Name ______________________ Class ______________ Date ________

16.1 Properties of Logarithms

Essential Question: What are the properties of logarithms?

Resource Locker

Explore 1 Investigating the Properties of Logarithms

You can use a scientific calculator to evaluate a logarithmic expression.

(A) Evaluate the expressions in each set using a scientific calculator.

Set A	Set B
$\log \frac{10}{e} \approx$ ☐	$\frac{1}{\log e} \approx$ ☐
$\ln 10 \approx$ ☐	$1 + \log e \approx$ ☐
$\log e^{10} \approx$ ☐	$1 - \log e \approx$ ☐
$\log 10e \approx$ ☐	$10 \log e \approx$ ☐

(B) Match the expressions in Set A to the equivalent expressions in Set B.

$\log \frac{10}{e} =$ ☐

$\ln 10 =$ ☐

$\log e^{10} =$ ☐

$\log 10e =$ ☐

Reflect

1. How can you check the results of evaluating the logarithmic expressions in Set A? Use this method to check each.

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2. **Discussion** How do you know that $\log e$ and $\ln 10$ are reciprocals? Given that the expressions are reciprocals, show another way to represent each expression.

Explore 2 Proving the Properties of Logarithms

A logarithm is the exponent to which a base must be raised in order to obtain a given number. So $\log_b b^m = m$. It follows that $\log_b b^0 = 0$, so $\log_b 1 = 0$. Also, $\log_b b^1 = 1$, so $\log_b b = 1$. Additional properties of logarithms are the Product Property of Logarithms, the Quotient Property of Logarithms, the Power Property of Logarithms, and the Change of Base Property of Logarithms.

Properties of Logarithms	
For any positive numbers *a*, *m*, *n*, *b* $(b \neq 1)$, and *c* $(c \neq 1)$, the following properties hold.	
Definition-Based Properties	$\log_b b^m = m$ $\quad \log_b 1 = 0 \quad$ $\log_b b = 1$
Product Property of Logarithms	$\log_b mn = \log_b m + \log_b n$
Quotient Property of Logarithms	$\log_b \frac{m}{n} = \log_b m - \log_b n$
Power Property of Logarithms	$\log_b m^n = n\log_b m$
Change of Base Property of Logarithms	$\log_c a = \frac{\log_b a}{\log_b c}$

Given positive numbers *m*, *n*, and *b* $(b \neq 1)$, prove the Product Property of Logarithms.

(A) Let $x = \log_b m$ and $y = \log_b n$. Rewrite the expressions in exponential form.

$m =$ ☐

$n =$ ☐

(B) Substitute for m and n.

$\log_b mn = \log_b (\;☐\;)$

(C) Use the Product of Powers Property of Exponents to simplify.

$\log_b (b^x \cdot b^y) = \log_b b^{☐}$

(D) Use the definition of a logarithm $\log_b b^m = m$ to simplify further.

$\log_b b^{x+y} =$ ☐

(E) Substitute for x and y.

$x + y =$ ☐

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Reflect

3. Prove the Power Property of Logarithms. Justify each step of your proof.

Explain 1 Using the Properties of Logarithms

Logarithmic expressions can be rewritten using one or more of the properties of logarithms.

Example 1 **Express each expression as a single logarithm. Simplify if possible. Then check your results by converting to exponential form and evaluating.**

(A) $\log_3 27 - \log_3 81$

$\log_3 27 - \log_3 81 = \log_3\left(\frac{27}{81}\right)$	Quotient Property of Logarithms
$= \log_3\left(\frac{1}{3}\right)$	Simplify.
$= \log_3 3^{-1}$	Write using base 3.
$= -1\log_3 3$	Power Property of Logarithms
$= -1$	Simplify.

Check:

$$\log_3\left(\frac{1}{3}\right) = -1$$
$$\frac{1}{3} = 3^{-1}$$
$$\frac{1}{3} = \frac{1}{3}$$

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(B) $\log_5\left(\frac{1}{25}\right) + \log_5 625$

$\log_5\left(\frac{1}{25}\right) + \log_5 625 = \log_5\left(\frac{1}{25} \square 625\right)$ ________ Property of Logarithms

$= \log_5 \square$ Simplify.

$= \log_5 \square$ Write using base 5.

$= \square \log_5 5$ Power Property of Logarithms

$= \square$ Simplify

Check:

$\log_5 25 = \square$

$25 = 5^{\square}$

$25 = \square$

Your Turn

Express each expression as a single logarithm. Simplify if possible.

4. $\log_4 64^3$

5. $\log_8 18 - \log_8 2$

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Explain 2 Rewriting a Logarithmic Model

There are standard formulas that involve logarithms, such as the formula for measuring the loudness of sounds. The loudness of a sound $L(I)$, in decibels, is given by the function $L(I) = 10\log\left(\frac{I}{I_0}\right)$, where I is the sound's intensity in watts per square meter and I_0 is the intensity of a barely audible sound. It's also possible to develop logarithmic models from exponential growth or decay models of the form $f(t) = a(1 + r)^t$ or $f(t) = a(1 - r)^t$ by finding the inverse.

Example 2 **Solve the problems using logarithmic models.**

(A) During a concert, an orchestra plays a piece of music in which its volume increases from one measure to the next, tripling the sound's intensity. Find how many decibels the loudness of the sound increases between the two measures.

Let I be the intensity in the first measure. So $3I$ is the intensity in the second measure.

Increase in loudness $= L(3I) - L(I)$	Write the expression.
$= 10\log\left(\frac{3I}{I_0}\right) - 10\log\left(\frac{I}{I_0}\right)$	Substitute.
$= 10\left(\log\left(\frac{3I}{I_0}\right) - \log\left(\frac{I}{I_0}\right)\right)$	Distributive Property
$= 10\left(\log 3 + \log\left(\frac{I}{I_0}\right) - \log\left(\frac{I}{I_0}\right)\right)$	Product Property of Logarithms
$= 10\log 3$	Simplify.
≈ 4.77	Evaluate the logarithm.

So the loudness of sound increases by about 4.77 decibels.

(B) The population of the United States in 2012 was 313.9 million. If the population increases exponentially at an average rate of 1% each year, how long will it take for the population to double?

The exponential growth model is $P = P_0(1+r)^t$, where P is the population in millions after t years, P_0 is the population in 2012, and r is the average growth rate.

$P_0 = 313.9$

$P = 2P_0 = \square$

$r = 0.01$

Find the inverse model of $P = P_0(1+r)^t$.

$P = P_0(1+r)^t$	Exponential model
$\frac{P}{P_0} = (1+r)^t$	Divide both sides by P_0.
$\log_{1+r}\left(\frac{P}{P_0}\right) = \log_{\square}(1+r)^t$	Take the log of both sides.

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$$\log_{1+r}\left(\frac{\square}{\square}\right) = t$$ Definition of a logarithm

$$\frac{\log\left(\frac{\square}{\square}\right)}{\log\left(\square\right)} = t$$ Change of Base Property of Logarithms

Substitute and solve for *t*.

$$t = \frac{\log\left(\frac{\square}{313.9}\right)}{\log\left(1 + \square\right)}$$ Substitute.

$$= \frac{\log \square}{\log \square}$$ Simplify.

$$= \frac{\square}{\square}$$ Evaluate the logarithms.

$$= \square$$ Simplify.

The population of the United States will double in ☐ years from 2012, or in the year ☐.

Your Turn

6. A bank account earns 6% annual interest compounded annually. The balance B of the account after t years is given by the equation $B = B_0(1.06)^t$, where B_0 is the starting balance. If the account starts with a balance of \$250, how long will it take to triple the balance of the account?

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Elaborate

7. On what other properties do the proofs of the properties of logarithms rely?

8. What properties of logarithms would you use to rewrite the expression $\log_7 x + \log_7 4x$ as a single logarithm?

9. Explain how the properties of logarithms are useful in finding the inverse of an exponential growth or decay model.

10. **Essential Question Check-In** State each of the Product, Quotient, and Power Properties of Logarithms in a simple sentence.

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Evaluate: Homework and Practice

- Online Homework
- Hints and Help
- Extra Practice

Express each expression as a single logarithm. Simplify if possible.

1. $\log_9 12 + \log_9 546.75$

2. $\log_2 76.8 - \log_2 1.2$

3. $\log_{\frac{2}{5}} 0.0256^3$

4. $\log_{11} 11^{23}$

5. $\log_5 5^{x+1} + \log_4 256^2$

6. $\log\left(\log_7 98 - \log_7 2\right)^x$

7. $\log_{x+1}\left(x^2 + 2x + 1\right)^3$

8. $\log_4 5 + \log_4 12 - \log_4 3.75$

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Solve the problems using logarithmic models.

9. **Geology** Seismologists use the Richter scale to express the energy, or magnitude, of an earthquake. The Richter magnitude of an earthquake M is related to the energy released in ergs E shown by the formula $M = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$. In 1964, an earthquake centered at Prince William Sound, Alaska registered a magnitude of 9.2 on the Richter scale. Find the energy released by the earthquake.

10. **Astronomy** The difference between the apparent magnitude (brightness) m of a star and its absolute magnitude M is given by the formula $m - M = 5\log\frac{d}{10}$, where d is the distance of the star from the Earth, measured in parsecs. Find the distance d of the star Rho Oph from Earth, where Rho Oph has an apparent magnitude of 5.0 and an absolute magnitude −0.4.

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11. The intensity of the sound of a conversation ranges from 10^{-10} watts per square meter to 10^{-6} watts per square meter. What is the range in the loudness of the conversation? Use $I_0 = 10^{-12}$ watts per square meter.

12. The intensity of sound from the stands of a football game is 25 times as great when the home team scores a touchdown as it is when the away team scores. Find the difference in the loudness of the sound when the two teams score.

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13. **Finance** A stock priced at $40 increases at a rate of 8% per year. Write and evaluate a logarithmic expression for the number of years that it will take for the value of the stock to reach $50.

14. Suppose that the population of one endangered species decreases at a rate of 4% per year. In one habitat, the current population of the species is 143. After how long will the population drop below 30?

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15. The population P of bacteria in a culture after t minutes is given by the equation $P = P_0\left(1.12\right)^t$, where P_0 is the initial population. If the number of bacteria starts at 200, how long will it take for the population to increase to 1000?

16. Chemistry Most swimming pool experts recommend a pH of between 7.0 and 7.6 for water in a swimming pool. Use $\text{pH} = -\log\left[H^+\right]$ and write an expression for the difference in hydrogen ion concentration over this pH range.

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17. Match the logarithmic expressions to equivalent expressions.

a. $\log_2 4x$	________	$2x$
b. $\log_2 \frac{x}{4}$	________	$2 + \log_2 x$
c. $\log_2 4^x$	________	$\frac{\log x}{\log 2}$
d. $\log_2 x^4$	________	$4\log_2 x$
e. $\log_2 x$	________	$\log_2 x - 2$

18. Prove the Quotient Property of Logarithms. Justify each step of your proof.

19. Prove the Change of Base Property of Logarithms. Justify each step of your proof.

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H.O.T. Focus on Higher Order Thinking

20. Multi-Step The radioactive isotope Carbon-14 decays exponentially at a rate of 0.0121% each year.

a. How long will it take 250 g of Carbon-14 to decay to 100 g?

b. The half-life for a radioactive isotope is the amount of time it takes for the isotope to reach half its initial value. What is the half-life of Carbon-14?

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21. Explain the Error A student simplified the expression $\log_2 8 + \log_3 27$ as shown. Explain and correct the student's error.

$$\log_2 8 + \log_3 27 = \log(8 \cdot 27)$$
$$= \log(216)$$
$$\approx 2.33$$

22. Communicate Mathematical Ideas Explain why it is not necessary for a scientific calculator to have both a key for common logs and a key for natural logs.

23. Analyze Relationships Explain how to find the relationship between $\log_b a$ and $\log_{\frac{1}{b}} a$.

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Lesson Performance Task

Given the population data for the state of Texas from 1920–2010, perform exponential regression to obtain an exponential growth model for population as a function of time (represent 1920 as Year 0).

Obtain a logarithmic model for time as a function of population two ways: (1) by finding the inverse of the exponential model, and (2) by performing logarithmic regression on the same set of data but using population as the independent variable and time as the dependent variable. Then confirm that the two expressions are equivalent by applying the properties of logarithms.

Year	U.S. Census Count
1920	4,663,228
1930	5,824,715
1940	6,414,824
1950	7,711,194
1960	9,579,677
1970	11,196,730
1980	14,229,191
1990	16,986,335
2000	20,851,820
2010	25,145,561

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