ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Dynamic Systems and Control
IDSC
Institut für Dynamische Systeme und Regelungstechnik

Exam **August 17, 2017**

Control Systems II **(151-0590-00L)** **Dr. G. Ducard**

Exam - Solutions

Exam Duration: **120 minutes + 15 minutes reading time**

Number of Problems: **35**

Number of Points: **42**

Permitted aids: **40 pages (20 sheets) of A4 notes**

Important:

Questions must be answered on the provided **answer sheet** using a **pen** (NOT pencil); answers given in the booklet will not be considered.

There exist multiple versions of the exam, where the order of the answers has been permuted randomly.

One *unique* answer has to be marked for each question. If the correct answer is marked, all the points of that question are awarded, otherwise none of the points will be awarded. Giving multiple answers to a question will invalidate the answer.

No negative points will be given for incorrect answers.

You do not need to justify your answers; your calculations will not be considered or graded.

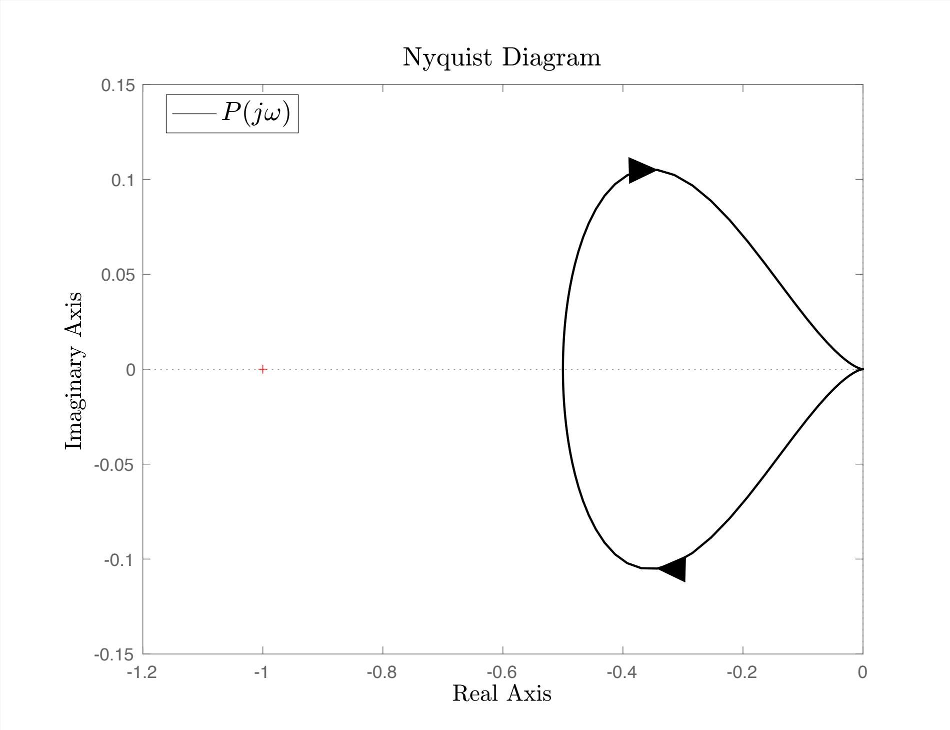
Use only the provided paper for your calculations; additional paper is available from the supervisors.

Good luck!

1 SISO Control Design

Question 1

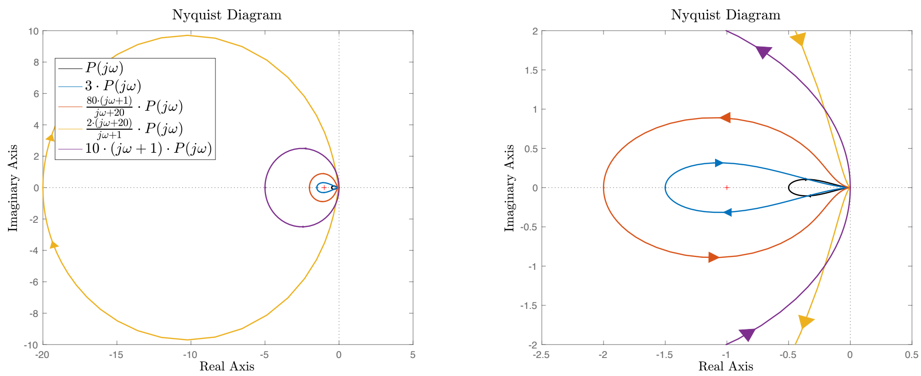
Description: Consider the plant $P(s) = \frac{1}{(s-2)\cdot(s+1)}$. The Nyquist diagram of the plant is shown below.



Question 1 *(1 Point)*
Which realizable controller leads to a stable closed-loop system?

- A $C(s) = 3$
- ■ $C(s) = \frac{80\cdot(s+1)}{s+20}$
- C $C(s) = \frac{2\cdot(s+20)}{s+1}$
- D $C(s) = 10 \cdot (s+1)$

Explanation: The poles of the plant are 2 and -1. The plant therefore has one unstable pole and no poles at zero. According to the Nyquist theorem the closed-loop system is stable if and only if the loop gain $L(j\omega)$ encircles the -1 point in a counterclockwise direction once ($n_c = n_+ + n_0/2 = 1 + 0 = 1$). Since none of the proposed controllers have non-negative poles, the number of required encirclements in a counterclockwise direction is the same for all controllers, i.e. one. The Nyquist diagram showing the resulting loop gains is given below.

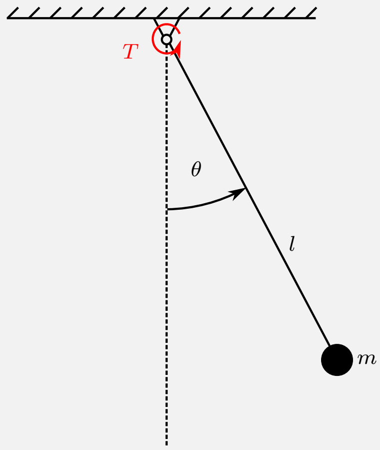


- $C(s) = 3$ – the -1 point is encircled once, but in a clockwise direction. The closed-loop system therefore remains unstable.
- $C(s) = \frac{80\cdot(s+1)}{s+20}$ – The controller has a zero at -1 and a pole at -20 (Lead Element). This results in an increase in the phase of the loop gain (now above -180°) and causes the loop gain to encircle the -1 point once in a counterclockwise direction, stabilizing the system. As the relative degree of the controller is 0, this controller is also realizable.
- $C(s) = \frac{2\cdot(s+20)}{s+1}$ – The controller has a zero at -20 and a pole at -1 (Lag Element). This results in a decrease in the phase of the loop gain (now even more below -180°) and results in the -1 point being encircled once in a clockwise direction. The closed-loop system therefore remains unstable.
- $C(s) = 10 \cdot (s+1)$ – The zero at -1 leads to an increase in the phase by 90° and results in a stable closed-loop system. However, as the relative degree of the controller is -1, this controller is not realizable.

The correct answer is $C(s) = \frac{80\cdot(s+1)}{s+20}$, as this is the only controller that is realizable and stabilizes the plant.

Question 2

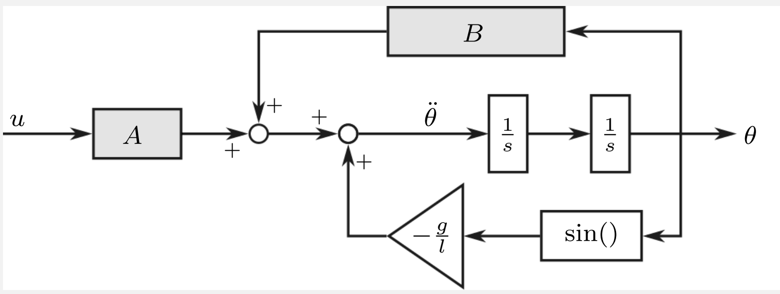
Description: The suspended pendulum depicted below, consist of a weightless, rigid rod of length l connected to a mass m placed at its end. The rod is anchored in its pivot. The pendulum is neither damped, nor susceptible to friction or aerodynamic drag. To control the pendulum's orientation (angle θ) an electric motor, which can apply a torque T to the rod of the pendulum, is mounted in its pivot.



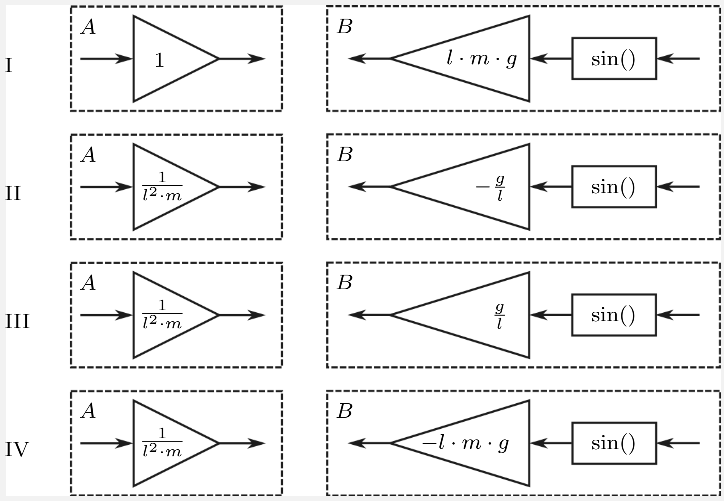
The resulting system equation is given as

$$l^2 \cdot m \cdot \ddot{\theta} = T - l \cdot m \cdot g \cdot \sin(\theta) \ .$$

To facilitate the task of designing a controller for the system you decide to use output feedback linearization and design the following system representation, where u is the input to the new, linearized system.



Given is the following selection to fill the grey boxes labeled A and B.



Question 2 *(1 Point)*
What is the correct content for the grey boxes labeled A and B?

- [A] I
- [B] II
- [■] III
- [D] IV

Explanation: Rewriting the system equation yields

$$\ddot{\theta} = \frac{T}{l^2 \cdot m} - \frac{g}{l} \cdot \sin(\theta) \ .$$

In order to remove the effect of the nonlinear term $-l \cdot m \cdot g \cdot \sin(\theta)$ using output feedback linearization, we choose

$$T = u + l \cdot m \cdot g \cdot \sin(\theta) \ .$$

Inserting this into the rewritten system equation gives us the linearized system

$$\ddot{\theta} = \left(\frac{u}{l^2 \cdot m} + \frac{g}{l} \cdot \sin(\theta) \right) - \frac{g}{l} \cdot \sin(\theta) = \frac{u}{l^2 \cdot m} \ .$$

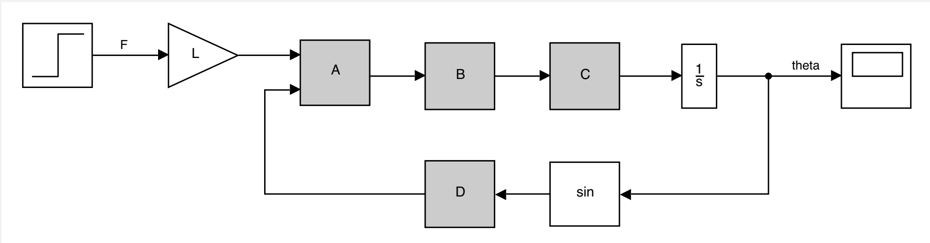
Implementing the equation above into the given system representation, leads to III.

Question 3

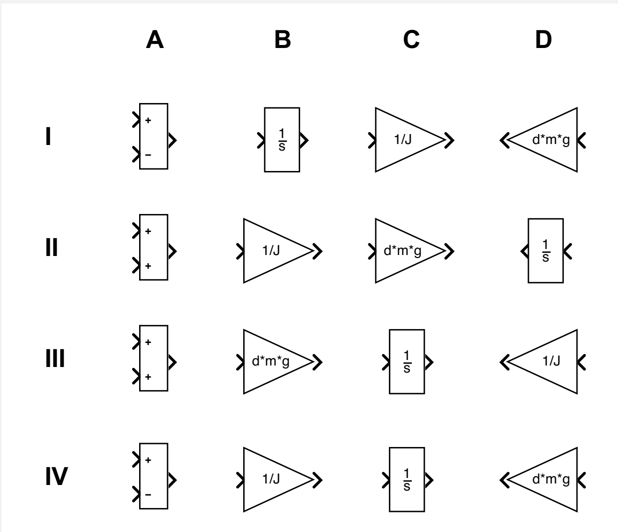
Description: Using a Simulink model, you want to simulate the step response of the flying arm presented during the lecture. The system equation of the flying arm is

$$\ddot{\theta} = \frac{1}{J} \cdot \Big(L \cdot F - d \cdot m \cdot g \cdot \sin(\theta) \Big) \quad .$$

Your incomplete Simulink model has the following form.



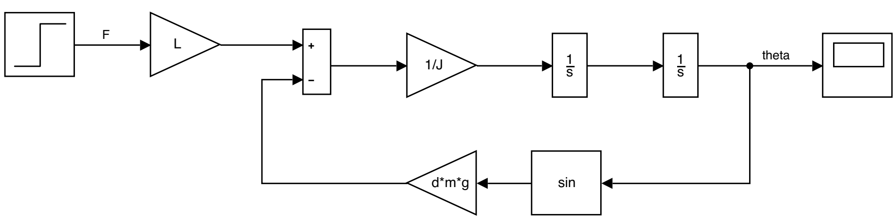
Given are the following four choices for filling the gaps labeled A through D in the Simulink model.



Question 3 *(1 Point)*
Which is the correct set of Simulink blocks to fill the gaps in the model labeled A through D?

- [A] I
- [B] II
- [C] III
- [■] IV

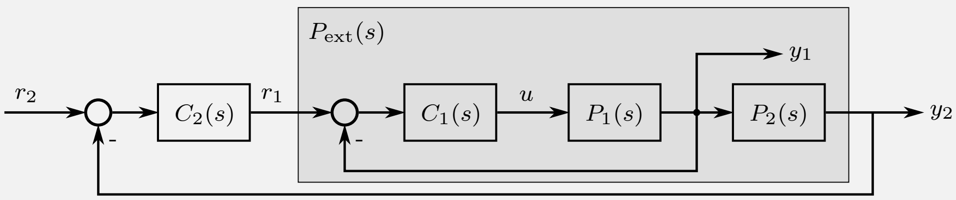
Explanation: Implementing the given system equation in Simulink yields



This means, that the correct answer is IV.

Question 4

Description: The following control architecture for a SIMO system is given.



The plant P_1 is given as

$$P_1(s) = \frac{1}{(s+1)} \quad ,$$

and the inner controller is

$$C_1(s) = 5 \quad .$$

To design the outer controller C_2, calculate the transfer function of the extended plant P_{ext} which includes the inner control loop and P_2, where

$$P_2(s) = \frac{1}{(5s+1)} \quad .$$

Question 4 *(1 Point)*
Which is the correct representation of P_{ext}?

■ $P_{\text{ext}} = \frac{5}{(s+6)\cdot(5s+1)}$

B $P_{\text{ext}} = \frac{s+1}{(s+1)\cdot(5s+1)}$

C $P_{\text{ext}} = \frac{s}{(s+6)\cdot(5s+1)}$

D $P_{\text{ext}} = \frac{s+1}{(s+6)\cdot(5s+1)}$

E $P_{\text{ext}} = \frac{s}{(s+1)\cdot(5s+1)}$

F $P_{\text{ext}} = \frac{5}{(s+1)\cdot(5s+1)}$

Explanation: The complementary sensitivity of the inner control loop is calculated as

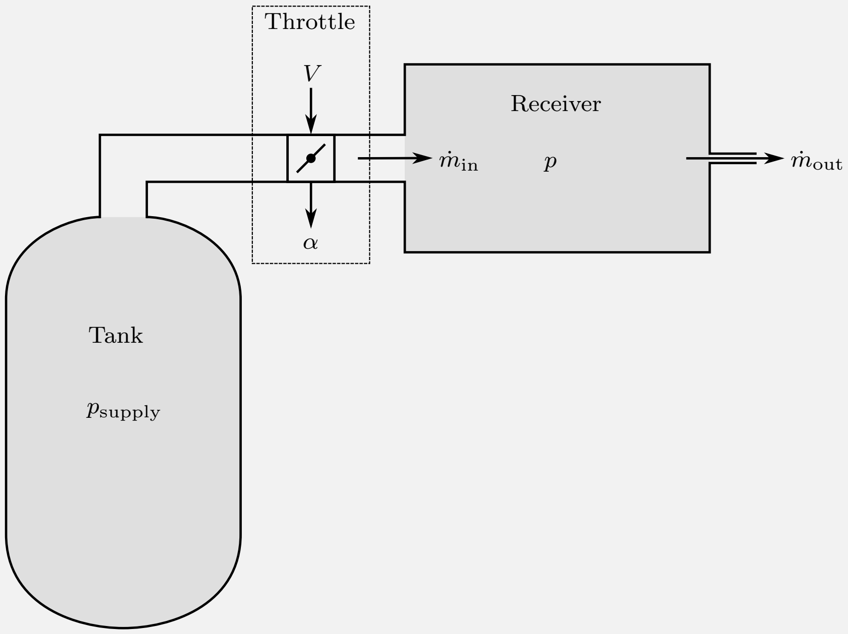
$$T_1(s) = \frac{L_1(s)}{1+L_1(s)} = \frac{C_1(s)\cdot P_1(s)}{1+C_1(s)\cdot P_1(s)} = \frac{5}{s+6} \quad .$$

The transfer function of the extended plant is then calculated as

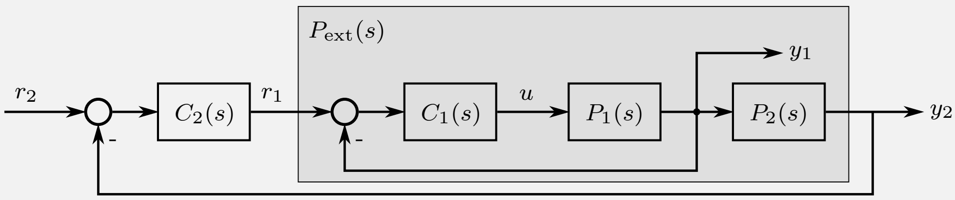
$$P_{\text{ext}}(s) = T_1(s)\cdot P_2(s) = \frac{5}{(s+6)\cdot(5s+1)} \quad .$$

Question 5

Description: The system depicted below consists of a tank containing pressurized air, a throttle and a receiver. As the pressure tank is much larger than the receiver, the pressure in the tank can be assumed constant. Your goal is to regulate the pressure in the receiver, which is subject to an unknown leakage flow $\dot{m}_{\text{out}}$.



You have installed sensors to measure the pressure p in the receiver as well as the position α of the throttle valve. The voltage V is applied to the electronically controlled throttle and $\dot{m}_{\text{in}}$ is the mass flow into the receiver. The following control architecture is to be used.



Question 5 *(1 Point)*
Which is the correct signal mapping?

- [A] $r_1 = \alpha_{\text{ref}}$ $r_2 = p_{\text{ref}}$ $y_1 = \alpha$ $y_2 = p$ $u = \dot{m}_{\text{in}}$
- ■ $r_1 = \alpha_{\text{ref}}$ $r_2 = p_{\text{ref}}$ $y_1 = \alpha$ $y_2 = p$ $u = V$
- [C] $r_1 = p_{\text{ref}}$ $r_2 = \alpha_{\text{ref}}$ $y_1 = p$ $y_2 = \alpha$ $u = V$
- [D] $r_1 = p_{\text{ref}}$ $r_2 = \alpha_{\text{ref}}$ $y_1 = p$ $y_2 = \alpha$ $u = \dot{m}_{\text{in}}$

Explanation: Since the pressure p is to be controlled, p must be the control variable of the outer loop. Furthermore, the dynamics of the throttle are faster than those of the pressure in the receiver, hence the throttle must be controlled by the inner loop. It follows that $r_1 = \alpha_{\text{ref}}$, $r_2 = p_{\text{ref}}$, $y_1 = \alpha$, $y_2 = p$. The plant P_1 describes the throttle. Its input is the voltage V and its output is the mass flow into the receiver. The correct choice for the input is therefore $u = V$.

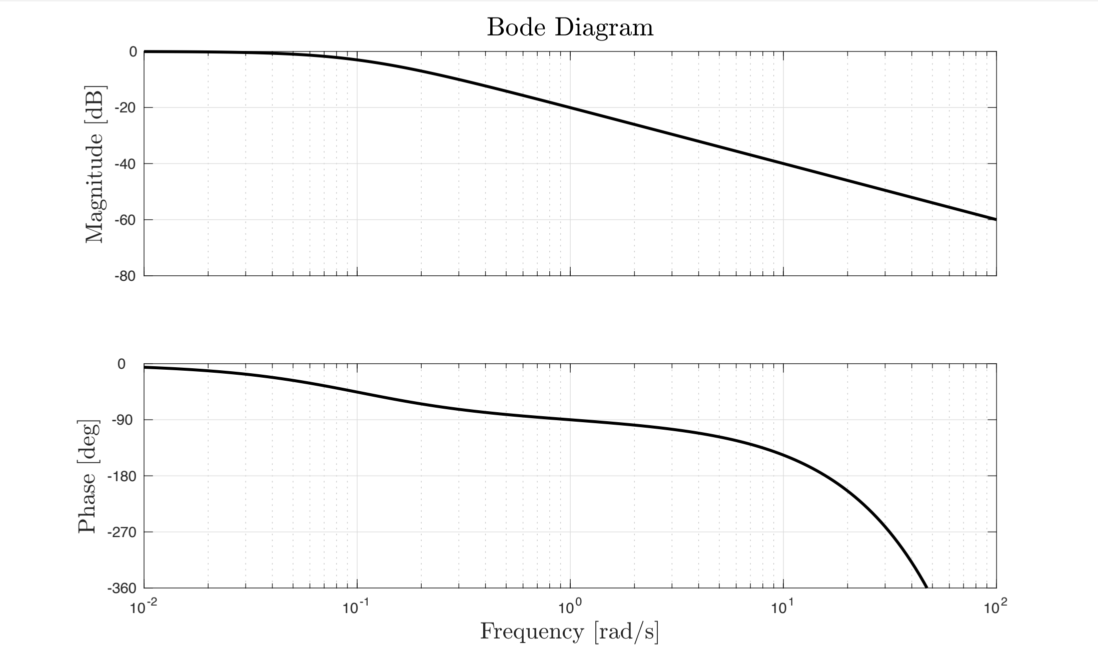
2 Controller Implementation

Question 6

Description: Your task is to design a controller for the plant

$$P(s) = \frac{1}{\tau \cdot s + 1} \cdot e^{-T \cdot s} \quad .$$

You have obtained the following Bode plot for the plant at hand



Question 6 *(1 Point)*
Is the use of a predictive controller mandatory? (Use the criterion presented during the lecture.)

- [A] Yes, the delay is significant and therefore a predictive controller should be used.
- [B] Not enough information is given to answer conclusively.
- ■ No, it is not necessary in this case.
- [D] Yes, if a delay time is present a predictive controller should always be used.

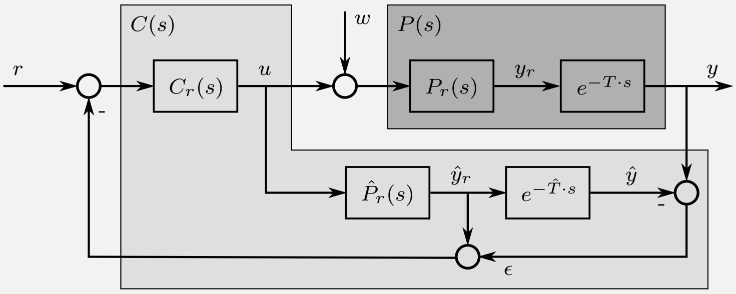
Explanation: From the Bode diagram the following system parameters can be identified:

$$\tau = 10 \text{ s} \qquad \text{and} \qquad T = 0.1 \text{ s} \quad .$$

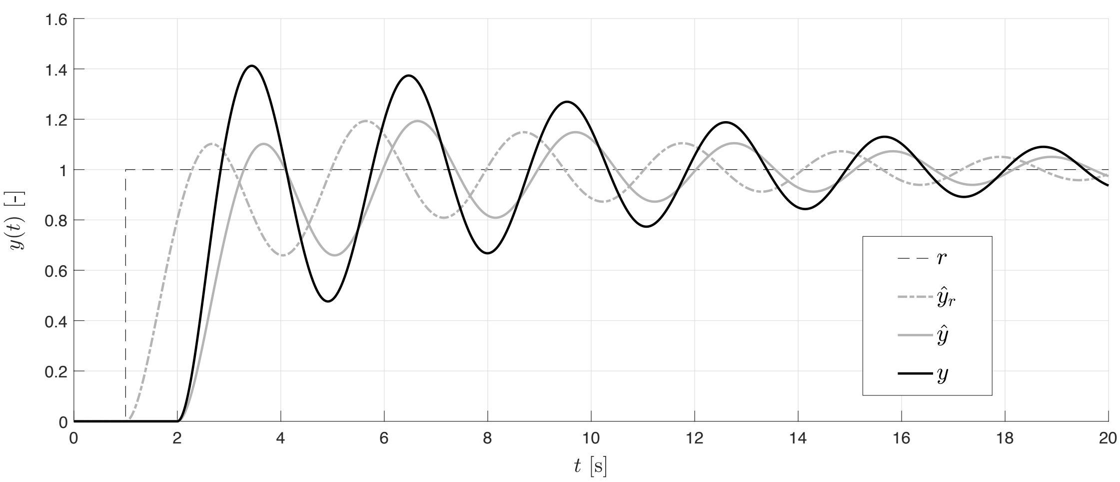
A predictive controller should be used if $\frac{T}{\tau + T} > 0.3$. For the given plant $\frac{T}{\tau+T} \approx 0.01$. A predictive controller is clearly not necessary.

Question 7

Description: You designed a Smith Predictor, as depicted below, to control a system with a time delay.



Testing your design by applying a step to the reference signal you obtain the following result.



The result is clearly not satisfying.

Question 7 *(1 Point)*
Which one of following steps do you expect to significantly improve performance?

■ Developing a more exact model $\hat{P}_r$ of the plant P_r.

B Tuning C_r to reduce the system inherent delay T of the system $P(s)$.

C Filtering the input to P_r, to reduce the effects of the high frequency disturbance w.

D Obtaining a more exact approximation $\hat{T}$ of the time delay T.

Explanation:

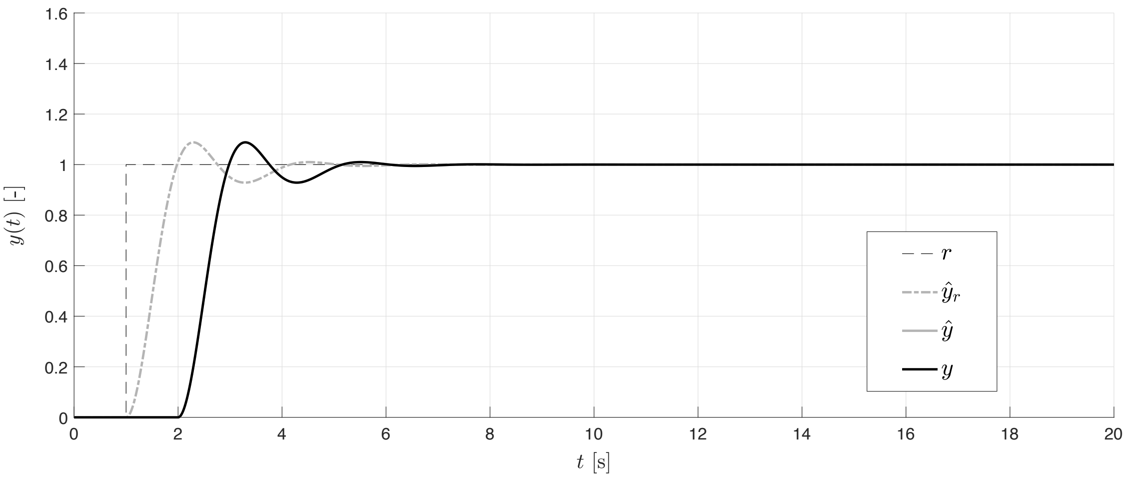
- When comparing the signals $\hat{y}$ and y, it can be seen that the dynamics of the systems are significantly different. This means that the plant P_r and the model $\hat{P}_r$ differ (in fact, both are second order systems, however, the model $\hat{P}_r$ has a higher characteristic frequency and a lower damping coefficient than the real plant P_r). This causes the mismatch of $\hat{y}$ and y. Improving the model $\hat{P}_r$ will therefore increase the performance of the control system.

- While the Smith Predictor can get rid of the adverse effect of the time delay on the feedback control, it cannot remove the time delay altogether. As a result, the system's inherent time delay T can never be reduced, no matter how C_r is chosen.

- As no high frequency content is visible in y, it can be assumed that the disturbance caused by w is either small or effectively damped by the plant. As a result, filtering the signal will not significantly improve the system's performance.

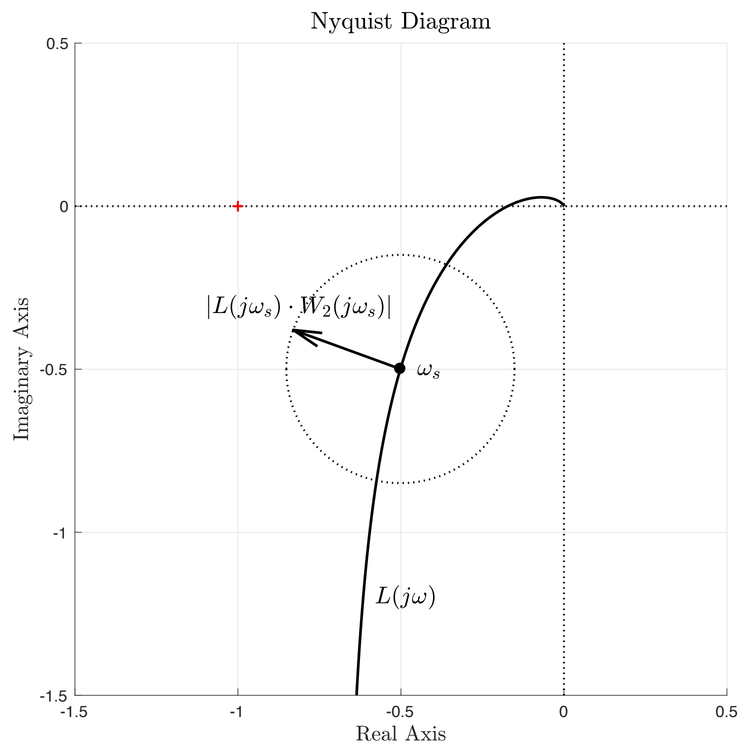
- By comparing the signals y and r, the time delay T of the real system is found to be 1 second. By comparing the signals $\hat{y}$ and r, the estimated time delay $\hat{T}$ is found to be 1 second as well. As there is no significant discrepancy, obtaining a more exact approximation of the time delay will not improve the system's performance.

As a result, the only way to significantly improve the performance of the Smith Predictor, in this case, is by improving the accuracy of $\hat{P}_r$. Obtaining an accurate model $\hat{P}_r$ results in $\hat{y}$ and y coinciding and gives the step response depicted below.



Question 8

Description: The Nyquist diagram of the loop gain $L(j\omega)$, as well as the absolute value of $L(j\omega) \cdot W_2(j\omega)$ at the frequency ω_s are depicted below.



Question 8 *(1 Point)*
From the following selection, which is the largest value of $|W_1(j\omega_s)|$ that fulfills the criterion for robust performance?

- [A] $|W_1(j\omega_s)| = 0.1$
- [■] $|W_1(j\omega_s)| = 0.25$
- [C] $|W_1(j\omega_s)| = 0.5$
- [D] $|W_1(j\omega_s)| = 0.75$
- [E] $|W_1(j\omega_s)| = 1$
- [F] $|W_1(j\omega_s)| = 2$

Explanation: The value $|W_1(j\omega_s)|_{\max}$ is given as the radius of the circle around the point (-1,0) which touches, but does not cross, the circle with radius $|L(j\omega_s) \cdot W_2(j\omega_s)|$ around $L(j\omega_s)$. From the Nyquist plot it can be seen that $|W_1(j\omega_s)| = 0.5$ clearly violates this condition, while it holds for 0.25, making this the largest of the given values to fulfill the criterion for robust performance (in fact $|W_1(j\omega_s)|_{\max} = 0.357$).

Question 9

Description: Given are four statements regarding robust performance.

Question 9 *(1 Point)*
Which of the following statements is correct?

- [A] If nominal performance is guaranteed for a control loop and the robust Nyquist theorem is fulfilled, robust performance is also guaranteed.
- [B] The function $W_2(j\omega)$ can be determined using an estimate of the measurement noise only.
- [■] For $|W_1(j\omega \to \infty)| \geq 1$ no real system can achieve nominal performance.
- [D] When evaluating a system's robust performance, the phase information of $T(j\omega)$ and $S(j\omega)$ must be considered.

Explanation:

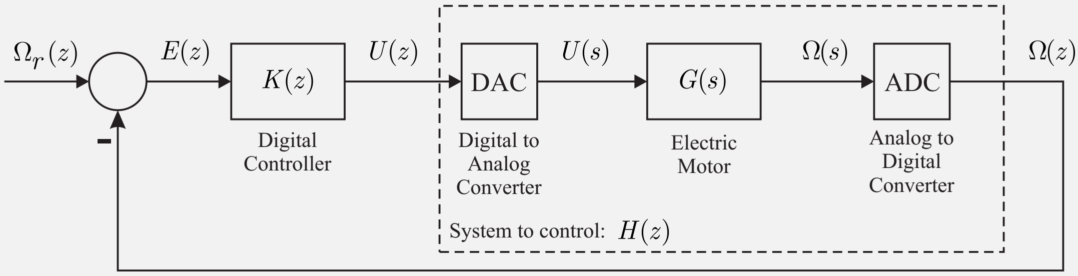
- If nominal performance is guaranteed for a control loop and the robust Nyquist theorem is fulfilled, robust performance is also guaranteed. False: Even if nominal performance is guaranteed and the system fulfills the robust Nyquist theorem, the two circles can still cross, resulting in the loss of robust performance.
- The function $W_2(j\omega)$ can be determined using an estimate of the measurement noise only. False : $W_2(j\omega)$ incorporates all types of uncertainties (i.e. measurement noise, process noise, disturbances, parameter uncertainties, etc). As a result, $W_2(j\omega)$ cannot be determined from the measurement noise alone.
- For $|W_1(j\omega \to \infty)| \geq 1$ no real system can achieve nominal performance. True: For all real systems $|L(j\omega \to \infty)| = 0$. Hence the maximal value of $|W_1(j\omega \to \infty)|$ has to be less than 1 for a real system to be able to fulfill the nominal performance criterion.
- When evaluating a system's robust performance, the phase information of $T(j\omega)$ and $S(j\omega)$ must be considered. False: The condition for robust performance reads $|S(j\omega)| \cdot |W_1(j\omega)| + |T(j\omega)| \cdot |W_2(j\omega)| < 1$. Only the magnitudes of $S(j\omega)$ and $T(j\omega)$ are required to achieve this condition, meaning that robust performance does not depend on the phase of either $S(j\omega)$ or $T(j\omega)$.

3 Digital Control

Questions 10, 11, 12, 13

Description: We consider a DC electric motor coupled to a load. The corresponding transfer function is denoted by $G(s) = \frac{\Omega(s)}{U(s)} = \frac{G_0}{s+a}$, where the input voltage to the motor is $U(s)$ and its rotational speed is $\Omega(s)$ in the Laplace domain.

The objective is to design a digital controller to control the rotational speed of the motor. As shown in the following figure, the control system is composed of a digital controller $K(z)$, a zero-order hold digital to analog converter (DAC) and a zero-order hold analog to digital converter (ADC), which are all running with a clock period T. The time-discrete reference rotational speed is denoted by $\Omega_r(z)$ in the z domain.



Question 10 *(2 Points)*
After calculation, the discrete-time transfer function $H(z) = \frac{\Omega(z)}{U(z)}$ is

- [ ] A $H(z) = \frac{G_0 \cdot T}{z-1+a\cdot T}$
- [ ] B $H(z) = \frac{G_0\cdot(z+1)}{z\cdot(\frac{2}{T}+a)+a-\frac{2}{T}}$
- [ ] C $H(z) = \frac{G_0\cdot z\cdot T}{z\cdot(1+a\cdot T)-1}$
- [x] D $H(z) = \frac{G_0}{a}\frac{(1-e^{-a\cdot T})\cdot z^{-1}}{1-e^{-a\cdot T}\cdot z^{-1}}$

Explanation: Use the formula:

$$H(z) = (1-z^{-1})\cdot \mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\right\} \quad ,$$

where the $\mathcal{Z}$-transform is denoted by $\mathcal{Z}$ and the Laplace transform is denoted by $\mathcal{L}$.

First, transform $\frac{G(s)}{s}$ into simple elements as follows:

$$\frac{G(s)}{s} = \frac{G_0}{s\cdot(s+a)} = \frac{\alpha}{s} + \frac{\beta}{s+a} = \frac{\alpha\cdot a + s\cdot(\alpha+\beta)}{s\cdot(s+a)} \quad .$$

By identification, it is found that $\alpha = \frac{G_0}{a}$ and $\beta = -\frac{G_0}{a}$. Therefore, $\frac{G(s)}{s} = \frac{G_0}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right)$ and

$$\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right) = \frac{G_0}{a}\left(u_h(t) - e^{-a\cdot t}\cdot u_h(t)\right) \quad ,$$

where $u_h(t)$ is the Heaviside function.

The $\mathcal{Z}$-transform can be computed as follows:

$$\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\right\} = \frac{G_0}{a}\left(\sum_{k=0}^{+\infty} z^{-k} - \sum_{k=0}^{+\infty} e^{-a\cdot k\cdot T}\cdot z^{-k}\right) = \frac{G_0}{a}\left(\sum_{k=0}^{+\infty} z^{-k} - \sum_{k=0}^{+\infty}\left(e^{-a\cdot T}z^{-1}\right)^k\right) \quad .$$

Under the conditions of radius of convergence: $|z|>1$ and $|z|>|e^{-a\cdot T}|$, the previous $\mathcal{Z}$-transform is:

$$\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\right\} = \frac{G_0}{a}\left(\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-a\cdot T}\cdot z^{-1}}\right).$$

Finally, the desired transfer function $H(z)$ is found to be

$$H(z) = (1 - z^{-1})\frac{G_0}{a}\left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-a \cdot T} \cdot z^{-1}}\right) \quad ,$$

which simplifies to

$$H(z) = \frac{G_0}{a}\left(1 - \frac{1 - z^{-1}}{1 - e^{-a \cdot T} \cdot z^{-1}}\right) = \frac{G_0}{a}\frac{(1 - e^{-a \cdot T}) \cdot z^{-1}}{1 - e^{-a \cdot T} \cdot z^{-1}} \quad .$$

Question 11 *(1 Point)*
A discrete-time step input signal with amplitude A is given by

[A] $\Omega_r(z) = \frac{A}{z}$ ■ $\Omega_r(z) = \frac{A}{1-z^{-1}}$

[B] $\Omega_r(z) = A \cdot (1 - z^{-1})$ [D] $\Omega_r(z) = A$

Explanation: This result comes directly from the definition of the $\mathcal{Z}$-transform when applied to a sampled step input.

$$\mathcal{Z}\left(\{\Omega_r(kT) = 1\}\right) = \sum_{k=0}^{+\infty} z^{-k}.$$

Under the condition of convergence $|z| > 1$, then the $\mathcal{Z}$-transform is $\Omega_r(z) = \frac{A}{1-z^{-1}}$.

Question 12 *(1 Point)*
The error signal $E(z)$ can be computed according to:

■ $E(z) = \frac{\Omega_r(z)}{1+H(z) \cdot K(z)}$ [C] $E(z) = \frac{\Omega(z)}{1+H(z) \cdot K(z)}$

[B] $E(z) = \Omega(z) \cdot (1 + H(z) \cdot K(z))$ [D] $E(z) = \Omega_r(z) \cdot (1 + H(z) \cdot K(z))$

Explanation: The error signal is the difference between the reference signal and the system's output, as follows:

$$\begin{aligned} E(z) &= \Omega_r(z) - \Omega(z) = \Omega_r(z) - H(z) \cdot K(z) \cdot E(z) \\ E(z) \cdot (1 + H(z) \cdot K(z)) &= \Omega_r(z) \\ E(z) &= \frac{\Omega_r(z)}{1 + H(z) \cdot K(z)} \quad . \end{aligned}$$

Question 13 *(1 Point)*
Choose the digital controller which corresponds to a proportional-integral controller, where K_0 and K_i are the gains associated with the proportional and integral actions, respectively.

[A] $K(z) = \frac{K_0}{K_i \cdot z}$ [C] $K(z) = \frac{K_0}{K_i \cdot (1-z^{-1})}$

■ $K(z) = K_0 + \frac{K_i}{1-z^{-1}}$ [D] $K(z) = K_0 + \frac{K_i}{z}$

Explanation: A proportional-integral controller is the sum of two control actions: a proportional K_0 and integral $\frac{K_i}{1-z^{-1}}$ actions.

Question 14

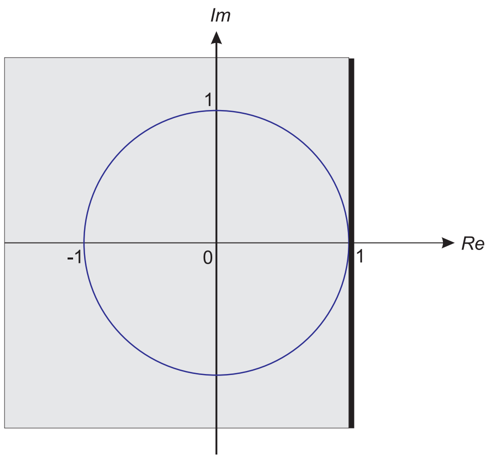
Description: Using numerical discretization methods a continuous-time controller can be emulated on a microprocessor.

Question 14 *(1 Point)*
The discretization method which can transform a stable continuous-time controller into an unstable discrete-time controller is

- [A] Euler Backward
- [B] None of them
- [C] Tustin
- [■] Euler Forward

Explanation: The discretization method which can transform a stable continuous-time controller into an unstable discrete-time controller is the one that maps a part of the stable continuous-time pole region ($\Re(s) \leq 0$) to the unstable discrete-time pole region ($|z| > 1$).

Using the Euler Forward method $s = \frac{z-1}{T}$, the z variable can be obtained as $z = s \cdot T + 1$. The stable continuous-time poles $\pi_{ct} = \sigma + j\omega$ with $\sigma < 0$ are mapped to the discrete-time poles $\pi_{dt} = \sigma \cdot T + 1 + j\omega \cdot T$. In other words, the region $\Re(s) < 0$ is mapped to $\Re(z) < 1$ by the Euler Forward transformation. The region $\Re(z) < 1$ is represented by the grey area in the image below. As some of the mapped poles can lie outside of the unit circle, the resulting discrete-time controller can be rendered unstable.



Using the same reasoning, it can be shown, that both the Euler Backward and the Tustin transformations map the entire stable continuous-time pole region to the stable discrete-time pole region, thereby maintaining controller stability.

Questions 15, 16

Description: Consider the continuous-time controller $C(s) = K_p \cdot \left(1 + \frac{1}{T_i \cdot s}\right)$.

Question 15 *(1 Point)*
The emulation of the controller $C(s)$ with the Tustin discretization method leads to a discrete controller K(z) as follows:

- A $K(z) = K_p \cdot \frac{z\cdot(1+\frac{T}{T_i})-1}{z-1}$
- ■ $K(z) = K_p \cdot \frac{z\cdot\left(1+\frac{T}{2T_i}\right)+\frac{T}{2\,T_i}-1}{z-1}$
- B $K(z) = K_p + \frac{\frac{T}{2T_i}}{z-1}$
- D $K(z) = K_p \cdot \frac{z+\frac{T}{T_i}-1}{z-1}$

Explanation: The Tustin approximation is $s \approx \frac{2}{T}\frac{z-1}{z+1}$. Substituting s with its Tustin approximation and factorizing the terms in z leads to the result above.

Question 16 *(2 Points)*
Suppose that the discrete proportional-integral controller has the form $K(z) = K_p \cdot \left(1 + \frac{1}{\alpha\cdot\frac{z-1}{z+1}}\right)$.
The recursive equation which can be implemented in a microprocessor takes the form:

- A $u[k] = u[k-1] + K_p \cdot (1 + \frac{1}{\alpha}) \cdot u[k-1] + K_p \cdot (\frac{1}{\alpha} - 1) \cdot e[k]$
- B $u[k] = u[k-1] + K_p \cdot (1 + \frac{1}{\alpha}) \cdot e[k-1] + K_p \cdot (\frac{1}{\alpha} - 1) \cdot e[k-2]$
- ■ $u[k] = u[k-1] + K_p \cdot (1 + \frac{1}{\alpha}) \cdot e[k] + K_p \cdot (\frac{1}{\alpha} - 1) \cdot e[k-1]$
- D $u[k] = u[k-2] + K_p \cdot (1 + \frac{1}{\alpha}) \cdot e[k] + K_p \cdot (\frac{1}{\alpha} - 1) \cdot e[k-1]$

Explanation: The discrete-time transfer function of the controller $K(z)$ is also written as $K(z) = \frac{U(z)}{E(z)}$. Therefore, $\frac{U(z)}{E(z)} = K_p \cdot \left(1 + \frac{1}{\alpha\cdot\frac{z-1}{z+1}}\right) = K_p \cdot \frac{z\cdot(1+\alpha^{-1})+\alpha^{-1}-1}{z-1}$. By performing a cross product, one gets:

$$\begin{aligned} U(z)\cdot(z-1) &= E(z)\cdot K_p \cdot \left(z\cdot(1+\alpha^{-1}) + \alpha^{-1} - 1\right) \\ U(z)\cdot(1-z^{-1}) &= E(z)\cdot K_p \cdot \left((1+\alpha^{-1}) + (\alpha^{-1}-1)\cdot z^{-1}\right) \end{aligned} \quad .$$

Finally, the recursive equation is obtained by using the *shift operator*:

$$u[k] = u[k-1] + K_p \cdot (1+\alpha^{-1}) \cdot e[k] + K_p \cdot (\alpha^{-1} - 1) \cdot e[k-1] \quad .$$

Question 17

Description: Suppose that a digital controller $K(z)$ possesses a pole of the form $\pi_{dt} = 1 - \frac{T}{\tau}$, where $\tau > 0$.

Question 17 *(1 Point)*
What is the condition on the period T (at which the controller is being executed) such that the controller remains stable?

- [A] $T > 2\tau$
- ■ $T < 2\tau$
- [C] $T > \frac{1}{2\tau}$
- [D] $T < 2\tau + 1$

Explanation: The stability criterion for discrete-time systems is that the stable poles are contained in the unit circle, this means that the norm of the stable discrete pole π_{dt} should be such that $|\pi_{dt}| < 1$. With $\pi_{dt} = 1 - \frac{T}{\tau}$, it is equivalent to saying $-1 < 1 - \frac{T}{\tau} < 1$. Therefore,

$$\begin{aligned} 1 - \frac{T}{\tau} &< 1 \\ T &> 0 \quad , \end{aligned}$$

which is given by default (the clock period T cannot be negative) and

$$\begin{aligned} -1 &< 1 - \frac{T}{\tau} \\ T &< 2\tau \quad . \end{aligned}$$

4 Relative Gain Array

Question 18

Description: Consider the following MIMO system

$$P(s) = \begin{pmatrix} \dfrac{-1}{s+1} & \dfrac{2}{s+1} \\ \dfrac{s}{s+3} & \dfrac{2s+1}{s+3} \end{pmatrix} .$$

Question 18 *(1 Point)*
Which of the following matrices is the RGA matrix of $P(s)$?

■

$$RGA(s) = \begin{pmatrix} \dfrac{2s+1}{4s+1} & \dfrac{2s}{4s+1} \\ \dfrac{2s}{4s+1} & \dfrac{2s+1}{4s+1} \end{pmatrix}$$

B

$$RGA(s) = \begin{pmatrix} \dfrac{3s+1}{4s+1} & \dfrac{s}{4s+1} \\ \dfrac{s}{4s+1} & \dfrac{3s+1}{4s+1} \end{pmatrix}$$

C

$$RGA(s) = \begin{pmatrix} \dfrac{5s-1}{6s-1} & \dfrac{s}{6s-1} \\ \dfrac{s}{6s-1} & \dfrac{5s-1}{6s-1} \end{pmatrix}$$

D

$$RGA(s) = \begin{pmatrix} \dfrac{4s-1}{6s-1} & \dfrac{2s}{6s-1} \\ \dfrac{2s}{6s-1} & \dfrac{4s-1}{6s-1} \end{pmatrix}$$

E

$$RGA(s) = \begin{pmatrix} \dfrac{4s+1}{6s+1} & \dfrac{2s}{6s+1} \\ \dfrac{2s}{6s+1} & \dfrac{4s+1}{6s+1} \end{pmatrix}$$

F

$$RGA(s) = \begin{pmatrix} \dfrac{2s+1}{6s+1} & \dfrac{4s}{6s+1} \\ \dfrac{4s}{6s+1} & \dfrac{2s+1}{6s+1} \end{pmatrix}$$

Explanation: The RGA matrix of a 2x2 system is defined as

$$RGA(s) = \begin{pmatrix} RGA_{11}(s) & RGA_{12}(s) \\ RGA_{21}(s) & RGA_{22}(s) \end{pmatrix}$$

where

$$RGA_{11}(s) = RGA_{22}(s) = \frac{P_{11} \cdot P_{22}}{P_{11} \cdot P_{22} - P_{12} \cdot P_{21}}$$

$$RGA_{12}(s) = RGA_{21}(s) = -\frac{P_{12} \cdot P_{21}}{P_{11} \cdot P_{22} - P_{12} \cdot P_{21}} \quad .$$

Inserting the transfer functions of the plant $P(s)$, we get

$$RGA_{11}(s) = RGA_{22}(s) = \frac{2s+1}{4s+1}$$

$$RGA_{12}(s) = RGA_{21}(s) = \frac{2s}{4s+1}$$

and therefore the RGA matrix is

$$RGA(s) = \begin{pmatrix} \dfrac{2s+1}{4s+1} & \dfrac{2s}{4s+1} \\ \\ \dfrac{2s}{4s+1} & \dfrac{2s+1}{4s+1} \end{pmatrix} \quad .$$

Question 19

Description: Consider the following RGA matrix:

$$RGA(s) = \begin{pmatrix} \dfrac{1}{8s+1} & \dfrac{8s}{8s+1} \\[2ex] \dfrac{8s}{8s+1} & \dfrac{1}{8s+1} \end{pmatrix} .$$

You would like to decouple the system and (if possible) control it as two SISO systems.

Question 19 *(1 Point)*
Which control coupling should you choose?

- A The system can be decoupled at all frequencies.
- ■ $u_1 \to y_2$ and $u_2 \to y_1$ at high frequencies, while $u_1 \to y_1$ and $u_2 \to y_2$ at low frequencies.
- C $u_1 \to y_1$ and $u_2 \to y_2$ at high frequencies, while $u_1 \to y_2$ and $u_2 \to y_1$ at low frequencies.
- D $u_1 \to y_2$ and $u_2 \to y_1$ at all frequencies.
- E $u_1 \to y_1$ and $u_2 \to y_2$ at all frequencies.

Explanation:
In order to decide which coupling to choose depending on the frequency, the transfer functions contained in the RGA matrix are analyzed in the table below.

TF	Description	$s \to 0$	$s \to \infty$
$\dfrac{1}{8s+1}$	Low-pass filter	1	0
$\dfrac{8s}{8s+1}$	High-pass filter	0	1

At high frequencies, the RGA matrix is

$$RGA(s \to \infty) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

while at low frequencies the RGA entries change to

$$RGA(s \to 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

These results show that $u_1 \to y_2$ and $u_2 \to y_1$ at high frequencies, while $u_1 \to y_1$ and $u_2 \to y_2$ at low frequencies, is the best choice.

5 MIMO Systems Analysis

Question 20

Description: The following MIMO system is considered:

$$P(s) = \begin{pmatrix} P_{11}(s) & P_{12}(s) \\ 0 & P_{22}(s) \end{pmatrix} \qquad C(s) = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$$

The closed-loop matrix $T(s)$ of the plant $P(s)$ with the proportional controller $C(s)$ is calculated. The values K_1 and K_2 are scalar and real.

Question 20 *(2 Points)*
Choose the correct closed-loop matrix $T(s)$. (The variable s is omitted to improve readability.)

[A]

$$T(s) = \begin{pmatrix} \frac{P_{11}\cdot K_1}{1+P_{11}\cdot K_1} & \frac{P_{12}\cdot K_1 + P_{12}\cdot K_2}{(1+P_{11}\cdot K_1)\cdot(1+P_{22}\cdot K_2)} \\ 0 & \frac{P_{22}\cdot K_2}{1+P_{22}\cdot K_2} \end{pmatrix}$$

[B]

$$T(s) = \begin{pmatrix} \frac{P_{11}\cdot K_1}{1+P_{11}\cdot K_1} & \frac{P_{12}\cdot K_1 - P_{12}\cdot K_2}{(1+P_{11}\cdot K_1)\cdot(1+P_{22}\cdot K_2)} \\ 0 & \frac{P_{22}\cdot K_2}{1+P_{22}\cdot K_2} \end{pmatrix}$$

■

$$T(s) = \begin{pmatrix} \frac{P_{11}\cdot K_1}{1+P_{11}\cdot K_1} & \frac{P_{12}\cdot K_2}{(1+P_{11}\cdot K_1)\cdot(1+P_{22}\cdot K_2)} \\ 0 & \frac{P_{22}\cdot K_2}{1+P_{22}\cdot K_2} \end{pmatrix}$$

[D]

$$T(s) = \begin{pmatrix} \frac{P_{11}\cdot K_1}{1+P_{11}\cdot K_1} & \frac{-P_{12}\cdot K_2}{(1+P_{11}\cdot K_1)\cdot(1+P_{22}\cdot K_2)} \\ 0 & \frac{P_{22}\cdot K_2}{1+P_{22}\cdot K_2} \end{pmatrix}$$

Explanation: The closed-loop transfer function $T(s)$ for a SISO system is definied as

$$T(s) = \frac{L(s)}{1+L(s)} \ ,$$

while for a MIMO system, the complementary sensitivity is given as

$$T(s) = \left(\mathbb{I} + L(s)\right)^{-1} \cdot L(s) \ ,$$

where

$$L(s) = P(s)\cdot C(s) \ .$$

The closed-loop transfer function $T(s)$ is found to be equal to

$$T(s) = \begin{pmatrix} \frac{P_{11}\cdot K_1}{1+P_{11}\cdot K_1} & \frac{P_{12}\cdot K_2}{(1+P_{11}\cdot K_1)\cdot(1+P_{22}\cdot K_2)} \\ 0 & \frac{P_{22}\cdot K_2}{1+P_{22}\cdot K_2} \end{pmatrix} \ .$$

Question 21

Description: Consider the following MIMO system

$$P(s) = \begin{pmatrix} \dfrac{s+2}{s-1} & \dfrac{1}{s^2-2s+1} & \dfrac{1}{s} \\ \\ 0 & \dfrac{-s}{s+2} & \dfrac{s+1}{s+2} \end{pmatrix} \; .$$

Question 21 *(2 Points)*

Choose the correct poles π and zeros ζ of $P(s)$.

- [A] Poles $\pi = \{0, 1, 1, -2\}$ and zeros $\zeta = \{-2, 0\}$
- [B] Poles $\pi = \{0, 1, 1, 1, -2, -2\}$ and zeros $\zeta = \{-2, 0, -1\}$
- ■ Poles $\pi = \{0, 1, 1, -2\}$ and zeros $\zeta = \{0\}$
- [D] Poles $\pi = \{0, 1, 1, 1, -2\}$ and zeros $\zeta = \{0\}$

Explanation: The poles π of $P(s)$ are the roots of the least common denominator of all minors of $P(s)$. The minors of first order are

$$\frac{s+2}{s-1}, \quad \frac{1}{s^2-2s+1}, \quad \frac{1}{s}, \quad \frac{-s}{s+2}, \quad \frac{s+1}{s+2} \; ,$$

while the minors of second order are

$$\frac{-s}{s-1}, \quad \frac{s+1}{s-1}, \quad \frac{s^2-s+2}{(s-1)^2 \cdot (s+2)} \; .$$

The pole polynomial is therefore

$$s \cdot (s+2) \cdot (s-1)^2 \; ,$$

yielding the poles

$$\pi_1 = 0, \qquad \pi_2 = 1, \qquad \pi_3 = 1, \qquad \pi_4 = -2 \; .$$

The zeros ζ of $P(s)$ are the roots of the greatest common divisor of the numerators of the maximum minors of $P(s)$ after normalization to the pole polynomial of $P(s)$ as denominators. Normalizing the minors of second order with the denominator $s \cdot (s+2) \cdot (s-1)^2$ we get

$$\frac{-s^2 \cdot (s-1) \cdot (s+2)}{s \cdot (s+2) \cdot (s-1)^2}, \quad \frac{s \cdot (s+1) \cdot (s-1) \cdot (s+2)}{s \cdot (s+2) \cdot (s-1)^2}, \quad \frac{s \cdot (s^2-s+2)}{s \cdot (s+2) \cdot (s-1)^2} \; .$$

The greatest common divisor is s, yielding the only zero

$$\zeta_1 = 0 \; .$$

Question 22

Description: Consider the following MIMO system

$$P(s) = \begin{pmatrix} \dfrac{1}{s+1} & \dfrac{2}{(s+1)\cdot(s-2)} \\ \\ \dfrac{s}{2s+1} & \dfrac{3s}{5s+4} \end{pmatrix} .$$

You would like to analyze the behavior of the plant $P(s)$ at the frequency $\omega = 0\,\text{rad/s}$. For this reason, you decide to calculate the singular values at that specific frequency.

Question 22 *(2 Points)*
Choose the correct singular values.

- [A] $\sigma = \{\sqrt{2}, \sqrt{1.5}\}$
- [B] $\sigma = \{\sqrt{3}, \sqrt{2}\}$
- [C] $\sigma = \{2, 0\}$
- [D] $\sigma = \{3, \sqrt{2}\}$
- [E] $\sigma = \{2, \sqrt{1.5}\}$
- ■ $\sigma = \{\sqrt{2}, 0\}$

Explanation: The plant $P(s)$ excited at a frequency ω reads as follows

$$P(j\omega) = \begin{pmatrix} \dfrac{1}{j\omega+1} & \dfrac{2}{(j\omega+1)\cdot(j\omega-2)} \\ \\ \dfrac{j\omega}{2j\omega+1} & \dfrac{3j\omega}{5j\omega+4} \end{pmatrix}$$

Inserting the frequency $\omega = 0\,\text{rad/s}$ we get

$$P(j\cdot 0) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} =: M \ .$$

Since the singular values of a matrix M are defined as

$$\sigma_i = \sqrt{\lambda_i(\overline{M}^\top \cdot M)} \ ,$$

where $\overline{M}$ is the complex conjugate of the matrix M, we get

$$\overline{M}^\top \cdot M = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} .$$

The eigenvalues of $\overline{M}^\top \cdot M$ are found to be $\lambda_1 = 2$ and $\lambda_2 = 0$. Therefore the singular values of $P(j\omega = 0)$ are $\sigma_1 = \sqrt{2}$ and $\sigma_2 = 0$.

Question 23

Description: Consider a system, which singular value decomposition is

$$P(j\omega_0) = U \cdot \Sigma \cdot \overline{V}^T \ ,$$

with the following matrices:

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1.5 \end{bmatrix} \ , \qquad V = \begin{bmatrix} \sqrt{3/8} + \sqrt{3/8}j & \sqrt{2}/2 \\ 1/2 & \sqrt{2}j/2 \end{bmatrix} \ , \qquad U = \begin{bmatrix} 1/2 + \sqrt{3}j/6 & 0 \\ 1/\sqrt{3} + j/\sqrt{3} & 1 \end{bmatrix} \ .$$

The system is excited at the frequency ω_0 and is considered after the transient phase.

Hint: the argument in the cosine function is given in radians. Therefore the following table can be helpful:

Angle	
Degree	Radians
0°	0
30°	0.5236
45°	0.7854
60°	1.0472
90°	1.5708

Question 23 *(2 Points)*
Choose the right input-output pairing for which the maximum amplification is achieved:

☐ A

$$u(t) = \begin{bmatrix} 0.5 \cdot \cos(\omega_0 t + 0.7854) \\ 0.5 \cdot \cos(\omega_0 t) \end{bmatrix} \qquad y_\infty(t) = \begin{bmatrix} \sqrt{3} \cdot \cos(\omega_0 t + 1.0472) \\ \sqrt{6} \cdot \cos(\omega_0 t + 1.5708) \end{bmatrix}$$

☐ B

$$u(t) = \begin{bmatrix} \sqrt{0.75} \cdot \cos(\omega_0 t + 0.7854) \\ 0.5 \cdot \cos(\omega_0 t) \end{bmatrix} \qquad y_\infty(t) = \begin{bmatrix} \sqrt{3} \cdot \cos(\omega_0 t + 0.7854) \\ \sqrt{6} \cdot \cos(\omega_0 t + 0.5236) \end{bmatrix}$$

☐ C

$$u(t) = \begin{bmatrix} 0.5 \cdot \cos(\omega_0 t + 0.7854) \\ 0.5 \cdot \cos(\omega_0 t) \end{bmatrix} \qquad y_\infty(t) = \begin{bmatrix} \sqrt{1/3} \cdot \cos(\omega_0 t + 0.5236) \\ \sqrt{2/3} \cdot \cos(\omega_0 t + 0.7854) \end{bmatrix}$$

■

$$u(t) = \begin{bmatrix} \sqrt{0.75} \cdot \cos(\omega_0 t + 0.7854) \\ 0.5 \cdot \cos(\omega_0 t) \end{bmatrix} \qquad y_\infty(t) = \begin{bmatrix} \sqrt{3} \cdot \cos(\omega_0 t + 0.5236) \\ \sqrt{6} \cdot \cos(\omega_0 t + 0.7854) \end{bmatrix}$$

Explanation: A plant $P(s)$ at a specific frequency ω can be decomposed into three matrices using the singular value decomposition

$$P(j\omega) = U \cdot \Sigma \cdot \overline{V}^T$$

where $\overline{V}^T$ is the complex conjugate transpose of the matrix V. The maximum amplification is achieved if the system is excited in the direction of the largest singular value (in this case $\sigma_{\max} = 3$). Since the largest singular value is in the first column of the matrix Σ, the input and the output direction for the maximum amplification are in the first column of the matrices V and U respectively. Therefore,

$$V_{\max} = \begin{bmatrix} \sqrt{3/8} + \sqrt{3/8}j \\ 1/2 \end{bmatrix} \qquad U_{\max} = \begin{bmatrix} 1/2 + \sqrt{3}j/6 \\ 1/\sqrt{3} + j/\sqrt{3} \end{bmatrix} \ ,$$

with

$$|V_{\max}| = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} \qquad \angle V_{\max} = \begin{bmatrix} 45^o \\ 0^o \end{bmatrix} = \begin{bmatrix} 0.7854 \\ 0 \end{bmatrix}$$

$$|U_{\max}| = \begin{bmatrix} 1/\sqrt{3} \\ \sqrt{2/3} \end{bmatrix} \qquad \angle U_{\max} = \begin{bmatrix} 30^o \\ 45^o \end{bmatrix} = \begin{bmatrix} 0.5236 \\ 0.7854 \end{bmatrix} \ ,$$

yielding the input signal

$$u(t) = \begin{bmatrix} \sqrt{0.75} \cdot \cos(\omega_0 t + 0.7854) \\ 0.5 \cdot \cos(\omega_0 t) \end{bmatrix}$$

and the output signal

$$y_\infty(t) = \sigma_{\max} \cdot \begin{bmatrix} 1/\sqrt{3} \cdot \cos(\omega_0 t + 0.5236) \\ \sqrt{2/3} \cdot \cos(\omega_0 t + 0.7854) \end{bmatrix} = \begin{bmatrix} \sqrt{3} \cdot \cos(\omega_0 t + 0.5236) \\ \sqrt{6} \cdot \cos(\omega_0 t + 0.7854) \end{bmatrix} \ .$$

Question 24

Description: Consider an asymptotically stable system $P(s)$, the input signal $u(t)$, the output signal $y(t)$ and the steady-state output signal $y_\infty(t)$, i.e. $y(t)$ after the transient phase:

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \\ P_{31}(s) & P_{32}(s) \end{bmatrix} \quad , \quad u(t) = \begin{bmatrix} \mu_1 \cdot \cos(\omega t + \varphi_1) \\ \mu_2 \cdot \cos(\omega t + \varphi_2) \end{bmatrix} \quad , \quad y_\infty(t) = \begin{bmatrix} \nu_1 \cdot \cos(\omega t + \psi_1) \\ \nu_2 \cdot \cos(\omega t + \psi_2) \\ \nu_3 \cdot \cos(\omega t + \psi_3) \end{bmatrix} \quad ,$$

where

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad , \quad \nu = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} \quad .$$

The system is excited at the frequency ω in its maximum amplification direction.

Question 24 *(1 Point)*
Choose the correct statement:

- [A] Both the norms $\|y(t)\|$ and $\|y_\infty(t)\|$ can exceed the value $\sigma_{\max} \cdot \|\mu\|$.
- ■ The norm $\|y(t)\|$ can exceed the value $\sigma_{\max} \cdot \|\mu\|$.
- [C] The norm $\|y_\infty(t)\|$ can exceed the value $\sigma_{\max} \cdot \|\mu\|$.
- [D] The norm $\|y(t)\|$ cannot exceed the value $\sigma_{\max} \cdot \|\mu\|$.

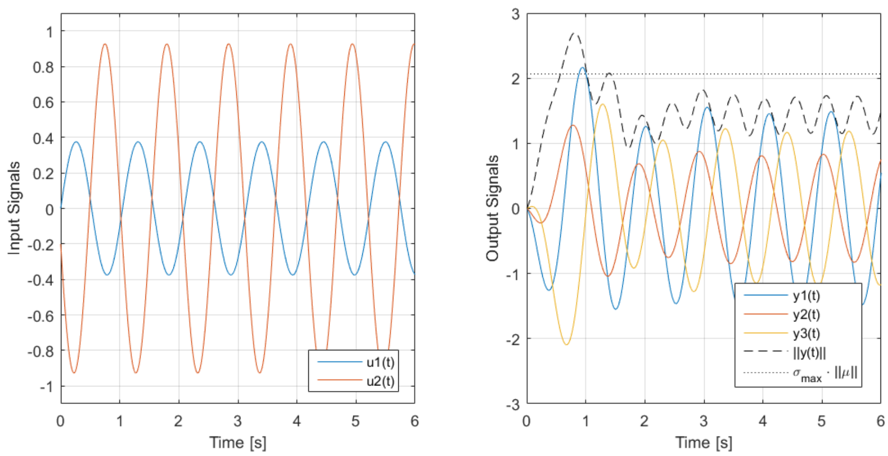
Explanation: At steady-state conditions, if the system is excited in its maximum amplification direction, it holds

$$\|\nu\| = \sigma_{\max} \cdot \|\mu\| \quad .$$

In general, the steady-state ouput signals $[y_{\infty,1}(t), y_{\infty,2}(t), \dots]$ are not in phase and, therefore,

$$\|y_\infty(t)\| \leq \|\nu\| = \sigma_{\max} \cdot \|\mu\| \quad .$$

On the other hand, in the transient phase, $\|y(t)\|$ can exceed $\sigma_{\max} \cdot \|\mu\|$. An example is shown in the figure below.



6 LQR and LQRI

Question 25

Description: Consider the following first order system

$$\dot{x}(t) = 3x(t) + 0.5u(t)$$
$$y(t) = x(t) \ .$$

You would like to control it using an infinite-horizon LQR controller with the cost function

$$J = \int_0^\infty 64x(t)^2 + u(t)^2 \, \mathrm{d}t \ .$$

Question 25 *(1 Point)*
Choose the correct LQR gain K.

- A $K = 32$
- B $K = 4$
- C $K = -4$
- D $K = -32$
- ■ $K = 16$
- F $K = -16$

Explanation: The problem is solved using the continuous-time algebraic Riccati equation (CARE) which is given below.

$$0 = \Phi \cdot B \cdot R^{-1} \cdot B^\top \cdot \Phi - \Phi \cdot A - A^\top \cdot \Phi - Q$$

The state-space matrices (in this case scalars) are $A = 3$ and $B = \frac{1}{2}$. Furthermore, $Q = 64$ and $R = 1$. Inserting these values into the CARE, we obtain

$$0 = \frac{1}{4}\Phi^2 - 6\Phi - 64$$

which solutions are

$$\Phi_1 = 32 \ \text{ and } \ \Phi_2 = -8 \ .$$

Since Φ must be positive definite, $\Phi_1 = 32$ is the only correct solution. The LQR gain K is then found as

$$K = R^{-1} \cdot B^\top \cdot \Phi = 16 \ .$$

Questions 26, 27

Description: Consider the scalar infinite-horizon LQR problem

$$\min \int_0^\infty Q \cdot x(t)^2 + R \cdot u(t)^2 \, \mathrm{d}t$$
$$\text{s.t. } \dot{x}(t) = x(t) + u(t) \ .$$

The problem was already solved by the legendary control engineer John McRitz. However, he has forgotten the values of Q and R. Nevertheless, he remembers that the closed-loop pole was $\pi_{\mathrm{CL}} = -2$ and that the scalar solution of the continuous-time algebraic Riccati equation (CARE) Φ was either 3, 0 or -1.

Question 26 *(1 Point)*
What is the value of Q as a function of R?

- ■ $Q = \frac{9}{R} - 6$
- B $Q = \frac{4}{R} - 8$
- C $Q = \frac{R}{6} - 4$
- D $Q = \frac{12}{R} - 8$
- E $Q = \frac{R}{12} - 6$
- F $Q = \frac{R}{4} + 4$

Explanation: From the state-space description we get $A = 1$ and $B = 1$. We know that Φ must be symmetric and positive definite, which in the scalar case means $\Phi > 0$. Hence, $\Phi = 3$ is the only meaningful value. Using the scalar version of the CARE, we get

$$0 = \frac{B^2}{R} \cdot \Phi^2 - 2 \cdot A \cdot \Phi - Q = \frac{9}{R} - 6 - Q \ .$$

Therefore, $Q = \frac{9}{R} - 6$.

Question 27 *(1 Point)*
What is the value of R?

- A $R = \frac{1}{2}$
- B $R = \frac{3}{2}$
- C $R = \frac{8}{3}$
- D $R = 2$
- ■ $R = 1$
- F $R = 3$

Explanation: From the state-space description we get $A = 1$ and $B = 1$. We know that Φ must be symmetric and positive definite, which in the scalar case means $\Phi > 0$. Hence, $\Phi = 3$ is the only meaningful value. Using the scalar LQR gain equation

$$K = \frac{B}{R} \cdot \Phi = \frac{3}{R} \quad \left(\Longrightarrow R = \frac{3}{K} \right)$$

and the fact that the closed-loop pole can be computed as

$$-2 = \pi_{\mathrm{CL}} = A - B \cdot K = 1 - K \quad (\Longrightarrow K = 3) \ ,$$

we get that

$$R = \frac{3}{3} = 1 \ .$$

7 Optimal Control

Questions 28, 29, 30

Description: After a brilliant career in sailing, Capitán Salazar has recently decided to join the Formula E community and has asked you to become part of his dream team. Since the legendary driver Millo Branzer Balerna is already responsible of following the fastest path on the circuit, the Captain has assigned you the task of controlling the electric motor of his race car. The system is described in space-domain using the position variable s as the independent variable. You would like to minimize the lap time, i.e. the time needed to drive one lap, while not discharging the battery below the minimum terminal battery level $E_{\mathrm{b,min}}$. Therefore, you formulate the corresponding optimal control problem as

$$\begin{aligned}
\min_{u(s)} \; & \int_0^S \frac{1}{v(s)} \, \mathrm{d}s \\
\text{s.t. } & \frac{\mathrm{d}}{\mathrm{d}s} E_{\mathrm{kin}}(s) = F_{\mathrm{el}}(s) - F_{\mathrm{drag}}(s, v(s)) \\
& \frac{\mathrm{d}}{\mathrm{d}s} E_{\mathrm{b}}(s) = -F_{\mathrm{el}}(s) \\
& v(0) = v_0 \\
& E_{\mathrm{b}}(0) = E_{\mathrm{b},0} \\
& E_{\mathrm{kin}}(s) = \frac{1}{2} \cdot m \cdot v(s)^2 \\
& v(s) \le v_{\max}(s) \\
& |F_{\mathrm{el}}(s)| \le F_{\mathrm{el,max}} \\
& E_{\mathrm{b}}(S) \ge E_{\mathrm{b,min}} \; ,
\end{aligned}$$

where the state variables are the kinetic and the battery energy, $x = (x_1, x_2) = (E_{\mathrm{kin}}, E_{\mathrm{b}})$, the input is the force exerted by the electric motor on the car, $u = F_{\mathrm{el}}$, and the speed v is an algebraic function of the state and input variables. Additionally, the nonlinear function $F_{\mathrm{drag}}(\cdot)$ represents the total drag force acting on the race car, m is the mass of the car, $v_{\max}(s)$ is the maximum speed profile arising from the track curvature, whereas $F_{\mathrm{el,max}}$ is the maximum force that can be exerted by the electric motor. Finally, v_0 and $E_{\mathrm{b},0}$ are the speed and the battery level at the beginning of the lap, and S is the length of the circuit. Formulate the whole problem as a function of state and input variables by eliminating the algebraic variable v, using appropriate substitutions, and answer the following.

Question 28 *(1 Point)*
What are the stage cost function $l(x, u)$ and the terminal cost function $m(x(S))$?

- ■ $l(x, u) = \sqrt{\frac{m}{2x_1}}$ and $m(x(S)) = 0$
- [B] $l(x, u) = \frac{1}{x_1}$ and $m(x(S)) = 0$
- [C] $l(x, u) = \frac{1}{x_1}$ and $m(x(S)) = -x_1(S)$
- [D] $l(x, u) = \sqrt{\frac{m}{2x_1}}$ and $m(x(S)) = (x_2(S) - E_{\mathrm{b,min}})^2$

Explanation: Observe that in this case the independent variable is not the time t as usual, but the position s. In order to eliminate the lifting variable v, we use the fact that $E_{\mathrm{kin}} = \frac{m}{2} \cdot v^2$ and substitute it in the optimal control problem as

$$
\begin{aligned}
\frac{1}{v(s)} &= \sqrt{\frac{m}{2 \cdot E_{\text{kin}}(s)}} \\
F_{\text{drag}}(s, v(s)) &= F_{\text{drag}}\left(s, \sqrt{\frac{2 \cdot E_{\text{kin}}(s)}{m}}\right) \\
E_{\text{kin}}(0) &= \frac{m}{2} \cdot v_0^2 \\
0 \le E_{\text{kin}}(s) &\le \frac{m}{2} \cdot v_{\max}(s)^2 \quad .
\end{aligned} \tag{1}
$$

This way, since $E_{\text{kin}} = x_1$, we get $l(x, u) = \frac{1}{v(x,u)} = \sqrt{\frac{m}{2x_1}}$ and $m(x(S)) = 0$.

Question 29 *(1 Point)*
What are the state constraint set $\mathcal{X}$ and the input constraint set $\mathcal{U}$?

■ $\mathcal{X} = [-\infty, \frac{m}{2} \cdot v_{\max}(s)^2] \times \mathbb{R}$ and $\mathcal{U} = [-F_{\text{el,max}}, F_{\text{el,max}}]$

B $\mathcal{X} = [-\infty, \frac{m}{2} \cdot v_{\max}(s)^2] \times [E_{\text{b,min}}, E_{\text{b},0}]$ and $\mathcal{U} = [-F_{\text{el,max}}, F_{\text{el,max}}]$

C $\mathcal{X} = [-\infty, \frac{m}{2} \cdot v_{\max}(s)^2] \times \mathbb{R}$ and $\mathcal{U} = [0, F_{\text{el,max}}]$

D $\mathcal{X} = [-\infty, v_{\max}(s)] \times \mathbb{R}$ and $\mathcal{U} = [-F_{\text{el,max}}, F_{\text{el,max}}]$

Explanation: The first state variable x_1 is the kinetic energy and is limited along the lap, since speed is limited (see the solution of the previous question), as $x_1 \le \frac{m}{2} \cdot v_{\max}(s)^2$. The second state variable $x_2 = E_{\text{b}}$ is the battery energy and is not constrained during the lap, but only at the end by the terminal constraint $E_{\text{b}}(S) \ge E_{\text{b,min}}$. Therefore, $(x_1,\ x_2) \in \left[-\infty, \frac{m}{2} \cdot v_{\max}(s)^2\right] \times \mathbb{R} = \mathcal{X}$. The only input variable u is the force of the electric motor F_{el}, which is constrained through the lap as $F_{\text{el}} \in [-F_{\text{el,max}}, F_{\text{el,max}}] = \mathcal{U}$.

Question 30 *(1 Point)*
Suppose that, instead of limiting the battery discharge, you want to minimize it, while not exceeding a maximum lap time of $T_{\max}$, and suppose you have modeled time as a function of space using a third state variable $x_3(s) = T(s)$. Moreover, you set the stage cost function to zero, i.e. $l(x, u) = 0$. How should you choose the terminal cost function $m(x(S))$ and the terminal constraint set $\mathcal{X}_{\text{f}}$ now?

■ $m(x(S)) = -x_2(S)$ and $\mathcal{X}_{\text{f}} = \mathbb{R} \times \mathbb{R} \times [-\infty, T_{\max}]$

B $m(x(S)) = x_3(S)$ and $\mathcal{X}_{\text{f}} = \mathbb{R} \times \mathbb{R} \times \{T_{\max}\}$

C $m(x(S)) = x_3(S)$ and $\mathcal{X}_{\text{f}} = \mathbb{R} \times [E_{\text{b,min}}, \infty] \times [-\infty, T_{\max}]$

D $m(x(S)) = -x_2(S)$ and $\mathcal{X}_{\text{f}} = \mathbb{R} \times \mathbb{R} \times \{T_{\max}\}$

Explanation: Since we want to minimize the amount of battery discharge at the end of the lap $-E_{\text{b}}(S) = -x_2(S)$, we get $m(x(S)) = -x_2(S)$. The fact that the lap time $T(S) = x_3(S)$ cannot exceed $T_{\max}$, can be expressed as $x_3(S) \le T_{\max} \iff x_3(S) \in [-\infty, T_{\max}]$. Since the battery energy is no longer limited at the end of the lap and we have no terminal constraints on the kinetic energy, i.e. $x_i(S) \in \mathbb{R}$ for $i = 1, 2$, we get $x(S) \in \mathbb{R} \times \mathbb{R} \times [-\infty, T_{\max}] = \mathcal{X}_{\text{f}}$.

8 MPC

Questions 31, 32

Description: Consider the scalar MPC optimization problem $\mathcal{P}(x_0)$

$$\begin{aligned} \min \quad & \int_0^T Q \cdot x(t)^2 + R \cdot u(t)^2 \, \mathrm{d}t \\ \text{s.t.} \quad & \dot{x}(t) = a \cdot x(t) + b \cdot u(t) \\ & x(0) = x_0 \\ & x \in \mathcal{X}, \ u \in \mathcal{U} \ , \end{aligned}$$

whereby there are no terminal state constraints and the terminal cost function is zero.

Question 31 *(1 Point)*
Suppose that $\mathcal{X} = \mathcal{U} = \mathcal{X}_\mathrm{f} = \mathbb{R}$ and $T = \infty$, such that the problem is now an infinite-horizon LQR, which you still want to implement as an MPC. Mr. Ritzmann has solved the LQR problem for you and gives you the value of the scalar static gain K. What is the optimal input trajectory $u^\star(t)$ for the first iteration interval?

- [A] $u^\star(t) = -K \cdot x_0 \cdot e^{-K \cdot t}$
- [B] $u^\star(t) = K \cdot x(t)$
- ■ $u^\star(t) = -K \cdot x_0 \cdot e^{(a-b\cdot K)\cdot t}$
- [D] $u^\star(t) = -K \cdot x(t)$

Explanation: A MPC scheme is a repetitive feedforward optimization problem, whereby we implement only the first piece of the optimal input trajectory. In the scalar LQR case, the optimal solution is computed as

$$\begin{aligned} \dot{x}^\star(t) &= a \cdot x^\star(t) + b \cdot u^\star(t) = (a - b \cdot K) \cdot x^\star(t) \\ x^\star(0) &= x_0 \\ u^\star(t) &= -K \cdot x^\star(t) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x^\star(t) &= x_0 \cdot e^{(a-b\cdot K)\cdot t} \\ u^\star(t) &= -K \cdot x_0 \cdot e^{(a-b\cdot K)\cdot t} \end{aligned} \ .$$

Observe that this solution corresponds to $-K \cdot x(t)$ only if the model is exact and there are no disturbances, which we cannot guarantee.

Question 32 *(2 Points)*
Suppose that $a = 0$, $b = 1$, $\mathcal{X} = [-5, 5]$, $\mathcal{U} = [-\frac{1}{10}, 1]$, $\mathcal{X}_\mathrm{f} = [-1, 1]$ and $T = 10$. What is the feasible set $\mathcal{X}_T = \{x_0 \in \mathbb{R} |\ \mathcal{P}(x_0) \text{ admits a feasible solution}\}$?

- [A] $\mathcal{X}_T = [-2, 2]$
- [B] $\mathcal{X}_T = [-11, 11]$
- [C] $\mathcal{X}_T = [-5, 5]$
- [D] $\mathcal{X}_T = [-11, 2]$
- [E] $\mathcal{X}_T = [-2, 11]$
- ■ $\mathcal{X}_T = [-5, 2]$

Explanation: The feasible set $\mathcal{X}_T$ is defined by all the initial conditions x_0 for which the optimization problem $\mathcal{P}(x_0)$ can be solved. Since we have a terminal set $\mathcal{X}_\mathrm{f} = [-1, 1]$, we first need to compute all the initial conditions from which we can reach $\mathcal{X}_\mathrm{f}$ in $T = 10$ using $u \in \mathcal{U} = [-1/10, 1]$. Let's define this set as $\mathcal{X}_T^\mathrm{f}$. As the system is a simple integrator, we know that the maximum distance that can be reached in $T = 10$ is obtained with a maximum or minimum input.

If our initial condition is such that $x_0 \geq x_\mathrm{f}^{\max} = 1$, we would reach $x_\mathrm{f}^{\max}$ using $u = u_{\min} = -1/10$ if x_0 is at most $x_0^{\max} = x_\mathrm{f}^{\max} - u_{\min} \cdot T = 1 + 1 = 2$.

With the same reasoning, If our initial condition is such that $x_0 \leq x_{\mathrm{f}}^{\min} = -1$, we would reach $x_{\mathrm{f}}^{\min}$ using $u = u_{\max} = 1$ if x_0 is at most $x_0^{\min} = x_{\mathrm{f}}^{\min} - u_{\max} \cdot T = -1 - 10 = -11$. This way, we get $\mathcal{X}_T^{\mathrm{f}} = [-11, 2]$.

As the system is a simple integrator, we can guarantee that we can keep the state variable inside the stage state constraint $\mathcal{X}$ if and only if all the initial conditions are already inside, i.e. $x_0 \in \mathcal{X} = [-5, 5]$. Therefore, the feasible set is given by all the initial conditions inside both $\mathcal{X}$ and $\mathcal{X}_T^{\mathrm{f}}$, i.e.

$$\mathcal{X}_T = \mathcal{X} \cap \mathcal{X}_T^{\mathrm{f}} = [-5, 5] \cap [-11, 2] = [-5, 2] \ .$$

9 LQG and LTR

Questions 33, 34, 35

Description: Consider the scalar SISO system

$$\dot{x}(t) = 4x(t) + 2u(t) \ ,$$

where you measure $y(t) = 3x(t)$. Your colleague Nick has designed a LQG, i.e. a LQR with an observer, but has forgotten the static gain L of the observer. He only remembers that the closed-loop poles are $\pi_{\text{CL}} = \{-3, -1\}$ and that the static LQR gain is $K = \frac{5}{2}$.

Question 33 *(1 Point)*
What is the value of L?

- ■ $L = \frac{7}{3}$
- B $L = \frac{3}{5}$
- C $L = \frac{7}{2}$
- D $L = \frac{2}{3}$
- E $L = \frac{5}{2}$
- F $L = \frac{5}{3}$

Explanation: By the separation principle, we know that $\pi_{\text{CL}} = \pi\{A - B \cdot K\} \cup \pi\{A - L \cdot C\} = \{-1, -3\}$. Since $\pi\{A - B \cdot K\} = A - B \cdot K = 4 - 2 \cdot \frac{5}{2} = -1$, we have that $-3 = \pi\{A - L \cdot C\} = A - L \cdot C = 4 - 3 \cdot L$. From this, we obtain $L = \frac{7}{3}$.

Question 34 *(1 Point)*
What can you say about stability and robustness of the closed-loop system?

- A No conclusion can be drawn.
- ■ The closed-loop system is asymptotically stable but there are no robustness guarantees.
- C The closed-loop system is unstable.
- D The closed-loop system is asymptotically stable and robust, i.e. $\mu_{\min} \geq 1$.

Explanation: Since both closed-loop poles are in the left half-plane, the closed-loop system is asymptotically stable. Nevertheless, with a LQG controller, there are no robustness guarantees. See the paper "Guaranteed Margins for LQG Regulators" by J.C. Doyle for more information.

Question 35 *(1 Point)*
You decide to redesign the observer on your own using Matlab, setting the tuning knobs of the observer to $\bar{B} \cdot \bar{B}^\top = 4$ and $q = 10$. What is the correct command to obtain the observer gain L?

- A `L = observer(4,2,3,10)`
- ■ `L = lqr(4,3,4,10)`
- C `L = lqr(4,3,10,4)`
- D `L = lqr(4,2,10,4)'`

Explanation: Since we need to solve the LQR problem for $L = K_{\text{LQR}}^\top$ with $A_{\text{LQR}} = A^\top = 4$, $B_{\text{LQR}} = C^\top = 3$, $Q_{\text{LQR}} = \bar{B} \cdot \bar{B}^\top = 4$ and $R_{\text{LQR}} = q = 10$, we type `L = lqr(4,3,4,10)`. Observe that the system is scalar. Therefore, there is no need to transpose the solution.