$ is said to be non-singular if $s(m)$ is a non-singular linear transformation for each $m \in M$.

LEMMA 2.2. *There is a natural equivalence* θ:*non-singular sections of* $\mathrm{Hom}(E, V^t) \to$ *bundle maps of* E *into* $\mu_k(V^t)$. *Under this equivalence, equivariant sections correspond to equivariant bundle maps.*

Proof. Almost a tautology. If s is a non-singular section of $\mathrm{Hom}(E, V^t)$ then $s(m)(\pi^{-1}(m))$ is a k-plane in V^t and if $e \in \pi^{-1}(m)$ then $s(m)(e)$ is a point in that k-plane. Hence s defines a bundle map $\theta(s): E \to \mu_k(V^t)$. Moreover, if s is equivariant, $s(gm)(ge) = \bar{g}\cdot s(m)\cdot \bar{g}^{-1}(ge) = g\cdot s(m)(e)$, hence $\theta(s)$ is equivariant. Similarly, if $f: E \to \mu_k(V^t)$ is a bundle map then $x \to f|\pi^{-1}(x)$ defines a non-singular section of $\mathrm{Hom}(E, V^t)$ which is equivariant if f is equivariant.

Let $\Gamma_G(E)$ denote the G equivariant sections of $\mathrm{Hom}(E, V^t)$ with the C^0 topology and let $\mathscr{N}_G(A, M) \subset \Gamma_G(E)$ denote those sections which are non-singular at points of $A \subset M$. Note that $\Gamma_G(E)$ is of the second category.

LEMMA 2.3. $\mathscr{N}_G(M, M)$ *is dense in* $\Gamma_G(E)$ *if* $t \geq n + k$.

Proof. Note that $\mathscr{N}_G(A, M)$ is open in $\Gamma_G(E)$ if A is compact; hence, by Baire's theorem, it is sufficient to find a countable number of compact sets C_i such that $\cup C_i = M$ and $\mathscr{N}_G(C_i, M)$ is dense in $\Gamma_G(E)$ and hence $\cap \mathscr{N}_G(C_i, M) = \mathscr{N}(_G \cup C_i, M)$ is dense in $\Gamma_G(E)$.

By the induction metatheorem of [13], we may assume the lemma true for all proper closed subgroups of G; in particular, if $x \in M - M_G$ we may assume that $\mathscr{N}_{G_x}(S_x, S_x)$ is dense in $\Gamma_{G_x}(E|S_x)$ where S_x is a slice at x. Moreover, the restriction map $\rho: \Gamma_G(E) \to \Gamma_{G_x}(E|S_x)$ is open and hence $\rho^{-1}(\mathscr{N}_{G_x}(S_x, S_x)) = \mathscr{N}_G(GS_x, M)$ is open and dense in $\Gamma_G(E)$.

Now let $y \in M_G$, U a neighborhood of y in M_G, and let $v_1, \ldots, v_k$ be sections of $E|U$ such that $v_1(y), \ldots, v_k(y)$ spans $\pi^{-1}(y) = F$. Let $T: U \times F \to \pi^{-1}(U)$ by $T(u, \sum a_i v_i(y)) = \sum a_i v_i(u)$; averaging over the group yields an equivariant homomorphism $T^*: U \times F \to \pi^{-1}(U)$ which is an isomorphism at y and hence in some compact neighborhood $B(y)$ of y; i.e., $E|B(y)$ is equivariantly isomorphic to $B(y) \times F$. Thus $\Gamma_G(E|B(y))$ is homeomorphic to $C^0(B(y), \mathrm{Hom}_G(F, V^t))$. Let $N_j = \{T \in \mathrm{Hom}_G(F, V^t) | \mathrm{rank}\, T = j\}$; N_j is a disjoint union of submanifolds of $\mathrm{Hom}_G(F, V^t)$ and each component has codimension at least $t - j$ and hence codimension greater than n for $j < k$. Since $\mathscr{N}_G(B(y), B(y))$ consists of those sections which are transverse regular to $\bigcup_{j<k} N_j$, i.e., avoid $\bigcup_{j<k} N_j$, $\mathscr{N}_G(B(y), B(y))$ is open and dense in $\Gamma_G(E)$.
Since the restriction map $\rho: \Gamma_G(E) \to \Gamma_G(E|B(y))$ is open $\mathscr{N}_G(B(y), M)$ is open and dense in $\Gamma_G(M)$. Covering M_G by a countable number of sets $B(y_i)$ and $M - M_G$ by a countable number of sets GS_{x_i}, the lemma follows.

Now let $\Gamma_G(E, s_0) \subset \Gamma_G(E)$ denote those sections which extend s_0. $\Gamma_G(E, s_0)$ is non-empty since any extension of s_0 may be averaged over the group to get an equivariant extension; moreover, $\Gamma_G(E, s_0)$ is of the second category. If $A \subset M$ is compact and $A \cap C = \varnothing$ then $\rho:\Gamma_G(E, s_0) \to \Gamma_G(E|A)$ is open hence $\mathscr{N}_G(A, M) \cap \Gamma_G(E, s_0)$ is dense in $\Gamma_G(E, s_0)$. Covering $M - C$ by a countable number of compact sets $B(y_i)$, GS_{x_i} with $B(y_i) \cap C = \varnothing$, $GS_{x_i} \cap C = \varnothing$ we have $\mathscr{N}_G(M, M) \cap \Gamma_G(E, s_0)$ is dense in $\Gamma_G(E, s_0)$.

Remark. See [1] for a quick proof of the following theorem when M is compact.

THEOREM 2.4. *Let $\pi:E \to M \times I$ be a differentiable G-vector bundle. Then there is an equivariant bundle equivalence $(E|M \times 0) \times I \to E$.*

Proof. We may assume that the structural group of E has been reduced to $O(k)$. Let $\pi:P \to M \times I$ be the principal bundle of E. P is a G-bundle with *compact* fibre. It is clearly sufficient to show that there is an equivariant bundle equivalence $(P|M \times 0) \times I \to P$. To that end let X^* be an invariant vector field on P projecting onto d/dt, i.e., $X^*_{gp} = gX^*_p$ and $d\pi(X_p) = d/dt|_{\pi(p)}$. We may obtain such a vector field directly using an equivariant partition of unity or alternatively define $X^* = \text{grad}(p_2 \circ \pi)$ where $p_2: M \times I \to I$ is the projection and the gradient is defined with respect to some invariant Riemannian metric for $T(P)$. Next let

$$X_p = \int_{O(k)} d\gamma^{-1} X^*_{\gamma p}\, d\gamma.$$

Since the actions of $O(k)$ and G on P commute and since $\pi(\gamma p) = \pi(p)$ we have that X is a G equivariant and an $O(k)$ equivariant vector field on P projecting onto d/dt. Let $\sigma_p(t)$ denote the unique maximal solution curve to the vector field X with initial condition p. By the G-equivariance of X we have that $g\sigma_p(t) = \sigma_{gp}(t)$. Let $U \subset P|(M \times 0) \times I$ be the maximum domain of the equivariant map $\theta: U \to P$ given by $\theta(p, t) = \sigma_p(t)$. We wish to show that $U = P|(M \times 0) \times I$. But if $p \in P|M \times 0$, $\pi\sigma_p(t) = (m, t)$ and hence $\sigma_p(t) \in \pi^{-1}(m \times t)$ for all $(m, t) \in U$ since $d\pi(X) = d/dt$. Hence, to determine the domain of σ_p we need only consider the bundle $\pi^{-1}(m \times I) \to m \times I$. But $\pi^{-1}(m \times I)$ is compact and hence σ_p is defined for all $t \in I$. Thus $U = (P|M \times 0) \times I$. Since X is an $O(k)$ invariant vector field, θ is a bundle map. Hence θ is an equivariant bundle equivalence.

COROLLARY 2.5. *If $\pi:E \to M$ is a differentiable G-vector bundle and $f, g:N \to M$ are homotopic then $f^*\pi$ is equivalent to $g^*\pi$.*

Proof. Let $h:N \times I \to M$ be the homotopy. Let $U \subset M \times M$ be an invariant neighborhood of the diagonal such that if $(x, y) \in U$ then there exists a unique minimal geodesic γ_{xy} with $\gamma_{xy}(0) = x$ and $\gamma_{xy}(1) = y$. Let $p: U \times I \to M$ by $p(x, y, t) = \gamma_{xy}(t)$. Let $\bar{h}:N \times I \to M$ be a differentiable approximation to h such that $w(n, t) = (\bar{h}(n, t), h(n, t)) \in U$ for all $(n, t) \in N \times I$. Then $p_0^*\pi$ is equivalent to $p_1^*\pi$ by Theorem 2.4. Hence $w^*p_0^*\pi = (p_0 \circ w)^*\pi = \bar{h}^*\pi$ is equivalent to $w^*p_1^*\pi = h^*\pi$. But $\bar{h}^*\pi$ is a product by the theorem since $\bar{h}$ is differentiable, hence $h^*\pi$ is a product, i.e., $f^*\pi \approx g^*\pi$.

COROLLARY 2.6. *The equivalence classes of k-dimensional G-vector bundles over M^n subordinate to V are isomorphic to the equivariant homotopy classes of maps of M into $G_k(V^t)$ if $t \geq n + k + 1$.*

Proof. Follows formally as in [16].

§3. COBORDISM AND EQUIVARIANT HOMOTOPY GROUPS

Let W be a finite dimensional orthogonal representation of G and let $D(W)$ (resp $S(W)$) denote the unit ball (resp unit sphere) in W. If X is a G-space, let X^W denote the space of continuous equivariant maps of $S(W)$ into X with the compact open topology. If $f: X \to Y$ is equivariant there is an obvious induced map $f^W: X^W \to Y^W$; the assignment $X \to X^W$, $f \to f^W$ is a covariant functor from the category of G-spaces and equivariant maps to the category of topological spaces and continuous maps.

Definition. A G-homotopy triple (X, A, a) is a G-space X, an invariant subspace A, and a fixed point a (i.e. $G_a = G$) in that subspace. If (X, A, a) is a homotopy triple and $n \geq 1$ we define the nth W-homotopy group of (X, A, a) by $\pi_n^W(X, A, a) = \pi_n(X^W, A^W, a^W)$. If $f:(X, A, a) \to (Y, B, b)$ is an equivariant map of triples the induced homomorphism $f_*: \pi_n^W(X, A, a) \to \pi_n^W(Y, B, b)$ is defined by $f_*^W: \pi_n(X^W, A^W, a^W) \to \pi_n(Y^W, B^W, b^W)$.

If $A = \{a\}$ we denote $\pi_n^W(X, A, a)$ by $\pi_n^W(X, a)$; $\pi_0^W(X, a)$ is defined to be $\pi_0(X^W, a^W)$.

Remark. If G is the trivial group and $W = \mathbf{R}$, then $S(W) = S^0$ and the above definition reduces to $\pi_n^W(X, A, a) = \pi_n(X^W, A^W, a^W) = \pi_n(X \times X, A \times A, (a, a)) \approx \pi_n(X, A, a) \oplus \pi_n(X, A, a)$.

$\pi_n^W(X, A, a)$ may alternatively be defined as equivariant homotopy classes of maps $(D(W \times \mathbf{R}^{n-1}), S(W \times \mathbf{R}^{n-1}), D(\mathbf{R}^{n-1})) \to (X, A, a)$. In particular $\pi_1^W(X, a)$ is the set of homotopy classes of maps $S(W \times \mathbf{R}) \to X$ which carry both "north" and "south" poles to a.

If $G_a \neq G$ then $(X|G_a, A|G_a, a)$ is a G_a homotopy triple and one can consider the G_a equivariant homotopy groups $\pi_n^{W'}(X|G_a, A|G_a, a)$ where W' is any representation of G_a (not necessarily of the form $W|G_a$). Note, however, that any G_a equivariant map $W' \to X$ extends uniquely to a G equivariant map $W' \times_{G_a} G \to X$ where $W' \times_{G_a} G$ is a G-vector bundle over G/G_a. Moreover, if $\pi: E \to G/G_a$ is any G-vector bundle over G/G_a such that the representation of G_a on $\pi^{-1}(\{e\})$ is equivalent of W', the equivalence $W' \to \pi^{-1}(\{e\})$ extends by equivariance to a G-bundle equivalence $W' \times_{G_a} G \to E$. Thus, E is determined by the representation of G_a on $\pi^{-1}(\{e\})$. Hence we may define the groups $\bar{\pi}_n^{W'}(X, A, a)$ as G-equivariant homotopy classes of maps $\{D(E \oplus \mathbf{R}^{n-1}), S(E \oplus \mathbf{R}^{n-1}), *\}$ into X, A, a where $\pi: E \to G/G_a$ is the unique G-vector bundle with fibre equivalent to W', $E \oplus \mathbf{R}^{n-1}$ denotes the Whitney sum of E with a trivial bundle of dimension $n-1$ and $* = \{x \in E \oplus \mathbf{R}^{n-1} | x = (0, y)$ and $\pi(x) = \{e\} \in G/G_a\}$. Clearly $\bar{\pi}^{W'}(X, A, a) = \pi_n^{W'}(X|G_a, A|G_a, a)$.

Let V be a finite dimensional orthogonal representation of G. We wish to develop a cobordism theory for $\mathscr{G}(V)$.

Definition. The compact G-manifolds M_1^n, M_2^n are said to be V-cobordant, $M_1 \underset{V}{\sim} M_2$ (or cobordant, $M_1 \sim M_2$ if no confusion will result), if there exists a compact G-manifold N^{n+1} in $\mathscr{G}(V)$ with ∂N_n^{n+1} equivariantly diffeomorphic to $M_1 \cup M_2$.

PROPOSITION 3.0. *$\underset{V}{\sim}$ is an equivalence relation.*

Proof. Symmetry and reflexivity are obvious and transitivity follows from the fact that there is an equivariant diffeomorphism of $\partial N \times [0, 1)$ onto an open neighborhood of ∂N in N.

Definition. $\mathfrak{n}_n(V)$ will denote the unoriented cobordism group of equivalence classes of n-dimensional compact G-manifolds in $\mathcal{G}(V)$. The group operationis given by $[M_1]+[M_2]=[M_1 \cup M_2]$, i.e. disjoint union. Similarly one can consider the oriented cobordism groups $\Omega_n(V)$.

Remark. Appropriate choices of G, V yield the equivariant cobordism groups considered by Conner and Floyd in [3] and [4].

Let $T_k(W)$ denote the Thom space of the bundle $\mu_k(W) \to G_k(W)$. $T_k(W)$ may be thought of as $\mu_k(W)(\varepsilon)/\dot{\mu}_k(W)(\varepsilon)$; G-acts on $T_k(W)$ in the obvious way and the fixed point $\{\dot{\mu}_k(W)(\varepsilon)\}$ will be denoted by ∞. We wish to define a homomorphism $\theta\colon \mathfrak{n}_n(V) \to \pi_1^{V^{n+h}}(T_k(V^{n+h} \oplus \mathbf{R}), \infty)$ where $h \geq n+3$ and $k+n=(n+h)\dim V$.

Let $[M] \in \mathfrak{n}_n(V)$ and $i\colon M \to V^{2n+1} \subset V^{n+h}$ be an imbedding with $0 \notin i(M)$ (cor 1.10). There is a bundle monomorphism $\nu(M) \to T(V^{n+h})|i(M) = MxV^{n+h}$ via the invariant metric on V^{n+h} and hence a bundle *map* $b\colon \nu(M) \to \mu_k(V^{n+h}) \to \mu_k(V^{n+h} \oplus R)$. Let $E\colon T(V^{n+h}) \to V^{n+h}$ be the end-point map; i.e. $E(v, x) = v + x$ where $x \in V^{n+h}$ and v is a tangent vector at x. Then $\bar{E} = E|\nu(M)(\delta) \to V^{n+h}$ is an equivariant diffeomorphism onto a neighborhood U of $i(M)$ for some $\delta > 0$; choose δ small enough so that $0 \notin U$. Let $f_{M,i}\colon V^{n+h} \to T_k(V^{n+h} \oplus \mathbf{R})$ be defined by $f_{M,i}|U = q \circ b \circ E^{-1}$, $f_{M,i}(V^{n+h} - U) = \infty$, where $q\colon \mu_k(V^{n+h} \oplus \mathbf{R}) \to \mu_k(V^{n+h} \oplus \mathbf{R})(\varepsilon)/\dot{\mu}_k(V^{n+h} \oplus \mathbf{R})$ is the identification map and $\varepsilon < \delta$. Extending $f_{M,i}$ to the one point compactification of V^{n+h}, i.e. to $S(V^{n+h} \oplus \mathbf{R})$, we get, via the above Thom construction an element $\theta([M]) \in \pi_1^{V^{n+h}}(T_k(V^{n+h} \oplus \mathbf{R}), \infty)$.

PROPOSITION 3.1. *θ is a well defined homomorphism.*

Proof. Let Q^{n+1} be a compact manifold in $\mathcal{G}(V)$, $\partial Q = M_1 \cup M_2$, and let $i_j\colon M_j \to V^{2n+1} - \{0\}$ $j = 1, 2$ be imbeddings. We must show that f_{M_1,i_1} is equivariantly homotopic to f_{M_2,i_2} and hence that $\theta([M])$ is independent of the choice of representative or imbedding.

If $c > 0$ then $ci_1\colon M \to V^{2n+1}$ is an imbedding and f_{M_1,ci_1} is clearly homotopic to f_{M_1,i_1} ∘ hence we may assume, by choosing c large enough, that $i_1(M_1) \cap i_2(M_2) = \varnothing$. Let U_j, $j = 1, 2$, be an equivariant collaring of M_j in Q, i.e. U_j is an invariant neighborhood of M_j, with equivariant diffeomorphism $\psi_j M_j \times [0, 2) \to U_j$ such that $\psi_j|M_j \times \{0\}$ is the identity. Let $i_3\colon U_1 \cup U_2 \to V^{n+h} \times [0, 5] \subset V^{n+h} \oplus R$ by

$$i_3(q) = \begin{Bmatrix} (i_1(x), t) & \text{if} \quad q = \psi_1(x, t) \\ (i_2(x), 5 - t) & \text{if} \quad q = \psi_2(x, t) \end{Bmatrix}$$

and extend i_3 differentiably to $i_4\colon Q \to V^{n+h} \times [0,5]$ so that $i_4(Q - U_1 \cup U_2) \subset V^{n+h} \times [2,3]$. If $Q_G \neq \varnothing$ we insist that $i_4|Q_G$ be transverse regular to $\{0\} \times [0, 5]$ in $V_g^{n+h} \times [0, 5]$, i.e. $i_4(Q_G) \cap \{0\} \times [0, 5] = \varnothing$. Then i_4 may be averaged over G to get an equivariant differentiable map $i_5\colon Q \to V^{n+h} \times [0, 5]$. Since $h \geq n+3$, i_5 may be approximated by an equivariant 1–1 immersion (and hence an embedding) $i\colon Q \to V^{n+h} \times [0, 5]$ with $i|U_1 \cup U_2 = i_3$ (Corollary 1:10). Note that $i(Q) \cap \{0\} \times [0, 5] = \varnothing$. [If $x \notin Q_g$ this follows since i is an imbedding; for $x \in Q_g$ we note that $i_5(Q_g) \cap \{0\} \times [0, 5] = \varnothing$ and hence for a sufficiently close approximation i, $i(Q_g) \cap \{0\} \times [0, 5] = \varnothing$]. Then we apply the Thom construction as before to get an equivariant homotopy $f_{Q,i}\colon S(V^{n+h} \oplus \mathbf{R}) \times [0, 5] \to T_k(V^{n+h} \oplus \mathbf{R})$ with

$f_{Q,i}|S(V^{n+h} \oplus \mathbf{R}) \times \{0\} = f_{M_1,i_1}$ and $f_{Q,i}|S(V^{n+h} \oplus \mathbf{R}) \times \{5\} = f_{M_2,i_2}$. Note that for each $t \in [0, 5]$, $f_{Q,i}|S(V^{n+h} \oplus \mathbf{R}) \times \{t\} : (S(V^{n+h} \oplus \mathbf{R}), \{0, \infty\}) \to (T_k(V^{n+h} \oplus \mathbf{R}), \infty)$ if the neighborhood U of $i(Q)$ in $V^{n+h} \times [0, 5]$ used in the Thom construction is chosen small enough so that $U \cap 0 \times [0, 5] = \emptyset$. Hence $[f_{M_1 i_1}] = [f_{M_2 i_2}] \in \pi_1^{n+h}(T_k(V^{n+h} \oplus \mathbf{R}), \infty)$ and thus θ is well defined. Clearly θ is a homomorphism.

If G is trivial, i.e. $G = e$, then it is well known that θ is an isomorphism [9]. One defines a map $\lambda : \pi_1^{V^{n+h}}(T_k(V^{n+h} \oplus \mathbf{R}), \infty) \to \boldsymbol{\eta}_n(v)$ by $\lambda[f] = f_1^{-1}(G_k(V^{n+h} \oplus \mathbf{R}))$ where (i) f_1 is homotopic to f, (ii) "differentiable," and (iii) transverse regular (TR) to $G_k(V^{n+h} \oplus \mathbf{R})$. If f_2 is any other such map, then there exists a homotopy $F : S(V^{n+h} \oplus \mathbf{R}) \times [0, 5] \to T_k(V^{n+h} \oplus \mathbf{R})$ such that $F_0 = f_1$, $F_5 = f_2$ and F is TR to $G_k(V^{n+h} \oplus \mathbf{R})$; hence $F^{-1}(G_k(V^{n+h} \oplus \mathbf{R}))$ is a cobordism between $f_1^{-1}(G_k(V^{n+h} \oplus \mathbf{R}))$ and $f_1^{-1}(G_k(V^{n+h} \oplus \mathbf{R}))$ and λ is well defined. Clearly $\lambda \circ \theta =$ identity. One then shows that that λ is a monomorphism by using the fact that $\mu_k(V^{n+h} \oplus \mathbf{R})$ is $(n + 1)$ universal (since $G = e$, $V = \mathbf{R}$, and $k = h$).

Serious difficulties arise in trying to carry out this proof when $G \neq e$. First of all, if $f : M \to N$ is a differentiable equivariant map and $W \subset N$ a compact submanifold, it is not true, in general, that f can be approximated by a map $f_1 : M \to N$ which is TR to W. For example, let $G = Z_2$, $M =$ one point, $N = \tilde{R}$ the real line with Z_2 acting by reflection, $W = 0 \in \tilde{R}$ and $f(x) = 0$, $x \in M$. Clearly f is the only equivariant map $M \to N$ and is not TR to W.

However, in the special case we are considering, $M = S(V^{n+h} \times \mathbf{R})$, $N = T_k(V^{n+h} \oplus \mathbf{R})$, $W = G_k(V^{n+h} \oplus \mathbf{R})$ one can find in each equivariant homotopy class a map f which is TR to W if G is a "nice" group. However, if f_1 and f_2 are two such maps which are equivariantly homotopic there will not, in general, be an equivariant homotopy h between them satisfying (ii) and (iii). For example, let $G = Z_2$, $V = \mathbf{R} \oplus \tilde{R}$, $n = 0$, $h = 3$. Let M be a point, $i : M \to (\mathbf{R} + \tilde{R})^3$ and consider the maps $f_{M,i}$, $\tilde{g} \circ f_{M,i}$ where $e \neq g \in Z_2$; both maps are transverse regular to $G_k(V^3 \oplus \mathbf{R})$ and $f_{M,i}$ is equivariantly homotopic to $\tilde{g} \circ f_{M,i}$ but there is no TR homotopy between them as can be shown by a simple determinant argument. In addition $\pi : \mu_k(V^{n+h} + \mathbf{R}) \to G_k(V^{n+h} \oplus \mathbf{R})$ is not necessarily $(n + 1)$ universal. It turns out that the notion of "consistent transverse regularity" (CTR) is sufficient to overcome these difficulties.

The following lemmas are preparatory to proving the transversality theorem.

LEMMA 3.2. *Let M, N be G-manifolds and $f : M \to N$ a differentiable equivariant map. If C is a closed invariant subspace of M and $h_t : C \to N$ is a differentiable equivariant homotopy of $f|C$ then h_t can be extended to a differentiable equivariant homotopy of f. Moreover, if U is an open neighborhood of C, the extension F_t may be chosen so that $F_t|M - U = f|M - U$.*

Proof. By Proposition 1.66 of [13] and Corollary 1.11 of § 2, N is a $G-ANR$. Hence, the map $\bar{F} : M \times \{0\} \cup C \times I \to N$ given by $\bar{F}|M \times 0 = f$, $\bar{F}|C \times I = h$ can be extended to a map also called $\bar{F}$ defined in an invariant neighborhood V of $M \times \{0\} \cup C \times I$ in $M \times I$. V contains an open invariant set of the form $U_1 \times I$ where $U_1 \supset C$. Let $\alpha : M \to I$ be differentiable, invariant with support $\alpha \subset U_1 \cap U$ and $\alpha(C) = 1$. Define $F : M \times I \to N$ by $F(x, t) = \bar{F}(x, \alpha(x)t)$.

LEMMA 3.3. *Let $f: M \to N$ be differentiable and equivariant and $W \subset N$ a closed invariant submanifold. Let C be a closed subset of M_G and suppose that $f|M_G$ is transverse regular* (TR) *to W_G in N_G at points of C. Then there exists a homotopy f_t such that*

(i) $f_0 = f$,
(ii) $f_t|C = f|C$ *and*
(iii) $f_1|M_G$ *is* TR *to* W_G *in* N_G.

Proof. By the standard transversality lemma (§1.35 of [9]) there exists a homotopy $h_t: M_G \to N_G$ such that $h_0 = f|M_G$, $h_t|C = f|C$ and h_1 is TR to W_G in N_G. Since M_G is a closed subset of M the homotopy h_t may be extended to a differentiable equivariant homotopy f_t of f by Lemma 1.

LEMMA 3.4. *Let $f: M \to N$ be a differentiable equivariant map of G manifolds and let $C \subset U \subset M$ where C is closed and invariant and U is open in M. If $h: U \to N$ is a differentiable equivariant map with $f|C = h|C$ then there is an equivariant homotopy F_t and an open set V with $C \subset \bar{V} \subset U$ and*

(i) $F_0 = f$
(ii) $F_t|M - U = f|M - U$
(iii) $F_1|V = h|V$

Proof. Let $\mathcal{O} \subset N \times N$ be an invariant neighborhood of the diagonal in $N \times N$ such that for all $(x, y) \in \mathcal{O}$ there is a unique minimal geodesic $\gamma_t(x\ y)$ with $\gamma_0(x, y) = x$, $\gamma_1(x, y) = y$. Define $H: U \to N \times N$ by $H(\eta) = (f(\eta), h(\eta))$. Let $U' = H^{-1}(\mathcal{O})$ and choose an open set V in M so that $V \subset U'$. Let $\lambda: M \to [0, 1]$ be invariant and differentiable with $\lambda(M - U) = 0$ and $\lambda(V) = 1$ and define

$$F_t(\eta) = \begin{cases} \gamma_{\lambda(\eta)t}(f(\eta), h(\eta)) & \eta \in U' \\ f(\eta) & \eta \in M - U' \end{cases}$$

Clearly F_t has the desired properties.

Let $\pi: E \to B$ be a Riemarnian G-vector bundle. Then there is a canonical decomposition $T(E)|B \approx T(B) \oplus E$. If $\pi': E' \to B'$ is another differentiable G-vector bundle and $f: E \to E$ is a differentiable equivariant map preserving the zero-section, define $\widetilde{df}: E \to E'$ by the composition $E \to T(B) \oplus E \approx T(E)|B \xrightarrow{df} T(E')|B' \approx T(B') \oplus E' \to E'$, $\widetilde{df}$ is a bundle homomorphism, the linearization of f. f is said to be linear on $E(m)$ if $f|E(m) = \widetilde{df}|E(m)$.

LEMMA 3.5. *Let f be as above with B compact and suppose f linear on $(E|C)(\eta)$ where C is closed in B. Then there is a differentiable equivariant homotopy F_t of f such that*

(i) $F_0 = f$
(ii) F_1 *is linear on* $E(\delta)$ *for some* $\delta > 0$
(iii) $F_t|E - E(2\delta) = f|E - E(2\delta)$
(iv) $F_t|(E|C) = f|(E|C)$

Proof. Apply Lemma 3.4 with $h = \widetilde{df}$, $U = E(2\delta)$; (iv) follows by choosing $2\delta < \eta$.

Let $V \subset W$ be orthogonal representations of G and let M be a compact G-manifold equivariantly imbedded in the representation space V with $p: \nu(M) \to M$ the normal bundle of this imbedding. Let $\pi: G_k(W) \to G_{r-k}(W)$ be the equivariant diffeomorphism defined by $D(y) = Id - y$; here $r =$ dimension W and a point $y \in G_k(W)$ is regarded as an orthogonal projection: $W \to W$ with nullity k, $D(y)$ clearly is an orthogonal projection with nullity $r - k$.

Definition. Let $M \subset V$ and $p: \nu(M) \to M$; an equivariant bundle *epimorphism* $f: \nu(M) \to \mu_k(W)$ is said to be consistent (with respect to the inclusion of V in W) at $x \in M$ if the following diagram is commutative:

$$\begin{array}{ccc} W & \xrightarrow{D(f(x))} & W \\ \cup & & \\ V & & \cup \\ \cup & & \\ p^{-1}(x)^{G_x} & \xrightarrow{f} & \pi^{-1}(f(x))^{G_x}. \end{array}$$

The symbol U^H denotes the orthogonal complement of the fixed point set in the representation space U of the group H, i.e. $U^H = (U_H)^\perp$. f is said to be consistent on $C \subset M$ if f is consistent at each $x \in C$.

PROPOSITION 3.6. *Let $N^{n+1} \subset V^{n+h} \oplus \mathbf{R}$ and let $f: \nu(N)|U \to \mu_k(V^{n+h} \oplus \mathbf{R})$ $(k + n + 1 = \dim(V^{n+h} \oplus \mathbf{R})$ be a consistent bundle map where U is a neighborhood of the closed invariant set $C \subset U \subset N$. Then $f|(\nu(N)|C)$ may be extended to a consistent bundle map $\nu(N) \to \mu_k(V^{n+h} \oplus \mathbf{R})$.*

COROLLARY 3.7. *Let $M^n \subset V^{n+h}$ and let $f_i: \nu(M) \to \mu_k(V^{n+h} \oplus \mathbf{R})$ $i = 1, 2$, be consistent bundle maps. Then there is a homotopy $F: \nu(M)x[0, 5] \to \mu_k(V^{n+h} \oplus \mathbf{R})$ such that*

(i) $F_0 = f_1$
(ii) $F_5 = f_2$
(iii) F_t *is a consistent bundle map for each t.*

Proof. Apply the above theorem to $M \times [0, 5] \subset V^{n+h} \oplus \mathbf{R}$, $U = M \times [0, 1)$, $\bigcup M \times (4, 5]$, $C = M \times 0 \cup M \times 5$ and $f: \nu(M) \times [0, 5]|U \to \mu_k(V^{n+h} \oplus \mathbf{R})$ defined by

$$f(v, t) = \begin{cases} f_1(v) & t < 1 \\ f_2(v) & t > 4 \end{cases} \quad v \in \nu(M)$$

Remark. The corrollary may be paraphrased, $\pi: \mu_k(V^{n+h} \oplus \mathbf{R}) \to G_k(V^{n+h} \oplus \mathbf{R})$ is $(n + 1)$ universal for *consistent* bundle maps.

Proof of Proposition. By Lemma 2.2 we must find a non-singular equivariant section of the G-vector bundle $\mathrm{Hom}(\nu(N), V^{n+h} \oplus \mathbf{R})$ which extends the section, s_f over C defined by f. A section, s, is said to be consistent at x if

$$\begin{array}{ccc} p^{-1}(x) & \xrightarrow{s(x)} & V^{n+h} \oplus \mathbf{R} \\ \cup & & \cup \\ p^{-1}(x)^{G_x} & \xrightarrow{id} & p^{-1}(x)^{G_x} \end{array}$$

is commutative, i.e. if $s(x)|p^{-1}(x)^{G_x}$ is the identity. Note that a consistent non-singular section defines a consistent bundle map and vice-versa. For $A \subset B \subset N$, $H \subset G$, let $\Gamma_H(B, A)$ denote the consistent H equivariant sections of $\mathrm{Hom}(\nu(N), V^{n+h} \oplus \mathbf{R})$ over $B \cup C$ which extend the section s_f and are non-singular on A. Note that $\Gamma_G(N, \varnothing)$ is a closed subset of the space of equivariant sections of $\mathrm{Hom}(\nu(N), V^{n+h} \oplus \mathbf{R})$ with the $C-0$ topology) and hence is a complete metric space. We shall prove

(i) $\Gamma_G(N, \varnothing) \neq \varnothing$

(ii) for each $x \in N - C$, there is a compact invariant set C_x such that $x \in C_x$, $C_x \cap C = \varnothing$ and there exists a countable number of C_{x_i} such that $UC_{x_i} \cup C = N$

(iii) $\zeta : \Gamma_G(N, \varnothing) \to \Gamma_G(C_x, \varnothing)$ is open

(iv) $\Gamma_G(C_x, C_x)$ is open and dense in $\Gamma_G(C_x, \varnothing)$.

Then by (iii) and (iv) $\Gamma_G(N, C_x)$ is open and dense in $\Gamma_G(N, \varnothing)$; by (ii) and Baire's theorem $\cap \Gamma_G(N, C_{x_i}) = \Gamma_G(N, N)$ is open and dense in $\Gamma_G(N, \varnothing)$ and hence by (i) there exists a non-singular equivariant consistent section and thus a consistent bundle map $\nu(N) \to \mu_k(V^{n+h} \oplus \mathbf{R})$ extending f.

(i) let s_f be the section over U defined by f and let s_N be the section over N defined by $\nu(N) \subset T(V^{n+h} \oplus \mathbf{R})|N = (V^{n+h} \oplus \mathbf{R}) \times N \to V^{n+h} \oplus \mathbf{R}$. Let $\lambda : N \to I$ be differentiable and invariant with $\lambda(N - U) = 0$, $\lambda(C) = 1$. Then $s(x) = \lambda(x)s_f(x) + (1 - \lambda(x))s_N(x)$ is clearly consistent hence $\Gamma_G(N, \varnothing) \neq \varnothing$.

(ii) for each $x \in N - C$, there is a slice S_x in N such that $S_x \cap C = \varnothing$; then let $C_x = G((S_x)_{G_x})$, i.e. $C_x = \{y \in GS_x | G_y \text{ is conjugate to } G_x\}$. If $P \subset V^{n+h} \oplus \mathbf{R}$ is a submanifold (not necessarily compact) then P may be covered by a countable number of C_{x_i}. Note that if there is only one orbit type in P, i.e. all G_x, $x \in P$, are conjugate then C_x contains a neighborhood of x and hence a countable number of C_{x_i} will cover P. If there are r orbit types in P, let (H) be the minimal orbit type, i.e. $H = G_x$ for some $x \in P$ and there does *not* exist a $y \in P$ with $G_y \supset H$; then $P_0 = \{x \in P | G_x \text{ is conjugate to } H\}$ is a closed submanifold of P with only one orbit type and hence can be covered by a countable number of C_{x_i} (it is immaterial whether one chooses a slice in P_0 or a slice in P to define C_x). Moreover, $P - P_0$ has only $r - 1$ orbit types and hence by induction may be covered by a countable number of C_{x_i}; therefore P may be so covered.

(iii) to show that $\zeta : \Gamma_G(N, \varnothing) \to \Gamma_G(C_x, \varnothing)$ is open, it is sufficient to show that if $s \in \Gamma_G(N, \varnothing)$ and $s' \in \Gamma_G(C_x, \varnothing)$ with $\|s' - s|C_x\| < \varepsilon$ then there exists a $s'' \in \Gamma_G(N, \varnothing)$ with $\zeta(s'') = s''|C_x = s'$ and $\|s'' - s\| < 3\varepsilon/2$. Suppose that s' can be extended to a consistent section s''' in a neighborhood U of C_x; then since $\|s'''|C_x - s|C_x\| < \varepsilon$ there exists a neighborhood V of C_x with $\|s|V - s'''|V\| < 3/2\varepsilon$.

Let $\lambda : N \to I$ be invariant and differentiable with $\lambda(N - V) = 0$, $\lambda(C_x) = 1$ and let $s''(x) = \lambda(x)s'''(x) + (1 - \lambda(x))s(x)$ then s'' clearly has the desired property.

To establish the neighborhood extension property for consistent sections and the set C_x we first note that $\Gamma_G(C_x, \varnothing) = \Gamma_{G_x}((S_x)_{G_x}, \varnothing)$ by equivariance. Moreover, $S_x(2)$ (the slice of radius 2 at x) is equivariantly contractible and hence by Corollary 2.6 $\nu(M)|S_x(2) \simeq S_x(2) \times W \times \mathbf{R}^a$ where W is a representation space of G_x and $k = a + \dim W$. Let $\theta : \nu(M)|S_x(2) \to S_x(2) \times W \times \mathbf{R}^a$ be an equivalence. Via θ an element $s \in \Gamma_{G_x}(S_x(2), \varnothing)$ may be regarded as a pair of G_x equivariant maps $s_1 : S_x(2) \to \mathrm{Hom}(W, V^{n+h} \oplus \mathbf{R})_1 s_2 : s_x(2) \to \mathrm{Hom}(\mathbf{R}^a, V^{n+h} \oplus \mathbf{R})$. If $s' \in \Gamma_{G_x}((S_x)_{Gx}, \varnothing)$ then $s_2' : (S_x)_{Gx} \to \mathrm{Hom}(\mathbf{R}^a, V^{n+h} \oplus \mathbf{R})$ may clearly be extended to a map $s_2'' : S_x(2) \to \mathrm{Hom}(\mathbf{R}^a, V^{n+h} \oplus \mathbf{R})$ since $(S_x)_{Gx}$ is a G_x equivariant retract of $S_x(2)$. To show that s_1' may be extended so as to be consistent we note that s_1' is defined on $(S_x)_{G_x}$ by $s_1'(y): y \times W \times 0 \subset (S_x)_{G_x} \times W \times \mathbf{R}^a \overset{\theta}{\cong} \nu(N)|(S_x)_{G_x} \subset T(V^{n+h} \oplus \mathbf{R})|(S_x)_{G_x} \approx (S_x)_{Gx} \times V^{n+h} \oplus \mathbf{R} \to V^{n+h} \oplus \mathbf{R}$ and hence s_1' may be extended to $s_1': S_x(2) \to \mathrm{Hom}(W, V^{n+h} \oplus \mathbf{R})$ by $s_1'': S_x(2) \times W \times O \subset S_x(2) \times W \times \mathbf{R}^a \approx \nu(M)|S_x(2) \subset S_x(2) \times V^{n+h} \oplus \mathbf{R} \to V^{n+h} \oplus \mathbf{R}$. Hence the section $s''(y) = s_1''(y) + s_2''(y)$ defined by s_1'' and s_2'' clearly extends s' and is consistent. Thus $\Gamma_{G_x}(S_x(2), \varnothing) \to \Gamma_{G_x}((S_x)_{G_x}, \varnothing)$ and hence $\Gamma_G(GS_x(2), \varnothing) \to \Gamma_G(C_x, \varnothing)$ is onto:

(iv) to show that $\Gamma_G(C_x, C_x)$ is open and dense in $\Gamma_G(C_x, \varnothing)$ or equivalently that $\Gamma_{G_x}((S_x)_{G_x}, (S_x)_{G_x})$ is open and dense in $\Gamma_{G_x}(S_x)_{G_x}, \varnothing)$ let $s \in \Gamma_{G_x}((S_x)_{G_x}, \varnothing)$ $s = s_1 + s_2$ as before where $s_1: (S_x)_{G_x} \to \mathrm{Hom}(W, V^{n+h} \oplus \mathbf{R})$, $s_2: (S_x)_{G_x} \to \mathrm{Hom}(\mathbf{R}^a, V^{n+h} \oplus \mathbf{R})$. Note that $s_1(y)$ is a monomorphism for each y by consistency, hence it is sufficient to show that s_2 can be approximated by a map s_2' with $s_2'(y)$ a monomorphism for each y. Since G_x acts trivially on $(S_x)_{G_x}$ and s_2 is G_x equivariant, we may regard s_2 as a map into $\mathrm{Hom}(\mathbf{R}^a, (V^{n+h} \oplus \mathbf{R})_{G_x})$.

Letting $F_j = \{T \in \mathrm{Hom}(\mathbf{R}^a, (V^{n+h} \oplus \mathbf{R})_{Gx} | \mathrm{rank}\, T = j\}$ $j = 0, 1 \ldots, a-1$, we see that codimension $F_j < \dim(S_x)_{G_x}$ (Lemma 2.3) since $\dim(S_x)_{G_x} + \dim R^a + \dim T(Gx)_{G_x} = \dim(V^{n+h} \oplus R)_{G_x}$ and hence $\dim(S_x)_{G_x} + a \leq \dim(V^{n+h} \oplus \mathbf{R})_{G_x}$. Thus, s_2 may be approximated arbitrarily closely by a map transversal to $F_0 \cup F_1 \ldots \cup F_{a-1}$, i.e. by a map s_2' with $s_2'(y)$ a monomorphism for each y. Then $s' = s_1 + s_2'$ is a non-singular approximation showing that $\Gamma_G(C_x, C_x)$ is dense in $\Gamma_G(C_x, \varnothing)$. Clearly $\Gamma_G(C_x, C_x)$ is open.

Definition. Let $W \subset V^{n+h} \oplus \mathbf{R}$ and let $f: W \to \mu_k(V^{n+h} \oplus \mathbf{R})$ be a differentiable equivariant map. Then f is said to be consistently transverse regular (CTR) at $0 \in W$ if $f(0) \notin G_k(V^{n+h} \oplus \mathbf{R})$ or if $f(0) \in G_k(V^{n+h} \oplus \mathbf{R})$ then

(i) $f|W_G: W_G \to \mu_k(V^{n+h} \oplus \mathbf{R})_G$ is transverse regular to $G_k(V^{n+h} \oplus \mathbf{R})_G$ at 0, and if $F = (f|W_G)^{-1}(G_k(V^{n+h} \oplus \mathbf{R})_G)$ then
(ii) f is locally linear at F and
(iii) $f: \nu(F) \to \mu_k(V^{n+h} \oplus \mathbf{R})$ is consistent.

f is said to be CTR at $w \in W$ if $f|S_w$ is CTR as a G_w map where S_w is the slice at w defined by the end point map. f is said to be CTR on $C \subset W$ if f is CTR at each $x \in C$.

LEMMA 3.8. *If f is* CTR *on a neighborhood of $W_G(1)$ in W_G then there is a neighborhood of $W_G(1)$ in W on which f is* CTR.

Proof. Follows immediately from local linearity.

LEMMA 3.9. *Let $f: W \to \mu_k(V^{n+h} \oplus \mathbf{R})$ be* CTR *in a neighborhood U of the closed set C. If $(V^{n+h} \oplus \mathbf{R})^G \subset W$ then there is a homotopy $F_t: W \to \mu_k(V^{n+h} \oplus \mathbf{R})$ such that*

(i) $F_0 = f$
(ii) $F_t|W - W(2) = f|W - W(2)$
(iii) $F_t|C = f|C$
(iv) F_1 *is* CTR *on a neighborhood of $C \bigcup W_G(1)$.*

Proof. By Lemma 3.3 we may assume that $f|W_G$ is TR to $G_k(V^{n+h} \oplus \mathbf{R})_G$ in $\mu_k(V^{n+h} \oplus \mathbf{R})_G$ at points of $W_G(2)$. Let $F = (f|W_G(2))^{-1}(G_k(V^{n+h} \oplus \mathbf{R})_G)$. Then by Lemma 3.5 we may assume that $f|W_G(2)$ is linear on $\nu(F, W_G)(\delta)$ for some $\delta > 0$. There is at most one CTR map

$h: \nu(F)(\delta) \cup U \to \mu_k(V^{n+h} \oplus \mathbf{R})$ such that $h|U = f$ and $h|\nu(F, W_G)(\delta) = f$ and, since $W \supset (V^{n+h} \oplus \mathbf{R})^G$ exactly one; hence, by Lemma 3.4 there is a homotopy F_t satisfying (i), (ii) and (iii) with $F_1|\ \nu(F, W_G)(\delta) = h$, i.e. with F_1 satisfying (iv).

Let V^{n+h} be identified with $S(V^{n+h} \oplus \mathbf{R})$ − {north pole} in some fixed way; it then makes sense to talk of a map $f: (S(V^{n+h} \oplus \mathbf{R})) \to (T_k) V^{n+h} \oplus \mathbf{R}), \infty)$ being CTR.

LEMMA 3.10. *Let* $X = S(V^{n+h} \oplus \mathbf{R})$ *or* $S(V^{n+h} \oplus \mathbf{R}) \times I$ *and let* $f: X \to T_k(V^{n+h} \oplus \mathbf{R})$ *be an equivariant differentiable map which is* CTR *on a neighborhood of the closed set* $C \subset X$. *If* G_x *acts trivially on* $T(G/G_x)_e$ *for each* $x \in X$, *then* f *is homotopic to a map* $\bar{f}$ *which is* CTR *on* X. *Moreover,* $\bar{f}$ *may be chosen so that* $\bar{f}|C = f|C$.

Proof. Note that if $x \in X$, the G_x space S_x satisfies the hypothesis of Lemma 3.9, $S_x \supset (V^{n+h} \oplus R)^{G_x}$, since G_x acts trivially on $T(G/G_x)_e$; and $S_x + T(G|G_x)_e = V^{n+h} \oplus \mathbf{R}|G_x$.

If H is an isotropy group in X, define the level of H by level $G = 0$; level $H \geq s$ if $H \subsetneqq H'$ where H' is an isotropy group with level $H' = s - 1$; level $H = s$ if level $H \geq s$ and level $H \ngeq s + 1$. Let $X_r = \{x \in X | \text{level } G_x \leq r\}$. Then $X_{-1} = \varnothing$ and $X_0 = X_G$. Suppose that $X = X_s$ and that $f_r: X \to T_k(V^{n+h} \oplus \mathbf{R})$ is defined so that

(i) f_r is homotopic to f,

(ii) $f_r|C = f|C$,

(iii) f_r is CTR on U_r where U_r is an open neighborhood of $C \cup X_r$.

If $f_{-1} = f$ then (i), (ii), (iii) above are satisfied and hence we proceed by induction. Since $X_{r+1} - (U_r \cap X_{r+1})$ is compact we may choose a finite number of slices $S_{x_i}(3)$, $i = 1, \ldots, m$, $x_i \in X_{r+1} - (U_r \cap X_{r+1})$ such that $\bigcup_{L=1}^{m} GSx_i$ covers $X_{r+1} - (U_r \cap X_{r+1})$ and $GSx_i(3) \cap (C \cup X_r = \varnothing$. Let $f_{r+1}^{-1} = f_r$ and suppose inductively that f_{r+1}^l has been defined so that

(i) f_{r+1}^l is homotopic to f,

(ii) $f_{r+1}^l|C = f|C$,

(iii) f_{r+1}^l is CTR on $U_r - G\left(\bigcup_{i=1}^{l} S_{x_i}(3)\right)$,

(iv) f_{r+1}^l is CTR on $G\left(\bigcup_{i=1}^{l} (S_{x_i})_{G_{x_i}}\right)$:

Applying Lemma 3.9 to $f_{r+1}^l|\mathring{S}_{x_{l+1}}(3)$ and the closed subset $\mathring{S}_{x_{l+1}}(3) \bigcap G\left(\bigcup_{i=1}^{l} (S_{x_i})_{G_{x_i}}\right)$ we get a map f_{r+1}^{l+1} such that (i) to (iv) are satisfied with $l + 1$ replacing l. Finally let $f_{r+1} = f_{r+1}^m$. Then f_{r+1} is homotopic to f and $f_{r+1}|C = f|C$ by construction. Moreover, f_{r+1} is CTR on $U_r - G\left(\bigcup_{i=1}^{m} S_{x_i}(3)\right) \supset C$. Since f_{r+1} is also CTR on X_{r+1}, by Lemma 3.8 there is a neighborhood U_{r+1} of $C \cup X_{r+1}$ on which f_{r+1} is CTR. Hence, the inductive hypothesis is satisfied and $\bar{f} = f_s$ has the required properties.

Note that if G is finite or if G is abelian then G_x acts trivially on $T(G/G_x)_e$ and hence lemma 3.10 holds.

THEOREM 3.11. *If G is finite or abelian then* θ: $\eta_n(V) \to \pi_1^{V^{n+h}}(T_k(V^{n+h} \oplus R), \infty)$ *is an isomorphism.*

Proof. θ is onto : let $[f]\in\pi_1^{V^{n+h}}T_k(V^{n+h}\oplus\mathbf{R}),\infty)$. By Lemma 3.10 there is a CTR map $\bar{f}$ with $[\bar{f}]=[f]$. Let $\bar{f}^{-1}(G_k(V^{n+h}\oplus\mathbf{R}))=M$. Then $\theta([M])=[f_{M,i}]=[\bar{f}]$ since the bundle maps $\nu(M)\to\mu_k(V^{n+h}\oplus\mathbf{R})$ defined by $\bar{f}$ and $f_{M,i}$ are consistent and, therefore, homotopic by Corollary 3.7; the Thom construction applied to $M\times I\subset S(V^{n+h}\oplus\mathbf{R})\times I$ then yields a homotopy between $\bar{f}$ and $f_{M,i}$. Hence θ is onto.

To show that θ is a monomorphism suppose $\theta([M])=0$, i.e. suppose $f_{M,i}$ is equivariantly homotopic to $[0]$. If we knew that $f_{M,i}$ was a CTR map Lemma 3.10 would imply that there was a CTR homotopy $F:S(V^{n+h}\oplus\mathbf{R})\times I\to T_k(V^{n+h}\oplus\mathbf{R})$ with $F_0=f_{M,i}$ and $F_1=[0]$ and hence $F^{-1}(G_k(V^{n+h}\oplus\mathbf{R}))$ would provide a cobordism between M and $\varnothing$, i.e. would show that $[M]=0$. The only difficulty is that $f_{M,i}$ need not be locally linear.

Let $i:M\subset V^{n+h}$ and let $x\in M$. Then the G_x space, $T(M)_x$, splits as the direct sum of $T(Gx)_x$, the tangent space to the orbit and its orthogonal complement W (orthogonal with respect to the metric on M induced by i). Recall that any slice S_x is the image of a G_x equivariant diffeomorphism $\psi:W(\varepsilon)\to M$; $\psi(W(\varepsilon))=S_x$.

Definition. The imbedding $i:M\to V^{n+h}$ is said to be straight at $x\in M$ if, for some slice S_x at x, $S_x=\psi(W(\varepsilon))$, there is a $\delta>0$ such that the map $\psi':W(\delta)\to V^{n+h}$ given by $W(\delta)\subset W_{G_x}(\delta)xW^{G_x}(\delta)\xrightarrow{\psi'}V^{n+h}$, $\psi'(y,z)=i\circ\psi(y)+di_{\psi(y)}(z)$ for $y\in W_{G_x}(\delta)$; $z\in W^G{}_x(\delta)$ defines a slice at $i(x)\in i(M)$, i.e., $\psi'(W(\delta))\subset i(M)$ and $i^{-1}\circ\psi':W(\delta)\to M$ defines a slice at $x\in M$.

Remark 1. It is clear that this condition is independent at the particular slice S_x or map ψ.

Remark 2. If i is straight at x, then i is straight on a neighborhood of x, in fact, on. $G(\psi'(W)(\delta))$.

Remark 3. The map $f_{M,i}$ is CTR in a neighborhood of x if and only if i is straight at xi Hence to complete the proof of Theorem 3.11 we need only show there exists an imbedding $:M\to V^{n+h}$ such that i is straight at each $x\in M$.

LEMMA 3.12. *Let $i:M\to V^{n+h}$ be an imbedding which is straight on a neighborhood U at the closed invariant set C. Let $S_x(2)$ be a slice of radius 2 at $x\in M$. Then there is an imbedding $\bar{i}:M\to V^{n+h}$ such that $\bar{i}|C=\bar{i}|C$ and $\bar{i}$ is straight on $C\cup G((S_x)_{G_x})$.*

Proof. Let $\psi:W(\varepsilon)\to S_x(2)$ be as above and define $h:S_x(2)\to V^{n+h}$ by the composition. $S_x(2)\xrightarrow{\psi^{-1}}W(\varepsilon)\subset W_{G_x}(\varepsilon)\times W^{G_x}(\varepsilon)\xrightarrow{\psi'}V^{n+h}$ where $\psi'(y,z)=i\circ\psi(y)+di_{\psi(y)}(z)$ for $y\in W_{G_x}(\varepsilon)$ $z\in W^{G_x}(\varepsilon)$ and extend h to $GS_x(2)$ by equivariance. Let $\lambda:M\to[0,1]$ be an invariant differentiable map with $\lambda(C\cup M-GS_x(2))=0$, $\lambda(S_x(1)-U)=1$. Let $i_1:M\to V^{n+h}$ be defined by $i_1(p)=(1-\lambda(p))i(p)+\lambda(p)h(p)$. Note that $i_1|C\cup G(S_x)_{G_x}=i|C\cup G(S_x)_{G_x}$ and $di_1|C\cup G(S_x)_{G_x}=di|C\cup G(S_x)_{G_x}$ and hence that i_1 is an imbedding of a closed neighborhood Q of $C\cup G(S_x)_{G_x}$. By construction $i_1|Q$ is straight on $C\cup(S_x)_{G_x}$. Let $\bar{i}:M\to V^{n+h}$ be an imbedding with $\bar{i}|Q=i|Q$ (Corrollary 1.10). Then $\bar{i}$ satisfies the stated conditions.

Remark. Note that the metric induced from V^{n+h} by $\bar{i}$ and that induced by i agree on $C\cup G(S_x)_{G_x}$ and hence $\bar{i}$ is straight on C since i was straight on C.

To show that M admits a straight imbedding in V^{n+h} one proceeds by induction on the level sets $M_r = \{x \in M | \text{level } G_x \leq r\}$ as in Lemma 3.10. Lemma 3.12 justifies the inductive step.

§4. EQUIVARIANT MORSE THEORY

In this section we extend the results of *R*. Palais in [14] to study an invariant C^∞ function $f: M \to \mathbf{R}$ on a complete Riemannian G-space M.

Definition. At a critical point p of f, i.e., where $\Delta f_p = 0$, we have a bounded, self-adjoint operator, the hessian operator, $\varphi(f)_p = T(M)_p \to T(M)_p$ defined by $\langle \varphi(f)_p v, w \rangle = H(f)p$ (v, w) where $H(f)_p$ is the hessian bilinear form [14, §7]. A closed invariant submanifold V of M will be called a *critical manifold* of f if $\partial V = \emptyset$, $V \cap \partial M = \emptyset$ and if each $p \in V$ is a critical point of f. It follows that $T(V)_p \subseteq \ker \varphi(f)_p$ and so there is an induced bounded self-adjoint operator $\bar{\varphi}(f)_p: T(M)_p/T(V)_p \to T(M)_p/T(V)_p$. If $\bar{\varphi}(f)_p$ is an isomorphism for each $p \in V$ then V is called a *non-degenerate critical manifold of f.*

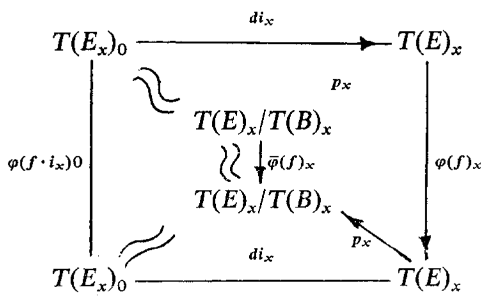
Recall that f is said to satisfy condition (C) [14, §10] if, for each closed subset S of M on which f is bounded, $\|\Delta f\|$ is bounded away from zero *or* there is a critical point $p \in S$.

Definition. The invariant C_∞ function $f: M \to \mathbf{R}$ is called a *Morse function* for the Riemannian G-manifold M if it satisfies condition (C) and if the critical locus of f is a union of non-degenerate critical manifolds without interior.

The behavior of a function near a critical manifold is specified by the Morse Lemma.

LEMMA 4.1. *Let $\pi: E \to B$ be a Riemannian G-vector bundle and f a Morse function on E having B (i.e., the zero section) as a non-degenerate critical manifold. If B is compact there is an equivariant diffeomorphism $\theta: E(r) \to E$ for some $r > 0$ such that $f(\theta(e)) = \|Pe\|^2 - \|(1-P)e\|^2$ where P is an equivariant orthogonal bundle projection.*

Proof. Let $E_x = \pi^{-1}(x)$ and let $i_x: E_x \to E$, $p_x: T(E)_x \to T(E)_x/T(B)$ then from the commutative diagram we see that



$\varphi(f \circ i_x)_0$ is an isomorphism. Hence, in each fibre, 0 is a non-degenerate critical point of the function $f \circ i_x$ and hence, by the results of [12], there is an origin preserving diffeomorphism $\theta_x: E_x \to E_x$ and a projection P_x such that $f \circ i_x \circ \theta_x(e) = \|P_x(e)\|^2 - \|(1-P_x)(e)\|^2$ in a neighborhood of the origin. To complete the proof, we must show that θ_x and P_x are smooth functions of x and that the resulting maps $\theta: E \to E$, $P: E \to E$ are equivariant.

Let Hom(E, E) denote the G-vector bundle over B with fibre Hom(E_x, E_x) at x where Hom(E_x, E_x) denotes the bounded linear operators on E_x and the action of G on Hom(E, E) is given by $gT = \bar{g} \cdot T \cdot \overline{g^{-1}}$ where $T \in \mathrm{Hom}(E_x, E_x)$ and $gT \in \mathrm{Hom}(E_{gx}, E_{gx})$. We regard $B \subset E$ via the zero section. We shall define an equivariant fibre preserving map $A: E \to \mathrm{Hom}(E, E)$ such that

(i) $A(e)$ is a self-adjoint operator for each $e \in E$,
(ii) $f(e) = \langle A(e)e, e\rangle$,
(iii) if $x \in B$, $\bar{\varphi}(f)_x \circ p_x \circ di_x = 2p_x \circ di_x \circ A(x)$

$$\begin{array}{ccccccc} E_x = T(E_x)_0 & \xrightarrow{di_x} & T(E)_x & \xrightarrow{p_x} & T(E)_x/T(B)_x \\ {\scriptstyle 2A(x)}\downarrow \quad \downarrow{\scriptstyle \varphi(f\cdot i_x)_0} & & \downarrow{\scriptstyle \varphi(f)_x} & & \downarrow{\scriptstyle \bar{\varphi}(f)_x} \\ E_x = T(E_x) & \xrightarrow{di_x} & T(E)_x & \xrightarrow{p_x} & T(E)_x/T(B)_x \end{array}$$

A is given by

$$\langle A(e)v_1 v_2\rangle = \int_0^1 (1-t)d^2(f \circ \psi^{-1})_{\varphi(te)}(d\psi_{te}(\bar{v}_1), d\psi_{te}(\bar{v}_2))dt,$$

where $\psi: \pi^{-1}(U) \to U \times F$ is any bundle chart for E at $\pi(e)$ and $\bar{v}_i$ denotes the tangent vector at te corresponding to $v_i \in E$, i.e., $\bar{v}_i = (di_{(\pi e)})_{te}(v_i)$. Property (i) follows from the symmetry of $d^2(f \cdot \psi^{-1})$ and (iii) follows from the fact that $tx = x$ for $x \in B$ and $\int_0^1 (1-t)dt = 1/2$.

Since $f(B) = 0$ and $df|B = 0$ Taylor's formula for f with $n = 1$ yields the remainder term

$$f(e) = \int_0^1 (1-t)d^2(f \circ \psi^{-1})_{\psi(te)}(d\psi_{te}(\bar{e}), d\psi_{te}(\bar{e}))dt = \langle A(e)e, e\rangle$$

and hence (ii). To show that A is well-defined we apply the chain rule to $(f \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1}) = f \circ \psi^{-1}$ where $\varphi: \pi^{-1}(U) \to U \times F$ is another bundle chart, noting that $\varphi \circ \psi^{-1}$ is linear in each fibre and hence $d^2(\varphi \circ \psi^{-1})_e(\bar{v}_1, \bar{v}_2) = 0$ for $v_1, v_2 \in E$. Then

$$\begin{aligned} d^2(f \circ \psi^{-1})_{\psi(e)}(d\psi_e(\bar{v}_1), d\psi_e(\bar{v}_2)) &= d^2(f \circ \varphi^{-1})_{\varphi(e)}(d\varphi_e(\bar{v}_1), d\varphi_e(\bar{v}_2)) \\ &\quad + d(f \circ \varphi^{-1})_{\varphi(e)}[d^2(\varphi \circ \psi^{-1})_{\psi(e)}(d\psi_e(\bar{v}_1), d\psi_e(\bar{v}_2))] \\ &= d^2(f \circ \varphi^{-1})_{\varphi(e)}(d\varphi_e(\bar{v}_1), d\varphi_e(\bar{v}_2)) \end{aligned}$$

and hence A is well-defined. To demonstrate the equivariance of A we note that if ψ is a bundle chart at $\pi(e)$ then

$$(\bar{g} \times id) \circ \psi \circ \bar{g}^{-1}: \pi^{-1}(gU) \xrightarrow{g^{-1}} \pi^{-1}(U) \longrightarrow U \times F \xrightarrow{g \times id} gU \times F$$

is a bundle chart at $\pi(ge)$. Then $\langle A(ge)gv_1, gv_2\rangle \overset{\text{def}}{=} d^2(f \circ \bar{g} \circ \psi^{-1} \circ (\bar{g}^{-1} \times id))^{(\bar{g} \times id)\psi(e)}$ $(d(\bar{g} \times id)d\psi_e(\bar{v}_1), d(\bar{g} \times id)d\psi_e(\bar{v}_2)$ and by the invariance of f and the chain rule this equals

$$d^2(f \circ \psi^{-1})_{\psi(e)}(d\psi_e(\bar{v}_1), d\psi_e(\bar{v}_2)) + d(f \circ \psi^{-1})_{\psi(e)}(d^2(\bar{g}^{-1} \times id)(d\bar{g}(\bar{v}_1), d\bar{g}(\bar{v}_2)).$$

Since $\bar{g}^{-1} \times id$ is linear in each fibre $d^2(\bar{g}^{-1} \times id) = 0$ and hence $\langle A(ge)gv_1, gv_2\rangle = \langle A(e)v_1, v_2\rangle$ and thus $A(ge) = g \circ A(e) \circ g^{-1}$ since the metric is invariant. The maps θ, P are limits of polynomials in A and hence are equivariant and differentiable. The rest of the proof follows formally as in [14, §7].

An important property of Morse functions is given by:

PROPOSITION 4.2. *If f is a Morse function the critical locus of f in $f^{a,b} = f^{-1}[a, b]$ is the union of a finite number of disjoint, compact, non-degenerate critical manifolds of f.*

Proof. Let $\{a_n\}$ be a sequence at points in the critical set. Since, by assumption, a_n is in a non-degenerate critical manifold without interior we may choose points $\{b_n\}$ such that

$$\text{(i) the distance } \rho(a_n, b_n) < \frac{1}{n}$$
$$\text{(ii) } a - 1 < f(b_n) < b + 1$$
$$\text{(iii) } 0 < \|\Delta f_{b_n}\| < \frac{1}{n}.$$

Then by condition (C) there is a critical point p adherent to $\{b_n\}$ and hence $\{b_n\}$ has a subsequence which converges to p. The corresponding subsequence of $\{a_n\}$ will also converge to p, thus proving the compactness of the critical set in $f^{a,b}$.

We also have the Diffeomorphism Theorem.

THEOREM 4.3. *Let f be a Morse function on M, $\partial M = \emptyset$, with no critical value in the bounded interval $[a, b]$. If $f^{a-\delta,b+\delta}$ is complete for some $\delta > 0$ then $f^a = f^{-1}(-\infty, a]$ is equivariantly diffeomorphic to f^b.*

Proof. Essentially, this theorem is Proposition 2, Section 10 of [14]. We need only verify that the map defined there is equivariant. The map is given by $p \to \sigma_p(\alpha(f(p)))$ where $\alpha: \mathbf{R} \to \mathbf{R}$ is C_∞; hence

$$gp \to \sigma_{gp}(\alpha(f(gp))) = \sigma_{gp}(\alpha(f(p))) = g\sigma_p(\alpha(f(p))).$$

COROLLARY 4.4. (Palais and Stewart [13]). *Every differentiable deformation ψ_t of a G-manifold M is trivial.*

Proof. Recall that a differentiable deformation is a one-parameter family of actions $\psi_t: G \times M \to M$ such that the action $\psi: G \times M \times \mathbf{R} \to M \times \mathbf{R}$ given by $\psi(g, m, t) = (\psi_t(g, m), t)$ is differentiable. ψ_t is trivial if there is a one-parameter family of diffeomorphisms θ_t of M such that $\psi_t(g, m) = \theta_t \psi_0(g, \theta_t^{-1}(m))$. Let $M \times \mathbf{R}$ have a complete invariant metric with respect to ψ and let $f: M \times \mathbf{R} \to \mathbf{R}$ be the projection onto the second factor. Since f is a Morse function and has no critical points, the map $\theta_t(p) = \sigma_p(t)$ has the required properties.

Definition. Let V, W be Riemannian G-vector bundles over B. The bundle $V(1) \oplus W(1) = \{(x, y) \in V \oplus W \mid \|x\| \leq 1, \|y\| \leq 1\}$ (not a manifold) is called a handle-bundle of type (V, W) with index = dimension of W. Let N, M be G-manifolds with boundary, $N \subset M$ and $\bar{F}: V(1) \oplus W(1) \to M$ a homeomorphism onto a closed subset H of M. Let $F = \bar{F} | V(1) \oplus \dot{W}(1)$. We shall write $M = N \cup_F H$ and say that M arises from N by attaching a handle-bundle of type (V, W) if

(i) $M = N \cup H$
(ii) F is an equivariant diffeomorphism onto $H \cap \partial N$
(iii) $\bar{F} | V(1) \oplus (1) \mathring{W}$ is an equivariant diffeomorphism onto $M - N$.

LEMMA 4.5 (Attaching Lemma). *Let $\pi: E \to B$ be a Riemannian G-vector bundle and P an orthogonal bundle projection. Let $V = P(E)$, $W = (1 - P)(E)$ and define $f, g: E \to \mathbf{R}$ by $f(e) = \|Pe\|^2 - \|(1-P)e\|^2$, $g(e) = f(e) - 3\varepsilon/2\lambda(\|Pe\|^2/\varepsilon)$ where $\varepsilon > 0$ and λ is a positive C^∞ function which is monotone decreasing, $\lambda([0, 1/2]) = 1$ and $\lambda(1) = 0$. Then $\{x \in E(2\varepsilon) | g(x) \leq -\varepsilon\}$ arises from $\{x \in E(2\varepsilon) | f(x) \leq -\varepsilon\}$ by attaching a handle-bundle of type (V, W).*

Proof. Let $\sigma(s)$ be the unique solution of $\lambda(\sigma)/1 + \sigma = 2/3(1 - s)$ for $s \in [0, 1]$. Define $\bar{F}: V(1) \oplus W(1) \to E$ by $\bar{F}(x, y) = (\varepsilon\sigma(\|x\|^2)\|y\|^2 + \varepsilon)^{1/2}x + (\varepsilon\sigma(\|x\|^2))^{1/2}y$. It is shown in Section 11 of [14] that $\bar{F}$ has the required properties.

Note that B is a non-degenerate critical manifold of f. By the Morse lemma we can choose coordinates for $\pi: E \to B$ and a projection P such that $f(e) = \|Pe\|^2 - \|(1-P)e\|^2$ in a neighborhood of B for any function f having B as a non-degenerate critical manifold. Hence, by abuse of notation, we shall also refer to the handle-bundle of type $(P(E), (1-P)E)$ as a handle-bundle of type (B, f).

THEOREM 4.6. *Let f be a Morse function on the complete Riemannian G-space M. If f has a single critical value $a < c < b$ in the bounded interval $[a, b]$ then the critical locus of f in $[a, b]$ is the disjoint union of a finite number of compact submanifolds $N_1, \ldots, N_s$. f^b is equivariantly diffeomorphic to f^a with s handle-bundles of type (N_i, f) disjointly attached.*

Proof. Only the last statement remains. Let $\{U_i\}_{i=1,\ldots,}$ be disjoint tubular neighborhoods of the critical submanifolds $\{N_i\}$ given by the maps $T_i: \nu(N_i)(2\delta) \to U_i$ where $\nu(N_i)$ is the normal bundle of N_i in M with the induced Riemannian metric. We may assume $c = 0$ and by the Morse Lemma that $f \circ T_i(x) = \|P_i x\|^2 - \|(1-P_i)x\|^2$ where P_i is an orthogonal bundle projection in $\nu(N_i)$. Choose ε so that $0 < \varepsilon < \delta^2$ and $a < -3\varepsilon$, $3\varepsilon < b$.

Let $Q = f^{-2\varepsilon,\infty}$ and define $g: Q \to R$ by

$$g(x) = \begin{cases} f(x) & x \notin \bigcup_{i=1}^{s} U_i \\ f(x) - 3\varepsilon/2\lambda(\|P_i T_i^{-1}(x)\|^2/\varepsilon) & x \in U_i \end{cases}$$

where λ is the function defined in the Attaching Lemma. It is shown in (14, §11) that g is C^∞ and $g^\varepsilon = (f|Q)^\varepsilon$. Moreover, by the Attaching Lemma, $g^{-\varepsilon}$ is equivariantly diffeomorphic to $(f|Q)^{-\varepsilon} \cup s$ handle-bundles of type (N_i, f). Since f has no critical value in $[a, -\varepsilon]$ or $[\varepsilon, b]$ it is sufficient to show that $g^{-\varepsilon} \approx g^\varepsilon$. To that end we apply the Diffeomorphism Theorem to the manifold without boundary $g^{-1}(-5\varepsilon/4, 5\varepsilon/4)$ and the function g. We note that $g^{-}C^{9\varepsilon/8, 9\varepsilon/8}$ is complete and hence we need only show that g is a Morse function, i.e., $\|\nabla g\|$ is bounded away from zero for $x \in g^{-1}(-5\varepsilon/4, 5\varepsilon/4)$. Since $g(N_i) = -3\varepsilon/2$, $N_i \cap g^{-1}(-5\varepsilon/4, 5\varepsilon/4) = \varnothing$. Hence there is an $\alpha > 0$ such that $T_i(\nu(N_i)(\alpha)) \cap g^{-1}(-5\varepsilon/4, 5\varepsilon/4) = \varnothing$. Moreover, $f(g^{-1}(-5\varepsilon/4, 5\varepsilon/4)) \subset [-5\varepsilon/4, 5\varepsilon/4]$ and hence, since f has no critical points in $g^{-1}[-5\varepsilon/4, 5\varepsilon/4]$, $\|\nabla f_x\|$ must be bounded away from zero, say $\|\nabla f_x\| \geq \eta > 0$. But

$$g|Q - \bigcup_{i=1}^{s} U_i = f|Q - \bigcup_{i=1}^{s} U_i \quad \text{and hence} \quad \|\nabla g_x\| \geq \eta > 0 \quad \text{for} \quad x \in Q - \bigcup_{i=1}^{s} U_i.$$

Thus we need only show that $\|\nabla g\| \,|U_i \cap g^{-1}(-5\varepsilon/4, 5\varepsilon/4)$ is bounded away from zero. To compute $\|\nabla g\|$ we first construct a Riemannian metric $\langle , \rangle^*$ for $T(\nu(N_i))$ such that $\langle \bar{v}_1, v_2 \rangle =$

$\langle v_1, v_2 \rangle$ where $v_i \in v(N_i)$, $\langle , \rangle$ denotes the metric in $v(N_i)$ and $\bar{v}_i$ denotes the tangent vector at $x \in v(N_1)$ corresponding to v_i. Then if $x \in v(N_i)$, let $\bar{w} = P_i(x) - (1 - P_i)(x) \in T(v(N_i))_x$. We have $d(g \circ T)_x(\bar{w}) = 2[\langle Px, \bar{w} \rangle - \langle 1 - Px, \bar{w} \rangle] - 3\lambda'(\|Px\|^2/\varepsilon)\langle Px, \bar{w} \rangle$ since $g \circ T(x) = \|Px\|^2 - \|1 - Px\|^2 - 3\varepsilon/2\lambda(\|Px\|^2/\varepsilon)$. Since $\lambda'(t) \leq 0$, $d(g \circ T)_x(\bar{w}) = 2\|x\|^2 - 3\lambda'(\|Px\|^2 \geq 2\|x\|^2$. But $d(g \circ T)_x(\bar{w}) = dg_{Tx}(dT_x(\bar{w})) = \langle \nabla g_{Tx}, dT_x(\bar{w}) \rangle \leq \|\nabla g_{Tx}\| \, \|dT_x(\bar{w})\| \leq \|\nabla g_{Tx}\| \, \|dT_x\| \, \|\bar{w}\| = \|\nabla g_{Tx}\| \, \|dT_x\| \, \|x\|$. Since $\|x\| \geq \alpha$ we see that $\|\nabla g_{Tx}\| \geq \| 2\alpha/\|dT_x\|$. We need only show that $\|dT_x\|$ is bounded. Since N_i is compact $\|dT\|$ is bounded on $N_i \subset v(N_i)$ and hence in a neighborhood $v(N_i)(\beta)$ of N_i. Hence since δ was arbitrary we assume $2\delta < \beta$. Finally, we have $(f|Q)^b \approx (f|Q)^\varepsilon = g^\varepsilon \approx g^{-\varepsilon} \approx (f|Q)^{-\varepsilon} \cup s$ handle-bundles of type (N_i, f) and therefore $f^b \approx f^{-\varepsilon} \cup s$ handle-bundles $\approx f^a \cup s$ handle-bundles. The homology implications of the above theorem are contained in

COROLLARY 4.7 (Bott [2]). *Let $N_1, \ldots, N_r$ be those critical manifolds in $f^{a,b}$ with index $(N_i, f) = k_i < \infty$. Then*

$$H_n(f^b, f^a; Z_2) \approx \sum_{i=1}^{t} H_{n-k_i}(N_i; Z_2).$$

Proof. By the above theorem $f^b \approx f^a \cup s$ handle-bundles of type (N_i, f). Let $H_i = V_i(1) \oplus W_i(1)$ denote the ith handle-bundle and let $P_i \colon V_i \oplus W_i \to V_i \oplus W_i$ denote the projection onto V_i. Then by excising out the interior of f^a we have

$$H_n(f^b, f^a; Z_2) \approx \sum_{i=1}^{s} H_n(H_i, V_i(1) \oplus \dot{W}_i(1); Z_2).$$

But $H_n(H, V(1) \oplus \dot{W}(1); Z_2) = H_n(W(1), \dot{W}(1); Z_2)$ since the fibre of H is convex and we have an equivariant fibre preserving retraction, ρ, of H onto $V(1) \oplus \dot{W}(1) \cup 0 \oplus W(1)$ given by

$$\rho(h) = \rho(P(h), (1 - P)(h)) = \rho(x, y)$$
$$= \begin{cases} \left(\dfrac{2x}{2 - \|y\|}, 0\right) & \text{if } \|x\| \leq 1 - \dfrac{\|y\|}{2} \\ \left(\dfrac{x}{\|x\|}, (2\|x\| + \|y\| - 2)\dfrac{y}{\|y\|}\right) & \text{if } \|x\| \geq 1 - \dfrac{\|y\|}{2}. \end{cases}$$

Hence

$$H_n(f^b, f^a; Z_2) \approx \sum_{i=1}^{s} H_n(W_i(1), \dot{W}_i(1); Z_2)$$
$$\approx \sum_{i=1}^{t} H_{n-k_i}(N_i; Z_2) + \sum_{i=t+1}^{s} H_n(W_i(1), \dot{W}_i(1); Z_2)$$

where the last isomorphism is the Thom isomorphism for $i \leq t$. It only remains to show that $H_n(W(1), \dot{W}(1); Z_2) = 0$ if $\dim W = \infty$ or even strong that $\pi_m(W(1), \dot{W}(1)) = 0$ for all m. Let $\alpha \colon D^n, S^{n-1} \to W(1), \dot{W}(1)$ represent an element of $\pi_m(W(1), \dot{W}(1))$. We may approximate α by a map α' which is homotopic to α, differentiable and transverse regular to N, the zero section. Since codimension $N = \infty$, $\alpha'(D^n) \cap N = \varnothing$ and we can deform α' into $\dot{W}(1)$ and hence $[\alpha' =]0$. Thus critical manifolds of infinite index do not affect the homology of (f^b, f^a).

Now let a, b be arbitary regular values of f, $a < b$, and again denote the critical manifolds of finite index k_i by $\{N_i\}$, $i = 1, \ldots, t$. Let $R_n(X) =$ dimension of $H_n(X; Z_2)$ and $\chi(X)$ the Euler characteristic of X. Then we have the Morse inequalities.

$$\text{(i)}\ \chi(f^b, f^a) = \sum_{i=1}^{t} (-1)^{k_i} \chi(N_i)$$

$$\text{(ii)}\ R_n(f^b, f^a) \le \sum_{i=1}^{t} R_{n-k_i}(N_i)$$

$$\text{(iii)}\ \sum_{l=0}^{n} (-1)^{n-l} R_l(f^b, f^a) \le \sum_{i=1}^{t} \sum_{l=0}^{n} (-1)^{n-l} R_{l-k_i}(N_i).$$

The statements follow from the above corollary and the fact that χ is additive, and R_n, $\sum_{n \le k} (-1)^{k-n} R_n$, are subadditive ([14], §15).

Remark. If every critical manifold of finite index in $f^{a,b}$ has an orientable normal bundle then equations (i), (ii), (iii), are valid with integer coefficients.

We now show that there exist Morse functions on any finite-dimensional G-manifold, M. To that end let $\mathscr{M}_G(A, M) \subset C_G(M, \mathbf{R})$ denote those functions whose critical locus in A is a union of non-degenerate critical orbits. Clearly $\mathscr{M}_G(A, M)$ is open if A is compact.

Density Lemma 4.8. *For any finite-dimensional G-manifold M, $\mathscr{M}_G(M, M)$ is dense in $C_G(M, \mathbf{R})$.*

Proof. Let $x \in M - M_G$. By the induction metatheorem of [13] we may assume that $\mathscr{M}_{G_x}(S(x), S(x))$ is dense in $C_{G_x}(S(x), \mathbf{R})$, where $S(x)$ is a slice at x. Since the restriction map $\rho: C_G(M, R) \to C_{G_x}(S(x), \mathbf{R})$ is open, $\rho^{-1}(\mathscr{M}_{G_x}(S(x), S(x)) = \mathscr{M}_G(B(x), M)$ is dense in $C_G(M, \mathbf{R})$. Now let $y \in M_G$ and let $A = B_y \cap M_G$. We show that $\mathscr{M}_G(A, M)$ is dense in $C_G(M, R)$ and then complete the proof with Baire's theorem. Let $f: M \to \mathbf{R}$. We must find a C^k approximation, f', such that f' has only non-degenerate critical points in A. We note that $\mathscr{M}(A, M_G)$ is dense in $C(M_G, \mathbf{R})$ (10, p. 37] and that the restriction map $C_G(M, R) \to C(M_G, \mathbf{R})$ is open. Hence, we may assume that $f|M_G$ has only non-degenerate critical points and by induction that y is the only critical point in A which is degenerate for f (y is non-degenerate for $f|M_G$). This problem is local and is settled by the following.

Lemma 4.9. *Let W be an Euclidean G-space and $f: W \to \mathbf{R}$ an invariant C^∞ function such that $f|W_G$ has only non-degenerate critical points and such that $0 \in W$ is the only degenerate critical point of f in $W(1)$. Then there exists a C^∞ invariant function $f': W \to R$ such that*

(i) $f'|W - W(2) = f|W - W(2)$

(ii) *f' has only non-degenerate critical points in $W_G(1)$*

(ii) *f' is a C^k approximation to f.*

Proof. Let $P: W \to W$ denote the internal projection onto W_G. Define f' by $f'(w) = f(w) + \varepsilon\lambda(\|w/c\|^2)\|(1-P)w\|^2$, where ε, c are constants to be chosen and λ is the function of Lemma 4.5. We choose $c < 2$ such that if $x \in W_G$ is a critical point of f, then $\|x\| > c$ or $x = 0$; this is clearly possible since $f|W_G$ has only isolated critical points by the Morse Lemma. Then note that $f'|W_G = f|W_G$ and $f'|W - W(c) = f|W - W(c)$ which proves (i) and shows that f' has at most 0 as a degenerate critical point. By definition of f', $\varphi_0(f')(v) = \varphi_0(f)(v) + 2\varepsilon(1-P)v$ or in matrix form

$$\varphi_0(f') = \begin{bmatrix} \varphi_0(f|W_G) & C \\ D & B + 2\varepsilon I \end{bmatrix}$$

where B, C, D are determined by f and φ_0 is the Hessian operator. But $\varphi_0(f|W_G)$ is non-singular since $f|W_G$ has only non-degenerate critical points and hence det $\varphi_0(f')$ is a non-zero polynomial in ε with roots $\varepsilon_1, \ldots, \varepsilon_n$; (iii) can then be satisfied by choosing ε small enough and (ii) be demanding that $\varepsilon \neq \varepsilon_i$.

Remark. Let $f \in \mathscr{M}_G(C, M)$ where C is closed and $\varepsilon: M \to \mathbf{R}$ a positive function. Let $C_G(f, C, \varepsilon) = \{h \in C_G(M) | h|C = f|C$ and $|h(x) - f(x)| < \varepsilon(x)\}$. Then $C_G(f, C, \varepsilon)$ is of the second category and the same argument as above shows that $\mathscr{M}_G(M, M) \cap C_G(f, C, \varepsilon)$ is dense in $C_G(f, C, \varepsilon)$.

COROLLARY 4.10. *There exists a Morse function on M.*

Proof. Let $\{\psi_i\}$ be a countable partition of unity with compact support. Then $f(x) = \sum_{i=1}^{\infty} i\psi_i(x)$ is proper. Uniformly approximating f by a function in $C_G(f, \varphi, 1) \cap \mathscr{M}_G(M, M)$ yields a Morse function.

COROLLARY 4.11. *If M is compact then M is equivariantly diffeomorphic to* $(N_1, f) \cup_{g_2} (N_2, f) \ldots \cup_{g_k} (N_k, f)$ *where the* (N_i, f) *are handle-bundles over orbits. M has the equivariant homotopy type of* $(V_1(1) \times_{H_1} G) \cup_{g_2} (V_2(1) \times_{H_2} G) \ldots \cup_{g_n} (V_n(1) \times_{H_n} G)$ *where* $V_i(1) \times_H G$ *is a disc bundle over* G/H_i *and the* g_i *are attaching maps.*

Proof. Let $f \in \mathscr{M}_G(M, M)$ and apply the main theorem to f and the interval $[\min f - 1, \max f + 1]$ to get the first statement. The second follows from the deformation defined in Corollary 4.7.

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