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Error Analysis of 8th Graders' Reasoning and Proof of Congruent Triangles in China

Zhiling Wang
East China Normal University, Shanghai, China
Zhifeng Wang
Yuci Jinhua Elementary School, Shanxi, China
Shuhua An
California State University, Long Beach, USA

This article aimed at investigating what kind of errors Chinese 8th graders have in congruent triangles reasoning and proof. The participants were 102 students in two eighth-grade classes in junior high school in China. The results showed that they were confused with the connotation and form of five congruent triangle theorems. Students had standard graph set. They had difficulty in graphic analysis. They had difficulty in transformation among representations, math language, natural language and figures. Students were unable to give correct proof process. In addition, this article developed a rubric for evaluating eight graders' congruent reasoning and proof.

Keywords: error analysis, congruent triangles, reasoning and proof, geometry.

Mathematics educators face an increasingly significant challenge in getting students to understand the roles of reasoning and proving in mathematics. The National Council of Teachers of Mathematics (NCTM) Principles and Standards (2000) document and many worldwide mathematics curricular documents have stated the prominent role of reasoning and proof in school mathematics at all levels. "Proof has played a major role in the development of mathematics, from the Euclidean geometry of the Greeks, through various forms of proofs in different cultures, to twentieth-century formal mathematics based on set-theory and logical deduction" (Hanna & De Vulliers, 2012, p. 443).

Similarly, reasoning is one of the mathematical key competencies for K-12 school students nationally and internationally. Some national standards emphasize on mathematical reasoning. For example, the standards for Mathematical Practice (MP) in Common Core State Standards for Mathematics (CCSSM) have different requirements for the development of the reasoning ability (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). From the acquirements of basic knowledge and skills (MP1) to reviewing others' arguments (MP2), and inductively reasoning ability to make a plausible argument from the data (MP3). One of NCTM Process Standards highlighted reasoning and proof (NCTM, 2000), and National Research Council's (NRC) report *Adding It Up* also called for adaptive reasoning (NRC, 2001).

Geometric reasoning plays an important role on developing students' mathematical reasoning ability. According to Piaget's cognitive developmental theory (1970), the developmental sequence of children is sensorimotor stage (birth to age 2), preoperational stage (2 to 7 years), concrete operations stage (7 to 11 or 12 years of age), formal operations stage (11 or 12 years and up). The cognitive level of eight grade students（13 or 14 years）is the beginning of formal operations stage. Thus, geometric reasoning and proof development in this stage lays solid foundation for their future geometry learning and mathematical reasoning and proof development. From the perspective of development of geometry content, congruent triangles reasoning and proof is the beginning of formal mathematical reasoning and proof, because students start to use formal language that contains "∵" "∴"to prove congruent triangles.

Many studies (Bao & Zhou, 2009; Charalambos, 1997; Healey & Hoyles, 1998; Huang & Chen, 2003; Koedinger & Anderson, 1993; Lin, 2005; Long, 2013; Lu, 2011; Qian, 2008; Tian, 2006; Zhou, 2000; Zhu, 2011) indicated that students have difficulties in geometric reasoning and proof nationally and internationally. However, there has been little research aimed at analyzing the errors of congruent triangles reasoning and proof. It is imperative in investigating what kinds of errors students have in triangles congruent reasoning and proof and finding out why they have these errors.

This paper aimed at investigating the following research question: What kinds of errors Chinse 8^{th} graders have when they learn congruent triangles reasoning and proof? Research question ought to provide reliable and valid evidence to support the errors Chinese 8^{th} graders have in congruent triangle reasoning and proof.

Theoretical Framework

Mathematical Reasoning and Proof

The development of students' mathematical reasoning (MR) is a goal of several curricula and an essential element of the culture of the mathematics education research community. Various researchers gave the definition of mathematical reasoning from their own perspectives. Bao and Xu, (2013) had a definition that mathematical proof is the process of drawing conclusions through logical reasoning based on the basic concepts and basic assumptions. "For mathematicians' proof is much more than a sequence of correct steps; it is also and, perhaps most importantly, a sequence of ideas and insights with the goal of mathematical understanding-specifically, understanding why a claim is true" (Hanna & De Villiers, 2012, p. 444). However, for students in learning reasoning, "mathematical reasoning is no less than a basic skill" (Ball & Bass, 2003, p.28). An and Wu (2017) called to focus on fostering reasoning skills throughout the components of model, strategy, and application (MSA) tasks and defined the reasoning is a process of MSA explained with academic language and deep understanding. Similarly, Johan (2008) pointed out that reasoning involves solving tasks that produce assertions and reach conclusions. “It is not necessarily based on formal logic; thus, not restricted to proof, and may even be incorrect as long as there are some kinds of sensible (to the reasoner) reasons backing it” (Johan, 2008, p. 257).

As a verb, mathematical proof refers to an argumentation process that starts from the hypothesis and draws a conclusion through strict logical deduction (NCTM, 2002). “For mathematicians, proof varies according to the discipline involved, although one essential principle underlies all its varieties: to specify clearly the assumptions made and to provide an appropriate argument supported by valid reasoning so as to draw necessary conclusions” (Hanna & De Villiers, 2012, p. 443).

Reasoning was a thinking form to get a new judgment from one or several judgments. Mathematical reasoning was divided into plausible reasoning and deductive reasoning. Plausible reasoning was divided into inductive reasoning and analogical reasoning.

According to the above-mentioned researches， mathematical reasoning and proof are considered “as a compound word that describes an overall activity: recognize pattern, form conjectures, provide arguments for non-mathematical proof, mathematical proof” (Bao & Xu, 2013, p. 267) in this article.

Congruent Triangles Reasoning and Proof

Reasoning and proof of congruent triangles involves proving congruence of at least two triangles according to five congruent triangles theorems (Secondary School Mathematics Section by People’s Education Press, 2013). The five congruent triangles theorems have the following types:

1. Side-Side-Side (SSS), if three sides of a triangle are congruent to three sides of another triangle, the triangles are congruent.
2. Side-Angle-Side (SAS), if two sides and the included angle of one triangle are congruent to the corresponding parts of another triangle, the triangles are congruent.
3. Angle-Angle-Side (AAS or SAA), if two angles and the non-included side of one triangle are congruent to the corresponding parts of another triangle, the triangles are congruent.
4. Angle-Side-Angle (ASA), if two angles and the included side of one triangle are congruent to the corresponding parts of another triangle, the triangles are congruent.
5. Hypotenuse-Leg (HL), if the hypotenuse and leg of one right triangle are congruent to the corresponding parts of another right triangle, the right triangles are congruent.

Ruthmae and Óscar (2015) carried out a descriptive study to examine 1936 students' performance on a proof task related to congruent triangles, their findings indicated that students generally experience difficulty with construction of the proof. In fact, the rigorous proof of a congruent triangle was very difficult in itself. For over 2000 years, the side angle side (SAS) had remained analytically unproved. Dokai (2014) found a proof of the elusive side angle side theorem of Euclid on the congruence of triangles. The congruent triangle had a significant position in the geometry curriculum, because (1) "the three conditions for triangle congruency can be used to prove many more propositions" and (2) triangle congruency links to similarity (Jones, Fujita & Miyazaki, 2013, p. 31). In addition, the proof of congruent triangle had rich educational values. Luo and Lin (2007) stated that "the congruent triangle has four educational values: the congruent triangle is the basic knowledge of plane geometry teaching; the best material for introductory teaching of proof; contain rich resources for geometric transformation ideas; special content for cultivating students' geometric intuition" (p. 20).

In general, the congruent triangle is the beginning of the rigorous deductive reasoning proof in learning geometry. It is a typical material for cultivating students' geometric reasoning and proof. Congruent triangle reasoning and proof learning will lay a solid foundation for further geometric proof.

Errors in Geometry Reasoning and Proof

Many researchers conducted empirical studies to investigate the errors of geometry reasoning and proof of school students (Bao & Zhou, 2009; Charalambos, 1997; Healey & Hoyles, 1998； Huang & Chen, 2003; Koedinger & Anderson,1993; Lin, 2005; Long, 2013; Lu, 2011; Qian, 2008; Riess et al., 2001; Tian, 2006; Zhou, 2000; Zhu, 2011). Several studies indicated that students lack deep understanding of concept and property in geometry reasoning and proof (Gökkurt, Erdem, Basibüyük, Sahin, & Soylu, 2017; Long, 2013; Sisman & Aksu, 2016; Wijaya, Heuvel-Panhuizen, Doorman & Robitzsch, 2014; Zhou, 2000). Tian (2006) claimed that students were not able to build relationship with relevant concepts. Some researches stated that learners had visual problems in geometry learning. They were not able to decompose a graph into substructures and analyze, summarize the relation and nature of each element of the substructures (Huang & Chen, 2003; Long, 2013; Lu, 2011; Qian, 2008; Usman, 2017; Wijaya et al., 2014; Zhou, 2000; Zou, 2009). Zou (2009) observed that students cannot understand math terms.

Some studies indicated that students failed to eliminate irrelevant conditions and find underlying conditions (Huang & Chen, 2003; Long, 2013; Lu, 2011; Zhu, 2011). Students were not able to build a relationship between condition and conclusion (Huang & Chen, 2003; Watson, 1980; Zhu, 2011). Other studies pointed out that certain errors in geometry reasoning and proof were caused by problem-solving habits, for example, bad handwriting (Long, 2013; Lu, 2011; Zhou, 2000; Zhu, 2011). Similarly, Zhu (2011) found that students were not able to analyze the particular mathematical problem, they usually used one argument (theorem, conclusion, etc.) by rote. Qian (2008) indicated that students had a standard graphic set in geometry reasoning and proof. Several studies agreed that students were not able to smoothly translate multiple representations of the geometric concept, the graphics, the natural language, and the symbolic language (Huang & Chen, 2003; Long, 2013; Lu, 2011; Qian, 2008; Tian, 2006; Watson, 1980; Zhou, 2000; Zou, 2009). Some researchers found that students lacked control ability and error correction ability for the problem solving process, etc. (Huang & Chen, 2003; Long, 2013; Watson, 1980; Zhu, 2011). This view was supported by Zou (2009) that students lacked analysis ability in geometry reasoning and proof. In addition, students had difficulty in modelling from real word problem (Wijaya et al., 2014).

Methodology

Site and Subjects

The study was conducted at a junior high school in Fujian, China in 2014. The school was located in a low socioeconomic area that served low-income migrant worker families. Most parents of the students were from other cities in China. They were not able to pay attention to their children' math learning because of their heavy work load.

The participants were 102 students from two eight-grade classrooms. Students in class A were overall advanced proficient in mathematics according to their year-end assessment. While students in class B were overall not proficient in mathematics. There were 63 girls and 39 boys (see Table 1). The test was completed within 50 minutes. Subjects were asked to finish independently.

Table 1

Demographic Information of Participants

Class #	# of Students	Boy	Girl
A	51	17	34
B	51	22	29

Data Collection

The participants were given Congruent Triangles Reasoning and Proof Test (CTRPT) during math classes. As shown in Table 2, a total of 102 test sheets were distributed, and a total of102 test sheets were collected in the end. There were 99 (97.1%) test sheets were valid. Invalid sheets contained blank responses for all items. Participants in each of the two classes were asked to complete the test within fifty minutes.

Table 2
Test Information of Classes

Class#	Distributed	Collected	Valid	Test Time
A	51	51	51	50 minutes
B	51	51	48	50 minutes
Total	102	102	99	100 minutes

Instrument

Theoretical framework for development of CTRPT. According to relevant literatures, errors in the congruent triangle reasoning and proof were sorted out into nine types: (1) conceptual and property understanding, (2) visual process, (3) item understanding, (4) problem-solving habit, (5) thinking set, (6) representation conversion, (7) metacognition, (8) analysis ability, and (9) modeling (see Table 3).

Students' possible errors in congruent triangle reasoning and proof in Table 3 was developed to provide researchers with a tool that classifies errors in reasoning and proof of congruent triangles. At the same time, the types of errors in Table 3 provided an initial theoretical framework for CTRPT test design and development in this study. Each test item had a clear target to ensure the operability and accuracy of the test.

These possible students' errors in congruent triangle reasoning and proof in Table 3 were developed according to numerous existing relevant studies (Gökkurt et al., 2017; Huang & Chen, 2003; Long, 2013; Lu, 2011; Qian, 2008; Sisman & Aksu 2016; Tian, 2006; Usman, 2017; Watson, 1980; Wijaya et al., 2014; Zhou, 2000; Zhu, 2011; Zou, 2009) as clarified in "Errors in Geometry Reasoning and Proof" part in this article. All errors addressed in the above-mentioned studies were generalized, re-described, and re-categorized into nine types as shown in Table 3.

Congruent triangles reasoning and proof test (CTRPT). Based on the analysis of several relevant literatures, and a table of possible errors in congruent triangle reasoning and proof (see Table 3) was developed as an initial Congruent Triangles Reasoning and Proof Test (CTRPT) with 12 items. The format of the test included true or false items, multiple choice questions, short answer items, real world items and open-ended items. The items were written in an objective-test format based on relevant literature and research on mathematical reasoning and proof. According to interviews, reviews with experts and practicing math teachers (Mrs. Liu and Mrs. Zhao), as well as a group discussion with peers, the initial 12 items on the test were modified and honed to a 7-item test. The experts, math teachers and peers, provided input as to the relevancy, adequacy, accuracy, and wording of items to establish content validity of the test. The 7-item sheet included two multiple choice questions, one short answer question, one real-world question and three open-ended questions. Appendix B illustrated the items used during the study. Table 4 shows the information of the test.

Table 3

Errors Students Might Have in Congruent Triangle Reasoning and Proof

Type of Error	Explanation	Research
Conceptual and property understanding	Without deep understanding of concept and property.	Long, 2013; Sisman & Aksu 2016; Gökkurt, Erdem, Basibüyük, Sahin & Soylu, 2017; Wijaya, Heuvel-Panhuizen, Doorman & Robitzsch, 2014; Zhou, 2000
	Students are not able to build relationship with relevant concepts.	Tian, 2006
Visual process	Students are not able to decompose a graph into substructures to analyze and summarize the relation and nature of each element of the substructures.	Huang & Chen, 2003; Long, 2013; Lu, 2011; Qian, 2008; Usman, 2017; Wijaya, Heuvel-Panhuizen, Doorman & Robitzsch, 2014; Zhou, 2000; Zou, 2009
Item understanding	Cannot understand math terms.	Zou, 2009
	Students fail to eliminate irrelevant conditions and find out underlying conditions.	Huang & Chen, 2003; Long, 2013；Lu, 2011; Zhu, 2011
	Students are not able to build a relationship between condition and conclusion.	Huang & Chen, 2003; Watson, 1980; Zhu, 2011
Problem-solving habit	Errors caused by problem-solving habits, for example, bad handwriting.	Long, 2013； Lu, 2011; Zhou, 2000; Zhu, 2011
Thinking set	Students are not able to analyze the particular mathematical problem, they usually use one argument (theorem, conclusion, etc.) by rote.	Zhu, 2011
	Standard graphic set.	Qian, 2008
Representation conversion	Students are able to smoothly translate multiple representations of the geometric concept, the graphics, the natural language, and the symbolic language.	Huang & Chen, 2003; Long, 2013; Lu, 2011; Qian, 2008; Tian, 2006; Watson, 1980; Zhou, 2000; Zou, 2009
Metacognition	Lack of control ability and error correction ability for the problem solving process, etc.	Huang & Chen, 2003; Long, 2013； Watson, 1980; Zhu, 2011
Analysis ability	Students lack analysis ability in geometry reasoning and proof.	Zou, 2009
Modeling	Students have difficulty in modelling from real word problem.	Wijaya, Heuvel-Panhuizen, Doorman & Robitzsch, 2014

Table 4
Information of CTRPT

Item #	1	2	3	4	5	6	7
Type of Item	Multiple Choice	Multiple Choice	Short Answer	Real-World Problem	Open-ended Problem	Open-ended Problem	Open-ended Problem
Score	3	3	8	10	10	12	14
Total Score	60						

Interviews

Twelve semi-structured interviews in total were conducted both for students and teachers in spring semester in 2015. The purpose of ten students' interviews was to obtain and analyze the reasons of the errors. Ten interviewees were 8th graders who had typical errors in the CTRPT test. According to the amount of their errors, each of them participated in 15 to 30 minutes' face-to-face interviews after the test. The entire interview progress was recorded by notes. Students' interviews were analyzed qualitatively to further explain what types of errors they had and why in congruent triangles reasoning and proof.

The goals of the teachers' interviews were to investigate what kinds of errors students had in congruent triangle reasoning and proof. Two, 20-minute, face-to-face, interviews were conducted with Mrs. Liu who taught math in class A and Mrs. Zhao who was the math teacher of class B. The entire interview process was audiotaped. Both Mrs. Liu and Mrs. Zhao were math teacher specialists with rich experience in teaching math in junior high school. Mrs. Liu had thirty-years of math teaching experience in junior high school. Mrs. Zhao had taught math for fifteen years. Many errors that they pointed out were consistent with those errors from above item analysis. The data from teachers' interview were cross-checked with the data from the item analysis as a means of establishing validity and reliability.

Data Analysis

Data were analyzed both quantitatively and qualitatively. A qualitative table was developed to allow generalizability to educational practice when appropriately conducted (see Table 3). The development of this table provided researchers with a qualitative tool that was woefully lacking in the literature. At the same time, the table provided an initial theoretical framework for the test (Congruent Triangles Reasoning and Proof Test, CTRPT) design and development. Each item had a clear target which ensures the operability and accuracy of the test.

Table 5
A Rubric for Evaluating Congruent Reasoning and Proof

First level index	Second level index	Level		
		Informal	Formal	Rigor
Deductive Reasoning	Analysis and abstraction	Unable to understand the relationships. Incapable of excluding irrelevant conditions…	Is able to partially understand the relationships. Is able to partially exclude irrelevant conditions…	Completely understands the relationships. Is able to completely exclude irrelevant conditions…
	Drawing visualization construction	Unable to draw appropriate figures. Unable to decompose graphics into sub-structures.	Is able to draw appropriate figures. Is able to partially decompose graphics into sub-structures. Unable to solve the problem correctly.	Is able to completely draw appropriate figures. Effectively decomposes graphics into different sub-structures…
	Representation and transformation	Unable to use multiple representations. Has difficulty in transformation among representations.	Is able to partially use multiple representations. Is able to partially transform among representations.	Is able to completely use multiple representations. Is able to flexibly transform among representations.
	Proof	Uses informal and imprecise mathematics language to prove. Uses blind reasoning, and tedious steps; Unable to find mistakes.	Uses formal and standard mathematics language to prove. Can partially find and correct mistakes.	Uses formal and precise mathematics language to prove. Can completely find mistakes.
	Verification and adjustment	Students are completely unaware of reviewing their proof process.	Students are aware of reviewing their proof process. Can partially correct mistakes in others' proof process.	Aware of reviewing their proof process. Can completely correct all mistakes in others' proof process.
	Conclusion generalization	Unable to generalize and specialize on concepts, procedures, properties, and propositions …	Is able to partially generalize and specialize on concepts, procedures, properties, and proposition …	Is able to strictly generalize and specialize on concepts, procedures, properties, and propositions…

Note: See details in Table 20 in Appendix A

Rubric of Evaluating 8th Graders' Congruent Reasoning and Proof

This paper emphasized deductive reasoning (informal proof). Therefore, deductive reasoning was the first level index in the rubric as shown in Table 5 (Table 5 was a brief version of Table 20). Based on Van Hiele Model of geometric thinking for the levels of deductive reasoning (Crowley, 1987), cognitive development of the 8th graders in this study was informal. Thus, the visualization (level 0) and analysis (level 1) of Van Hiele Model were excluded in this rubric. The second level index was developed according to PISA 2012 proficiency scale descriptions for mathematics (organization for Economic Co-operation and Development [OECD], 2013). Although it did not have a deep explanation of geometric reasoning and proof, the cognitive activities of geometric reasoning and proof from PISA framework were generalized: analysis and abstract, representation and transformation, proof, conclusion and generalization. In addition, verification and adjustment (metacognitive activities) and drawing-visualization-construction which was the specific cognitive activity of geometry were added in the rubric (see Table 5). The rubric was finalized after revision and adjustment according to comments and feedback from experts and practical teachers.

Results

To answer the research question on errors in congruent triangles reasoning and proof by the 8th graders, this study analyzed the participants' responses in CTRPT Test, and teachers' responses from the interviews. The major error types from students' congruent triangle reasoning and proof were summarized in Table 6.

Table 6

Major Error Types in CTRPT Test

	Errors students had in congruent triangle reasoning and proof
Item One	Did not understand these theorems (SSS, AAS, SAS, ASA, and HL) in essence.
	Could not transform between math symbols and graphics.
Item Two	Did not understand these theorems (SSS, AAS, SAS, ASA, and HL) in essence.
	Difficulties in complex graphics processing.
Item Three	Difficulties in complex graphics processing.
	Errors in concrete proof steps.
	Lack of necessary prerequisites.
	Condition and conclusion reversed in the proof process.
	Conditions could not be enough or correct to put forward conclusions.
Item Four	Not able to analyze the question by combining conditions and graphs.
	Confusion about irrelevant conditions in the graph.
	Could not transform symbols, natural language and graphs.
	Had logical problem, such as difficulty in organizing proof steps, or using rigorous math language.
Item Five	Standard graph set.
	Difficulties in complex graphics processing.
	Lack of self-adjustment and self-reflection consciousness
Item Six	Lack of ability to understand and evaluate peers' proof process.
	Errors in establishing relations between congruent triangles and conclusions.
Item Seven	Not able to understand the intention of auxiliary lines.
	Not able to understand these theorems (SSS, AAS, SAS, ASA, HL) in essence.

Students' responses for each item of the test were analyzed by the rubric and descriptive statistics as follows:

Item One

Item one was a four-choice question. According to which one of the following given conditions could draw only one triangle ΔABC ( )

A. $AB=3$， $BC=4$， $CA=8$ B. $AB=4$, $BC=3$, $\angle A=30^o$

C. $\angle C=60^o$， $\angle B=45^o$， $AB=4$ D. $\angle C=90^o$， $AB=6$

Choice C ($\angle C=60^o, \angle B=45^o, AB=4$) was the correct answer. Table 6 shows that 68.6% and 66.7% of students in class A and class B selected the correct choice respectively.

Table 7

Response Distribution between Class A and Class B

Class #	Choice A	Choice B	Choice C (Correct)	Choice D
A	8	8	35	0
Percentage	15.7%	15.7%	68.6%	0%
B	5	10	32	1
Percentage	10.4%	20.8%	66.7%	2.1%

In terms of learners who choices A, B and D, we gave deep analysis according to given conditions, conclusions and rubric for evaluating eight graders' congruent reasoning and proof. This question mainly involved two aspects of students' cognitive activities, analysis and abstraction, representation and transformation. Cognitive activity level was divided into three in the rubric, informal, formal, and rigor. However, because the item was a multiple-choice question, there were two types of responses: correct and incorrect. Therefore, this item mainly involved two levels: informal and rigor. The students who selected choice C demonstrated that their responses were at rigor level. Students who selected the incorrect answer demonstrated that their responses were at the informal level (see Table 8).

According to the given conditions in the item, we could know that those who selected A ($AB=3$, $BC=4$, $CA=8$) wanted to apply SSS theorem to prove two triangles congruent, while they neglected the three-side relation of a triangle (the sum of the two sides is greater than the third side, and the difference between the two sides is less than the third side) which was an implicit condition. According to the three-side relation of a triangle, AB+BC< CA. Thus, it was not a triangle in choice A. There were two reasons why students selected B ($AB=4$, $BC=3$, $\angle A=30^o$). They intended to apply SAS theorem to prove two triangles congruent. However, they did not understand the connotation of the SAS theorem. The angle was the one between two sides. That's why A was written between S. Otherwise, SSA theorem cannot prove two triangles congruent. The other reason was that they could not transform between math symbols and graphics. Therefore, they were not able to discover that A was not the angle between AB and BC. For learners who selected D ($\angle C = 90^{\circ}, AB = 6$), they were supposed to apply HL theorem to prove two triangles congruent. Nevertheless, they didn't understand the connotation of the HL theorem.

Table 8

Errors in Item One

Choice	Second Level Index	Level	Errors
A	Analysis and Abstraction	Informal	Applied SSS theorem incorrectly to prove two triangles congruent.
B	Analysis and Abstraction Representation and Transformation	Informal	Applied SAS theorem to incorrectly prove two triangles congruent y.
D	Analysis and Abstraction	Informal	Applied HL theorem incorrectly to prove two triangles congruent.

Item Two

Item two was a four-choice question. As shown in Figure 1, $\angle 1 = \angle 2$, $AC = AD$. Add one more condition as follows: ① $AB = AE$; ② $BC = ED$; ③ $\angle C = \angle D$; ④ $\angle B = \angle E$. How many above conditions in total could prove $\Delta ABC \cong \Delta AED$ with the given conditions? ( )

A. 4 B. 3 C. 2 D. 1

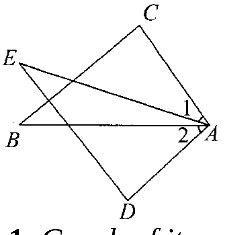


Figure 1. *Graph of item two.*

Choice B (three) was the correct answer. As shown in Table 9, about 84.3% of students in class A and 75% of students in class B selected the correct choice.

Table 9

Response Distribution Between Class A and B

Class #	Choice A	Choice B(Correct)	Choice C	Choice D
A	5	43	3	0
Percentage	9.8%	84.3%	5.9%	0%
B	9	36	3	0
Percentage	18.75%	75%	6.25%	0%

This question mainly involved two aspects of students' cognitive activities, analysis and abstraction, and drawing-visualization-construction. Cognitive activity level was divided into three in the rubric, informal, formal, and rigor. However, because the item was a multiple-choice question, there were two types of responses: correct and incorrect. Therefore, this item mainly involved two levels: informal and rigor. The students who selected choice B demonstrated that their responses were at the rigor level. Students who selected incorrectly demonstrated that their responses were at the informal level (see Table 10). According to the random interviews during the test, we found that some of the students who selected the correct answer had incorrect reasons.

Table 10
Errors in Item Two

Choice	Second Level Index	Level	Errors
A B C D	Analysis and Abstraction	Informal	Applied SSA theorem to prove two triangles congruent
	Drawing-visualization-Construction		Difficulties in complex graphics processing.

This question gave two conditions ($\angle 1 = \angle 2$, $AC = AD$) to students. They needed to select all possible conditions from the choices (① $AB = AE$; ② $BC = ED$; ③ $\angle C = \angle D$; ④ $\angle B = \angle E$) to make sure $\Delta ABC \cong \Delta AED$. In the end, the item asked how many conditions were selected. In analysis and abstraction, the errors were similar to item one. The reason why students had the errors was that they understood the connotation of five (SSS, ASA, SAS, AAS, HL) theorem to prove that the triangle was congruent. In drawing-visualization-construction, learners had difficulty in complex graphics processing. To analyze the figure of the item, students needed to divide it into two individual triangles. Then they could find the relationship between them. ∠BAE was the common angle, which was an implicit condition to solve the problem. Because ∠1=∠2, so ∠1+∠BAE=∠2+∠BAE, ∠BAC=∠EAD. Therefore, students lacked the capability of figure analysis. Because geometry problems usually have figures (triangle, rectangle, square…etc.), graphic analysis was an important proficiency for geometric reasoning and proof.

Item Three

This item was a short answer question. As shown in Figure 2, in △ABE and △CDF, AB=CD，AE=DF，CE=FB. Prove AF=DE. There were three ways to prove it (see Table 11).

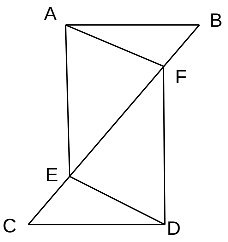


Figure 2. *Graph of item three.*

Table 11
Solutions of Item Three

Method One： Proof process omitted. $\therefore \Delta ABE \cong \Delta DCF\,(SSS)$ $\therefore \angle B = \angle C$ $\Theta\ AB = DC, BF = CE$ In ΔABF and ΔDCE $\begin{cases} AB = DC \\ \angle B = \angle C \\ BF = CE \end{cases}$ $\therefore \Delta ABF \cong \Delta DCE\,(SAS)$ $\therefore AF = DE$	Method Two： Proof process omitted. $\therefore \Delta ABE \cong \Delta DCF\,(SSS)$ $\therefore \angle AEF = \angle DFE$ $\Theta\ AE = DF, EF = FE$ In ΔAEF and ΔDFE 中 $\begin{cases} AE = DF \\ \angle AEF = \angle DFE \\ EF = FE \end{cases}$ $\therefore \Delta AEF \cong \Delta DFE\,(SAS)$ $\therefore AF = DE$	Method Three： Proof process omitted. $\therefore \Delta ABE \cong \Delta DCF\,(SSS)$ $\therefore \angle AEF = \angle DFE$ $\therefore AE \,//\, DF$ (Equal Alternate Interior Angles implies Parallel Line) $\Theta\ AE = DF$ $\therefore$ Quadrilateral AEDF is parallelogram. $\therefore AF = DE$

Participants' responses were coded into five types: blank, incorrect, incomplete proof, proof, and other. Further explanation for each type was presented in Table 11. As shown in Table 11, class A had 35 students give complete proof, accounting for 68.63% of the total number. In class B, 20 students provided complete proof, accounting for 41.67% of the total number. Therefore, the rest students in class A and B had errors in solving the problem.

This question mainly involved two cognitive activities: drawing-visualization-construction and proof. According to student interviews, students' errors in this item belonged to two above-mentioned cognitive activities. Some students demonstrated a lack of graphic analysis ability, which manifests as the chaos of the corresponding angle and the corresponding side. Most errors that students made in this question were in proof. The main manifestations were: reasoning errors in concrete proof steps; lack of necessary prerequisites; condition and conclusion reversed in the proof process; conditions insufficient or incorrect to support conclusions.

According to the rubric of evaluating the 8th graders' congruent reasoning and proof (see Table 2). Learners' responses with complete proof were at the rigor level. Incomplete proof were at the formal level. Other responses, or no responses were at the informal level. Table 13 shows the level distribution in class A and B.

Table 12
Solutions Distribution of Item Three

Type of Response	Explanation	Class A # of Students	Class A Percentage	Class B # of Students	Class B Percentage
Blank	Blank or copy the conditions, conclusion directly, no proof process.	2	3.92%	15	31.25%
Incorrect	Proof process is totally incorrect.	1	1.96%	5	10.42%
Incomplete proof	Only part of proof process was correct.	6	11.76%	7	14.58%
Complete proof	Proof process was completely correct, precise and clear.	35	68.63%	20	41.67%
Other	Proof process (logic) was correct. There were symbol errors, step repetition, etc.; however, the errors did not influence the proof process.	7	13.73%	1	2.08%

Table 13
Reasoning and Proof Level Distribution of Item Three in Classes A and B

Class #	Informal		Formal		Rigor	
	# of Student	Percentage	# of Student	Percentage	# of Student	Percentage
A	7	13.73%	6	11.76%	35	68.63%
B	1	2.08%	7	14.58%	20	41.67%
Total	8	8.08%	13	13.13%	55	55.55%

As presented in Table 13, there were 35 (68.63%) students at rigor level, 6 (11.76%) learners were at formal level, and 7 (13.73%) students were at informal level in class A. For class B, there were 20 (41.67%) students at rigor level, 7 (14.58%) learners were at formal level, and 1 (2.08%) students were at informal level. In addition, there were total 55 (55.55%) subjects were at rigor level, 13 (13.13%) learners were at formal level, and 8 (8.08%) participants were at informal level in both classes. Therefore, over half of students could give rigor proof for routine problems, which indicated that most students met the basic requirement for congruent triangles. They were able to apply basic knowledge and skills to prove triangles are congruent. However, there were 23% of learners below the informal level.

Item Four

This was a real-world problem. As shown in Figure 3, A and B were located at the ends of a pond. Jack wanted to measure the distance between A and B with a rope, but the rope was not long enough. He came up with an idea: first, he took a point C on the ground that could reach A or B at the same time, linked AC and extended to D, made DC = AC; linked BC and extended to E, made EC = BC; linked DE and measured its length, the length of DE was the distance between A and B. Why? Draw your graph according to the above description and explain why.

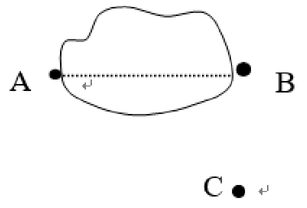


Figure 3. *Graph of item four.*

Table 14
Solutions Distribution of Item Four

Response Type		Class A		Class B	
Code	Explanation	# of Student	Percentage	# of Student	Percentage
00	Blank or incorrect graph only	2	3.92%	5	10.42%
01	Correct graph， without proof process	0	0%	1	2.08%
11	Incorrect graph, part of proof process is incorrect	5	9.80%	8	16.67%
21	Correct graph， part of proof process is incorrect	25	49.02%	23	47.92%
22	Correct graph， proof process is totally correct.	19	37.25%	11	22.92%

Students' responses were coded into five types: 00, 01, 11, 21, and 22. Further explanation for each type was presented in Table 13. As shown in Table 14, class A had 19 students give complete proof, accounting for 37.25% of the total number. In class B, 11 students provided complete proof, accounting for 22.92% of the total number. Therefore, compared with item four, this item was more challenge for the 8th graders. In addition, 25 students in class A and 23 students in class B were capable of drawing the correct graph, while part of proof process was incorrect.

This question mainly involved four cognitive activities: analysis and abstraction; drawing-visualization-construction; representation and transformation; and, proof. In analysis and abstraction, students were not able to analyze the question by combining conditions and graphs. In drawing-visualization-construction activity, students were unable to distinguish between relevant and irrelevant conditions in the graph. In representation and transformation, they could not transform symbols, natural language and graphs. In the proof activity, students had problems with logic, such as they had difficulty in organizing proof steps, or using rigorous math language.

Table 15

Reasoning and Proof Level Distribution of Item Four in Classes A and B

Class #	Informal		Formal		Rigor	
	# of Student	Percentage	# of Student	Percentage	# of Student	Percentage
A	5	9.80%	25	49.02%	19	37.25%
B	8	16.67%	23	47.92%	11	22.92%
Total	13	13.13%	48	48.48%	30	30.30%

According to the rubric of evaluating the 8th graders' congruent reasoning and proof (Table 15). Students who provided an incorrect graph with part of the proof process (11) were at the informal level. Those who provided a correct graph with part of proof process incorrect (21) were at formal level. Students who provided a correct graph with a correct proof process were at the rigor level (22). The rest of the subjects were not at the reasoning and proof level. Table 15 shows the level distribution in class A and B.

As illustrated in Table 15, there are 19 (37.25%) students at rigor level, 25 (49.02%) learners were at formal level, and 5 (9.80%) students were at the informal level in class A. For class B, there were 11 (22.92%) students at rigor level, 23 (47.92%) students were at the formal level, and 8 (16.67%) students were at the informal level. In addition, there were a total of 30 (30.30%) students at the rigor level, 48 (48.48%) of students were at the formal level, and 13 (13.13%) students were at the informal level in both classes. Therefore, more than half of the students could give a rigor proof for routine problems, which indicated that most students met the basic requirement for congruent triangles. They were able to apply basic knowledge and skills to prove congruent triangles. However, there were 9% of learners below the informal level.

Item Five

This item was an open-ended problem with two sub-questions. As presented in Figure 4, in △ABC，D is on AB, E is on BC, and BD=BE. **Question one**: please add one condition to make △BEA ≌△BDC， then give your proof process. **Question two**: according to the condition you add, write down another pair of congruent triangles in the graph (No adding segments, no marking or using other letters, no need to the proof process).

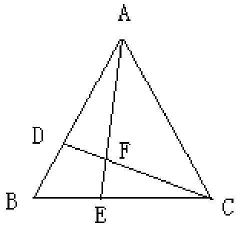


Figure 4. *Graph of item five.*

The researcher interviewed student 1 (S1) and student 2 (S2) who had typical errors as representations of all subjects (see Figure 5 and Figure 6).

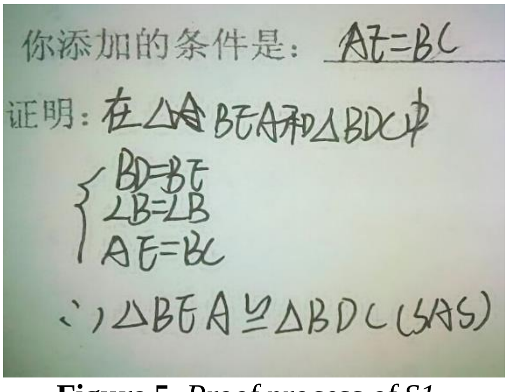


Figure 5. *Proof process of S1.*

In the following interviews, the researcher is referred to a R. The following showed the conversation between researcher and student 1 during the interview.

R: （Gave him a blank paper）Could you please tell me which two triangles you need to prove are congruent?

S1：△BEA≌△BDC, he pointed to the item.

R：Could you draw these two triangles separately on draft paper？

After 5 minutes, he finished it, while the position was same as which in the original graph.

R：If the position and direction of these two triangles were changed, without changing their size and shape, will they be congruent？

S1：He thought for a while and said “yes”.

R：OK, then could you adjust their positions to make them be more “good-looking” and “stand up” so that the two triangles could be overlapped after translation.

S1：After one minute, he drew it (these two triangles were like the congruent triangles which we saw most in math class).

R：If you want to apply SAS theorem to prove these two triangles congruent, which condition will you add?

S1：AB=BC, he said immediately.

Then researcher showed him the response he submitted (see Figure 5). He could not believe that the condition he added was different from the present one. The researcher asked him what he was thinking at that time. He said that he was confused about the graph which was different from usual ones in class. According to the interview, the student made mistakes because "the graph is not the same as usual ones in classroom." It can be seen that students' strong dependence on standard graphs which were presented by math teachers in class. Once the position of the graph was changed, students will have difficulty in analyzing it. For example, the two graphs with overlapping part in this item was a challenge for learners. They cannot be able to divide it into two substructures. This was a visual problem in geometry reasoning and proof.

Thus, the errors in this item were standard graph set, graph analysis (visual).

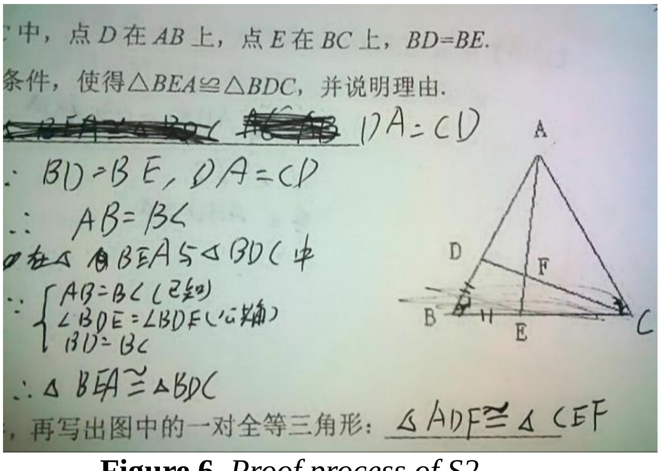


Figure 6. *Proof process of S2.*

R：Could you tell me how many conditions are needed to prove two triangles congruent except right triangles?

S2： “Three” he said immediately.

R：If you want to apply SAS theorem to prove two triangles congruent，which condition is needed except the condition you added and given condition (BD=BE)?

S2：He thought for a while and said one common angle ∠B.

R：We ha D two conditions now, please add one condition to apply SAS theorem to prove two triangles congruent.

S2：He said quickly “AB=BC”. However, BD=BE. He thought for a while and said “I add DA=CE.”

R: This is your paper and look at the response for this item (The researcher handed him his response shown in Figure 6).

S2: He checked it.

R: What you think about it？

S2: It was wrong.

R: Why?

S2：I saw AB and BC seem have the same length from the graph at that time. At the same time, AB and DC also seemed have equal length. So I was confused about it. I didn’t know which was correct, I just wrote down one of them.

R：Why didn’t you check the item for a clue?

S2：He blushed and whispered “I didn’t see it”.

R：Your congruence symbol (≅) was very special, the correct one should be (≌), why did you write it like that？

S2：Many classmates wrote like that. I thought it was not a big deal. Both of them meant congruence.

It can be seen from the above interview that he made a subjective assumption. He determined the two segments were equal because visually they ‘looked alike’. In fact, these errors could have been avoided by analyzing the given conditions again. Nevertheless, he was unaware of checking the item again. In addition, he wrote the condition BD=BE as BD=BC incorrectly. Therefore, he lacked self-adjustment and self-reflection consciousness.

Item Six

This was an open-ended problem with two questions (see Appendix B). The analysis of the responses of question one are presented first, followed by the analysis of the responses of question two.

As shown in Figure 7，in △ABC，∠BAC=60°，AD is the bisector of ∠BAC, $AC = AB + BD$ .What’s the degree of $\angle ABC$?

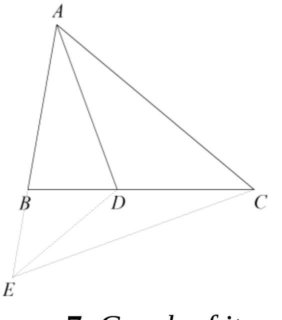


Figure 7. *Graph of item six.*

Question One: The teacher presented a student named Tony’s proof process. Students were asked to explain rationales for three steps in Tony’s proof process. The purpose of question one was to assess the 8th graders’ ability to understand and evaluate others’ proof process.

Students’ responses were categorized into four types (see Table 16). Blank meant learners didn’t write anything in answer area. Incorrect meant students gave incorrect explanations. Inappropriate meant students wrote down an answer which might be correct, while it was not the correct response. All above three response types demonstrated that learners lacked the ability to understand and evaluate their peers' proof processes. Nevertheless, correct explanation was correct answer, which indicated that students were able to understand and evaluate others' proof process.

Table 16

Response Distribution of Question One in Class A and Class B

Step #	Class #	Blank	Incorrect Explanation	Inappropriate Explanation	Correct Explanation
1	A	2	3	4	42
	B	12	7	8	21
2	A	1	5	3	42
	B	10	6	5	27
3	A	0	3	0	48
	B	5	5	4	34
Total		30	29	24	214
Percentage		10.1%	9.76%	8.08%	72.05%

The first sub-question (step 1) was about the intention of the auxiliary line. According to Table 16 above, 42 (82.35%) students in class A and 21 (43.75%) students in class B gave the correct explanation. The second sub-question (step 2) was to provide an explanation. In class A, 42 (82.35%) students gave the correct explanation. In Class B, 27 (56.25%) students gave the correct explanation. The third sub-question (step 3) asked participants to write down which theorem two congruent triangles were based on. Class A had 48 (94.12%) students give the correct explanation. In class B, 34 (70.83%) students gave the correct explanation. Most learners in class A and B were able to provide the correct explanation for question one. A total of 30 (10.1%) students submitted blank response for this question. In both classes, 29 (9.76%) 8^{th} graders gave incorrect explanations. About 24 (8.08%) students wrote down inappropriate responses. In addition, 214 (72.05%) students solved the problem correctly.

Generally speaking, nearly half or more than half of the students were able to give explanations correctly, indicating that most of the students were capable of giving the correct explanation for Tony's proof process. Therefore, most students had mastered the basic knowledge and skills of congruent triangles. The mastery of the theorem was still quite optimistic. According to the reasoning and proof level scale, these students were at the "formal" level, but this didn't mean that they understood Tony's proof process in essence. That was the strict level.

Question Two: John selected E on AC, made AE=AB， and linked DE as shown in Figure 8. Please continue and complete the proof process according to above thought. John’s proof method was a variant of Tony’s method. The methods they used to prove congruency were essentially the same. Students who solved the problem correctly totally understood Tony’s proof process, which was the foundation for solving this question. This question was more difficult than question one.

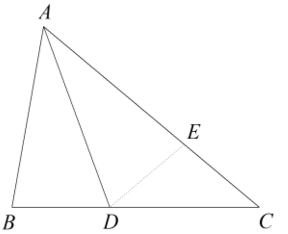


Figure 8. *Graph of item six.*

The students’ responses were coded into five categories as follows (see Table 17). Those who submitted response 00 meant they wrote nothing for this question. Students with response 11 gave a part of the proof process. 12 means these students proved △ABD≌△AED, while they didn’t solve the degree of $\angle ABC$. There were students who used other methods (13) to solve the problem; however, they provided incomplete steps. All four codes were indicators that students were incapable of understanding Tony’s proof process. Code 21 meant that students provided totally correct responses which was a result of positive transfer. They were essentially able to understand others’ proof process.

Table 17

Response Code of Question Two

Code	Explanation
00	Blank
11	Part of proof process
12	Prove △ABD≌△AED， while the degree of $\angle ABC$ was not solved.
13	Other methods with incomplete steps.
21	Complete proof.

Table 18 shows that 18 (35.29%) and 28 (58.33%) students wrote nothing for this item in class A and class B respectively. There were 15 (29.41%) and 11 (22.92%) learners in class A and class B respectively, who provided a proof for △ABD≌△AED; but, were unable to solve the degree of $\angle ABC$. Only 12 (23.53%) students in class A and 5 (10.42%) students in Class B that solved the problem correctly. Furthermore, there were merely 17 (17.17%) students in total, who gave the correct proof. It was worth mentioning that 26% of the total students proved △ABD≌△AED, while they didn’t solve the degree of the angle. Therefore, these students had a partial migration. They had errors in searching for the quantitative relationship of the angles.

Table 18
Response Distribution of Question Two in Class A and Class B

Code	Class A		Class B		Total	
	# of Student	Percentage	# of Student	Percentage	# of Student	Percentage
0	18	35.29%	28	58.33%	46	46.46%
1	3	5.88%	4	8.33%	7	7.07%
2	15	29.41%	11	22.92%	26	26.26%
3	3	5.88%	0	0.00%	3	3.03%
1	12	23.53%	5	10.42%	17	17.17%

Almost half (46.46%) of the learners submitted a blank answer, which indicated that this item was a great challenge for the 8th graders. They didn't have positive migration from Tony's proof process. During the interviews with these students. Most of them said "I understood Tony's proof process, while John's method was totally different. I didn't know how to prove it." Because they thought John's method was distinct. It can be seen that they were at the formal level in understanding Tony's proof process. They didn't analyze the common in essence.

According to above-mentioned analysis, students' errors in this item could be summarized as follows: (a) using congruent triangles as a premise to deduce other quantitative relationships to prove the conclusion. The congruent triangles proof process was not the difficulty. Instead, the8th graders had errors in establishing relations between congruent triangles and conclusions. (b) Learners could understand others' proof process, which didn't mean they learnt it.

Item Seven

This item was an open-ended problem. E was the midpoint of BC， A was on DE, ∠BAE=∠CDE, prove AB=CD. Three students used three methods to prove the conclusion (see Figure 9). However, some proof steps were missing and incorrect. Some steps required explanations. There were six sub-questions in total. The first sub-question was aimed at investigating 8th graders knowledge and skills about auxiliary lines. Adding auxiliary lines was one of important skills for solving geometric problems.

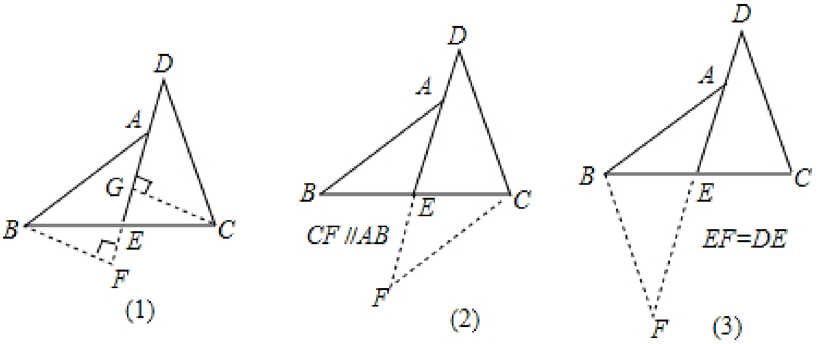


Figure 9. *Graph of item seven.*

Understanding the purpose of auxiliary lines was the key to understand others' proof process. There were 12% of students who did not demonstrate an understanding of the purpose of auxiliary lines. There were .36% of students who gave blank responses. There were also 53% of students who demonstrated an understanding of the purpose of auxiliary lines. The indication was that students had errors in understanding the intention of auxiliary lines.

The purpose of second sub-question was to investigate students' ability to evaluate and revise others' proof process. Students' responses were categorized into six types: (1) Blank; (2) AAA theorem can prove two triangles congruent; (3) AAA theorem can't prove two triangles congruent; (4) AAA theorem can't prove two triangles congruent and gave an incorrect explanation; (5) AAA theorem can't prove two triangles congruent and gave a correct explanation; and (6) AAA theorem can't prove two triangles congruent, and gave a correct explanation and revised the proof. Students who gave the sixth response type were at the rigor level. In total, there were 15 responses at the rigor level. It can be seen that learners had misconceptions about congruent theorems. They were not able to understand these theorems (SSS, AAS, SAS, ASA, and HL) in essence. For example, ZJ wrote that AAA could prove △ABF≌△DCG (see Figure 10). Therefore, he had the misconception that AAA could prove two triangles congruent.

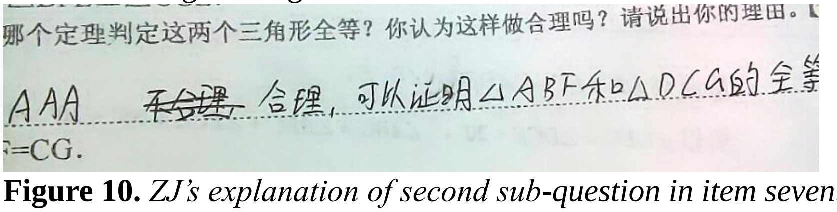


Figure 10. *ZJ's explanation of second sub-question in item seven.*

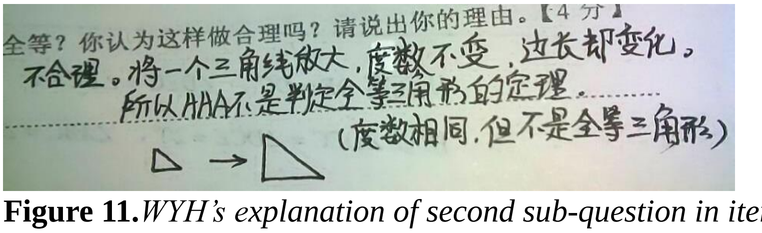


Figure 11. *WYH's explanation of second sub-question in item seven.*

Figure 11 shows another example by WYH who thought that "AAA couldn't prove these two triangles congruent because the degree of each angle doesn't change if the triangle is enlarged, while side length changes." He drew a graph to present why and wrote a sentence to explain "degree of angles was same, but they were not congruent triangles."

Sub-questions third to sixth were intended to study students' ability to understand and revise others' proof process. Only 15% of the students were able to write all the proof processes correctly and rigorously enough to reach the rigor level. Additionally, 30% of the students were not able to write the proof process at all. While there were obvious mistakes, 55% of the students completed the proof process. For instance, some students did not reason strictly.

Interview

Many errors the teachers pointed out were consistent with those errors shown in the item analysis. Different errors were as follows: Students had difficulty in analyzing relationships between conditions and conclusions; students were usually not able to find the appropriate theorem to prove two triangles congruent; students omitted certain steps of proof process; and, it was difficult for some students to use formal math language to write the proof process.

Discussion

The results of this study showed that 8th graders had many errors in their reasoning and proof of congruent triangles. According to the item analysis, student interviews and teacher interviews, these errors could be summarized into five categorizes.

Superficial Understanding of Five Congruent Triangle Theorems

The 8th graders in this study had superficial understanding of the connotation and form of the five congruent triangle theorems. In particular, the students made the most frequent errors in proofs of SAS and SSA. They were confused with the relationship between sides and angles. Students knew that three conditions could prove two triangles congruent, while their relationship was ignored. Students did not understand why AAA could not prove two congruent triangles. The reason why they had above-mentioned errors is that they did not understand these theorems in essence. These results were in agreement with previous studies in students' geometry learning (Long, 2013; Tian, 2006). Tian (2006) found that students had superficial understanding of concepts and theorems in geometric learning. In Long's (2013) study, it indicated that one of the typical mistakes that junior high school students made in geometry proof was their superficial understanding of geometric concepts and they ignored some conditions that applied to theorems.

Standard Graph Set

According to student interviews, a number of students said that “this graph is different from the one which the math teacher drew in classes,” “I haven’t seen this graph before,” and so on. Most teachers used acute triangles frequently to improve their teaching efficiency and help students understand. This practice led to students' strong dependence on standard graphs. Once they saw non-standard graphs, they felt unfamiliar and did not know how to analyze them. These results were in agreement with Qian’s (2008) findings, which showed that students had the standard graph set in geometry learning.

Difficulty in Graphic Analysis

Questions about congruent triangle reasoning and proof usually contain graphs, some of which are complicated. Students need to have the ability to separate them into sub-triangles and manipulate them (usually finding the corresponding sides and angles). However, this study, students were not able to decompose them into sub-triangles when they encountered complex graphics. For example, the researchers provided a student with a blank test paper in the interview and let the student to decompose one of the graphs into two sub-structures. After a series of adjustments, it became a standard graph. Afterwards, the student was able to find the corresponding side and angle correctly. This finding was consistent with Qian’s (2008) findings, which indicated that complex graphs cause errors in students’ geometry learning.

Difficulty in Transformation Among Representations

The findings from this study show that students had difficulty with transformation among representations, using formal math language, natural language, and figures in reasoning and proof of congruent triangles. These results were supported by several studies done by Qian (2008), Tian (2006), Long (2013) and Lu (2011).

Difficulty in Proof Process

This study also found that students had difficulties in the proof process of congruent triangles. For example, several students lacked the ability to use rigor in their writing language, and had difficulty in organizing the steps of their proof. In the process of student geometric reasoning and proof, many mistakes were caused by bad habits. For example, students had symbol errors, omission of necessary steps, and reversal sequence of proof steps. In addition, they lacked the ability to understand and evaluate their peers’ proof process.

The first and second error types were related to specific geometric content, which was congruent triangle reasoning and proof. The third to fifth error types belonged to geometric reasoning and proof, which was consistent with the existing research discussed in the theoretical framework of this article. The above findings were consistent with Long’s (2013) study results, which indicated that one of the typical errors in junior high school students’ geometry learning was that they could not write the proof process in clear and rigorous language.

Conclusion

In conclusion, congruent triangle reasoning and proof is the beginning of mathematics formal reasoning and proof learning. The findings on congruent triangle reasoning and proof in this study provide meaningful insights for mathematics teaching and learning at the junior high school level. The findings also contribute to a better understanding of existing research on congruent triangles. This study suggests that including more samples and using different objects with various attributes would be beneficial in future studies. The present study confirmed previous findings and contributed additional evidence that added substantially to our understanding of congruent triangle reasoning and proof; and suggested the need to assess and support students' congruent triangle learning in multiple forms and approaches.

According to the above-mentioned results, the following teaching strategies and suggestions are proposed to improve the 8th graders' learning of congruent triangle reasoning and proof.

Teaching Based on Geometric Concept's Double Characteristics of "Concept-Process"

Teaching based on geometric concept's double characteristics of "concept-process" is a good teaching approach to avoid the above-mentioned mistakes that students often make in cognitive activities "drawing-visualization-construction," A geometry concept could not only be regarded as a process but also as a mathematical object. Therefore, teachers should provide students with specific examples, or graphics, at the introduction of the lesson to help students build initial understanding. Then, teachers guide students to explore, discover the concepts' connotation and relationships with other concepts through group cooperation.

Apply Multiple-Contact Representation Strategy

Mathematics teachers should try to apply the "multiple-contact representation" strategy in the specific classroom teaching. This refers to the use of multiple representations of the mathematical concept, so as to achieve a multi-angle understanding of mathematical concepts. The transformation among natural language, graphics, and mathematical language in geometry reasoning and proof is especially important. The transformation is of great significance for correct reason and proof.

Apply Variant Teaching to Break through Standard Graph Set

The problem of standard graph set can be improved through variant teaching. Variant teaching is the theoretical essence of mathematics education in China, which has important guiding significance for the actual mathematics teaching. According to Bao and Zhou (2009), Lingyuan Gu conducted two tasks on variant teaching in China: systematically restoring and sorting the "concept variant" in traditional teaching; Put forward "process variants." Conceptual variants are divided into conceptual variants (also divided into standard variants and non-standard variants) and non-conceptual variants. Conceptual variants teaching is helpful to reveal the essential attributes of concepts and define the extension of concepts. This helps students understand the nature of concepts, which is the structural characteristics of mathematical objects. Second, process variants help students to establish the internal relationship among knowledge, so as to form good knowledge structure. Therefore, these two variants teaching respectively promote the development of the two aspects of mathematics objects according to its duality. The interaction of these two aspects promotes the understanding of the nature of the learning object and the construction of a good cognitive structure (Bao, Huang, Yi, & Gu, 2003).

Specifically, two congruent triangles used most often by math teachers in the process of congruent triangle reasoning and proof are two acute triangles (see Figure 19). The following examples of variant graphs could be used in teaching to provide students with more options than just using the standard graph set.

Table 19

Graph Variants Examples of Congruent Triangles

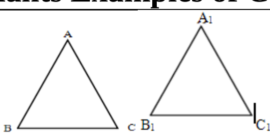
<table>
<tr>
<td>standard variants</td>
<td>$\Delta ABC \cong \Delta A_1B_1C_1$</td>
<td>


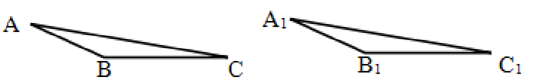
Figure12.*Standard variants graph of congruent triangles.*
</td>
</tr>
<tr>
<td rowspan="3">non-standard variants</td>
<td>$\Delta ABC \cong \Delta A_1B_1C_1$</td>
<td>


Figure13.*Non-standard variants graph of congruent triangles.*
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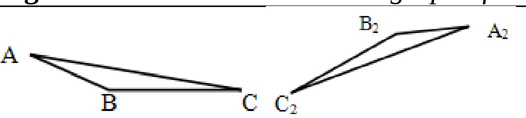
<tr>
<td>$\Delta ABC \cong \Delta A_2B_2C_2$</td>
<td>


Figure14.*Non-standard variants graph of congruent triangles.*
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<tr>
<td>$\Delta ABC \cong \Delta FEA$</td>
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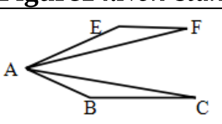

Figure15.*Non-standard variants graph of congruent triangles.*
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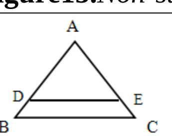
<tr>
<td>non-conceptual variants</td>
<td>ΔABC and ΔADE are not congruent</td>
<td>


Figure16.*Non-conceptualvariants graph of congruent triangles.*
</td>
</tr>
</table>

Focus on the Analysis the "Thinking Process" of Geometric Reasoning and Proof

According to results in this study that the 8th graders lack the ability to do graphic analysis in the process of reasoning and proof. Teachers could guide learners to think by themselves to get a conclusion or conjecture at first when they solve math problems. Afterwards, teachers help them to find their errors by asking questions. After that, guide them to correct errors. In the end, they analyze the thinking process (show how to solve the problem correctly) together with students. It helps to improve learners' analysis ability with practice.

Emphasize Students' Rigorousness and Develop Good Habits of Reasoning and Proof

Mathematics is a rigorous discipline. One of the findings in this study is that students make non-rigor errors, such as they intended to use SAS theorem to prove congruent triangles. Instead, they incorrectly wrote SSA theorem. Therefore, it is necessary to help students develop good habits of reasoning and proof.

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Appendix A

Table 20

A Rubric for Evaluating Congruent Reasoning and Proof

First level index	Second level index	Level: Informal	Formal	Rigor
Deductive Reasoning	Analysis and Abstraction	Unable to understand the relationships, structures, and properties in the math problems. Incapable of excluding irrelevant conditions, excavating hidden conditions, reorganizing conditions, building relationship between conditions and conclusions,	Is able to partially understand the relationships, structures, and properties in the math problems. Is able to partially exclude irrelevant conditions, excavate hidden conditions, reorganize conditions, build relationship between	Completely understand the relationships, structures, and properties in the math problems. Is able to completely exclude irrelevant conditions, excavate hidden conditions, reorganize conditions, build relationship between conditions and conclusions,

	and forming conjectures.	conditions and conclusions, and form initial conjectures.	and form effective conjectures.
Drawing-visualization-construction	Unable to draw appropriate figures according to the requirement of the math problem; Unable to decompose graphics into different sub-structures.	Is able to draw appropriate figures according to the requirement of the math problem; Is able to partially decompose graphics into different sub-structures. Unable to solve the problem correctly.	Is able to completely draw appropriate figures according to the requirement of the math problem; Effectively decompose graphics into different sub-structures, and continually adjust and reorganize them. Otherwise, repeat the above cycle until they are able to solve the problem smoothly.
representation and Transformation	Unable to use multiple representations to represent the relationships in problem. Has difficulty in transformation among these representations.	Is able to partially use multiple representations to represent the relationships in problem. Is able to partially transform among these representations.	Is able to completely use multiple representations to represent the relationships in problem. Is able to flexibly transform among these representations.
Proof	Uses informal and imprecise mathematics language to prove. Uses blind reasoning, and tedious steps; Unable to find mistakes.	Uses formal and standard mathematics language to prove. Can partially find and correct mistakes in others' proof process.	Uses formal and precise mathematics language to prove. Can completely find mistakes in others' proof process and correct them.
Verification and Adjustment	Students are completely unaware of reviewing their proof process.	Students are aware of reviewing their proof process. Can partially find and correct mistakes in others' proof process.	Students are aware of reviewing their proof process. Can completely find and correct all mistakes in others' proof process.
Conclusion Generalization	Unable to generalize and specialize on existing mathematical concepts, procedures, properties, and propositions according to features and relationships of mathematical structures.	Is able to partially generalize and specialize on existing mathematical concepts, procedures, properties, and propositions according to features and relationships of mathematical structures.	Is able to strictly generalize and specialize on existing mathematical concepts, procedures, properties, and propositions according to features and relationships of mathematical structures.

Appendix B
Congruent Triangles Reasoning and Proof Test (CTRPT)

Item One: (3points) According to which one of the following given conditions could draw only one triangle ΔABC ( )

A. $AB=3$, $BC=4$, $CA=8$ B. $AB=4$, $BC=3$, $\angle A=30^{o}$

C. $\angle C=60^{o}$, $\angle B=45^{o}$, $AB=4$ D. $\angle C=90^{o}$, $AB=6$

Item Two: (3points) As shown in the right graph, $\angle 1=\angle 2$, $AC=AD$. Add one more condition as follows: ① $AB=AE$; ② $BC=ED$; ③ $\angle C=\angle D$; ④ $\angle B=\angle E$. How many above conditions in total could prove $\Delta ABC\cong\Delta AED$ with the given conditions? ( )

A. 4 B. 3 C. 2 D. 1

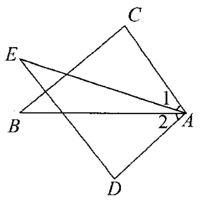

Figure 1. *Graph of item two*

Item Three: (8 points) As shown in Figure 2, in △ABE and △CDF, AB=CD，AE=DF，CE=FB. Prove AF=DE.

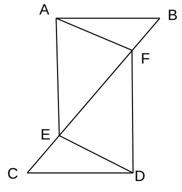

Figure 2. *Graph of item three*

Item Four: (10 points) As shown in Figure 3, A and B were located at the ends of a pond. Jack wanted to measure the distance between A and B with a rope, but the rope was not long enough. He came up with an idea: first took a point C on the ground that could reach A, B at the same time, linked AC and extended to D, made DC = AC; linked BC and extended to E, made EC = BC; linked DE and measured its length, the length of DE was the distance between A and B. Why? According to the above description, draw your graph and explain why.

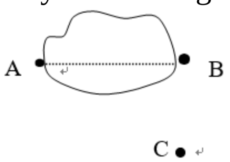

Figure 3. *Graph of item four*

Item Five: As presented in Figure 4, in △ABC，D is on AB, E is on BC, and BD=BE.

Question one: please add one condition to make △BEA ≌△BDC，then give your proof process.

Condition you add:

Proof:

Question two: according to the condition you added, write down another pair of congruent triangles in the graph (No adding segments, no marking or using other letters, no need to the proof process).

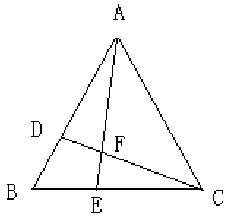

Figure 4. *Graph of item five*

Item Six: (12 points) As shown in Figure5, in △ABC, ∠BAC=60°, AD is the bisector of ∠BAC, $AC = AB + BD$ .What's the degree of $\angle ABC$? The teacher selected the proof processes of two students, Tony and John. Let's think about why they did it. Can you find the rationales for his (her) steps? Write your response on the corresponding line.

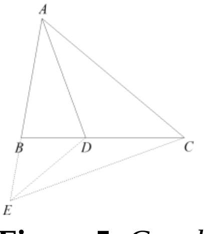

Figure 5. *Graph of item six*

【Tony】As shown in Figure5，extend AB to E , let $BE = BD$ ，link ED and EC .

Why extend AB to E , make $BE = BD$ ？ (2 points)

$\because AC = AB + BD \therefore AE = AC$,

Why explain $AE = AC$ ？ (2 points)

While $\angle BAC = 60^{\circ}$, so ΔAEC is an equilateral triangle.

$\angle EAD = \angle CAD$, $AD = AD$, $AE = AC$,

So $\Delta AED \cong \Delta ACD$.

Which theorem he applied to prove $\Delta AED \cong \Delta ACD$? (2 points)

$DE = DC$, $\angle DEC = \angle DCE$,

So $\angle DEC = \angle DCE = 20^{\circ}$, $\angle ABC = \angle BEC + \angle BCE = 60^{\circ} + 20^{\circ} = 80^{\circ}$.

【John】(6 points) As shown in Figure 6, John selected E on AC, make AE=AB，link DE. Please continue and complete the proof process according to above thought.

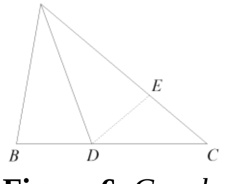

Figure 6. *Graph of item six*

Item Seven: (14 points) Read the following items and analyze proof process. If there are errors in the process, correct and explain why.
As shown in Figure 7，E is the midpoint of BC，A is on DE, ∠BAE=∠CDE, prove AB=CD.

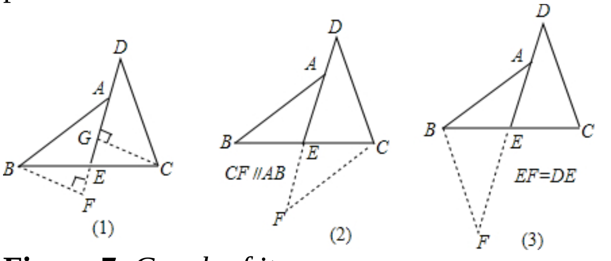

Figure 7. *Graph of item seven*

Method One：Draw BF⊥DE to F，CG⊥DE to G．What's the purpose of the auxiliary line?
(1 point)

∴∠F=∠CGE=90°
∴ $CG \parallel BF$
∴ $\angle GCE = \angle EBF$
∵∠BEF=∠CEG，
∠F=∠CGE=90°
∴△BFE≌△CGE．
Which theorem he applied to prove △BFE≌△CGE？Do you think it is reasonable? If reasonable, please give your explanation. Otherwise correct it. (3 points)

∴BF=CG．
Can you complete the missing steps? (3 points)

∴AB=CD．
Method Two：Draw CF∥AB，extend DE to F．
∴∠F=∠BAE．
∵∠ABE=∠D，
Can you complete the proof process? (3 points)

∴△ABE≌△FCE．
Which theorem he applied to prove △ABE≌△FCE? (1 point)

∴AB=CF．
∴AB=CD．
Method Three：Extend DE to F，let EF=DE．
Please complete the proof process on the lines as follows. (3 points)

Corresponding Author:

Zhiling Wang
East China Normal University, Shanghai, China
Email:wzling90@163.com