EXAM 3

Math 212, 2021 Fall, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything near the staple – this will be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(20 pts)* The surface S is the part of the surface with equation $3x^2 - 4y + z = 7$ inside of the cylinder $x^2 + z^2 = 9$. Write a single iterated integral that represents the surface area of S. (You do not have to evaluate the integral.)

$$y = \frac{3x^2 + z - 7}{4}$$ So we can parametrize with

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ (3u^2 + v - 7)/4 \\ v \end{pmatrix} \Rightarrow \vec{N} = \vec{X}_u \times \vec{X}_v = \begin{pmatrix} 1 \\ 3u/2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1/4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3u/2 \\ -1 \\ 1/4 \end{pmatrix}$$

u, v range the same as x, z, so $u^2 + v^2 \leq 9$.

Then

$$A = \iint \|\vec{N}\| \, du \, dv$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-v^2}}^{\sqrt{9-v^2}} \sqrt{\left(\frac{3u}{2}\right)^2 + (-1)^2 + \left(\frac{1}{4}\right)^2} \, du \, dv$$

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2. *(20 pts)* The curve C is the unit circle in the xy-plane, where we also have vector fields $\vec{F}(x,y) = (x-2y, 3x-4y)$ and $\vec{G}(x,y) = (4y-3x, x-2y)$.

(a) Use divergence to compute the flux $\int_C \vec{F}\cdot\vec{n}\,ds$ outward through C.

C here is the outward boundary of the unit disk D, so the 2-d divergence theorem gives us

$$\int_C \vec{F}\cdot\vec{n}\,ds = \iint_D \nabla\cdot\vec{F}\,dA = \iint (1)+(-4)\,dA = -3(\text{area})$$
$$= -3\pi$$

(b) Use Green's theorem to compute the line integral $\int_C \vec{G}\cdot\vec{T}\,ds$ counterclockwise around C.

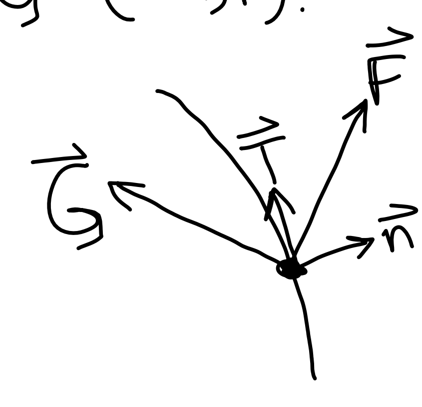
C here is the ccw boundary of the unit disk D, so Green's theorem gives us

$$\int_C \vec{G}\cdot\vec{T}\,ds = \iint_D \text{grn}\,\vec{G}\,dA = \iint_D (1)-(4)\,dA$$
$$= -3(\text{area}) = -3\pi$$

(c) Without referencing anything from either (a) or (b), explain why $\int_C \vec{F}\cdot\vec{n}\,ds$ and $\int_C \vec{G}\cdot\vec{T}\,ds$ should have the same value.

We can write $\vec{F}=(P,Q)$ and $\vec{G}=(-Q,P)$.

Then $\vec{G}$ and $\vec{T}$ are 90° ccw rotations of $\vec{F}$ and $\vec{n}$ (resp.).



So $\vec{G}\cdot\vec{T} = \vec{F}\cdot\vec{n}$.

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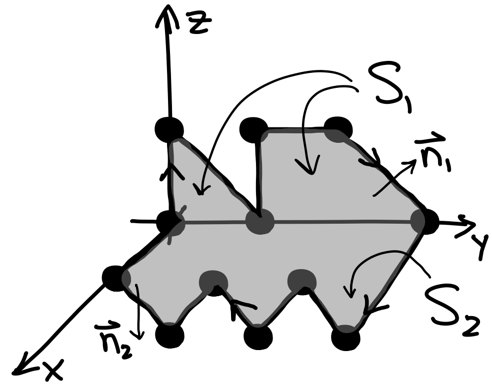
3. *(20 pts)* The curve C begins at the origin in $\mathbb{R}^3$, and then moves in straight line segments in succession to $(0,0,1)$, $(0,1,0)$, $(0,1,1)$, $(0,2,1)$, $(0,3,0)$, $(2,3,0)$, $(1,2,0)$, $(2,2,0)$, $(1,1,0)$, $(2,1,0)$, $(1,0,0)$, and then back to the origin.

Compute the line integral $\int_C \vec{F} \cdot d\vec{x}$ of $\vec{F}(x,y,z) = (e^x - z, 3y^3, x^3 - z^3)$. *(Hint: Every point on the curve is in either the xy-plane or the yz-plane.)*



C is the boundary of $S_1 \cup S_2$, oriented as drawn here:

Then by Stokes we have

$$\int_C \vec{F} \cdot d\vec{x} = \iint_{S_1} (\nabla \times \vec{F}) \cdot \vec{n}_1 \, dS + \iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n}_2 \, dS$$

$$= \iint_{S_1} \begin{pmatrix} 0 \\ -1-3x^2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} dS + \iint_{S_2} \begin{pmatrix} 0 \\ -1-3x^2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dS$$

$$= 0$$

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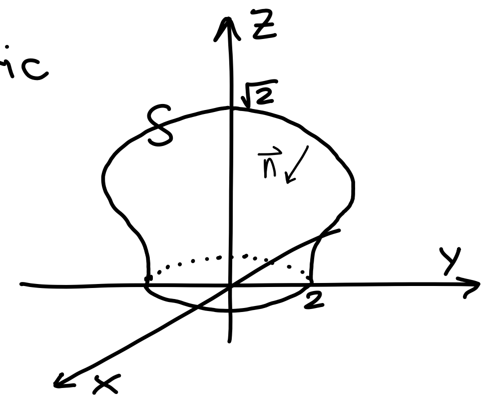
4. *(20 pts)* The surface S is the part of $x^2 + y^2 = 4 + 2z^2 - 2z^4$ that is above the xy-plane, oriented downward. (Starting from the bottom and going upward, the horizontal cross sections widen at first and then shrink, disappearing by $z = \sqrt{2}$.) Compute the flux through S of the vector field $\vec{F}(x, y, z) = (y^3 - xye^y, z^3 - xze^z, x^3 + yze^y - 3)$.

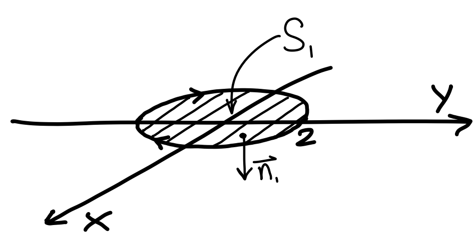
$$4 + 2z^2 - 2z^4 = -2(z^4 - z^2 - 2) = 2(2 - z^2)(z^2 + 1)$$

And S is rotationally symmetric around the z-axis, so it looks something like:



$$\nabla \cdot \vec{F} = (-ye^y) + (0) + (ye^y) = 0.$$

So $\vec{F}$ is surface independent. ∂S is the circle in the xy-plane, oriented as shown. S_1 shown has the same boundary. So



$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_{S_1} \vec{F} \cdot \vec{n}_1\, dS$$

$$= \iint_{S_1} \begin{pmatrix} y^3 - xye^y \\ z^3 - xze^z \\ x^3 + yze^y - 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dS = \iint_{S_1} \underbrace{-x^3}_{\text{(int = 0 by symm. through } yz\text{-plane)}} - \overbrace{yze^y}^{(z = 0 \text{ on } S_1!)} + 3\, dS$$

$$= \iint_{S_1} 3\, dS = 3(\pi(2)^2) = 12\pi$$

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5. *(20 pts)* The function f is defined by $f(x, y) = 3x^4 + 6x^2y - 2y^3$.

(a) Find all of the critical points of f.

f is differentiable everywhere.

$$\nabla f = \begin{pmatrix} 12x^3 + 12xy \\ 6x^2 - 6y^2 \end{pmatrix} = \vec{0} \Rightarrow x = \pm y$$

Case 1, $x=y$: $12x^3 + 12x^2 = 0 \Rightarrow x = 0, -1$

This gives critical points $(0,0), (-1,-1)$.

Case 2, $x=-y$: $12x^3 - 12x^2 = 0 \Rightarrow x = 0, 1$

This gives critical points $(0,0), (1,-1)$.

(b) Identify what the second derivative test concludes about each of the above critical points.

$$H = \begin{pmatrix} 36x^2 + 12y & 12x \\ 12x & -12y \end{pmatrix}$$

$(0,0)$: $\det H = \det \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \Rightarrow$ test fails!

$(-1,-1)$: $\det H = \det \begin{pmatrix} 24 & -12 \\ -12 & 12 \end{pmatrix} = 144 > 0$

$f_{xx} = 24 > 0 \Rightarrow$ local min

$(1,-1)$: $\det H = \det \begin{pmatrix} 24 & 12 \\ 12 & 12 \end{pmatrix} = 144 > 0$

$f_{xx} = 24 > 0 \Rightarrow$ local min

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