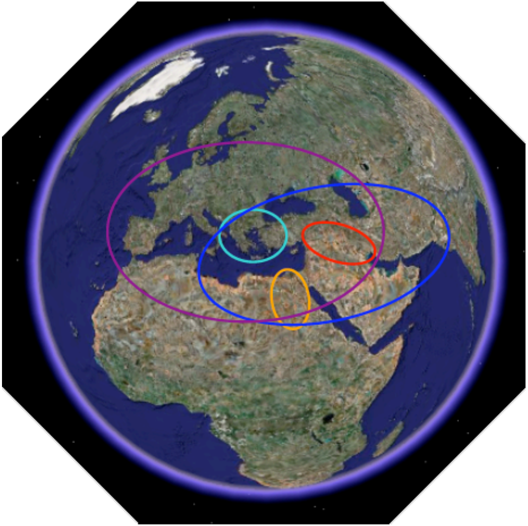
Euclid of Alexandria: Elementary Geometry

Waseda University, SILS,
History of Mathematics

Ancient cultures around the Mediterranean



Mesopotamia
(3000 B.C. - A.D. 100)

Egypt
(3000 B.C. - A.D. 300)

Greek states
(1000 B.C. - 330 B.C.)

Hellenistic kingdoms
(330 B.C. - 30 B.C.)

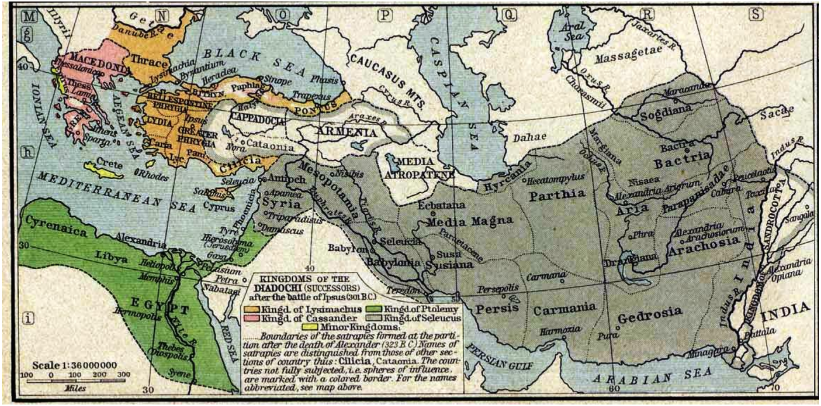
Roman Empire
(30 B.C. - A.D. 400)

The Hellenistic Period (around 330–30 BCE)

- The Hellenistic Period began with the conquests of Alexander ("the Great") III of Macedon (356–323 BCE).
- At the age of 19, Alexander inherited a strong country that already controlled much of mainland Greece. Until his death at 32, near Babylon, he devoted himself to conquest.
- When he died, his generals fought over his vast empire and carved it up into a number of Greek-ruled monarchies that persisted until the arrival of the Roman armies.
- This led to a flourishing of Hellenic culture in the eastern Mediterranean and Middle East.
- With regards to sciences and mathematics, the Hellenistic period was a high point for Greek-speaking culture.

The Macedonian Empire, 336–323 BCE

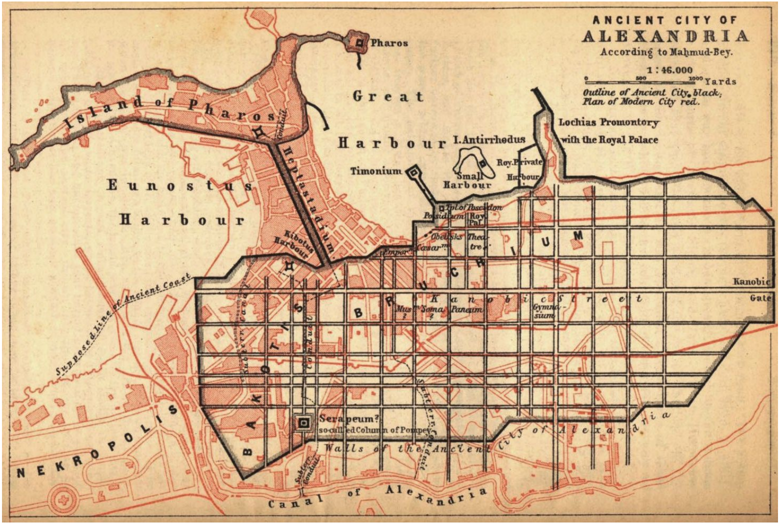
The Hellenistic kingdoms, 301 BCE



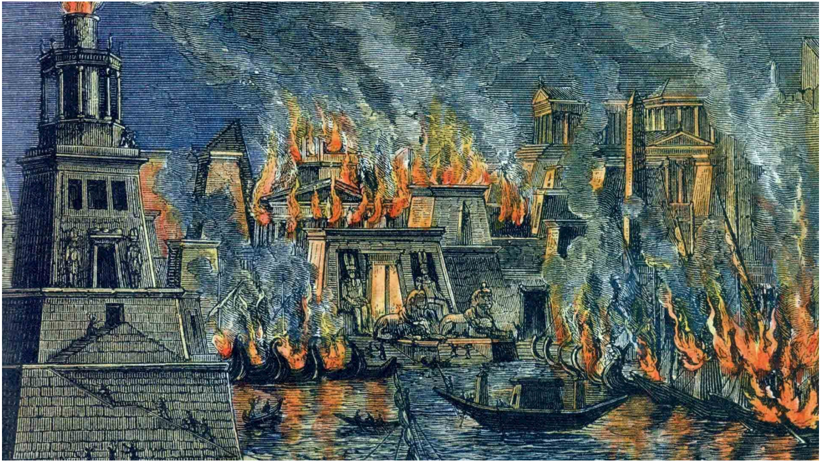
The city of Alexandria

- Founded by Alexander in the delta of the Nile in 331 BCE.
- Alexandria became a principal center of Greco-Roman culture.
- It had an important institute of higher learning (the Museum) and the largest library in Antiquity (the Library of Alexandria).
- Many famous philosophers, physicians, mathematicians, poets, etc., worked, or were educated, in Alexandria.
- One of the great legends of antiquity is the burning of the library. Everyone likes to blame it on their enemies (Pagans, Christians, Muslims, etc.), but what happened to all the other libraries?

Map of Ancient Alexandria



The Library in Flames



Hermann Göll, 1876

Euclid (early 3rd century BCE)

- We know almost nothing about Euclid.
- We are not even certain that he lived and worked in Alexandria, but we assume this is the case, since he is called Euclid of Alexandria by later authors.
- Nevertheless, we have reason to believe he lived before Archimedes (c. 287–212 BCE), and we are told (by Pappus in the 4th century CE) that Apollonius studied in Alexandria with Euclid's students.
- He wrote works in a number of the exact sciences: mathematics (plane and solid geometry, ratio theory, number theory, conic theory), astronomy (spherical astronomy), optics (geometric optics), and (perhaps) music theory.

Structure in the *Elements*

Euclid may not have been a brilliant mathematical discoverer but his works demonstrate great attention to the details of mathematical and logical structure. One of Euclid's principle concerns seems to have been to set mathematics on a secure logical foundation.

- Each book is arranged around a particular topic or theme.
- Each book begins with a series of (1) definitions, (2) permitted geometric or arithmetic procedures (or operations), and (3) common assumptions.
- It then proceeds by developing (1) theorems, which show what facts are true, and (2) "problems," which show what procedures are possible (and how to do them). We use the word *proposition* for both theorems and "problems."

The overall structure of the *Elements*

The goal of the *Elements* is to develop certain individual mathematical theories by relying on a limited set of assumptions (starting points), and deriving everything from these.

- Books I–IV: elementary plane geometry (I: congruence of triangles, theory of parallels, II: application of areas, III: circles, IV: regular figures)
- Book V: ratio theory
- Book VI: proportionality in geometric figures
- Book VII–IX: number theory
- Book X: incommensurable (irrational) lengths
- Book XI–XIII: three dimensional geometry

The structure of a theory

- A theory develops propositions that all relate to a particular property of some set of objects.
- The goal is to develop a full picture of the properties from a limited set of assumptions.
  - For example, we might want to be able to show how we can check for the existence (or absence) of a property (such as tangency, or parallelism, etc.), so that it can be used in further mathematical work.

- We will look at the examples of the theory of the *congruence of triangles*, and the theory of *parallel lines* and *equal area* in *Elements* I.

The structure of a Euclidean proposition

A Euclidean proposition has a very specific structure.

1. Enunciation: A general statement of what is to be shown.
2. Exposition: A statement setting out the assumed objects, and giving them letter names.
3. Specification: A restatement of what is to be shown in terms of the named objects.
4. Construction: Instructions for drawing new objects that will be required in the proof, but which were not mentioned in the enunciation. (Relies on postulates and previous construction propositions ("problems").)
5. Proof: A logical argument that the proposition holds. (Relies on definitions, common notions and previous theorems.)
6. Conclusion: A restatement, in general terms, of what has been shown.

The overall structure of *Elements* Book I

- The book begins with (1) definitions (ex. line, plane, circle, right angle, parallel line, etc.), (2) postulates (ex. to draw a line, circle, all right angles are equal, 5th post.[1]), and (3) common notions (equals to equals are equal, the whole is greater than the individual parts, etc.)
- The propositions start with a series of construction "problems" and develop a theory of the congruence of triangles (SAS (I.4), SSS (I.8), AAS (I.26)).
- Next, a theory of parallel lines, involving the 5th postulate, leads to a theory of the equality of certain areas.
- Finally, these two theories are combined to demonstrate the so-called Pythagorean Theorem (I.47).

[1] "If a straight line falling on two lines makes the interior angles on the same side less than two right angles, then the two lines, if produced, will meet on the side on which the angles are less than two right angles."

Theory of congruence

- We begin with some construction "problems" (equilateral triangle (I.1), moving a line (I.2 & I.3), etc.).
- We show SAS (side-angle-side) congruence (I.4) and SSS (side-side-side) congruence (I.8).
- Using the fact that a straight line is $2\mathbf{R}$ $(= 180^\circ)$ (I.13 & 14), we show that *vertical angles* are equal (I.15), and that the exterior angle of a triangle is greater than the two interior angles (I.16).[2]
- We use all this to show AAS (angle-angle-side) congruence (I.26).
- All these theorems hold with or without the 5th postulate.[3]

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[2]This theorem is not a theorem of "absolute geometry." Why?

[3]Aside from I.16 and I.26, the proofs of these theorems belong to "absolute geometry," because they hold in both Euclidean and non-Euclidean spaces.

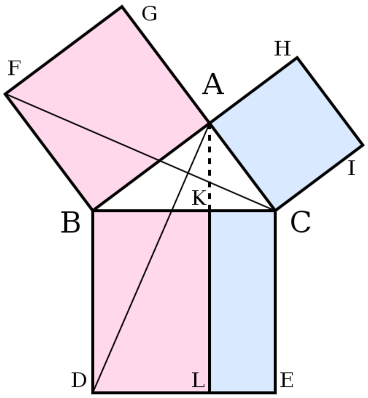
Theory of parallels and theory of area

- We develop some theorems about angles related to parallel lines (alternate angles (I.27), exterior and interior angles (I.28), and conversely (I.29)).
- We show that the exterior angle of a triangle is equal to the sum of the the two opposite interior angles, and the sum of the three angles is two right angles (I.32).[4]
- We show that parallelograms on the same base between two parallels are equal (I.35), which leads us to show that any parallelogram between two parallels is twice the area of a triangle on the same base between the same parallels (I.41).

[4]How is this related to I.16? Since I.32 is a stronger result, why did we bother to show I.16?

Elements I.47 (The so-called Pythagorean Theorem)

- We begin with $\triangle ABC$ and draw squares $BFGA$, $AHIC$ and $CEDB$ so that lines BAH and CAG are straight.
- We argue that $\triangle ABD = \triangle FBC$.
- Therefore, rectangle $▭\, BKLD = \square\, ABGF$.
- The same argument shows that $▭\, KCEL = \square\, ACIH$.



Final considerations

- Why does Euclid divide his approach into different theories?
- Why does he avoid using the 5th postulate until as late as possible.
- How can the *Elements* I be taken as a ideal for all mathematical practice? Until the 19th century, the *Elements* was still held out as a sort of model for mathematicians.