Estimating Option Prices with Heston's Stochastic Volatility Model

Robin Dunn[1*], Paloma Hauser[2*], Tom Seibold[3*], Hugh Gong[4†]

1. Department of Mathematics and Statistics, Kenyon College, Gambier, OH 43022
2. Department of Mathematics and Statistics, The College of New Jersey, Ewing, NJ 08628
3. Department of Mathematics, Western Kentucky University, Bowling Green, KY 42101
4. Department of Mathematics and Statistics, Valparaiso University, Valparaiso, IN 46383

Abstract

An option is a security that gives the holder the right to buy or sell an asset at a specified price at a future time. This paper focuses on deriving and testing option pricing formulas for the Heston model [3], which describes the asset's volatility as a stochastic process. Historical option data provides a basis for comparing the estimated option prices from the Heston model and from the popular Black-Scholes model. Root-mean-square error calculations find that the Heston model provides more accurate option pricing estimates than the Black-Scholes model for our data sample.

Keywords: Stochastic Volatility Model, Option Pricing, Heston Model, Black-Scholes Model, Characteristic Function, Method of Moments, Maximum Likelihood Estimation, Root-mean-square Error (RMSE)

1 Introduction

Options are a type of financial derivative. This means that their price is not based directly on an asset's price. Instead, the value of an option is based on the likelihood of change in an underlying asset's price. More specifically, an option is a contract between a buyer and a seller. This contract gives the holder the right but not the obligation to buy or sell an underlying asset for a specific price (strike price) within a specific amount of time. The date at which the option expires is called the date of expiration.

Options fit into the classification of call options or put options. Call options give the holder of the option the right to buy the specific underlying asset, whereas put options give the holder the right to sell the specific underlying asset.

Further, within the categories of call and put options, there are both American options and European options. American options give the holder of the option the right to exercise the option at any time before the date of expiration. In contrast, European options give the holder of the option the right to exercise the option only on the date of expiration. This research focuses specifically on estimating the premium of European call options.

In general, when a party seeks to buy an option, that party can easily research the history of the asset's price. Furthermore, both the date of expiration and strike price are contracted within a given option. With this, it becomes the responsibility of that party to take into consideration those known factors and objectively evaluate the value of a given option. This value is represented monetarily through the option's price, or premium.

As the market for financial derivatives continues to grow, the success of option pricing models at estimating the value of option premiums is under examination. If a participant in the options market can predict the value of an option before the value is set, that participant will have an advantage. Today, the

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Black-Scholes model is widely used in the asset pricing industry. Praised for its computational simplicity and relative accuracy, it treats the volatility of an underlying asset as a constant. Stochastic volatility models, on the other hand, allow for variation in both the asset's price and its price volatility, or standard deviation. This research focuses specifically on one stochastic volatility model: the Heston model [3].

This paper examines the Heston model's success at estimating European call option premiums and compares the estimates to those of the Black-Scholes model. Heston and Nandi [4] proposed a formula for the valuation of a premium; their formula incorporates the characteristic function of the Heston model. After solving for the explicit form of the Heston model's characteristic function, we use S&P 100 data from January 1991 to June 1997 to estimate the parameters of the Heston model's characteristic function, which we then use in the call pricing formula. We compare the Heston model's estimates, the Black-Scholes model's estimates, and the actual premiums of option data from June 1997.

In the remainder of this paper, Section 2 focuses on determining the explicit closed form of the Heston model's characteristic function. In Section 3, we discretize the Heston model and employ two separate parameter estimation methods - the method of moments and maximum likelihood estimation. We discuss the Black-Scholes model in Section 4, as it serves as an alternate option pricing method to the Heston model. Finally, in Section 5 we use sample data in a numerical example and evaluate the Heston model's success at premium estimation. Section 6 concludes by discussing our findings and suggesting topics for future research.

Due to the heavy computational nature of this research, Maple 16, R, and Microsoft Excel 2010 were all utilized in the development of this paper.

2 The Heston Model

In 1993, Steven Heston proposed the following formulas to describe the movement of asset prices, where an asset's price and volatility follow random, Brownian motion processes:

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_{1t} \tag{1}$$

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}dW_{2t} \tag{2}$$

The variables of the system are defined as follows:

- S_t: the asset price at time t

- r: risk-free interest rate - the theoretical interest rate on an asset that carries no risk

- $\sqrt{V_t}$: volatility (standard deviation) of the asset price

- σ: volatility of the volatility $\sqrt{V_t}$

- θ: long-term price variance

- k: rate of reversion to the long-term price variance

- dt: indefinitely small positive time increment

- W_{1t}: Brownian motion of the asset price

- W_{2t}: Brownian motion of the asset's price variance

- ρ: Correlation coefficient for W_{1t} and W_{2t}

Given the above terms in the Heston model, it is important to note the properties of Brownian motion as they relate to stochastic volatility. As stated in [10], Brownian motion is a random process $W_t, t \in [0, T]$, with the following properties:

- $W_0 = 0$.

- W_t has independent movements.

- W_t is continuous in t.

- The increments $W_t - W_s$ have a normal distribution with mean zero and variance $|t - s|$.

$$(W_t - W_s) \sim N(0, |t - s|)$$

Heston's system utilizes the properties of a no-arbitrage martingale to model the motion of asset price and volatility. In a martingale, the present value of a financial derivative is equal to the expected future value of that derivative, discounted by the risk-free interest rate.

2.1 The Heston Model's Characteristic Function

Each stochastic volatility model will have a unique characteristic function that describes the probability density function of that model. Heston and Nandi [4] utilize the characteristic function of the Heston model when proposing the following formula for the fair value of a European call option at time t, given a strike price K, that expires at time T:

$$
\begin{aligned}
C \quad = \quad & \frac{1}{2}S(t) + \frac{e^{-r(T-t)}}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\phi}f(i\phi+1)}{i\phi}\right]d\phi \\
& -Ke^{-r(T-t)}\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\phi}f(i\phi)}{i\phi}\right]d\phi\right).
\end{aligned}
\tag{3}
$$

The characteristic function for a random variable x is defined by the following equation:

$$f(i\phi) = E(e^{i\phi x})$$

In equation (3), the function $f(i\phi)$ represents the characteristic function of the Heston model. Therefore, in order to test the option pricing success of the Heston model, it is necessary to solve for the explicit form of the characteristic function.

To find the explicit characteristic function for the Heston model, we must use Ito's Lemma [6] - a stochastic calculus equivalent of the chain rule. For a two variable case involving a time dependent stochastic process of two variables, t and X_t, Ito's Lemma makes the following statement:

Assume that X_t satisfies the stochastic differential equation

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

If $f(t, X_t)$ is a twice differentiable scalar function, then,

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu_t\frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma_t\frac{\partial f}{\partial x}dW_t.$$

Since Heston's stochastic volatility model treats t, X_t, and V_t as variables, we extend Ito's Lemma to three variables. Assume that we have the following system of two standard stochastic differential equations, where $f(X_t, V_t, t)$ is a continuous, twice differentiable, scalar function:

$$dX_t = \mu_x dt + \sigma_x dW_{1t}$$

$$dV_t = \mu_v dt + \sigma_v dW_{2t}$$

Further, let W_{1t} and W_{2t} have correlation ρ, where $-1 \le \rho \le 1$. For a function $f(X_t, V_t, t)$, we wish to find $df(X_t, V_t, t)$. Using multivariable Taylor series expansion and the properties of Ito Calculus, we find that the derivative of a three variable function involving two stochastic processes equals the following expression:

$$
\begin{aligned}
df(X_t, V_t, t) = \quad & \left[\mu_x f_x + \mu_v f_v + f_t + f_{xv}\sigma_x\sigma_v\rho + \frac{1}{2}\left(f_{xx}\sigma_x^2 + f_{vv}\sigma_v^2\right)\right]dt \\
& +\left[\sigma_x f_x\right]dW_{1t} + \left[\sigma_v f_v\right]dW_{2t}.
\end{aligned}
$$

3

The complete details of the derivation of Ito's Lemma in three variables are available in Appendix A.

From Ito's Lemma in three variables, we know the form of the derivative of any function of X_t, V_t, and t, where X_t and V_t are governed by stochastic differential equations. The Heston model's characteristic function is a function of X_t, V_t, and t, so Ito's Lemma determines the form of the derivative of the characteristic function. Further, we know that the characteristic function for a three variable stochastic process has the following exponential affine form [5]:

$$f(X_t, V_t, t) = e^{A(T-t)+B(T-t)X_t+C(T-t)V_t+i\phi X_t}.$$

Letting $T - t = \tau$, the explicit form of the Heston model's characteristic function appears below. A full derivation of the characteristic function is available in Appendix B.

$$
\begin{aligned}
f(i\phi) &= e^{A(\tau)+B(\tau)X_t+C(\tau)V_t+i\phi X_t} \\
A(\tau) &= ri\phi\tau + \frac{k\theta}{\sigma^2}\left[-(\rho\sigma i\phi - k - M)\tau - 2\ln\left(\frac{1-Ne^{M\tau}}{1-N}\right)\right] \\
B(\tau) &= 0 \\
C(\tau) &= \frac{(e^{M\tau}-1)(\rho\sigma i\phi - k - M)}{\sigma^2(1-Ne^{M\tau})}
\end{aligned}
$$

Where
$$
\begin{aligned}
M &= \sqrt{(\rho\sigma i\phi - k)^2 + \sigma^2(i\phi + \phi^2)} \\
N &= \frac{\rho\sigma i\phi - k - M}{\rho\sigma i\phi - k + M},
\end{aligned}
$$

In the above characteristic function for the Heston model, the variables r, σ, k, ρ, and θ require numerical values in order to be used in the option pricing formula. Given an asset's history, parameter estimation techniques can estimate numerical values for those variables.

3 Parameter Estimation

In this section, we explain how to estimate the parameters of the Heston model from a data set of asset prices. The first step is to discretize the Heston model. To that end, we employ Euler's discretization method [12]. Once the discretized model is in place, one can use data to estimate the model's parameters.

3.1 Discretization of the Heston Model

The Heston model treats movements in the asset price as a continuous time process. Measurements of asset prices, however, occur in discrete time. Thus, when beginning the process of estimating parameters from the asset price data, it is crucial to obtain a discretized asset movement model.

We used the method of Euler discretization in order to discretize the Heston model. Given a stochastic model of the form

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t,$$

the Euler discretization of that model is

$$S_{t+dt} = S_t + \mu(S_t, t)dt + \sigma(S_t, t)\sqrt{dt}Z,$$

where Z is a standard normal random variable.

Applying Euler discretization to the Heston model, we wish to discretize the system given by (1) and (2). We will let $dt = 1$ to represent the one trading day between each of our asset price observations. In (1), $\mu(S_t, t) = rS_t$ and $\sigma(S_t, t) = \sqrt{V_t}S_t$. Thus, the Euler discretized form of (1) is

$$S_{t+1} = S_t + rS_t + \sqrt{V_t}S_t Z_s. \tag{4}$$

For future simplicity in terms of parameter estimation, it is useful to model the change in asset prices in terms of the change in asset returns, where a return is equal to $\frac{S_{t+1}}{S_t}$. We denote that quotient with the notation Q_{t+1}. Dividing both sides of (4) by S_t, the modified version of (4) is

$$Q_{t+1} = 1 + r + \sqrt{V_t} Z_s,$$

where $Z_s \sim N(0,1)$. Next, we must discretize the second system of the Heston model. In (2), let $\mu(V_t, t) = k(\theta - V_t)$ and $\sigma(V_t, t) = \sigma\sqrt{V_t}$. Using Euler's discretization method, we determine that the discretized form of the second equation is

$$V_{t+1} = V_t + k(\theta - V_t) + \sigma\sqrt{V_t} Z_v,$$

where $Z_v \sim N(0,1)$. In the continuous form of the Heston model, W_{1t} and W_{2t} are two Brownian motion processes that have correlation ρ. In the discrete form of the Heston model, Z_s and Z_v are two standard normal random variables with that same correlation ρ. To conclude the discretization process, we set $Z_v = Z_1$ and $Z_s = \rho Z_1 + \sqrt{1 - \rho^2} Z_2$. Where $Z_1, Z_2 \sim N(0,1)$ and are independent, we have the following discretized system:

$$Q_{t+1} = 1 + r + \sqrt{V_t}\left(\rho Z_1 + \sqrt{1 - \rho^2} Z_2\right) \tag{5}$$

$$V_{t+1} = V_t + k(\theta - V_t) + \sigma\sqrt{V_t} Z_1. \tag{6}$$

3.2 Method of Moments

One parameter estimation method that we employ is the method of moments. The jth moment of the random variable Q_{t+1} is defined as $E(Q_{t+1}^j)$. We use μ_j to denote the jth moment.

We can solve for method of moments parameter estimates according to the following process:

1. Write m moments in terms of the m parameters that we are trying to estimate.

2. Obtain sample moments from the data set. The jth sample moment, denoted $\hat{\mu}_j$ is obtained by raising each observation to the power of j and taking the average of those terms. Symbolically,

$$\hat{\mu}_j = \frac{1}{n}\sum_{t=1}^{n} Q_{t+1}^j.$$

   The `moments` package in R calculates the sample moments with ease.

3. Substitute the jth sample moment for the jth moment in each of the m equations. That is, let $\mu_j = \hat{\mu}_j$. Now we have a system of m equations in m unknowns.

4. Solve for each of the m parameters. The resulting parameter values are the method of moments estimates. We denote the method of moments estimate of a parameter α as $\hat{\alpha}_{MOM}$.

When working with a data set of stock values, we may be given values of S_t rather than values of Q_{t+1}. We can easily transform the data set into values of Q_{t+1} by solving for $\frac{S_{t+1}}{S_t}$ for each value of t. We wish to write five moments of Q_{t+1} in terms of the five parameters r, k, θ, σ, and ρ.

Letting μ_j represent the jth moment of Q_{t+1}, we express formulas for the first moment, the second moment, the fourth moment, and the fifth moment. We have excluded the third moment because it is in terms of only μ and θ; thus, it does not add any information to the system beyond the information available from the first two moments.

$$
\begin{aligned}
\mu_1 &= 1 + r \\
\mu_2 &= (r+1)^2 + \theta \\
\mu_4 &= \frac{1}{k(k-2)}(k^2 r^4 + 4k^2 r^3 + 6k^2 r^2 \theta - 2kr^4 + 6k^2 r^2 + 12k^2 r\theta \\
&\quad + 3k^2\theta^2 - 8kr^3 - 12kr^2\theta + 4k^2 r + 6k^2\theta - 12kr^2 - 24kr\theta \\
&\quad - 6k\theta^2 - 3\sigma^2\theta + k^2 - 8kr - 12k\theta - 2k) \\
\mu_5 &= \frac{1}{k(k-2)}(k^2 r^5 + 5k^2 r^4 + 10k^2 r^3\theta - 2kr^5 + 10k^2 r^3 + 30k^2 r^2\theta \\
&\quad + 15k^2 r\theta^2 - 10kr^4 - 20kr^3\theta + 10k^2 r^2 + 30k^2 r\theta + 15k^2\theta^2 \\
&\quad - 20kr^3 - 60kr^2\theta - 30kr\theta^2 - 15r\sigma^2\theta + 5k^2 r + 10k^2\theta - 20kr^2 \\
&\quad - 60kr\theta - 30k\theta^2 - 15\sigma^2\theta + k^2 - 10kr - 20k\theta - 2k)
\end{aligned}
$$

The complete derivation of the moments is available in Appendix C.

Note that we only have a system of 4 equations in 4 parameters since ρ does not appear in any of the given moments. We have shown that ρ does not appear in the formula for any of the moments up to the seventh-order moment. We conjecture that ρ will not appear in any of the formulas for higher order moments. That poses a drawback to the method of moments; however, in the Section 5, we will show that the value of ρ that we use in the call pricing formula has little effect on the estimated call prices in our data sample.

To solve for the method of moments parameter estimates for r, θ, k, and σ, replace μ_1 with $\hat{\mu}_1$, μ_2 with $\hat{\mu}_2$, μ_4 with $\hat{\mu}_4$, μ_5 with $\hat{\mu}_5$, r with $\hat{r}_{MOM}$, θ with $\hat{\theta}_{MOM}$, k with $\hat{k}_{MOM}$, and σ with $\hat{\sigma}_{MOM}$. Given the system of equations, Maple can assist in the calculation of $\hat{r}_{MOM}$, $\hat{\theta}_{MOM}$, $\hat{k}_{MOM}$, and $\hat{\sigma}_{MOM}$.

3.3 Maximum Likelihood Estimation

The second parameter estimation method that we use is maximum likelihood estimation. Maximum likelihood estimation involves optimizing the parameter estimates such that the data that we observe is the data that we are most likely to observe.

The follow algorithm allows us to solve for maximum likelihood estimation parameter estimates:

1. Find the likelihood function of the data in our data set. The likelihood function is defined as the product of the probability density functions of each observation of the random variable. Where $L(r, k, \theta, \sigma, \rho)$ is the likelihood function of the data and $f(Q_{t+1}, V_{t+1})$ is the joint probability density function of Q_{t+1} and V_{t+1},

$$
L(r, k, \theta, \sigma, \rho) = \prod_{t=1}^{n} f(Q_{t+1}, V_{t+1} | r, k, \theta, \sigma, \rho).
$$

2. For simplicity of calculation, solve for the natural logarithm of the likelihood function, denoted $\ell(r, k, \theta, \sigma, \rho)$. Optimizing $\ell(r, k, \theta, \sigma, \rho)$ is equivalent to optimizing $L(r, k, \theta, \sigma, \rho)$.

$$
\ell(r, k, \theta, \sigma, \rho) = \sum_{t=1}^{n} \log\left(f(Q_{t+1}, V_{t+1}) | r, k, \theta, \sigma, \rho\right)
$$

3. To optimize the parameters, take partial derivatives of $\ell(\cdot)$ with respect to each of the parameters. Set the partial derivatives equal to 0 and solve for the maximum likelihood estimation parameter estimates. The maximum likelihood estimate of a parameter α is denoted $\hat{\alpha}_{MLE}$.

In practice, when working with a large data set, it is ideal to use software to assist in maximum likelihood estimation. Specifically, the `nlminb` function in R is capable of optimizing the log likelihood function under parameter constraints.[1] As in [14], we set the constraints that $r \in \mathbb{R}$, $k \geq 0$, $\theta \geq 0$, $\sigma \geq 0$, and $-1 \leq \rho \leq 1$.

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[1] `nlminb` finds the parameter estimates that minimize a function. Thus, in order to perform maximum likelihood estimation, the user must provide `nlminb` with the negative of the log likelihood function.

To begin the process of maximum likelihood estimation, we solve for the joint probability density function $f(Q_{t+1}, V_{t+1})$. Recall the discretized form of the equations for Q_{t+1} (5) and V_{t+1} (6). Z_s and Z_v are standard normal random variables, so $Q_{t+1} \sim N(1 + r, V_t)$ and $V_{t+1} \sim N(V_t + k(\theta - V_t), \sigma^2 V_t)$. Further, since Z_s and Z_v have correlation ρ, Q_{t+1} and V_{t+1} have that same correlation ρ. Based on those properties of Q_{t+1} and V_{t+1},

$$
\begin{aligned}
f(Q_{t+1}, V_{t+1}) \quad = \quad & \frac{1}{2\pi\sigma V_t\sqrt{1 - \rho^2}}\exp\Bigg[-\frac{(Q_{t+1} - 1 - r)^2}{2V_t(1 - \rho^2)} \\
& + \frac{\rho(Q_{t+1} - 1 - r)(V_{t+1} - V_t - \theta k + kV_t)}{V_t\sigma(1 - \rho^2)} \\
& - \frac{(V_{t+1} - V_t - \theta k + kV_t)^2}{2\sigma^2 V_t(1 - \rho^2)}\Bigg].
\end{aligned}
$$

Since the likelihood function is

$$
L(r, k, \theta, \sigma, \rho) = \prod_{t=1}^{n} f(Q_{t+1}, V_{t+1}|r, k, \theta, \sigma, \rho),
$$

the log likelihood function is

$$
\begin{aligned}
\ell(r, k, \theta, \sigma, \rho) \quad = \quad & \sum_{t=1}^{n} \Bigg(-\log(2\pi) - \log(\sigma) - \log(V_t) - \frac{1}{2}\log(1 - \rho^2) \\
& - \frac{(Q_{t+1} - 1 - r)^2}{2V_t(1 - \rho^2)} + \frac{\rho(Q_{t+1} - 1 - r)(V_{t+1} - V_t - \theta k + kV_t)}{V_t\sigma(1 - \rho^2)} \\
& - \frac{(V_{t+1} - V_t - \theta k + kV_t)^2}{2\sigma^2 V_t(1 - \rho^2)}\Bigg).
\end{aligned}
$$

The next step is to plug stock return and asset variance values into the log likelihood function in order to optimize the parameters. While the volatility is a latent variable, we can estimate a vector of variance values from our data. Recall that $Q_{t+1} \sim N(1 + r, V_t)$. That signifies that V_t is the variance of Q_{t+1}. Thus, in order to estimate V_t for any given time t, we determine the variance of the values of Q_{t+1} up to and including its value at the given time t.

The above process allows us to construct a data vector for values of V_t and, by extension, V_{t+1}. When substituting the log likelihood function into R, the new range of t will be all values of t for which we know Q_{t+1}, V_t, and V_{t+1}.

Once we know the log likelihood function and we have a data set with values of Q_{t+1}, V_t, and V_{t+1}, R can perform the optimization that returns the five maximum likelihood parameter estimates.

4   The Black-Scholes Model

The Black-Scholes model is a mathematical model used to price European options and is one of the most well-known and widely used option pricing models. In 1973, Fischer Black, Myron Scholes [2], and Robert Merton [8] proposed a formula to price European options, under the assumptions that an asset's price follows Brownian motion but the asset's price volatility is constant. The following formulas give the Black-Scholes price of a call option:

$$
\begin{aligned}
C \quad &= \quad \Phi(d_1)S_t - \Phi(d_2)Ke^{-r(T-t)} \\
d_1 \quad &= \quad \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}} \\
d_2 \quad &= \quad d_1 - \sigma\sqrt{T - t}.
\end{aligned}
$$

The system's variables are defined in the following manner:

- C: call premium

- t: current time

- T: time at which option expires

- S_t: stock price at time t

- K: strike price

- r: risk-free interest rate. We use the rate of return on three-month U.S. Treasury Bills.

- σ: volatility of the asset's price, given by the standard deviation of the asset's returns

- $\Phi(\cdot)$: standard normal cumulative distribution function

The Black-Scholes model is widely popular due to its simplicity and ease of calculation. The model, however, makes a strong assumption by treating the volatility as a constant. The use of newer models that treat the volatility as a stochastic process is growing, however, and it is possible that Heston's stochastic volatility model gives better option price estimates than the Black-Scholes model. Therefore, we wish to draw a comparison between the results of the Black-Scholes model and the results of the Heston model.

5 Data Example

In order to compare the accuracy of the Heston model and the Black-Scholes model, we test out how well they estimate the premiums of 36 call options on the S&P 100 exchange-traded fund from June 1997.

First, we must estimate the Heston model's parameters. Utilizing both the method of moments (MOM) and maximum likelihood estimation (MLE) detailed previously, we find two sets of parameter estimates. The data set that we use to perform parameter estimation contains daily S&P 100 data from January 1991 to June 1997. Table 1 gives the parameter estimates that we find using both MOM and MLE estimation methods.

Table 1: Parameter Estimates

	MOM	MLE
r	6.50×10^{-4}	6.40×10^{-4}
k	2.00	6.57×10^{-3}
θ	5.16×10^{-5}	6.47×10^{-5}
σ	2.38×10^{-3}	5.09×10^{-4}
ρ	NA	-1.98×10^{-3}

Since we are unable to solve for $\hat{\rho}_{MOM}$, we solve for the call estimates using 21 values of ρ on increments of 0.1 between -1 and 1. For our data set, the smaller the value of ρ, the closer the call estimates to the actual call data. Thus, $\rho = -1$ gives the most accurate price estimates, upon which we base our method of moments analysis. Granted, a person who wishes to estimate the call price before it is available will not know which value of ρ will give the best estimates. We find, however, that the call estimates we obtain from using $\rho = -1$ (the best choice of ρ) and the call estimates we obtain from using $\rho = 1$ (the worst choice of ρ) are within five cents of each other. Further, in order to measure the sensitivity of ρ, we compute the values of each of the 36 call options using all 21 values of ρ. That gives us a set of $36 \times 21 = 756$ data entries consisting of a correlation ρ and a call value. We find that the correlation between the values of ρ and the call estimates equals -6.44×10^{-5}. That correlation, which is close to 0, implies that the call estimates from our data set are not particularly sensitive to the value of ρ.

For the Black-Scholes model, σ, the standard deviation of the daily returns for the S&P100 index for January 2nd, 1991 through June 11th, 1997, is approximately 0.00723. Furthermore, for r, we use the June 11, 1997 three month T-bill interest rate of 4.85% [1]. Together, those values for r and σ are utilized in the Black-Scholes formula to yield call price estimates.

Finally, we examine a set of S&P 100 call option transaction data from June 1997, the end of the time period for which we have data on the S&P 100 index's values. This sample of data contains options with expiration dates that were 24 days, 87 days, and 115 days into the future. With this expiration separation, we test the abilities of the Heston model and the Black-Scholes model to accurately estimate the premiums of short-term, mid-term, and long-term options.

We observe the contracted strike prices and time until expiration for each of the option transactions. Then we use our parameter estimates to estimate the premiums according to the option pricing formula in equation (3). The option prices, Black-Scholes estimates, and Heston estimates are available in the proceeding table.
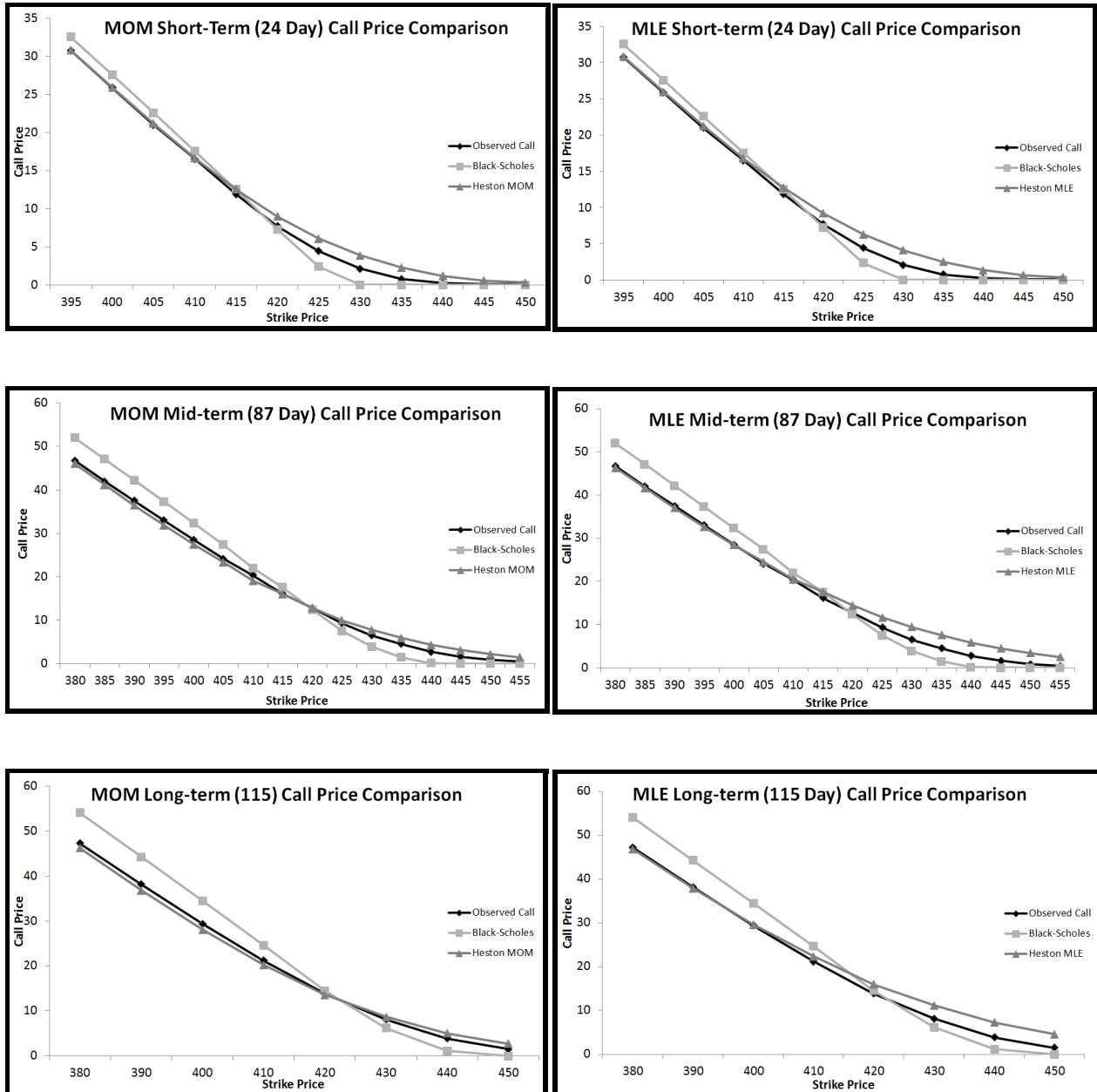
Table 2: Option Price Comparison

Expiration	Stock Price	Strike Price	Actual Call	Heston MOM Call[2]	Heston MLE Call	Black-Scholes Call
24	425.73	395	30.75	30.82	30.84	32.55
24	425.73	400	25.88	25.93	25.98	27.57
24	425.73	405	21.00	21.19	21.27	22.60
24	425.67	410	16.50	16.64	16.78	17.56
24	425.68	415	11.88	12.55	12.75	12.53
24	425.65	420	7.69	8.97	9.22	7.26
24	425.65	425	4.44	6.06	6.30	2.37
24	425.68	430	2.10	3.86	4.12	0.00
24	425.65	435	0.78	2.28	2.51	0.00
24	425.16	440	0.25	1.17	1.35	0.00
24	424.78	445	0.10	0.56	0.68	0.00
24	425.19	450	0.10	0.28	0.36	0.00
87	425.73	380	46.75	46.02	46.32	52.04
87	425.73	385	42.00	41.19	41.61	47.12
87	425.73	390	37.50	36.46	37.04	42.21
87	425.73	395	33.00	31.87	32.64	37.29
87	425.73	400	28.50	27.48	28.45	32.37
87	425.73	405	24.13	23.33	24.50	27.43
87	425.26	410	20.38	19.12	20.50	21.99
87	425.86	415	16.13	16.06	17.58	17.58
87	425.68	420	12.82	12.81	14.45	12.41
87	425.42	425	9.32	9.96	11.66	7.54
87	425.62	430	6.51	7.76	9.45	3.93
87	425.82	435	4.51	5.93	7.56	1.48
87	425.68	440	2.75	4.33	5.85	0.14
87	425.75	445	1.60	3.14	4.51	0.00
87	425.78	450	0.85	2.22	3.41	0.00
87	425.39	455	0.44	1.47	2.47	0.00
115	425.73	380	47.25	46.20	46.81	54.05
115	425.73	390	38.13	36.79	37.84	44.27
115	425.73	400	29.38	28.03	29.61	34.48
115	425.73	410	21.19	20.25	22.32	24.63
115	425.41	420	13.88	13.57	15.98	14.50
115	425.63	430	8.13	8.70	11.17	6.18
115	425.28	440	3.88	5.03	7.29	1.15
115	425.13	450	1.50	2.72	4.57	0.00

To begin our analysis of the results, we visually compare the estimates to the actual call prices. We perform a graphical comparison by separating the option transaction data into three groups organized by time until expiration. We plot the predicted call values from the above table with respect to their strike prices. All six

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[2]The MOM data uses a value of $\rho = -1$.

graphs containing the short-term, mid-term, and long-term estimates using the MOM and MLE parameter estimates are displayed.

As seen in the above graph for mid-term option call price comparison, the Heston model's call price estimates are closer than the Black-Scholes estimates to the observed call prices.

5.1 Root-Mean-Square Error

Root-mean-square error (RMSE) is a statistical tool used for the comparison between estimated $(\hat{C}_i)$ and observed (C_i) values. Defined by the following formula, RMSE provides a quantitative measurement for the comparison between two models. The smaller the value of RMSE, the closer the estimated values are to the actual values.

$$RMSE = \sqrt{\frac{\sum_{i=1}^{n}(\hat{C}_i - C_i)^2}{n}} \tag{7}$$

For our purposes, we use the RMSE to compare the Black-Scholes model's estimates and the Heston model's estimates to the actual premiums. In Table 3, we observe that the method of moments gives closer call price estimates than maximum likelihood estimation. Regardless of parameter estimation technique, however, the Heston model provides estimates that are closer than the Black-Scholes model's estimates to actual transaction data.

Table 3: Root-Mean-Square Error Results

	24 days	87 days	115 days
Black-Scholes Model	1.28	3.11	4.12
Heston Model (MOM)	0.968	1.08	1.05
Heston Model (MLE)	1.13	1.90	2.13

6 Conclusion

This paper provides promising results regarding the application of the Heston model to option price estimation. Both visual inspection and RMSE results show that the estimates of the Heston model are closer than the estimates of the popular Black-Scholes model to the actual call option prices in our data sample.

The results that we find in this research make us optimistic about the knowledge that could result from further exploration of the Heston model. Utilizing additional real world option transaction data would provide results of a broader scope. Furthermore, the use of data simulation to test the parameter estimation methods would permit measurement of the relative success of the estimation techniques. Although the method of moments provides closer price estimates for our data set, it remains unclear whether this result will continue. Finally, it would be beneficial to compare the results of the Heston model to those of other stochastic volatility models.

Acknowledgements

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Appendix

Appendix A: Three Variable Ito Lemma

In this section, we derive the form of the Ito Lemma in three variables. Assume that we have the following system of two standard stochastic differential equations:

$$
\begin{aligned}
dX_t &= \mu_x dt + \sigma_x dW_{1t} \\
dV_t &= \mu_v dt + \sigma_v dW_{2t}.
\end{aligned}
$$

Further, let W_{1t} and W_{2t} have correlation ρ, for some $-1 \leq \rho \leq 1$. For a function $f(X_t, V_t, t)$, we wish to find $df(X_t, V_t, t)$. Note that $df(X_t, V_t, t) = f(X_{t+dt}, V_{t+dt}, t) - f(X_t, V_t, t)$. Use Taylor series expansion to expand $f(X_{t+dt}, V_{t+dt}, t)$.

$$
\begin{aligned}
f(X_{t+dt}, V_{t+dt}, t) &= f(X_t, V_t, t) + f_x(dX) + f_v(dV) + f_t(dt) + \frac{1}{2}\Big[f_{xx}(dX)^2 + f_{vv}(dV)^2 \\
&\quad + f_{tt}(dt)^2 + f_{xv}(dX)(dV) + f_{xt}(dX)(dt) + f_{vx}(dV)(dX) + f_{vt}(dV)(dt) \\
&\quad + f_{tx}(dt)(dX) + f_{tv}(dt)(dV) \Big] \\
&= f(X_t, V_t, t) + f_x(dX) + f_v(dV) + f_t(dt) + \frac{1}{2}\Big[f_{xx}(dX)^2 + f_{vv}(dV)^2 + f_{tt}(dt)^2 \Big] \\
&\quad + f_{xt}(dX)(dt) + f_{vx}(dV)(dX) + f_{vt}(dV)(dt) \\
df(X_t, V_t, t) &= f_x(dX) + f_v(dV) + f_t(dt) + \frac{1}{2}\Big[f_{xx}(dX)^2 + f_{vv}(dV)^2 + f_{tt}(dt)^2 \Big] \\
&\quad + f_{xt}(dX)(dt) + f_{vx}(dV)(dX) + f_{vt}(dV)(dt) \\
&= f_x(\mu_x dt + \sigma_x dW_{1t}) + f_v(\mu_v dt + \sigma_v dW_{2t}) + f_t dt + \frac{1}{2}\Big[f_{xx}(\mu_x dt + \sigma_x dW_{1t})^2 \\
&\quad + f_{vv}(\mu_v dt + \sigma_v dW_{2t})^2 + f_{tt}(dt)^2 \Big] + f_{xt}(\mu_x dt + \sigma_x dW_{1t})(dt) \\
&\quad + f_{xv}(\mu_x dt + \sigma_x dW_{1t})(\mu_v dt + \sigma_v dW_{2t}) + f_{vt}(u_v dt + \sigma_v dW_{2t})(dt) \\
&= \mu_x f_x dt + \sigma_x f_x dW_{1t} + \mu_v f_v dt + \sigma_v f_v dW_{2t} + f_t dt \\
&\quad + \frac{1}{2}\Big[f_{xx}\big(\mu_x^2(dt)^2 + 2\mu_x\sigma_x(dt)(dW_{1t}) + \sigma_x^2(dW_{1t})^2 \big) \\
&\quad + f_{vv}\big(\mu_v^2(dt)^2 + 2\mu_v\sigma_v(dt)(dW_{2t}) + \sigma_v^2(dW_{2t})^2 \big) + f_{tt}(dt)^2 \Big] \\
&\quad + f_{xv}\big(\mu_x\mu_v(dt)^2 + \mu_x\sigma_v(dt)(dW_{2t}) + \mu_v\sigma_x(dt)(dW_{1t}) + \sigma_x\sigma_v(dW_{1t})(dW_{2t}) \big) \\
&\quad + f_{xt}\big(\mu_x(dt)^2 + \sigma_x(dW_{1t})(dt) \big) + f_{vt}\big(\mu_v(dt)^2 + \sigma_v(dW_{2t})(dt) \big)
\end{aligned}
$$

We can simplify further using the following properties of the products for increments in Ito calculus:

$$
\begin{aligned}
(dt)^2 &= 0 \\
(dW_t)(dt) &= 0 \\
(dW_{1t})(dW_{1t}) &= dt \\
(dW_{1t})(dW_{2t}) &= \rho(dt).
\end{aligned}
$$

$$
\begin{aligned}
df(X_t, V_t, t) &= \mu_x f_x dt + \sigma_x f_x dW_{1t} + \mu_v f_v dt + \sigma_v f_v dW_{2t} + f_t dt \\
&\quad + \frac{1}{2}\Big[f_{xx}\sigma_x^2 dt + f_{vv}\sigma_v^2 dt \Big] + f_{xv}(\sigma_x\sigma_v\rho dt) \\
&= \Big[\mu_x f_x + \mu_v f_v + f_t + f_{xv}\sigma_x\sigma_v\rho + \frac{1}{2}\big(f_{xx}\sigma_x^2 + f_{vv}\sigma_v^2 \big) \Big] dt \\
&\quad + \big[\sigma_x f_x \big] dW_{1t} + \big[\sigma_v f_v \big] dW_{2t}
\end{aligned}
$$

Therefore, we can conclude that the Ito Lemma for three variables is given by

$$df(X_t, V_t, t) = \left(f_t + \mu_x f_x + \mu_v f_v + \rho \sigma_x \sigma_v f_{xv} + \frac{1}{2}\sigma_x^2 f_{xx} + \frac{1}{2}\sigma_v^2 f_{vv}\right)dt + \sigma_x f_x dW_{1t} + \sigma_v f_v dW_{2t}.$$

Appendix B: Characteristic Function Derivation

We know that the characteristic function has the following exponential affine form [5]:

$$f(i\phi) = e^{A(T-t)+B(T-t)X_t+C(T-t)V_t+i\phi X_t}.$$

By the Ito Lemma in three variables, the drift (the portion preceding dt) of the derivative of characteristic function is

$$\left(r - \frac{1}{2}V_t\right)f_x + k(\theta - V_t)f_v + f_t + \frac{1}{2}V_t f_{xx} + \frac{1}{2}\sigma^2 V_t f_{vv} + \rho\sigma V_t f_{xv}.$$

According to the Fundamental Theorem of Asset Pricing, the drift equals 0. Setting the drift equal to 0 and substituting the partial derivatives,

$$0 = f\left[\left(r - \frac{1}{2}V_t\right)(B(T-t) + i\phi) + k(\theta - V_t)(C(T-t) - A'(T-t) - B'(T-t)X_t) - C'(T-t)V_t\right.$$

$$+ \frac{1}{2}V_t(B(T-t) + i\phi)^2 + \frac{1}{2}\sigma^2 V_t(C(T-t))^2 + \rho\sigma V_t(B(T-t) + i\phi)C(T-t)\bigg]$$

$$= [-B'(T-t)]X_t + \left[-\frac{1}{2}B(T-t) - \frac{1}{2}i\phi - kC(T-t) - C'(T-t) + \frac{1}{2}(B(T-t))^2 + i\phi(B(T-t))\right.$$

$$- \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2(C(T-t))^2 + \rho\sigma(B(T-t))(C(T-t)) + \rho\sigma i\phi(C(T-t))\bigg]V_t$$

$$+ [rB(T-t) + ri\phi + k\theta C(T-t) - A'(T-t)]$$

In the second step, we drop f because it will always be true that $f > 0$.

In order for the drift to equal 0 for all values of X_t and V_t, the coefficients in front of X_t must sum to 0, the coefficients in front of V_t must sum to 0, and the constant terms must sum to 0. That gives us the following system:

$$0 = -B'(T-t)$$

$$0 = -\frac{1}{2}B(T-t) - \frac{1}{2}i\phi - kC(T-t) - C'(T-t) + \frac{1}{2}(B(T-t))^2 + i\phi(B(T-t))$$

$$- \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2(C(T-t))^2 + \rho\sigma(B(T-t))(C(T-t)) + \rho\sigma i\phi(C(T-t))$$

$$0 = rB(T-t) + ri\phi + k\theta C(T-t) - A'(T-t).$$

Further, we have the initial condition that when $t = T, f(i\phi) = e^{i\phi X_T}$. As a result, we have initial conditions $A(0) = 0, B(0) = 0$, and $C(0) = 0$.

Since $-B'(T-t) = 0$ and $B(0) = 0$, it follows that $B(T-t) = 0$. Given that condition, we know that the characteristic function has the form

$$f(i\phi) = e^{A(T-t)+C(T-t)V_t+i\phi X_t}.$$

Further, since $B(T-t) = 0$, we have the following two equation system:

$$C'(T-t) = -\frac{1}{2}i\phi - kC(T-t) - \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2(C(T-t))^2 + \rho\sigma i\phi(C(T-t))$$

$$A'(T-t) = ri\phi + k\theta C(T-t)$$

with the initial conditions that $A(0) = 0$ and $C(0) = 0$.

Performing algebra and rearranging,

$$\frac{dC(T-t)}{dt} = -\frac{1}{2}\sigma^2\left[\frac{-i\phi}{\sigma^2} - \frac{\phi^2}{\sigma^2} + \left(\frac{2\rho i\phi}{\sigma} - \frac{2k}{\sigma^2}\right)C(T-t) + (C(T-t))^2\right]$$

$$\frac{dC(T-t)}{(C(T-t))^2 + \left(\frac{2\rho i\phi}{\sigma} - \frac{2k}{\sigma^2}\right)C(T-t) - \left(\frac{i\phi+\phi^2}{\sigma^2}\right)} = -\frac{1}{2}\sigma^2 dt$$

Via the quadratic formula, we determine that the roots of the denominator are

$$\frac{-(i\phi\rho\sigma - k)}{\sigma^2} \pm \sqrt{-\frac{\rho^2\phi^2}{\sigma^2} - \frac{2i\rho\phi k}{\sigma^3} + \frac{k^2}{\sigma^4} + \frac{i\phi + \phi^2}{\sigma^2}}.$$

Let $\alpha = \frac{-(i\phi\rho\sigma - k)}{\sigma^2}$ and let $\beta = \sqrt{-\frac{\rho^2\phi^2}{\sigma^2} - \frac{2i\rho\phi k}{\sigma^3} + \frac{k^2}{\sigma^4} + \frac{(i\phi+\phi^2)}{\sigma^2}}$. Factoring, we obtain

$$\frac{dC(T - t)}{[C(T - t) - (\alpha - \beta)][C(T - t) - (\alpha + \beta)]} = -\frac{1}{2}\sigma^2 dt.$$

Next, we solve for the partial fraction decomposition of the first fraction.

$$\frac{1}{[C(T - t) - (\alpha - \beta)][C(T - t) - (\alpha + \beta)]} = \frac{G}{C(T - t) - (\alpha - \beta)} + \frac{H}{C(T - t) - (\alpha + \beta)}$$
$$1 = G[C(T - t) - (\alpha + \beta)] + H[C(T - t) - (\alpha - \beta)]$$

$$\text{Let } C(T - t) = \alpha + \beta.$$
$$G(0) + H(2\beta) = 1$$
$$H = \frac{1}{2\beta}$$

$$\text{Let } C(T - t) = \alpha - \beta$$
$$G[(\alpha - \beta) - (\alpha + \beta)] + H(0) = 1$$
$$G(-2\beta) = 1$$
$$G = \frac{-1}{2B}$$

$$\frac{1}{(C(T - t))^2 + (\frac{2\rho i\phi}{\sigma} - \frac{2k}{\sigma^2})C(T - t) - (\frac{i\phi+\phi^2}{\sigma^2})} = \frac{-\frac{1}{2B}}{C(T - t) - (\alpha - \beta)} + \frac{\frac{1}{2\beta}}{C(T - t) - (\alpha + \beta)}$$
$$\frac{dC(T - t)}{(C(T - t))^2 + (\frac{2\rho i\phi}{\sigma} - \frac{2k}{\sigma^2})C(T - t) - (\frac{i\phi+\phi^2}{\sigma^2})} = \left(\frac{-\frac{1}{2B}}{C(T - t) - (\alpha - \beta)} + \frac{\frac{1}{2\beta}}{C(T - t) - (\alpha + \beta)}\right) dC(T - t)$$

Using the above equation, we are able to solve for $C(T - t)$.

$$-\frac{1}{2}\sigma^2 dt = \left(\frac{-\frac{1}{2\beta}}{C(T - t) - (\alpha - \beta)} + \frac{\frac{1}{2\beta}}{C(T - t) - (\alpha + \beta)}\right) dC(T - t)$$

$$\int_{s=t}^{s=T} -\frac{1}{2}\sigma^2 ds = \int_{s=t}^{s=T} \frac{-\frac{1}{2\beta}}{C(T - t) - (\alpha - \beta)} dC(T - s) + \int_{s=t}^{s=T} \frac{\frac{1}{2\beta}}{C(T - t) - (\alpha + \beta)} dC(T - s)$$

$$-\frac{1}{2}\sigma^2 s \Big|_{s=t}^{s=T} = -\frac{1}{2\beta}\Big[\ln(C(T - s) - (\alpha - \beta))\Big]_{s=t}^{s=T} + \frac{1}{2\beta}\Big[\ln(C(T - s) - (\alpha + \beta))\Big]_{s=t}^{s=T}$$

$$-\frac{1}{2}\sigma^2(T - t) = -\frac{1}{2\beta}\Big[\ln(C(0) - (\alpha - \beta)) - \ln(C(T - t) - (\alpha - \beta))\Big]$$

$$+ \frac{1}{2\beta}\Big[\ln(C(0) - (\alpha + \beta)) - \ln(C(T - t) - (\alpha + \beta))\Big]$$

Let $T - t = \tau$.

$$-\frac{1}{2}\sigma^2\tau = -\frac{1}{2\beta}\Big[\ln(\beta - \alpha) - \ln(C(\tau) - (\alpha - \beta))\Big] + \frac{1}{2\beta}\Big[\ln(-(\alpha + \beta)) - \ln(C(\tau) - (\alpha + \beta))\Big]$$

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$$-\frac{1}{2}\sigma^2\tau = -\frac{1}{2\beta}\ln(\beta-\alpha) + \frac{1}{2\beta}\ln(C(\tau)-(\alpha-\beta)) + \frac{1}{2\beta}\ln(-(\alpha+\beta)) - \frac{1}{2\beta}\ln(C(\tau)-(\alpha+\beta))$$

$$-\frac{1}{2}\sigma^2\tau = \frac{1}{2\beta}\left[\ln(C(\tau)-(\alpha-\beta)) + \ln(-(\alpha+\beta))\right] - \frac{1}{2\beta}\left[\ln(C(\tau)-(\alpha+\beta)) + \ln(\beta-\alpha)\right]$$

$$-\frac{1}{2}\sigma^2\tau = \frac{1}{2\beta}\ln\left[(C(\tau)-(\alpha-\beta))(-(\alpha+\beta))\right] - \frac{1}{2\beta}\ln\left[(C(\tau)-(\alpha+\beta))(\beta-\alpha)\right]$$

$$-\frac{1}{2}\sigma^2\tau = \frac{1}{2\beta}\ln\left[(C(\tau)-(\alpha-\beta))(-(\alpha+\beta))\right] + \frac{1}{2\beta}\ln\left[\frac{1}{(C(\tau)-(\alpha+\beta))(\beta-\alpha)}\right]$$

$$-\frac{1}{2}\sigma^2\tau = \frac{1}{2\beta}\ln\left[\frac{(C(\tau)-(\alpha-\beta))(-(\alpha+\beta))}{(C(\tau)-(\alpha+\beta))(\beta-\alpha)}\right]$$

$$-\beta\sigma^2\tau = \ln\left[\frac{(-\alpha-\beta)(C(\tau)-(\alpha-\beta))}{(\beta-\alpha)(C(\tau)-(\alpha+\beta))}\right]$$

$$e^{-\beta\sigma^2\tau} = \frac{(-\alpha-\beta)(C(\tau)-(\alpha-\beta))}{(\beta-\alpha)(C(\tau)-(\alpha+\beta))}$$

$$\frac{\beta-\alpha}{-(\beta+\alpha)}e^{-\beta\sigma^2\tau} = \frac{C(\tau)-(\alpha-\beta)}{C(\tau)-(\alpha+\beta)}$$

$$C(\tau)-(\alpha-\beta) = \left[\frac{\beta-\alpha}{-(\beta+\alpha)}e^{-\beta\sigma^2\tau}\right]\left[C(\tau)-(\alpha+\beta)\right]$$

$$C(\tau) + \frac{\beta-\alpha}{\beta+\alpha}e^{-\beta\sigma^2\tau}C(\tau) = (\alpha-\beta) + (\beta-\alpha)e^{-\beta\sigma^2\tau}$$

$$C(\tau)\left[1 + \frac{\beta-\alpha}{\beta+\alpha}e^{-\beta\sigma^2\tau}\right] = (\alpha-\beta) + (\beta-\alpha)e^{-\beta\sigma^2\tau}$$

$$C(\tau) = \frac{(\alpha-\beta) + (\beta-\alpha)e^{-\beta\sigma^2\tau}}{1 + \frac{\beta-\alpha}{\beta+\alpha}e^{-\beta\sigma^2\tau}}$$

Next, let us find an alternate expression for β.

$$\beta = \sqrt{\left(\frac{1}{\sigma^4}\right)\left[-\rho^2\phi^2\sigma^2 - 2i\rho\phi k\sigma + k^2 + i\phi\sigma^2 + \phi^2\sigma^2\right]}$$

$$\beta = \frac{1}{\sigma^2}\sqrt{-\rho^2\phi^2\sigma^2 - 2i\rho\phi k\sigma + k^2 + \sigma^2(i\phi + \phi^2)}$$

$$\beta = \frac{1}{\sigma^2}\sqrt{(\rho\sigma i\phi - k)^2 + \sigma^2(i\phi + \phi^2)}$$

Let $M = \sqrt{(\rho\sigma i\phi - k)^2 + \sigma^2(i\phi + \phi^2)}$

$$\beta = \frac{1}{\sigma^2}M$$

Simplify $C(\tau)$ using the new form of β.

$$C(\tau) = \frac{\left(\alpha - \frac{M}{\sigma^2}\right) + \left(\frac{M}{\sigma^2} - \alpha\right)e^{-M\tau}}{1 + \frac{\frac{M}{\sigma^2}-\alpha}{\frac{M}{\sigma^2}+\alpha}e^{-M\tau}}$$

$$= \frac{\left(\alpha - \frac{M}{\sigma^2}\right) + \left(\frac{M}{\sigma^2} - \alpha\right)e^{-M\tau}}{\frac{\frac{M}{\sigma^2}+\alpha}{\frac{M}{\sigma^2}+\alpha} + \frac{\left(\frac{M}{\sigma^2}-\alpha\right)e^{-M\tau}}{\frac{M}{\sigma^2}+\alpha}}$$

$$= \frac{\left(\alpha - \frac{M}{\sigma^2}\right) + \left(\frac{M}{\sigma^2} - \alpha\right)e^{-M\tau}}{\frac{\frac{M}{\sigma^2}+\alpha+\left(\frac{M}{\sigma^2}-\alpha\right)e^{-M\tau}}{\frac{M}{\sigma^2}+\alpha}}$$

$$= \frac{\left(\frac{M}{\sigma^2} + \alpha\right)\left[\left(\alpha - \frac{M}{\sigma^2}\right) + \left(\frac{M}{\sigma^2} - \alpha\right)e^{-M\tau}\right]}{\frac{M}{\sigma^2} + \alpha + \left(\frac{M}{\sigma^2} - \alpha\right)e^{-M\tau}}$$

$$= \frac{\left(-\frac{M^2}{\sigma^4} + \alpha^2\right) + \left(\frac{M^2}{\sigma^4} - \alpha^2\right) e^{-M\tau}}{\frac{M}{\sigma^2} + \alpha + \left(\frac{M}{\sigma^2} - \alpha\right) e^{-M\tau}} \left(\frac{e^{M\tau}}{e^{M\tau}}\right)$$

$$= \frac{\left(\alpha^2 - \frac{M^2}{\sigma^4}\right) e^{M\tau} + \left(\frac{M^2}{\sigma^4} - \alpha^2\right)}{\left(\frac{M}{\sigma^2} + \alpha\right) e^{M\tau} + \left(\frac{M}{\sigma^2} - \alpha\right)}$$

$$= \frac{\left(e^{M\tau} - 1\right) \left(\alpha^2 - \frac{M^2}{\sigma^4}\right)}{\left(\frac{M}{\sigma^2} + \alpha\right) e^{M\tau} + \left(\frac{M}{\sigma^2} - \alpha\right)}$$

$$= \frac{\left(e^{M\tau} - 1\right) \left(-\frac{\phi^2 \rho^2}{\sigma^2} - \frac{2i\phi\rho k}{\sigma^3} + \frac{k^2}{\sigma^4} - \frac{1}{\sigma^4}\left[-\phi^2 \rho^2 \sigma^2 - 2i\phi\rho\sigma k + k^2 + \sigma^2 i\phi + \sigma^2 \phi^2\right]\right)}{\left(\frac{M}{\sigma^2} + \alpha\right) e^{M\tau} + \left(\frac{M}{\sigma^2} - \alpha\right)}$$

$$= \frac{\left(e^{M\tau} - 1\right) \left(-\frac{\phi^2 \rho^2}{\sigma^2} - \frac{2i\phi\rho k}{\sigma^3} + \frac{k^2}{\sigma^4} + \frac{\phi^2 \rho^2}{\sigma^2} + \frac{2i\phi\rho k}{\sigma^3} - \frac{k^2}{\sigma^4} - \frac{i\phi}{\sigma^2} - \frac{\phi^2}{\sigma^2}\right)}{\left(\frac{M}{\sigma^2} + \alpha\right) e^{M\tau} + \left(\frac{M}{\sigma^2} - \alpha\right)}$$

$$= \frac{e^{M\tau} - 1}{\sigma^2} \left[\frac{-i\phi - \phi^2}{\left(\frac{M}{\sigma^2} - \frac{i\phi\rho\sigma - k}{\sigma^2}\right) e^{M\tau} + \left(\frac{M}{\sigma^2} + \frac{i\phi\rho\sigma - k}{\sigma^2}\right)}\right]$$

$$= \frac{e^{M\tau} - 1}{\sigma^2} \left[\frac{i\phi + \phi^2}{\left(\frac{i\phi\rho\sigma - k}{\sigma^2} - \frac{M}{\sigma^2}\right) e^{M\tau} - \left(\frac{M}{\sigma^2} + \frac{i\phi\rho\sigma - k}{\sigma^2}\right)}\right]$$

$$= \frac{e^{M\tau} - 1}{\sigma^2} \left[\frac{i\phi + \phi^2}{\frac{1}{\sigma^2}\left[(i\phi\rho\sigma - k - M) e^{M\tau} - (i\phi\rho\sigma - k + M)\right]}\right]$$

$$= \frac{\left(e^{M\tau} - 1\right) \left(i\phi + \phi^2\right)}{(i\phi\rho\sigma - k - M)e^{M\tau} - (i\phi\rho\sigma - k + M)} \cdot \frac{\left(\frac{1}{-(i\phi\rho\sigma - k + M)}\right)}{\left(\frac{1}{-(i\phi\rho\sigma - k + M)}\right)}$$

$$= \frac{\left(e^{M\tau} - 1\right) \left(\frac{-(i\phi + \phi^2)}{i\phi\rho\sigma - k + M}\right)}{1 - \frac{(i\phi\rho\sigma - k - M)}{(i\phi\rho\sigma - k + M)} e^{M\tau}}$$

Let $N = \dfrac{i\phi\rho\sigma - k - M}{i\phi\rho\sigma - k + M}$.

$$C(\tau) = \frac{\left(e^{M\tau} - 1\right) \left(\frac{-\phi^2 - i\phi}{i\phi\rho\sigma - k + M}\right)}{1 - N e^{M\tau}}$$

$$= \frac{\left(e^{M\tau} - 1\right) \left(\frac{-\sigma^2 \phi^2 - i\phi\sigma^2}{i\phi\rho\sigma - k + M}\right)}{\sigma^2 \left(1 - N e^{M\tau}\right)}$$

$$= \frac{\left(e^{M\tau} - 1\right) \left(\frac{\left(-\sigma^2 \phi^2 - i\phi\sigma^2\right)(\rho\sigma i\phi - k - M)}{(i\phi\rho\sigma - k + M)(\rho\sigma i\phi - k - M)}\right)}{\sigma^2 \left(1 - N e^{M\tau}\right)}$$

$$= \frac{\left(e^{M\tau} - 1\right) \left(\frac{\left(-\sigma^2 \phi^2 - i\phi\sigma^2\right)(\rho\sigma i\phi - k - M)}{-\sigma^2 \phi^2 - i\phi\sigma^2}\right)}{\sigma^2 \left(1 - N e^{M\tau}\right)}$$

$$C(\tau) = \frac{\left(e^{M\tau} - 1\right) (\rho\sigma i\phi - k - M)}{\sigma^2 \left(1 - N e^{M\tau}\right)},$$

where $M = \sqrt{(\rho\sigma i\phi - k)^2 + \sigma^2(i\phi + \phi^2)}$ and $N = \dfrac{\rho\sigma i\phi - k - M}{\rho\sigma i\phi - k + M}$.

Using that formula for $C(\tau) = C(T - t)$, we can solve for $A(\tau) = A(T - t)$.

$$A'(T - t) = ri\phi + k\theta C(T - t)$$

$$\int_{s=t}^{s=T} A'(T - s)ds = \int_{s=t}^{s=T} ri\phi\, ds + \int_{s=t}^{s=T} k\theta C(T - s)ds$$

$$-A(T - s)\Big|_{s=t}^{s=T} = ri\phi s\Big|_{s=t}^{s=T} + k\theta \int_{s=t}^{s=T} C(T - s)ds$$

$$-A(0) + A(T - t) = ri\phi(T - t) + k\theta \int_{s=t}^{s=T} C(T - s)ds$$

$$A(T - t) = ri\phi(T - t) + k\theta \int_{s=t}^{s=T} \frac{\left(e^{M(T-s)} - 1\right)\left(\rho\sigma i\phi - k - M\right)}{\sigma^2 \left(1 - Ne^{M(T-s)}\right)} ds$$

$$= ri\phi(T - t) + \frac{k\theta}{\sigma^2}(\rho\sigma i\phi - k - M) \int_{s=t}^{s=T} \frac{e^{M(T-s)} - 1}{1 - Ne^{M(T-s)}} ds$$

$$= ri\phi(T - t) + \frac{k\theta}{\sigma^2}(\rho\sigma i\phi - k - M)\left[-\frac{\ln(Ne^{M(T-s)} - 1)}{M} + \frac{\ln(Ne^{M(T-s)} - 1)}{MN} \right.$$

$$\left. + \frac{\ln(e^{M(T-s)})}{M} \right]_{s=t}^{s=T}$$

$$= ri\phi(T - t) + \frac{k\theta}{\sigma^2}(\rho\sigma i\phi - k - M)\left[\left(-\frac{\ln(N - 1)}{M} + \frac{\ln(N - 1)}{MN} + \frac{\ln(1)}{M} \right) \right.$$

$$\left. - \left(-\frac{\ln\left(Ne^{M(T-t)} - 1\right)}{M} + \frac{\ln\left(Ne^{M(T-t)} - 1\right)}{MN} + \frac{\ln\left(e^{M(T-t)}\right)}{M} \right) \right]$$

$$A(\tau) = ri\phi\tau + \frac{k\theta}{\sigma^2}(\rho\sigma i\phi - k - M)\left[-\frac{\ln(N - 1)}{M} + \frac{\ln(N - 1)}{MN} + \frac{\ln(Ne^{M\tau} - 1)}{M} \right.$$

$$\left. - \frac{\ln(Ne^{M\tau} - 1)}{MN} - \frac{\ln(e^{M\tau})}{M} \right]$$

$$= ri\phi\tau + \frac{k\theta}{\sigma^2}(\rho\sigma i\phi - k - M)\left[\frac{1}{M}\left(\ln(Ne^{M\tau} - 1) - \ln(N - 1) \right) \right.$$

$$\left. + \frac{1}{MN}\left(\ln(N - 1) - \ln(Ne^{M\tau} - 1) \right) - \frac{M\tau}{M} \right]$$

$$= ri\phi\tau + \frac{k\theta}{\sigma^2}(\rho\sigma i\phi - k - M)\left[\frac{1}{M}\ln\left(\frac{Ne^{M\tau} - 1}{N - 1}\right) + \frac{1}{MN}\ln\left(\frac{N - 1}{Ne^{M\tau} - 1}\right) - \tau \right]$$

$$= ri\phi\tau + \frac{k\theta}{\sigma^2}(\rho\sigma i\phi - k - M)\left[\frac{1}{M}\ln\left(\frac{1 - Ne^{M\tau}}{1 - N}\right) + \frac{1}{MN}\ln\left(\frac{1 - N}{1 - Ne^{M\tau}}\right) - \tau \right]$$

$$= ri\phi\tau + \frac{k\theta}{\sigma^2}(\rho\sigma i\phi - k - M)\left[\frac{N\ln\left(\frac{1 - Ne^{M\tau}}{1 - N}\right) + \ln\left(\frac{1 - N}{1 - Ne^{M\tau}}\right)}{MN} - \tau \right]$$

$$= ri\phi\tau + \frac{k\theta}{\sigma^2}\left[-(\rho\sigma i\phi - k - M)\tau + (\rho\sigma i\phi - k + M)\left[\frac{N\ln\left(\frac{1 - Ne^{M\tau}}{1 - N}\right) + \ln\left(\frac{1 - N}{1 - Ne^{M\tau}}\right)}{M} \right] \right]$$

$$= ri\phi\tau + \frac{k\theta}{\sigma^2}\left[-(\rho\sigma i\phi - k - M)\tau + (\rho\sigma i\phi - k + M)\left[\frac{N\ln\left(\frac{1 - Ne^{M\tau}}{1 - N}\right) - \ln\left(\frac{1 - Ne^{M\tau}}{1 - N}\right)}{M} \right] \right]$$

$$= ri\phi\tau + \frac{k\theta}{\sigma^2}\left[-(\rho\sigma i\phi - k - M)\tau + (\rho\sigma i\phi - k + M)\left(\frac{N - 1}{M}\right)\ln\left(\frac{1 - Ne^{M\tau}}{1 - N}\right) \right]$$

$$= ri\phi\tau + \frac{k\theta}{\sigma^2}\left[-(\rho\sigma i\phi - k - M)\tau + (\rho\sigma i\phi - k + M)\left(\frac{\frac{\rho\sigma i\phi - k - M}{\rho\sigma i\phi - k + M} - 1}{M}\right)\ln\left(\frac{1 - Ne^{M\tau}}{1 - N}\right)\right]$$

$$= ri\phi\tau + \frac{k\theta}{\sigma^2}\left[-(\rho\sigma i\phi - k - M)\tau + \left(\frac{\rho\sigma i\phi - k - M - \rho\sigma i\phi + k - M}{M}\right)\ln\left(\frac{1 - Ne^{M\tau}}{1 - N}\right)\right]$$

$$A(\tau) = ri\phi\tau + \frac{k\theta}{\sigma^2}\left[-(\rho\sigma i\phi - k - M)\tau - 2\ln\left(\frac{1 - Ne^{M\tau}}{1 - N}\right)\right]$$

Appendix C: MOM Moment Derivation

In this section, we will derive the first through fifth moment of Q_{t+1}. Before we are able to derive expressions for the moments of Q_{t+1}, we will derive formulas for the first six moments of standard normal random variables. Then, we will derive the first three moments of V_{t+1}. With those properties in place, we will derive the moments of Q_{t+1}.

Standard Normal Moments

Assume that we have a random variable Z, where $Z \sim N(0, 1)$. In order to solve for the moments of Z, we can use the moment generating function $M_Z(t)$ of a standard normal random variable.

$$M_Z(t) = e^{\frac{t^2}{2}}$$

We can use the following formula to extract the jth moment of Z from $M_Z(t)$:

$$E(Z^j) = \frac{d^j}{dt^j} e^{\frac{t^2}{2}} \Big|_{t=0}$$

Therefore,

$$
\begin{aligned}
E(Z) &= \frac{d}{dt} e^{\frac{t^2}{2}} \Big|_{t=0} \\
&= t e^{\frac{t^2}{2}} \Big|_{t=0} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
E(Z^2) &= \frac{d^2}{dt^2} e^{\frac{t^2}{2}} \Big|_{t=0} \\
&= e^{\frac{t^2}{2}} + t^2 e^{\frac{t^2}{2}} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
E(Z^3) &= \frac{d^3}{dt^3} e^{\frac{t^2}{2}} \Big|_{t=0} \\
&= 3t e^{\frac{t^2}{2}} + t^3 e^{\frac{t^2}{2}} \Big|_{t=0} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
E(Z^4) &= \frac{d^4}{dt^4} e^{\frac{t^2}{2}} \Big|_{t=0} \\
&= 3 e^{\frac{t^2}{2}} + 6t^2 e^{\frac{t^2}{2}} + t^4 e^{\frac{t^2}{2}} \Big|_{t=0} \\
&= 3
\end{aligned}
$$

$$
\begin{aligned}
E(Z^5) &= \frac{d^5}{dt^5} e^{\frac{t^2}{2}} \Big|_{t=0} \\
&= 15t e^{\frac{t^2}{2}} + 10t^3 e^{\frac{t^2}{2}} + t^5 e^{\frac{t^2}{2}} \Big|_{t=0} \\
&= 0
\end{aligned}
$$

$$E(Z^6) = \left.\frac{d^6}{dt^6} e^{\frac{t^2}{2}}\right|_{t=0}$$

$$= \left. 15e^{\frac{t^2}{2}} + 45t^2 e^{\frac{t^2}{2}} + 15t^4 e^{\frac{t^2}{2}} + t^6 e^{\frac{t^2}{2}}\right|_{t=0}$$

$$= 15.$$

In the following two sections, since Z_1 and Z_2 are two independent standard normal random variables, for any $a, b \in \mathbb{R}$,

$$E(Z_1^a Z_2^b) = E(Z_1^a)E(Z_2^b).$$

Moments of V_{t+1}

Recall the discretized formula for $V_{t+1}(6)$. The derivations of the moments of V_{t+1} will make use of the property that

$$E(V_{t+1}) = E(V_t).$$

$$E(V_{t+1}) = E(V_t + k(\theta - V_t))$$
$$E(V_{t+1}) = E(V_{t+1}) + k\theta - kE(V_{t+1})$$
$$kE(V_{t+1}) = k\theta$$
$$E(V_{t+1}) = \theta$$

$$E(V_{t+1}^2) = E\left((V_t + k(\theta - V_t) + \sigma\sqrt{V_t}Z_1)^2\right)$$

$$E(V_{t+1}^2) = E\big(V_t^2 - 2kV_t^2 + 2V_t k\theta + 2\sigma V_t^{\frac{3}{2}}Z_1 + k^2 V_t^2 - 2k^2 V_t \theta$$
$$- 2kV_t^{\frac{3}{2}}\sigma Z_1 + k^2\theta^2 + 2k\theta\sigma\sqrt{V_t}Z_1 + \sigma^2 V_t Z_1^2\big)$$

$$E(V_{t+1}^2) = E(V_{t+1}^2) - 2kE(V_{t+1}^2) + 2k\theta^2 + k^2 E(V_{t+1}^2) - 2k^2\theta^2$$
$$+ k^2\theta^2 + \sigma^2\theta$$

$$(2k - k^2)E(V_{t+1}^2) = -k^2\theta^2 + 2k\theta^2 + \sigma^2\theta$$

$$E(V_{t+1}^2) = \frac{-k^2\theta^2 + 2k\theta^2 + \sigma^2\theta}{2k - k^2}$$

$$E(V_{t+1}^3) = E\left((V_t + k(\theta - V_t) + \sigma\sqrt{V_t}Z_1)^3\right)$$

$$E(V_{t+1}^3) = E\big(V_t^3 - 3kV_t^3 + 3k^2 V_t^3 - V_t^3 k^3 + k^3\theta^3 + 3\sigma V_t^{\frac{5}{2}}Z_1$$
$$+ 3\sigma^2 V_t^2 Z_1^2 + 3V_t^2 k^3\theta + 3V_t^2 k\theta - 6k^2 V_t^2\theta - 3V_t k^3\theta^2$$
$$+ 3V_t k^2\theta^2 + \sigma^3 V_t^{\frac{3}{2}}Z_1^3 + 6k\theta\sigma V_t^{\frac{3}{2}}Z_1 - 6V_t^{\frac{3}{2}}k^2\theta\sigma Z_1$$
$$+ 3k^2\theta^2\sigma\sqrt{V_t}Z_1 + 3k\theta\sigma^2 V_t Z_1^2 - 6kV_t^{\frac{5}{2}}\sigma Z_1$$
$$+ 3V_t^{\frac{5}{2}}k^2\sigma Z_1 - 3V_t^2 k\sigma^2 Z_1^2\big)$$

$$E(V_{t+1}^3) = E(V_t^3) - 3kE(V_t^3) + 3k^2 E(V_t^3) - k^3 E(V_t^3) + k^3\theta^3$$
$$+ 3\sigma^2 E(V_t^2) + 3E(V_t^2)k^3\theta + 3E(V_t^2)k\theta - 6k^2 E(V_t^2)\theta$$
$$- 3k^3\theta^3 + 3k^2\theta^3 + 3k\theta^2\sigma^2 - 3E(V_t^2)k\sigma^2$$

$$(k^3 - 3k^2 + 3k)E(V_{t+1}^3) = -2k^3\theta^3 - \frac{15\sigma^2 k^2\theta^2}{-k^2 + 2k} + \frac{9\sigma^2 k\theta^2}{-k^2 + 2k} + \frac{3\sigma^4\theta}{k^2 + 2k}$$
$$- \frac{3k^5\theta^3}{-k^2 + 2k} + \frac{12k^4\theta^3}{-k^2 + 2k} + \frac{6k^3\theta^2\sigma^2}{-k^2 + 2k} - \frac{15k^3\theta^3}{-k^2 + 2k}$$

$$E(V_{t+1}^3) = \cfrac{+\cfrac{6k^2\theta^3}{-k^2+2k}+3k^2\theta^3+3k\theta^2\sigma^2-\cfrac{3k\sigma^4\theta}{-k^2+2k}}{k^2(k-2)(k^2-3k+3)}(5k^5\theta^2-13k^4\theta^2-3k^3\sigma^2\theta$$
$$+9k^3\theta^2+9k^2\sigma^2\theta+3k\sigma^4-6k^2\theta^2-9k\sigma^2\theta-3\sigma^4)$$

Moments of Q_{t+1}

Recall the discretized formula for Q_{t+1} (5).

$$E(Q_{t+1}) = E\left(1+r+\sqrt{V_t}\left(\rho Z_1+\sqrt{1-\rho^2}Z_2\right)\right)$$
$$= 1+r$$

$$E(Q_{t+1}^2) = E\left(\left(1+r+\sqrt{V_t}\left(\rho Z_1+\sqrt{1-\rho^2}Z_2\right)\right)^2\right)$$
$$E(Q_{t+1}^2) = E\left(1+2r+2\sqrt{V_t}\rho Z_1+2\sqrt{V_t}\sqrt{1-\rho^2}Z_2+r^2+2r\sqrt{V_t}\rho Z_1\right.$$
$$+2r\sqrt{V_t}\sqrt{1-\rho^2}Z_2+V_t\rho^2Z_1^2+2V_t\rho Z_1\sqrt{1-\rho^2}Z_2$$
$$\left.-\rho^2V_tZ_2^2+V_tZ_2^2\right)$$
$$E(Q_{t+1}^2) = 1+2r+r^2+E(V_t)\rho^2-E(V_t)\rho^2+E(V_t)$$
$$E(Q_{t+1}^2) = r^2+2r+1+\theta$$
$$E(Q_{t+1}^2) = (r+1)^2+\theta$$

$$E(Q_{t+1}^3) = E\left(\left(1+r+\sqrt{V_t}\left(\rho Z_1+\sqrt{1-\rho^2}Z_2\right)\right)^3\right)$$
$$E(Q_{t+1}^3) = E\left(1+3V_t\rho^2Z_1^2-3\rho^2V_tZ_2^2+3r+3r^2+3V_tZ_2^2\right.$$
$$+6rV_t\rho Z_1\sqrt{1-\rho^2}Z_2+6r\sqrt{V_t}\rho Z_1+6r\sqrt{V_t}\sqrt{1-\rho^2}Z_2$$
$$+3r^2\sqrt{V_t}\rho Z_1+3r^2\sqrt{V_t}\sqrt{1-\rho^2}Z_2+3rV_t\rho^2Z_1^2-3r\rho^2V_tZ_2^2$$
$$-3V_t^{\frac{3}{2}}\rho^3Z_1Z_2^2+3V_t^{\frac{3}{2}}\rho Z_1Z_2^2-V_t^{\frac{3}{2}}\sqrt{1-\rho^2}Z_2^3\rho^2+3rV_tZ_2^2$$
$$+6V_t\rho Z_1\sqrt{1-\rho^2}Z_2+3V_t^{\frac{3}{2}}\rho^2Z_1^2\sqrt{1-\rho^2}Z_2+V_t^{\frac{3}{2}}\rho^3Z_1^3$$
$$\left.+3\sqrt{V_t}\sqrt{1-\rho^2}Z_2+3\sqrt{V_t}\rho Z_1+V_t^{\frac{3}{2}}\sqrt{1-\rho^2}Z_2^3+r^3\right)$$
$$E(Q_{t+1}^3) = 1+3E(V_t)\rho^2-3E(V_t)\rho^2+3r+3r^2+3E(V_t)+3r\rho^2E(V_t)$$
$$-3r\rho^2E(V_t)+3rE(V_t)+r^3$$
$$E(Q_{t+1}^3) = 1+3r+3r^2+3E(V_t)+3rE(V_t)+r^3$$
$$E(Q_{t+1}^3) = (r+1)^3+3\theta+3r\theta$$

The third moment of Q_{t+1} uses the same two parameters that the first and second moments use, so the third moment will not be used when solving for the method of moments parameter estimates.

$$E(Q_{t+1}^4) = E\left(\left(1+r+\sqrt{V_t}\left(\rho Z_1+\sqrt{1-\rho^2}Z_2\right)\right)^4\right)$$
$$E(Q_{t+1}^4) = E\left(1-12rV_t^{\frac{3}{2}}\rho^3Z_1Z_2^2+12rV_t^{\frac{3}{2}}\rho Z_1Z_2^2+12rV_t\rho^2Z_1^2-12rV_tZ_2^2\rho^2\right.$$
$$-12V_t^{\frac{3}{2}}\rho^3Z_1Z_2^2+12V_t^{\frac{3}{2}}\rho Z_1Z_2^2+6r^2V_t\rho^2Z_1^2-6r^2V_tZ_2^2\rho^2$$
$$\left.+4rV_t^{\frac{3}{2}}\rho^3Z_1^3-6V_t^2\rho^4Z_1^2Z_2^2+6V_t^2\rho^2Z_1^2Z_2^2-2V_t^2Z_2^4\rho^2\right.$$

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$$+ 4rV_t^{\frac{3}{2}}\sqrt{1-\rho^2}Z_2^3 + 12r\sqrt{V_t}\rho Z_1 + 12r\sqrt{V_t}\sqrt{1-\rho^2}Z_2$$

$$+ 12r^2\sqrt{V_t}\rho Z_1 + 12r^2\sqrt{V_t}\sqrt{1-\rho^2}Z_2 - 4V_t^{\frac{3}{2}}\sqrt{1-\rho^2}Z_2^3\rho^2$$

$$+ 4r^3\sqrt{V_t}\rho Z_1 + 4r^3\sqrt{V_t}\sqrt{1-\rho^2}Z_2 + 4\sqrt{V_t}\rho Z_1$$

$$- 4V_t^2\rho^3 Z_1\sqrt{1-\rho^2}Z_2^3 + 4V_t^2\rho Z_1\sqrt{1-\rho^2}Z_2^3 - 4rV_t^{\frac{3}{2}}\sqrt{1-\rho^2}Z_2^3\rho^2$$

$$+ 4V_t^2\rho^3 Z_1^3\sqrt{1-\rho^2}Z_2 + 12V_t^{\frac{3}{2}}\rho^2 Z_1^2\sqrt{1-\rho^2}Z_2 + 12V_t\rho Z_1\sqrt{1-\rho^2}Z_2$$

$$+ 4\sqrt{V_t}\sqrt{1-\rho^2}Z_2 + 4V_t^{\frac{3}{2}}\sqrt{1-\rho^2}Z_2^3 + 4r^3 + r^4$$

$$+ 12r^2 V_t\rho Z_1\sqrt{1-\rho^2}Z_2 + 12rV_t^{\frac{3}{2}}\rho^2 Z_1^2\sqrt{1-\rho^2}Z_2$$

$$+ 24rV_t\rho Z_1\sqrt{1-\rho^2}Z_2 + 4V_t^{\frac{3}{2}}\rho^3 Z_1^3 + 6r^2 V_t Z_2^2 + V_t^2\rho^4 Z_1^4$$

$$+ V_t^2 Z_2^4\rho^4 + V_t^2 Z_2^4 + 6V_t\rho^2 Z_1^2 - 6V_t Z_2^2\rho^2 + 6V_t Z_2^2 + 12rV_t Z_2^2$$

$$+ 4r + 6r^2\big)$$

$$E(Q_{t+1}^4) = 1 + 6\rho^2 E(V_t) - 6\rho^2 E(V_t) + 4r + 6r^2\rho^2 E(V_t) - 6r^2\rho^2 E(V_t)$$

$$- 6E(V_t^2)\rho^4 + 6E(V_t^2)\rho^2 - 6E(V_t^2)\rho^2 + r^4 + 6r^2 + 6E(V_t)$$

$$+ 12r\rho^2 E(V_t) - 12r\rho^2 E(V_t) + 12rE(V_t) + 3E(V_t^2)\rho^4 + 6r^2 E(V_t)$$

$$+ 3E(V_t^2) + 3E(V_t^2)\rho^4 + 4r^3$$

$$E(Q_{t+1}^4) = 1 + 4r + r^4 + 6r^2 + 6E(V_t) + 12rE(V_t) + 6r^2 E(V_t) + 3E(V_t^2) + 4r^3$$

$$E(Q_{t+1}^4) = \frac{1}{k(k-2)}\big(k^2 r^4 + 4k^2 r^3 + 6k^2 r^2\theta - 2kr^4 + 6k^2 r^2 + 12k^2 r\theta + 3k^2\theta^2$$

$$- 8kr^3 - 12kr^2\theta + 4k^2 r + 6k^2\theta - 12kr^2 - 24kr\theta - 6k\theta^2 - 3\sigma^2\theta$$

$$+ k^2 - 8kr - 12k\theta - 2k\big)$$

$$E(Q_{t+1}^5) = E\left(\left(1 + r + \sqrt{V_t}\left(\rho Z_1 + \sqrt{1-\rho^2}Z_2\right)\right)^5\right)$$

$$E(Q_{t+1}^5) = E\big(1 + 5V_t^{\frac{5}{2}}\rho^4 Z_1^4\sqrt{1-\rho^2}Z_2 - 10V_t^{\frac{5}{2}}\rho^4 Z_1^2\sqrt{1-\rho^2}Z_2^3$$

$$- 10r^2 V_t^{\frac{3}{2}}\sqrt{1-\rho^2}Z_2^3\rho^2 - 30r^2 V_t^{\frac{3}{2}}\rho^3 Z_1 Z_2^2 + 30r^2 V_t^{\frac{3}{2}}\rho Z_1 Z_2^2$$

$$+ 10V_t^{\frac{5}{2}}\rho^2 Z_1^2\sqrt{1-\rho^2}Z_2^3 - 30rV_t^2\rho^4 Z_1^2 Z_2^2 + 30rV_t^2\rho^2 Z_1^2 Z_2^2 + r^5$$

$$+ 20V_t\rho Z_1\sqrt{1-\rho^2}Z_2 + 30V_t^{\frac{3}{2}}\rho^2 Z_1^2\sqrt{1-\rho^2}Z_2 - 30r\rho^2 V_t Z_2^2$$

$$- 30V_t^{\frac{3}{2}}\rho^3 Z_1 Z_2^2 + 30V_t^{\frac{3}{2}}\rho Z_1 Z_2^2 - 10V_t^{\frac{3}{2}}\sqrt{1-\rho^2}Z_2^3\rho^2$$

$$+ V_t^{\frac{5}{2}}\sqrt{1-\rho^2}Z_2^5\rho^4 - 30V_t^2\rho^4 Z_1^2 Z_2^2 + 30V_t^2\rho^2 Z_1^2 Z_2^2 + 30r^2 V_t\rho^2 Z_1^2$$

$$- 30r^2\rho^2 V_t Z_2^2 + 20rV_t^{\frac{3}{2}}\rho^3 Z_1^3 + 20rV_t^{\frac{3}{2}}\sqrt{1-\rho^2}Z_2^3 - 2V_t^{\frac{5}{2}}\sqrt{1-\rho^2}Z_2^5\rho^2$$

$$+ 20r^3\sqrt{V_t}\rho Z_1 + 20r^3\sqrt{V_t}\sqrt{1-\rho^2}Z_2 - 10V_t^{\frac{5}{2}}\rho^5 Z_1^3 Z_2^2$$

$$+ 10V_t^{\frac{5}{2}}\rho^3 Z_1^3 Z_2^2 - 10V_t^{\frac{5}{2}}\rho^3 Z_1 Z_2^4 + 5V_t^{\frac{5}{2}}\rho^5 Z_1 Z_2^4 + 5V_t^{\frac{5}{2}}\rho Z_1 Z_2^4$$

$$+ 5r^4\sqrt{V_t}\rho Z_1 + 5r^4\sqrt{V_t}\sqrt{1-\rho^2}Z_2 + 10r^3 V_t\rho^2 Z_1^2 - 10r^3\rho^2 V_t Z_2^2$$

$$+ 10r^2 V_t^{\frac{3}{2}}\rho^3 Z_1^3 + 10r^2 V_t^{\frac{3}{2}}\sqrt{1-\rho^2}Z_2^3 - 10rV_t^2 Z_2^4\rho^2 + 5r\rho^4 V_t^2 Z_2^4$$

$$+ 5rV_t^2\rho^4 Z_1^4 + 30r^2\sqrt{V_t}\sqrt{1-\rho^2}Z_2 + 30rV_t\rho^2 Z_1^2$$

$$+ 20r\sqrt{V_t}\sqrt{1-\rho^2}Z_2 + 30r^2\sqrt{V_t}\rho Z_1 + 20r\sqrt{V_t}\rho Z_1$$

$$+ 5\sqrt{V_t}\sqrt{1-\rho^2}Z_2 + 5\sqrt{V_t}\rho Z_1 + 10V_t^{\frac{3}{2}}\rho^3 Z_1^3 + 30r^2 V_t Z_2^2 + 5V_t^2\rho^4 Z_1^4$$

$$+ 5\rho^4 V_t^2 Z_2^4 + 10V_t^{\frac{3}{2}}\sqrt{1-\rho^2}Z_2^3 + 30rV_t Z_2^2 - 60rV_t^{\frac{3}{2}}\rho^3 Z_1 Z_2^2$$

$$+ 60rV_t^{\frac{3}{2}}\rho Z_1 Z_2^2 - 20rV_t^{\frac{3}{2}}\sqrt{1-\rho^2}Z_2^3\rho^2 + 60r^2 V_t\rho Z_1\sqrt{1-\rho^2}Z_2$$

$$+ 60rV_t^{\frac{3}{2}}\rho^2 Z_1^2\sqrt{1-\rho^2}Z_2 - 20V_t^2\rho^3 Z_1\sqrt{1-\rho^2}Z_2^3$$

$$+ 20V_t^2\rho^3 Z_1^3\sqrt{1-\rho^2}Z_2 + 20V_t^2\rho Z_1\sqrt{1-\rho^2}Z_2^3 + 10r^3 V_t Z_2^2 + V_t^{\frac{5}{2}}\rho^5 Z_1^5$$

$$- 10V_t^2 Z_2^4\rho^2 + V_t^{\frac{5}{2}}\sqrt{1-\rho^2}Z_2^5 + 10V_t\rho^2 Z_1^2 - 10\rho^2 V_t Z_2^2 + 5rV_t^2 Z_2^4$$

$$+ 5r^4 + 10V_t Z_2^2 + 5V_t^2 Z_2^4 + 10r^3 + 10r^2 + 5r + 60rV_t\rho Z_1\sqrt{1-\rho^2}Z_2$$

$$+ 20r^3 V_t\rho Z_1\sqrt{1-\rho^2}Z_2 + 30r^2 V_t^{\frac{3}{2}}\rho^2 Z_1^2\sqrt{1-\rho^2}Z_2$$

$$- 20rV_t^2\rho^3 Z_1\sqrt{1-\rho^2}Z_2^3 + 20rV_t^2\rho^3 Z_1^3\sqrt{1-\rho^2}Z_2$$

$$+ 20rV_t^2\rho Z_1\sqrt{1-\rho^2}Z_2^3)$$

$$E(Q_{t+1}^5) = 1 + 10\rho^2 E(V_t) - 10\rho^2 E(V_t) + 10r^3\rho^2 E(V_t) - 10r^3\rho^2 E(V_t)$$

$$- 30r\rho^2 E(V_t^2) + 15r\rho^4 E(V_t^2) + 15r\rho^4 E(V_t^2) + 5r + 30r^2\rho^2 E(V_t)$$

$$- 30r^2\rho^2 E(V_t) - 30\rho^4 E(V_t^2) + 30\rho^2 E(V_t^2) - 30\rho^2 E(V_t^2) + 5r^4$$

$$+ 10r^2 + 10E(V_t) + r^5 + 10r^3 E(V_t) + 15rE(V_t^2) - 30r\rho^4 E(V_t^2)$$

$$+ 30r\rho^2 E(V_t^2) + 30r\rho^2 E(V_t) - 30r\rho^2 E(V_t) + 30rE(V_t) + 15\rho^4 E(V_t^2)$$

$$+ 30r^2 E(V_t) + 15E(V_t^2) + 15\rho^4 E(V_t^2) + 10r^3$$

$$E(Q_{t+1}^5) = 1 - 30r\rho^2 E(V_t^2) + 5r + 5r^4 + 10r^2 + 10E(V_t) + r^5 + 10r^3 E(V_t)$$

$$+ 15rE(V_t^2) + 30r\rho^2 E(V_t^2) + 30rE(V_t) + 30r^2 E(V_t) + 15E(V_t^2)$$

$$+ 10r^3$$

$$E(Q_{t+1}^5) = 1 - 30r\rho^2\left(\frac{-k^2\theta^2 + 2k\theta^2 + \sigma^2\theta}{-k^2 + 2k}\right) + 5r + 5r^4 + 10r^2 + 10\theta$$

$$+ r^5 + 10r^3\theta + 15r\left(\frac{-k^2\theta^2 + 2k\theta^2 + \sigma^2\theta}{-k^2 + 2k}\right)$$

$$+ 30r\rho^2\left(\frac{-k^2\theta^2 + 2k\theta^2 + \sigma^2\theta}{-k^2 + 2k}\right) + 30r\theta + 30r^2\theta$$

$$+ 15\left(\frac{-k^2\theta^2 + 2k\theta^2 + \sigma^2\theta}{-k^2 + 2k}\right) + 10r^3$$

$$E(Q_{t+1}^5) = \frac{1}{k(k-2)}\big(k^2 r^5 + 5k^2 r^4 + 10k^2 r^3\theta - 2kr^5 + 10k^2 r^3 + 30k^2 r^2\theta$$

$$+ 15k^2 r\theta^2 - 10kr^4 - 20kr^3\theta + 10k^2 r^2 + 30k^2 r\theta + 15k^2\theta^2 - 20kr^3$$

$$- 60kr^2\theta - 30kr\theta^2 - 15r\sigma^2\theta + 5k^2 r + 10k^2\theta - 20kr^2 - 60kr\theta$$

$$- 30k\theta^2 - 15\sigma^2\theta + k^2 - 10kr - 20k\theta - 2k\big)$$