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Exam STAM Study Manual

2nd Edition | Volumes I, II & III

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Lesson 7

Loss Reserving: Basic Methods

Reading: *Introduction to Ratemaking and Loss Reserving for Property and Casualty Insurance* 3.1–3.6.4, STAM-24-18

7.1 Case reserves and IBNR reserves

Claims on first-party coverages tend to be paid out in a fairly short time. Liability claims, however, can take a long time to settle. The largest claims can take as much as 10 years to settle. Losses may not even be reported that rapidly. The insurer must hold a reserve for losses that have occurred and that have not been fully paid.

A company's incurred claim expense during a year is the amount the company has paid on the claim plus the change in reserve over the period. For example, consider a loss with the following history:

- Reported 11/1/2017
- Reserve on 12/31/2017: 15,000
- Payment of 4,000 made on 4/15/2018
- Reserve on 12/31/2018: 35,000
- Payment of 10,000 made on 6/12/2019
- Reserve on 12/31/2019: 30,000
- Payment of 30,000 made on 10/5/2020 and the claim is closed

Then the company's expense for this loss in 2017 is the reserve set up at the end of the year, 15,000. The expense in 2018 is the 4,000 payment plus the increase in reserve, or 4,000 + 35,000 − 15,000 = 24,000. The expense in 2019 is the 10,000 payment plus the increase in reserve, or 10,000 + 30,000 − 35,000 = 5,000. The expense in 2020 is the payment of 30,000 minus the release of the 30,000 reserve, or 0.

The claim adjuster estimates the future payments on a claim and sets up a reserve for the claim. Those reserves are called *case reserves*. However, additional reserves are needed for:

1. Provision for future adjustments to known claims
2. Provision for claim files that are closed but may reopen
3. Provision for incurred but not reported claims (pure IBNR)
4. Provision for reported but not recorded claims (RBNR)

These additional reserves are called *gross IBNR* reserves or *bulk* reserves. We'll call them IBNR reserves and use the qualifier "pure" if we are referring to the third provision of the list. As the "bulk" name indicates, these reserves cannot be developed on a claim-by-claim basis (after all, some of the claims are not even known) but are developed on a bulk basis, by analyzing development trends for blocks of business.

An old non-actuarial method for calculating IBNR reserves is *case reserves plus*. The IBNR reserve is set as a percentage of the case reserve in some judgmental fashion. This method is subject to manipulation, with percentages raised or lowered as needed to smooth a company's earnings. It is therefore rarely used.

Before we go discuss other methods for calculating the IBNR reserve, let's define a couple of terms.

Written premium and earned premium *Written premium* is the amount of premium on a policy sold during a period of time. If a one-year policy with annual premium 900 is issued on 6/1/2017, then the written premium on that policy during 2017 is 900.[1] However, the company provides coverage on that policy through 5/31/2018, so the company has not earned the entire premium in 2017. Only 7/12 of the premium is earned in 2017, while the remaining 5/12 of the premium is earned over the 5 months in 2018 that the policy is in force. Earned premium is 900(7/12) = 525 in 2017 and 900(5/12) = 375 in 2018. *Earned premium* for a calender period is the written premium for a policy currently in force regardless of when the premium was paid, times the portion of the policy's duration that is within the calendar period (year, month, etc.) being considered.

Example 7A An insurance company sells one-year policies from July 1, 2019 through December 31, 2019. Sales are uniformly distributed over the 6 month period. Written premium is 300,000.

Calculate earned premium on these policies for 2019.

Solution: On average, the policies are in force for 3 months in 2019, so earned premium is 0.25(300,000) = **75,000**. The remainder of the written premium is earned in 2020. □

Quiz 7-1 A six-month policy is sold on 11/15/2020. The written premium is 600.

Calculate earned premium in 2021 on this policy.

CY and AY We will be using the abbreviation CY for calendar year. Calendar year accounting refers to when a transaction occurs or is incurred. For example, calendar year earned premium would be calculated as described before Example 7A. Losses paid or incurred (through an increase in reserve) in a year would be associated with that calendar year, regardless of when the underlying accident occurred.

Later on, we will be using the abbreviation AY for accident year. Accident years pertain only to costs related to accidents. For an accident that occurs between January 1 and December 31 of a year, payments and reserve increases are associated with that accident year, regardless of when the payments made or the increases in reserve occur.

Both of these abbreviations (CY and AY) will be used on exams.

Loss ratio The loss ratio is the ratio of losses to earned premiums. This concept is used in at least three ways:

1. The *permissible loss ratio* is the loss ratio that is used for pricing. It is the complement of the ratio of expenses and provisions for contingencies and profit. The premium rate is computed so that expected future losses are equal to the premium rate times the permissible loss ratio.

2. The *expected loss ratio* is the expected losses divided by the premium. While this may be the same as the permissible loss ratio, it may be different if the premium rate was not based on the permissible loss ratio for whatever reason (marketing considerations, regulatory constraints).

3. The *experience loss ratio* is the actual loss ratio experienced on the block of business. Typically, it is calculated as losses for accident year x divided by earned premiums for calendar year x.

For loss reserving, we will be using the expected loss ratio. The other two loss ratios will be used for ratemaking.

7.2 Three methods for calculating IBNR reserves

To help you understand what the three methods we will discuss for calculating IBNR reserves do, I'd like to discuss an unrelated calculation I used to do as a life insurance actuary. I worked on financial projection, which included projecting mortality experience. Thus I might project 40 million of death claims for the next year, 10 million per quarter. After the first quarter of the year, after actual results came in, I then was expected to update my projection.

[1]The policy may allow the policyholder to pay the premium in periodic installments. For example, instead of paying 900 on 6/1/2017, the policyholder may pay 225 on 6/1/2017, 225 on 9/1/2017, 225 on 12/1/2017, and 225 on 3/1/2018. But even then, the written premium is the annual premium for the policy sold in 2017, not the amount of cash paid by the policyholder during 2017.

Suppose in the first quarter, the company incurred 11 million of death claims. Assuming that there was no other reason to update my mortality projection, what should my new forecast be? Well, there are three alternatives:

1. Continue to project 40 million in death claims for the full year. In other words, project 29 million for the remainder of the year. This is in fact what the CFO wanted me to do. But being an actuary, my rejoinder was "Do you think that worse-than-expected mortality experience in the first quarter implies better-than expected mortality experience for the rest of the year?"[2]

2. Keep my projection of 30 million in death claims for the rest of the year, so that total death claims for the year will be 41 million. As an actuary, who believes that death claims in each quarter are independent random variables, I found this method appealing.

3. Update my projection to 11 million per quarter, or 33 million in death claims for the rest of the year, 44 million total. This method would be appealing if I felt that my projection was weak and that actual experience was fully credible and the best guide for the future.[3]

The three methods we're about to study for IBNR reserving correspond to these three alternatives. The first alternative corresponds to the loss ratio method. The second alternative corresponds to the Bornhuetter-Ferguson method. And the third alternative corresponds to the chain ladder method.

7.2.1 Expected loss ratio method

Under the *expected loss ratio method* for calculating reserves, the reserve is equal to the expected loss ratio times earned premium minus the amount paid to date. The loss ratio may be the one originally assumed in pricing or may be judgmentally modified. In other words, you assume that at the end of the day (or actually, the end of many years) losses will be some amount that is projected based on some reasonable loss ratio, and if you paid more than expected so far, you'll pay less than expected in the future.

Example 7B In 2019, earned premiums were 5 million. For accidents occurring in 2019, 1 million was paid in 2019 and 0.8 million was paid in 2020. Case reserves on this block of business were 0.9 million on 12/31/2020. The expected loss ratio for this block of business is 60%.

For this block of business calculate

(a) Total reserves as of 12/31/2020

(b) IBNR reserves as of 12/31/2020

Solution: Expected losses are 5,000,000(0.6) = 3,000,000. Subtracting the 1,800,000 that was paid, the total reserve is **1,200,000**. The IBNR reserve is 1,200,000 − 900,000 = **300,000**. □

The loss ratio method is crude. If past cumulative payments are higher than expected, it isn't reasonable to assume that future payments will therefore be lower than expected. However, sometimes it is the only method available. For example, on a new line of business with no claims experience, it would be the only method to use.

7.2.2 The chain-ladder or loss development triangle method

The chain-ladder method ignores the loss ratio, and projects future payments purely off past payments.

For a claim with a long time to settlement, the cumulative amount paid usually grows from year to year. This growth is called *loss development*. The ratio of cumulative amount paid through year x to cumulative amount paid through year $x-1$ is called an *age-to-age development factor*, where the age of a claim is the amount of time since the loss occurred; the age is 0 in the year it occurred, 1 in the next year, and so on. This is the convention of the Brown/Lennox textbook, but other textbooks express the age in months and call the age "12 months" at the end of one year, "24 months" at the end of two years, and so on. (See exercise 7.14, a sample question from the SOA, which

[2]The CFO was an accountant. Accountants believe in offsetting errors—if an error is made in one direction, as I did here by under-forecasting mortality, there will be an offsetting error in the rest of the year.

[3]Incidentally, at the end of the year, the CFO turned out to be right—we had about 40 million in claims for the year.

follows this convention.) An *age-to-ultimate development factor* is the ratio of the ultimate payment to the cumulative payment at an age.

In the chain-ladder method, the actuary estimates the age-to-age development factors based on history, and uses these to develop immature claims to their ultimate cost.

Let's do an example. Suppose you are given the following triangle, showing cumulative payments for each accident year through various development years:

Table 7.1: Cumulative Payment Loss Triangle

Cumulative Payments					
Accident Year	Development Years				
	0	1	2	3	4
AY1	1000	1200	1300	1400	1435
AY2	1040	1280	1400	1540	
AY3	1100	1400	1470		
AY4	1200	1500			
AY5	1250				

Assume that losses are mature at the end of 4 years; cumulative payments at that time represent the total loss. It is now 12/31/CY5, and we wish to calculate the reserve. We calculate development factors by dividing cumulative payments in year t by cumulative payments in year $t-1$. For example, for AY1, the ratio of development year 1 to development year 0 is 1200/1000 = 1.2; the ratio of development year 2 to development year 1 is 1300/1200 = 1.083. We obtain the following table of development factors:

Loss Development Factors Based on Cumulative Payments				
Accident Year	Development Years			
	1/0	2/1	3/2	4/3
AY1	1.200	1.083	1.077	1.025
AY2	1.231	1.094	1.100	
AY3	1.273	1.050		
AY4	1.250			

We average these factors. We can simply take the arithmetic average of the factors. For example, for 1/0, the arithmetic average is

$$\frac{1.200 + 1.231 + 1.273 + 1.250}{4} = 1.238$$

We obtain the following table of average factors. These factors, regardless of how they are calculated, are called *link ratios*.

Link Ratios (Arithmetic Average)			
Development Years			
1/0	2/1	3/2	4/3
1.238	1.076	1.088	1.025

This method is called the *arithmetic average* method. It may also be called the *average* method or the *average factor* method.

An alternative method is to divide the sum of cumulative payments to year t by the sum of corresponding cumulative payments to year $t-1$. For example, for the 1/0 ratio,

$$\frac{1200 + 1280 + 1400 + 1500}{1000 + 1040 + 1100 + 1200} = 1.240$$

We obtain the following table of link ratios:

Link Ratios (Volume-Weighted Average)			
Development Years			
1/0	2/1	3/2	4/3
1.240	1.075	1.089	1.025

This method is called the *volume-weighted average* method. It may also be called the *mean* method or the *mean factor* method. Since volume usually increases by accident year, it has the effect of giving more weight to recent data.

To use more recent data, we may average only recent years. For example, a 2-year arithmetic average method would average the 2 most recent years to compute the link ratios. For 1/0, the link ratio would be $(1.273+1.25)/2 = 1.261$ and for 2/1, the link ratio would be $(1.094+1.05)/2 = 1.072$.

Once we have calculated link ratios, we project the bottom half of the payment triangle using these link ratios. For example, suppose we are using the arithmetic average method. For AY4, we would project cumulative payments as $1500(1.076) = 1614$, $1614(1.088) = 1756$, $1756(1.025) = 1800$. After similar computations for the other accident years, the filled in triangle looks like this:

Table 7.2: Projected Cumulative Payments

Cumulative Payments					
Accident Year	Development Years				
	0	1	2	3	4
AY1	1000	1200	1300	1400	1435
AY2	1040	1280	1400	1540	*1579*
AY3	1100	1400	1470	*1600*	*1640*
AY4	1200	1500	*1614*	*1756*	*1800*
AY5	1250	*1548*	*1665*	*1812*	*1857*

The total reserve is the excess of the final column over the cumulative payments to date, or

$$(1435+1579+1640+1800+1857) - (1435+1540+1470+1500+1250) = 1116$$

The IBNR reserve is 1116 minus the case reserves.

This method may also be used with an incurred losses triangle. Incurred losses include case reserves. We would expect link ratios to be lower, possibly even below 1, since case reserves increase the earlier numbers. After performing the calculation, the difference between the developed ultimate numbers and the incurred-to-date numbers is the IBNR reserve.

Quiz 7-2 You have developed the following link ratios based on cumulative payments.

Link Ratios			
Development Years			
1/0	2/1	3/2	4/3
1.500	1.215	1.180	1.050

Losses are mature at the end of 4 years.

It is now 12/31/CY5. Earned premium in AY3 is 1,000,000. For losses in AY3, paid-to-date is 600,000.

Let R be the reserve calculated using the chain-ladder method. Let R_{LR} be the reserve calculated using the expected loss ratio method.

Calculate the expected loss ratio that results in $R = R_{LR}$.

The chain ladder method is not stable. Large payments or large case reserves in a year usually generate even higher reserves, since paid-to-date or incurred-to-date is usually multiplied by factors greater than 1. In some sense it is the opposite of the loss ratio method, which lowers future projected payments in response to higher earlier payments.

7.2.3 The Bornhuetter-Ferguson method

The Bornhuetter-Ferguson method is a compromise between the loss ratio method and the chain ladder method. The idea of the method is: suppose based on the link ratios the development factor to ultimate levels is f_{ult}. Then $1/f_{ult}$ of the ultimate loss has been paid (or incurred) so far and $1-1/f_{ult}$ of the ultimate loss remains to be paid (or incurred). To calculate the amount remaining to be paid (or incurred), assume future payments or incurred losses are in accordance with the expected loss ratio. In other words,

$$\text{Reserve} = \text{Earned Premium} \times \text{Expected Loss Ratio} \times \left(1 - \frac{1}{f_{ult}}\right)$$

where $f_{ult} = \prod_j f_j$, and f_j are link ratios from year $j-1$ to year j. The product begins one year after the current year.

With this method higher past payments do not reduce future payments, as would occur with the loss ratio method. Nor do they raise future payments, as would occur with the chain ladder method. (However, if future link ratios are increased, that will lower the proportion paid so far and thus will still increase the reserve.)

As with the chain-ladder method, this method may be used on cumulative payment triangles to generate the total reserve or on incurred loss triangles to generate the IBNR reserve.

Let's redo the calculation of the reserve from Table 7.1 with arithmetic average factors. We need an assumption for earned premium and loss ratios. Let's assume earned premium was:

Calendar year	CY1	CY2	CY3	CY4	CY5
Earned premium	2100	2400	2700	3000	3300

and the expected loss ratio is 0.60.

For convenience, we'll repeat the link ratio table.

Link Ratios (Arithmetic Average)			
Development Years			
1/0	2/1	3/2	4/3
1.238	1.076	1.088	1.025

The ultimate link factors f_{ult} are $f_4 = 1.025$ for 4/3, $(1.025)(1.088) = 1.116$ for 4/2, $(1.116)(1.076) = 1.200$ for 4/1, and $(1.200)(1.238) = 1.486$ for 4/0. So the total reserve is

$$0.6(2400)\left(1-\frac{1}{1.025}\right) + 0.6(2700)\left(1-\frac{1}{1.116}\right) + 0.6(3000)\left(1-\frac{1}{1.200}\right) + 0.6(3300)\left(1-\frac{1}{1.486}\right) = 1150.98$$

For comparison, the loss ratio method, would generate

$$0.6(2400+2700+3000+3300) - (1540+1470+1500+1250) = 1080$$

No reserve is generated for AY1, since it is mature.

For any accident year, let f_{ult} be the ultimate development factor. Let

R_{LR} be the expected loss ratio method reserve
R_{CL} be the chain ladder method reserve
R_{BF} be the Bornhuetter-Ferguson reserve

Then

$$R_{BF} = \left(1-\frac{1}{f_{ult}}\right) R_{LR} + \frac{1}{f_{ult}} R_{CL} \tag{7.1}$$

This should be obvious, since the method assumes that future payments will be in accordance with the loss ratio method, and that is the first term, while past payments are whatever they were, and that is the second term since R_{CL} multiplies past payments by f_{ult}. The textbook proves this algebraically. Even though the formula is obvious, it does show that for each accident year the Bornhuetter-Ferguson reserve is a weighted average of the loss ratio and chain ladder reserves and states the weights.

7.2.4 Variance of forecasted losses

Let $L_{i,j}$ be the random variable for payment for AYi, development year j. Let K be the ultimate development year. We have data for $L_{i,j}$ for $j \le K - i + 1$. Let f_j be the link ratio, the ratio of losses in development year j to losses in development year $j - 1$. We will select the f_j with the lowest variance.

To do so, we make the following assumptions:

1. $\mathbf{E}[L_{i,j} \mid L_{i,0}, L_{i,1}, \ldots, L_{i,j-1}] = L_{i,j-1} f_j$. In other words, the expected value of the loss in the j^{th} development year is a constant times the loss in the $j - 1^{\text{st}}$ development year, with the constant not varying by accident year, although it may vary by development year.
2. $\operatorname{Var}(L_{i,j} \mid L_{i,0}, L_{i,1}, \ldots, L_{i,j-1}) = L_{i,j-1} \alpha_j^2$. In other words, the variance of the loss in the j^{th} development year is a constant times the loss in the $j - 1^{\text{st}}$ development year, with the constant not varying by accident year, although it may vary by development year.
3. Losses are independent of accident year.

If you are choosing weights for a weighted average of independent random variables, the variance of the average is minimized by setting the weights proportional to the reciprocals of the variances. We wish to select an f_j which is a weighted average of the known values of $L_{i,j}/L_{i,j-1}$. Mack showed that with the three assumptions above, the variance of f_j is minimized when the volume-weighted average is used. Other sets of assumptions can be used to derive other formulas.

Exercises

7.1. A one-year policy is sold on 9/1/CY1. The policyholder pays premiums on an installment basis: 800 on 9/1/CY1 and 400 on 3/1/CY2.

Calculate earned premiums for CY1.

7.2. A company sells one-year auto insurance policies. The amount of written premium in months of 2018 is

January	5,000	July	8,500
February	6,500	August	8,500
March	7,000	September	7,000
April	7,000	October	7,500
May	8,000	November	6,000
June	10,000	December	6,500

For December 2017, written premium is 6,000.

Assume that all policies are sold in the middle of the month.

Calculate earned premium for December 2018.

7.3. On September 1, 2018, a company introduces a 1-year policy. Written premium for this policy in 2018 is 240,000. Written premium is uniformly distributed over the last 4 months of 2018.

Calculate earned premium in 2018 for this policy.

Exercises continue on the next page . . .

7.4. For a block of business, the expected loss ratio is 0.6. All policies are issued for one year, with premiums payable at the beginning of the year. Policies are sold uniformly throughout the year.

You are given the following sales:

Year	Sales
2014	4000
2015	5000
2016	5500
2017	6000

For accident year 2015, cumulative incurred losses are:

Cumulative Incurred Losses				
	Development Year			
	0	1	2	3
AY2015	1500	1800	2000	2050

Calculate the IBNR reserve for accident year 2015 as of 12/31/2018 using the expected loss ratio method.

Use the following information for questions 7.5 and 7.6:

You are given the following paid claims triangle:

Cumulative Loss Payments				
Accident Year	Development Years			
	0	1	2	3
2014	1200	2400	2700	3000
2015	1500	2500	3000	
2016	1600	2400		
2017	1800			

There is no development after development year 3.

7.5. Calculate the loss reserve on 12/31/2017 using the chain ladder method with arithmetic average loss development factors.

7.6. Calculate the loss reserve on 12/31/2017 using the chain ladder method with volume-weighted average loss development factors.

Use the following information for questions 7.7 and 7.8:

You are given the following information for the development of cumulative loss payments:

Accident Year	Development Years				
	0	1	2	3	4
AY1	200	360	500	700	800
AY2	250	400	500	680	
AY3	300	500	700		
AY4	400	600			
AY5	500				

Losses are mature at the end of 4 years.

7.7. Calculate the loss reserve using the chain ladder method with volume-weighted average loss development factors.

Exercises continue on the next page . . .

7.7–8. *(Repeated for convenience) Use the following information for questions 7.7 and 7.8:*

You are given the following information for the development of cumulative loss payments:

Accident Year	Development Years				
	0	1	2	3	4
AY1	200	360	500	700	800
AY2	250	400	500	680	
AY3	300	500	700		
AY4	400	600			
AY5	500				

Losses are mature at the end of 4 years.

7.8. Earned premium for CY4 is 1500.

Assuming that future payments on AY4 losses equal the reserve calculated in exercise 7.7, calculate the expected loss ratio for AY4.

7.9. You are given the following information regarding incurred loss development:

Accident Year	Development Years				
	0	1	2	3	4
AY1	600	650	600	650	700
AY2	620	640	680	690	
AY3	650	670	690		
AY4	700	800			
AY5	800				

Losses are mature at the end of 4 years.

IBNR reserves are calculated using the chain ladder method with arithmetic average loss development factors.

Calculate IBNR reserves.

7.10. The following table shows link ratios for cumulative payments based on the chain ladder method:

Development Years	Link Ratio
1/0	1.5
2/1	1.2
3/2	1.1
4/3	1.1

Claims mature at the end of four years of development.

The following amounts were paid through 12/31/2018 for recent accident years:

Accident Year	Cumulative Payments Through 12/31/2018
2015	500
2016	450
2017	425
2018	400

Calculate the reserve under the chain ladder method on 12/31/2018.

Exercises continue on the next page . . .

7.11. The following table shows cumulative payments as of December 31, 2018 for losses by accident year, and the reserve held for each accident year.

Accident Year	Paid to Date	Reserve
2014	550	400
2015	450	500
2016	400	600
2017	400	720
2018	320	960

Reserves were calculated using the chain ladder method.

Determine the link ratios between development years 1 to 0, 2 to 1, 3 to 2, and 4 to 3.

7.12. You have the following information for cumulative paid losses:

Accident Year	Development Years			
	0	1	2	3
AY1	400	600	700	800
AY2	350	600	750	
AY3	450	800		
AY4	500			

You have the following information for incurred losses:

Accident Year	Development Years			
	0	1	2	3
AY1	600	650	725	800
AY2	600	650	775	
AY3	700	900		
AY4	800			

Losses are mature at the end of 3 years.

Use the chain ladder method with volume-weighted average loss development factors on the paid losses triangle.

Calculate the IBNR reserve.

7.13. You have computed the following link factors between development years:

1/0: 1.48 2/1: 1.23 3/2: 1.11 4/3: 1.05

Losses mature at the end of 4 years.

For AY2 development year 1, losses paid-to-date is 4 million.

Earned premium for CY2 is 10 million and the expected loss ratio is 70%.

Calculate the reserve for AY2 as of 12/31/CY3 using the Bornhuetter-Ferguson method.

Exercises continue on the next page . . .

7.14. [**STAM Sample Question #318**] You are given the following information:

			Cumulative Loss Payments through Development Month			
Accident Year	Earned Premium	Expected Loss Ratio	12	24	36	48
AY5	19,000	0.90	4,850	9,700	14,100	16,200
AY6	20,000	0.85	5,150	10,300	14,900	
AY7	21,000	0.91	5,400	10,800		
AY8	22,000	0.88	7,200			

There is no development past 48 months.

Calculate the indicated actuarial reserve using the Bornhuetter-Ferguson method and volume-weighted average loss development factors.

(A) 22,600 (B) 23,400 (C) 24,200 (D) 25,300 (E) 26,200

Use the following information for questions 7.15 and 7.16:

You are given the following information regarding the development of cumulative loss payments:

Accident Year	Earned Premium	Cumulative Loss Payments through Development Year				
		0	1	2	3	4
AY1	1400	200	360	500	700	800
AY2	1500	250	400	500	680	
AY3	1600	300	500	700		
AY4	1700	400	600			
AY5	1800	500				

Losses are mature at the end of 4 years.
The expected loss ratio is 60%.

7.15. Calculate the loss reserve using the Bornhuetter-Ferguson method with volume-weighted average loss development factors.

7.16. Calculate the loss reserve using the Bornhuetter-Ferguson method with arithmetic average loss development factors.

Exercises continue on the next page . . .

7.17. **[STAM Sample Question #320]** You are given:

(i)

Accident Year	Cumulative Paid Losses through Development Year						Earned premium
	0	1	2	3	4	5	
AY4	1,400	5,200	7,300	8,800	9,800	9,800	18,000
AY5	2,200	6,400	8,800	10,200	11,500		20,000
AY6	2,500	7,500	10,700	12,600			25,000
AY7	2,800	8,700	12,900				26,000
AY8	2,500	7,900					27,000
AY9	2,600						28,000

(ii) The expected loss ratio for each Accident Year is 0.550.

Calculate the total loss reserve using the Bornhuetter-Ferguson method and three-year arithmetic average paid loss development factors.

(A) 21,800 (B) 22,500 (C) 23,600 (D) 24,700 (E) 25,400

7.18. The following table shows cumulative payments as of December 31, 2018 for losses by accident year, and the reserve held on December 31, 2018 for each accident year.

Accident Year	Earned Premium	Paid to Date	Reserve
2014	1500	550	100
2015	1500	450	200
2016	1600	400	400
2017	2000	400	800
2018	2000	320	1000

Reserves were calculated using the Bornhuetter-Ferguson method. The expected loss ratio is 60%.

Losses mature at the end of year 5.

Determine the link ratios between development years 0 and 1, development years 1 and 2, development years 2 and 3, development years 3 and 4, and development years 4 and 5.

7.19. **[STAM Sample Question #321]** You are given:

(i) An insurance company was formed to write workers compensation business in CY1.
(ii) Earned premium in CY1 was 1,000,000.
(iii) Earned premium growth through CY3 has been constant at 20% per year (compounded).
(iv) The expected loss ratio for AY1 is 60%.
(v) As of December 31, CY3, the company's reserving actuary believes the expected loss ratio has increased two percentage points each accident year since the company's inception.
(vi) Selected incurred loss development factors are as follows:

12 to 24 months	1.500
24 to 36 months	1.336
36 to 48 months	1.126
48 to 60 months	1.057
60 to 72 months	1.050
72 to ultimate	1.000

Calculate the total IBNR reserve as of December 31, CY3 using the Bornhuetter-Ferguson method.

(A) 964,000 (B) 966,000 (C) 968,000 (D) 970,000 (E) 972,000

Exercises continue on the next page . . .

7.20. You have the following information for losses:

Accident Year	Paid to date 12/31/2018	Loss Reserve 12/31/2018
2014	200	150
2015	160	200
2016	150	250

The loss reserves in this table were computed using the expected loss ratio method.

Losses mature at the end of 5 years.

Link factors are 1/0: 1.6 2/1: 1.3 3/2: 1.2 4/3: 1.1 5/4: 1.05

Calculate the Bornhuetter-Ferguson method reserve on 12/31/2018 for accident years 2014–2016.

7.21. You have the following information for cumulative paid losses:

Accident Year	Development Years			
	0	1	2	3
AY1	400	600	700	800
AY2	350	600	750	
AY3	450	800		
AY4	500			

You have the following information for incurred losses:

Accident Year	Development Years			
	0	1	2	3
AY1	600	650	725	800
AY2	600	650	775	
AY3	700	900		
AY4	800			

Losses are mature at the end of 3 years.

Expected ultimate losses based on the loss ratios are 850 for AY2, 1000 for AY3, and 1200 for AY4.

Use the Bornhuetter-Ferguson method with volume-weighted average factors.

Calculate the IBNR reserve.

7.22. You are given two independent random variables X_1 and X_2, with $\mathrm{Var}(X_1) = 3\,\mathrm{Var}(X_2)$.

Consider the weighted average $Y = wX_1 + (1 - w)X_2$.

Determine the w that minimizes the variance of Y.

Solutions

7.1. The total premium for the year is 1200, and the policy is in force for 4 months in CY1, so earned premium is $(4/12)(1200) = \boxed{\mathbf{400}}$ in CY1.

7.2. 1/12 of each month's premium is earned in December 2018, except that 1/24 of December 2017 and December 2018 premium is earned since for policies sold in December 2017 the premium is only earned through December 15, 2018 (the middle of the month) and for policies sold in December 2018 the premium is only earned starting on December 15, 2018 when the policy is sold. Earned premium for December 2018 is therefore

$$\frac{\begin{array}{l}5{,}000 + 6{,}500 + 7{,}000 + 7{,}000 + 8{,}000 + 10{,}000\\ \quad + 8{,}500 + 8{,}500 + 7{,}000 + 7{,}500 + 6{,}000 + 0.5(6{,}000 + 6{,}500)\end{array}}{12} = \boxed{\mathbf{7{,}270.83}}$$

7.3. Earned premium is $(4-x)/12$ of written premium, where x is the number of months (including fractions) that the issue date is after September. Integrating this from 0 to 4, times the density of the uniform distribution (1/4), we get

$$\int_0^4 \frac{1}{4}\frac{(4-x)dx}{12} = \frac{1}{6}$$

1/6 of the written premium, or **40,000**, is earned.

Another way to get this result is to observe that since the earned proportion ranges from 1/3 to 0 uniformly, on the average 1/6 of the written premium is earned.

7.4. Since policies are sold uniformly throughout the year, half of premium written in 2014 is earned in 2015 and half of premium written in 2015 is earned in 2015. Earned premium in 2015 is $0.5(4000+5000) = 4500$. The expected loss ratio method sets the ultimate reserve equal to $0.6(4500) = 2700$. Since 2050 includes the case reserves, the IBNR reserve is $2700 - 2050 =$ **650**.

7.5. The average factors are

Accident Year	Development Years 1/0	2/1	3/2
2014	2	1.125	1.111
2015	1.667	1.2	
2016	1.5		
Average	1.722	1.163	1.111

The resulting ultimate losses are $3000(1.111) = 3333$ for 2015, $2400(1.163)(1.111) = 3100$ for 2016, and for 2017 they are $1800(1.722)(1.163)(1.111) = 4004$. They add up to 10,438. $3000 + 2400 + 1800 = 7200$ has been paid, so the reserve is **3238**.

7.6. The development factors are $(2400+2500+2400)/(1200+1500+1600) = 1.698$ for 1/0, $(2700+3000)/(2400+2500) = 1.163$ for 2/1, and 1.111 for 3/2. The resulting ultimate losses are 3333 for 2015, $2400(1.163)(1.111) = 3102$ for 2016, and $1800(1.698)(1.163)(1.111) = 3950$ for 2017, which add up to 10,385. Subtracting 7200 that was paid, the reserve is **3185**.

7.7. Loss development factors are:

$$1/0: \quad \frac{360+400+500+600}{200+250+300+400} = 1.617391$$

$$2/1: \quad \frac{500+500+700}{360+400+500} = 1.349206$$

$$3/2: \quad \frac{700+680}{500+500} = 1.38$$

$$4/3: \quad \frac{800}{700} = 1.142857$$

Thus the reserves are

$$\text{AY2:} \quad 680(1.142857-1) = 97.14$$

$$\text{AY3:} \quad 700\big((1.38)(1.142857)-1\big) = 404$$

$$\text{AY4:} \quad 600\big((1.349206)(1.38)(1.142857)-1\big) = 676.73$$

$$\text{AY5:} \quad 500\big((1.617391)(1.349206)(1.38)(1.142857)-1\big) = 1220.82$$

These four reserves sum up to **2398.69**.

7.8. Total expected payments are $600 + 676.73 = 1276.73$. The expected loss ratio is $1276.73/1500 =$ **0.851156**.

7.9. By taking quotients of numbers in each column over numbers in the preceding column, we get

Accident Year	1/0	2/1	3/2	4/3
AY1	1.083333	0.923077	1.083333	1.076923
AY2	1.032258	1.062500	1.014706	
AY3	1.030769	1.029851		
AY4	1.142857			

The column averages are 1.072304, 1.005143, 1.049020, 1.076923 respectively. The IBNR reserves for AY2–AY5, which are total projected payout minus incurred to date, are

$$\begin{aligned}
\text{AY2:} &\quad 690(1.076923-1)=53.0769\\
\text{AY3:} &\quad 690\big((1.049020)(1.076923)-1\big)=89.5023\\
\text{AY4:} &\quad 800\big((1.005143)(1.049020)(1.076923)-1\big)=108.4184\\
\text{AY5:} &\quad 800\big((1.072304)(1.005143)(1.049020)(1.076923)-1\big)=174.1011
\end{aligned}$$

The sum of these IBNR reserves is **425.10**.

7.10. For AY 2015, there is one more year of development to go, so the reserve is $500(1.1-1)=50$. For AY 2016, there are two years to go: $450\big((1.1)(1.1)-1\big)=94.5$. For AY 2017, $425\big((1.2)(1.1)(1.1)-1\big)=192.1$. For AY 2018, $400\big((1.5)(1.2)(1.1)(1.1)-1\big)=471.2$. The four numbers sum to **807.8**.

7.11. We are not given the development year at which the claims mature, but it doesn't matter. The ratio from development year 4 to maturity is $(550+400)/550=1.727273$. The ratio from development year 3 to maturity is $(450+500)/450=2.111111$, so link ratio 4/3 is $2.111111/1.727273=$ **1.222222**. The ratio from development year 2 to maturity is $1000/400=2.5$, so link ratio 3/2 is $2.5/2.111111=$ **1.184211**. The ratio from development year 1 to maturity is $1120/400=2.8$, so link ratio 2/1 is $2.8/2.5=$ **1.12**. The ratio from development year 0 to maturity is $1280/320=4$. so link ratio 1/0 is $4/2.8=$ **1.428571**.

7.12. The mean factors are

$$\begin{aligned}
1/0: &\quad \frac{600+600+800}{400+350+450}=1.666667\\
2/1: &\quad \frac{700+750}{600+600}=1.208333\\
3/2: &\quad \frac{800}{700}=1.142857
\end{aligned}$$

Total expected payments for AY2–AY4 are $750(1.142857)=857.1429$, $800(1.2083F33)(1.142857)=1104.7619$, and $500(1.666667)(1.208333)(1.142857)=1150.7937$. The sum is 3112.70. Subtracting incurred losses of $775+800+900=2475$, the IBNR reserve is **637.70**.

7.13. For AY2, development year 0 is CY2 and development year 1 is CY3. So the reserve is losses projected at the end of development year 1 to ultimate, development year 4.

Expected ultimate losses are 7 million. The link ratio from year 1 to 4 is $(1.23)(1.11)(1.05)=1.433565$. Using the Bornhuetter-Ferguson method, the reserve is

$$7{,}000{,}000\left(1-\frac{1}{1.433565}\right)=\boxed{2{,}117{,}068}$$

7.14. Link ratios are

$$f_1 = \frac{9{,}700 + 10{,}300 + 10{,}800}{4{,}850 + 5{,}150 + 5{,}400} = 2$$
$$f_2 = \frac{14{,}100 + 14{,}900}{9{,}700 + 10{,}300} = 1.45$$
$$f_3 = \frac{16{,}200}{14{,}100} = 1.148936$$

Accumulating products of these, development factors to ultimate are

$$f_{3,ult} = 1.148936$$
$$f_{2,ult} = 1.148936(1.45) = 1.665957$$
$$f_{1,ult} = 1.665957(2) = 3.331915$$

Expected losses are 0.85(20,000) = 17,000 for AY6, 0.91(21,000) = 19,110 for AY7, and 0.88(22,000) = 19,360 for AY8. The reserve is

$$17{,}000\left(1 - \frac{1}{1.148936}\right) + 19{,}110\left(1 - \frac{1}{1.665957}\right) + 19{,}360\left(1 - \frac{1}{3.331915}\right)$$
$$= 2{,}203 + 7{,}639 + 13{,}550 = \boxed{23{,}392} \qquad \textbf{(B)}$$

7.15. Development factors are

$$f_1 = \frac{360 + 400 + 500 + 600}{200 + 250 + 300 + 400} = 1.617391$$
$$f_2 = \frac{500 + 500 + 700}{360 + 400 + 500} = 1.349206$$
$$f_3 = \frac{700 + 680}{500 + 500} = 1.38$$
$$f_4 = \frac{800}{700} = 1.142857$$

Cumulative products of the factors are 1.142857, (1.142857)(1.38) = 1.577143, (1.577143)(1.349206) = 2.127891, and (2.127891)(1.617391) = 3.441633. Expected ultimate losses are 0.6(1500) = 900 for AY2, 0.6(1600) = 960 for AY3, 0.6(1700) = 1020 for AY4, and 0.6(1800) = 1080 for AY5. The reserve is

$$900\left(1 - \frac{1}{1.142857}\right) + 960\left(1 - \frac{1}{1.577143}\right) + 1020\left(1 - \frac{1}{2.127891}\right) + 1080\left(1 - \frac{1}{3.441633}\right)$$
$$= 112.5 + 351.3043 + 540.6522 + 766.1954 = \boxed{1770.65}$$

7.16. Average factors are:

Accident Year	Development Years			
	1	2	3	4
AY1	1.8	1.38889	1.4	1.142857
AY2	1.6	1.25	1.36	
AY3	1.66667	1.4		
AY4	1.5			
Average	1.641667	1.346296	1.38	1.142857
Cum Prod	3.485753	2.123302	1.577143	1.142857

The reserve is

$$900\left(1 - \frac{1}{1.142857}\right) + 960\left(1 - \frac{1}{1.577143}\right) + 1020\left(1 - \frac{1}{2.123302}\right) + 1080\left(1 - \frac{1}{3.485753}\right)$$

$$= 112.5 + 351.3043 + 539.6161 + 770.1674 = \boxed{\mathbf{1773.59}}$$

7.17. Losses are fully developed after 4 years, since the ratio of AY4 development year 5 to development year 4 is 1. So there is no reserve for AY4 and AY5.

The ratios of paid losses in each year to the previous year are

Accident Year	Ratios of Cumulative Paid Losses			
	1/0	2/1	3/2	4/3
AY4	3.7143	1.4038	1.2055	1.1136
AY5	2.9091	1.3750	1.1591	1.1275
AY6	3.0000	1.4267	1.1776	
AY7	3.1071	1.4828		
AY8	3.1600			

Three year averages are

1/0: $(3 + 3.1071 + 3.16)/3 = 3.0890$

2/1: $(1.375 + 1.4267 + 1.4828)/3 = 1.4281$

3/2: $(1.2055 + 1.1591 + 1.1776)/3 = 1.1807$

4/3: $(1.1136 + 1.1275)/2 = 1.1205$

Accumulating these, we get

$$\begin{aligned}
f_{3,ult} &= 1.1205 \\
f_{2,ult} &= 1.1205(1.1807) = 1.3230 \\
f_{1,ult} &= 1.3230(1.4281) = 1.8895 \\
f_{0,ult} &= 1.8895(3.0890) = 5.8367
\end{aligned}$$

Multiply the earned premiums by the loss ratio of 0.550 to obtain expected losses. The Bornhuetter-Ferguson reserve is

$$25{,}000(0.55)\left(1 - \frac{1}{1.1205}\right) + 26{,}000(0.55)\left(1 - \frac{1}{1.3230}\right) + 27{,}000(0.55)\left(1 - \frac{1}{1.8895}\right) + 28{,}000(0.55)\left(1 - \frac{1}{5.8367}\right) = \boxed{\mathbf{24{,}723}} \qquad \textbf{(D)}$$

7.18. Let f_i be the link ratio from year $i-1$ to year i. Expected losses for 2014 are $1500(0.6) = 900$, so

$$\begin{aligned}
900\left(1 - \frac{1}{f_5}\right) &= 100 \\
\frac{1}{f_5} &= \frac{8}{9} \\
f_5 &= \boxed{\mathbf{1.125}}
\end{aligned}$$

Continuing with the other link ratios,

$$\begin{aligned}
200 &= 900\left(1 - \frac{1}{f_4 f_5}\right) \\
f_4 f_5 &= \frac{9}{7} \\
400 &= 960\left(1 - \frac{1}{f_3 f_4 f_5}\right)
\end{aligned}$$

$$f_3f_4f_5 = \frac{1}{0.583333} = 1.714286$$
$$800 = 1200\left(1 - \frac{1}{f_2f_3f_4f_5}\right)$$
$$f_2f_3f_4f_5 = 3$$
$$1000 = 1200\left(1 - \frac{1}{f_1f_2f_3f_4f_5}\right)$$
$$f_1f_2f_3f_4f_5 = 6$$

Taking successive quotients,

$$f_4 = \frac{9/7}{9/8} = \boxed{\mathbf{1.142857}}$$
$$f_3 = \frac{1.714286}{9/7} = \boxed{\mathbf{1.333333}}$$
$$f_2 = \frac{3}{1.714286} = \boxed{\mathbf{1.75}}$$
$$f_1 = \frac{6}{3} = \boxed{\mathbf{2}}$$

7.19. The value you get depends on how much the intermediate numbers are rounded. However, you should get the same answer to the nearest 1000 no matter how you do it.

The cumulative factor from third development year to ultimate is 1.126(1.057)(1.050) = 1.250. The IBNR reserve for AY1 is

$$1{,}000{,}000(0.6)\left(1 - \frac{1}{1.25}\right) = 120{,}000$$

The cumulative factor from second development year to ultimate is 1.250(1.336) = 1.670. The IBNR reserve for AY2 is

$$1{,}000{,}000(1.2)(0.62)\left(1 - \frac{1}{1.670}\right) = 298{,}381$$

The cumulative factor from first development year to ultimate is 1.670(1.500) = 2.504. The IBNR reserve for AY3 is

$$1{,}000{,}000(1.2^2)(0.64)\left(1 - \frac{1}{2.504}\right) = 553{,}605$$

The IBNR reserve is 120,000 + 298,381 + 553,605 = $\boxed{\mathbf{971{,}867}}$. **(E)**

7.20. Loss reserves under the expected loss ratio method are total amount expected to be paid minus amount paid so far. The Bornhuetter-Ferguson reserve is the proportion of losses under the expected loss ratio method that hasn't been paid yet, or $1 - 1/f_{ult}$ times losses using the expected loss ratio. We compute those products:

$$(200 + 150)\left(1 - \frac{1}{1.05}\right) = 16.66667$$
$$(160 + 200)\left(1 - \frac{1}{(1.05)(1.1)}\right) = 48.31169$$
$$(150 + 250)\left(1 - \frac{1}{(1.05)(1.1)(1.2)}\right) = 111.3997$$

These sum up to $\boxed{\mathbf{176.38}}$.

7.21. The total reserve using Bornhuetter-Ferguson is calculated from the paid loss triangle. We calculated the link ratios from this triangle in exercise 7.12: 1.666667 for 1/0, 1.208333 for 2/1, and 1.142857 for 3/2. The cumulative products are 1.208333(1.142857) = 1.380952 for 3/1 and 1.380952(1.666667) = 2.301587 for 3/0. The reserve is

$$850\left(1-\frac{1}{1.142857}\right)+1000\left(1-\frac{1}{1.380952}\right)+1200\left(1-\frac{1}{2.301587}\right)=1060.73$$

The case reserve held as part of incurred losses is the excess of incurred over paid, or (775 − 750) + (900 − 800) + (800 − 500) = 425. The IBNR reserve is 1060.73 − 425 = **635.73**.

7.22. Let $v = \operatorname{Var}(X_2)$. Use the reciprocals of variances: $1/(3v)$ and $1/v$. The weights must add up to 1, so the first weight is

$$w=\frac{1/(3v)}{1/(3v)+1/v}=\frac{1}{1+3}=\boxed{\mathbf{\frac{1}{4}}}$$

Quiz Solutions

7-1. The policy is in force 4.5 months in 2021, so the earned premium in 2021 is 4.5/6 of the written premium, or (4.5/6)(600) = **450**.

7-2. It is now the end of development year 2, so there are 2 more years of development to go. Ultimate losses are estimated to be 600,000(1.180)(1.050) = 743,400. Dividing by earned premiums of 1,000,000, the expected loss ratio is **74.34%**.

Practice Exam 1

1. For a health insurance coverage, there are two types of policyholders.

75% of policyholders are healthy. Annual claim costs for those policyholders have mean 2,000 and variance 10,000,000.

25% of policyholders are in bad health. Annual claim costs for those policyholders have mean 10,000 and variance 50,000,000.

Calculate the variance of annual claim costs for a policyholder selected at random.

(A) 20,000,000 (B) 28,000,000 (C) 32,000,000 (D) 44,000,000 (E) 48,000,000

2. Bill Driver and Jane Motorist are involved in an automobile accident. Jane Motorist's car is totally destroyed. Its value before the accident was 8000, and the scrap metal after the accident is worth 500. Bill Driver is at fault.

Big Insurance Company insures Jane Motorist. Jane has liability insurance with a 100,000 limit and collision insurance with a 1000 deductible.

Standard Insurance Company insures Bill Driver. Bill has liability insurance with a 50,000 limit and collision insurance with a 500 deductible.

Jane files a claim with Big Insurance Company and receives 7000.

Calculate the net amount that Big Insurance Company receives (net of payment of subrogation proceeds to Jane) from subrogation.

(A) 6000 (B) 6500 (C) 7000 (D) 7500 (E) 8000

3. An excess of loss catastrophe reinsurance treaty covers the following layers, expressed in millions:

80% of 100 excess of 100
85% of 200 excess of 200
90% of 400 excess of 400

Calculate the reinsurance payment for a catastrophic loss of 650 million.

(A) 225 million (B) 475 million (C) 495 million (D) 553 million (E) 585 million

4. On a workers' compensation insurance, annual claim frequency for a small company with 30 employees follows a negative binomial distribution with mean 27 and variance 67.5. Claim sizes follow an inverse Pareto distribution with $\tau = 2$ and $\theta = 1000$.

The coverage pays the excess of each claim over 500.

Calculate the variance of the annual count of claims for which a nonzero insurance payment is made.

(A) 52 (B) 54 (C) 56 (D) 58 (E) 60

Exam questions continue on the next page . . .

5. A rate filing for six-month policies will be effective starting October 1, CY6 for 2 years.

Losses for this rate filing were incurred in AY1 and the amount paid through 12/31/CY4 is 3,500,000.

Trend is at annual effective rate of 6.5%.

Loss development factors are:

$$3/2\colon 1.50, \qquad 4/3\colon 1.05, \qquad \infty/4\colon 1.05$$

Calculate trended and developed losses for AY1.

(A) Less than 5,600,000
(B) At least 5,600,000, but less than 5,700,000
(C) At least 5,700,000, but less than 5,800,000
(D) At least 5,800,000, but less than 5,900,000
(E) At least 5,900,000

6. You are given

Accident Year	Cumulative Payments through Development Year 0	1	2	Earned Premium
AY1	25,000	41,000	48,000	120,000
AY2	30,000	45,000		140,000
AY3	33,000			150,000

The loss ratio is 60%.

Calculate the loss reserve using the loss ratio method.

(A) 100,000 (B) 105,000 (C) 110,000 (D) 115,000 (E) 120,000

7. You are given the following observations:

$$2, \qquad 10, \qquad 28, \qquad 64, \qquad 100$$

The observations are fitted to an inverse exponential distribution using maximum likelihood.

Determine the resulting estimate of the mode.

(A) 3.2 (B) 3.4 (C) 3.6 (D) 3.8 (E) 4.0

Exam questions continue on the next page . . .

8. The annual number of claims submitted by a policyholder has the following distribution:

Number of Claims	Probability
0	a
1	$0.2a$
2	$0.9 - 1.2a$
3	0.1

The distribution of a among policyholders is

Value of a	0.5	0.6
Probability	0.25	0.75

A policyholder submits 0 claims in a year.

Calculate the expected number of claims submitted by this policyholder in the next year.

(A) 0.83 (B) 0.85 (C) 0.87 (D) 0.89 (E) 0.91

9. At a company, the number of sick days taken by each employee in a year follows a Poisson distribution with mean λ. Over all employees, the distribution of λ has the following density function:

$$f(\lambda) = \frac{\lambda e^{-\lambda/3}}{9}$$

Calculate the probability that an employee selected at random will take more than 2 sick days in a year.

(A) 0.59 (B) 0.62 (C) 0.66 (D) 0.70 (E) 0.74

10. You are given:

Loss Size	Number of Losses	Average Loss Size
1–500	338	222
501–1,000	674	785
1,001–2,500	1,055	1,600
Over 2,500	933	4,800

The basic deductible is 500.

Calculate the indicated deductible relativity for a deductible of 2,500.

(A) 0.40 (B) 0.43 (C) 0.47 (D) 0.50 (E) 0.53

Exam questions continue on the next page . . .

11. Annual aggregate losses for each policyholder follow an inverse exponential distribution with parameter θ. The parameter θ varies by policyholder. The probability density function of θ is

$$\pi(\theta) = \frac{2}{\theta^3} \qquad \theta \geq 1$$

For one policyholder, you have three years of experience. The policyholder incurred the following aggregate losses in those three years:

$$10, \qquad 20, \qquad 40$$

Calculate the posterior expected value of θ.

(A) 5.7 (B) 6.7 (C) 7.7 (D) 8.7 (E) 9.7

12. For loss size X, you are given:

x	$\mathbf{E}[X \wedge x]$
1000	400
2000	700
3000	900
4000	1000
5000	1100
∞	2500

An insurance coverage has an ordinary deductible of 2000.

Calculate the loss elimination ratio after 100% inflation if the deductible is not changed.

(A) 0.08 (B) 0.14 (C) 0.16 (D) 0.20 (E) 0.40

13. You are given the following sample:

$$0.150, \qquad 0.200, \qquad 0.400, \qquad 0.550$$

You are to fit this sample to a distribution with probability density function

$$f(x) = (a+1)x^a \qquad 0 \leq x \leq 1$$

using maximum likelihood.

Calculate the asymptotic variance of the estimator for a, evaluated at the estimated value of a.

(A) 0.16 (B) 0.20 (C) 0.24 (D) 0.28 (E) 0.32

Exam questions continue on the next page . . .

14. For each exposure in a group, the hypothetical mean of aggregate losses is Θ and the process variance is $e^{0.3\Theta}$. Θ varies by group. Its probability distribution is exponential with mean 3.

For three years experience from a group, you have the following data:

Year	Exposures	Aggregate losses
1	20	70
2	25	90
3	30	110

There will be 35 exposures in the group next year.

Calculate the Bühlmann-Straub credibility premium for the group.

(A) 115.0 (B) 120.3 (C) 125.3 (D) 125.7 (E) 126.0

15. In a study on loss sizes on automobile liability coverage, you are given:

(i) 5 observations $x_1, \ldots, x_5$ from a plan with no deductible and a policy limit of 10,000.
(ii) 5 observations at the limit from a plan with no deductible and a policy limit of 10,000.
(iii) 5 observations $y_1, \ldots, y_5$ from a plan with a deductible of 10,000 and no policy limit.

Which of the following is the likelihood function for this set of observations?

(A) $\prod_{i=1}^{5} f(x_i) \prod_{i-1}^{5} f(y_i)$

(B) $\bigl(1 - F(10{,}000)\bigr)^5 \prod_{i=1}^{5} f(x_i) \prod_{i-1}^{5} f(y_i)$

(C) $\dfrac{\bigl(1 - F(10{,}000)\bigr)^5 \prod_{i=1}^{5} f(x_i) \prod_{i-1}^{5} f(y_i)}{\bigl(F(10{,}000)\bigr)^5}$

(D) $\dfrac{\prod_{i=1}^{5} f(x_i) \prod_{i=1}^{5} f(y_i)}{\bigl(1 - F(10{,}000)\bigr)^5}$

(E) $\dfrac{\bigl(F(10{,}000)\bigr)^5 \prod_{i=1}^{5} f(x_i) \prod_{i-1}^{5} f(y_i)}{\bigl(1 - F(10{,}000)\bigr)^5}$

16. You are given the following data on 940 losses from homeowner's insurance:

Range of losses	Number of losses
(0,1000)	380
[1000,2000)	220
[2000,3000)	162
[3000,4000)	84
4000 and over	94

The data are fitted to an exponential distribution using maximum likelihood. The fit is tested using the chi-square test.

At what significance level is the fit accepted?

(A) Reject at 0.5% significance.
(B) Reject at 1% significance but not at 0.5% significance.
(C) Reject at 2.5% significance but not at 1% significance.
(D) Reject at 5% significance but not at 2.5% significance.
(E) Accept at 5% significance.

Exam questions continue on the next page . . .

17. Loss sizes follow a lognormal distribution. You have estimated the parameters of the distribution as $\mu = 3$ and $\sigma = 0.5$. The information matrix is

$$\begin{pmatrix} 200 & 0 \\ 0 & 400 \end{pmatrix}$$

You estimate the mean of the lognormal distribution using the estimated parameters.

Approximate the asymptotic variance of the estimate of the mean using the delta method.

(A) 0.8 (B) 1.5 (C) 2.2 (D) 2.9 (E) 3.6

18. A study on claim sizes produced the following results:

Claim size	Number	Deductible	Limit
500	4	None	10,000
1000	4	500	None
2000	3	500	None
5000	2	None	10,000
At limit	5	None	10,000

A single-parameter Pareto with $\theta = 400$ is fitted to the data using maximum likelihood.

Determine the estimate of α.

(A) 0.43 (B) 0.44 (C) 0.45 (D) 0.61 (E) 0.62

19. You are given:

Accident Year	Incurred Losses through Development Year					Earned Premium
	0	1	2	3	4	
AY1	7,800	8,900	9,500	11,000	11,000	16,000
AY2	9,100	9,800	10,500	10,800		20,000
AY3	8,600	9,500	10,100			23,000
AY4	9,500	10,000				24,000
AY5	10,700					25,000

The expected loss ratio is 0.7.

Losses mature at the end of 3 years.

Calculate the IBNR reserve using the Bornhuetter-Ferguson method with volume-weighted average loss development factors.

(A) 7,100 (B) 7,200 (C) 7,300 (D) 7,400 (E) 7,500

20. On an automobile liability coverage, annual claim counts follow a negative binomial distribution with mean 0.2 and variance 0.3. Claim sizes follow a two-parameter Pareto distribution with $\alpha = 3$ and $\theta = 10$. Claim counts and claim sizes are independent.

Calculate the variance of annual aggregate claim costs.

(A) 22.5 (B) 25.0 (C) 27.5 (D) 32.5 (E) 35.0

21. For an insurance, Class A is the base class. Premium rates are 600 for Class A and 900 for Class B.

For Class B, the experience loss ratio is 0.8. Based on this experience, the indicated relativity of Class B is 1.8.

Calculate the experience loss ratio for Class A.

(A) 0.60 (B) 0.63 (C) 0.65 (D) 0.67 (E) 0.70

Exam questions continue on the next page . . .

22. You have the following experience for 2 group policyholders:

Group		Year 1	Year 2	Year 3	Total
A	Number of members	15	20	25	60
	Aggregate losses	150	100	170	420
B	Number of members		5	15	20
	Aggregate losses		50	200	250

Using non-parametric empirical Bayes credibility methods, determine the credibility given to the experience of Group A.

(A) 0.88 (B) 0.89 (C) 0.90 (D) 0.91 (E) 0.92

23. Annual claim counts for each insured follow a two-parameter Pareto distribution with parameters $\alpha = 4$ and θ. The parameter θ varies by insured, and its distribution over all insureds has density function

$$f(\theta) = \frac{(1000/\theta)^3 e^{-1000/\theta}}{2\theta}$$

Calculate the Bühlmann credibility assigned to 4 years of experience from a single insured.

(A) 0.38 (B) 0.44 (C) 0.50 (D) 0.56 (E) 0.62

24. Health insurance is sold to 500 individuals. The following table summarizes the number of claims submitted by these individuals in a year.

Number of Claims	Number of Policyholders
0	365
1	105
2	25
3	5
4 or more	0

Credibility is calculated using empirical Bayes semiparametric methods. Annual claim counts for each individual are assumed to follow a Poisson distribution.

Determine the estimate of the number of claims submitted in the next year by someone who submitted 2 claims in the current year.

(A) 0.53 (B) 0.58 (C) 0.63 (D) 0.68 (E) 0.73

25. You are using the following loss development factors for cumulative payments:

1/0: 2.61 2/1: 1.25 3/2: 1.11 4/3: 1.05

Cumulative payments for AY3 through the end of development year 1 are 8,000.

Assume all payments in each year are made in the middle of the year.

Calculate the loss reserve on 12/31/CY4 for AY3 losses discounted at $i = 0.05$ using the chain ladder method.

(A) 3382 (B) 3408 (C) 3439 (D) 3465 (E) 3489

Exam questions continue on the next page . . .

26. A sample of 5 losses is:

200 200 500 2500 5000

This sample is fitted to a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 1600$.

Calculate the Kolmogorov-Smirnov statistic for this fit.

(A) 0.21 (B) 0.25 (C) 0.28 (D) 0.31 (E) 0.33

27. A city purchases a snow cleaning contract for a winter season. The contract pays the costs of cleaning the snow, subject to an aggregate deductible of 30 for the season. You are given:

(i) The number of snowstorms is binomially distributed with parameters $m = 10$, $q = 0.3$.
(ii) The cost of cleaning for each snowstorm has the following distribution:

Probability	Amount
0.4	10
0.3	20
0.3	35

Determine the expected aggregate reimbursement the city receives during a season.

(A) 27.7 (B) 32.5 (C) 32.8 (D) 33.0 (E) 34.1

28. A minor medical insurance coverage has the following provisions:

(i) Annual losses in excess of 10,000 are not covered by the insurance.
(ii) The policyholder pays the first 1,000 of annual losses.
(iii) The insurance company pays 60% of the excess of annual losses over 1,000, after taking into account the limitation mentioned in (i).

Annual losses follow a two-parameter Pareto distribution with $\alpha = 4$ and $\theta = 8000$.

Calculate expected annual payments for one policyholder under this insurance.

(A) 983 (B) 1004 (C) 1025 (D) 1046 (E) 1067

29. Let X be the random variable with distribution function

$$F_X(x) = 1 - 0.6e^{-x/10} - 0.4e^{-x/20}$$

Calculate $\mathrm{TVaR}_{0.95}(X)$.

(A) 59 (B) 60 (C) 61 (D) 62 (E) 63

30. For a discrete probability distribution in the $(a, b, 0)$ class, you are given

(i) $p_2 = 0.0768$
(ii) $p_3 = p_4 = 0.08192$

Determine p_0.

(A) 0.02 (B) 0.03 (C) 0.04 (D) 0.05 (E) 0.06

Exam questions continue on the next page . . .

31. For a rate filing, there are three territories. You are given:

	Percentage of Business	Existing Differential	Indicated Differential
Territory I	60%	1.00	1.00
Territory II	30%	1.50	1.80
Territory III	10%	2.00	2.20

The current premium for Territory I is 500. An overall rate increase of 5% is indicated.

Calculate the indicated premium for Territory I.

(A) 483 (B) 488 (C) 493 (D) 498 (E) 503

32. Losses on an insurance coverage follow a distribution with density function

$$f(x) = \frac{3}{100^3}(100 - x)^2 \qquad 0 \le x \le 100$$

Losses are subject to an ordinary deductible of 15.

Calculate the loss elimination ratio.

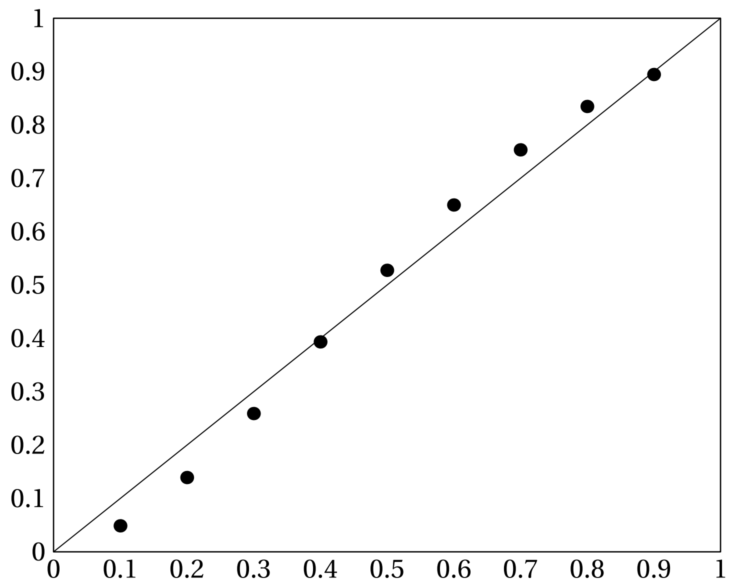
(A) 0.46 (B) 0.48 (C) 0.50 (D) 0.52 (E) 0.54

Exam questions continue on the next page . . .

33. You are given the following data:

1, 3, 6, 10, 15, 21, 28, 36, 45

The data are fitted to a distribution and the following p–p plot is drawn:



Which of the following is the fitted distribution?

(A) Uniform on $[1, 45]$
(B) Exponential with $\theta = 20$
(C) Normal with $\mu = 20$, $\sigma^2 = 100$
(D) Lognormal with $\mu = 2.4$, $\sigma = 1.4$
(E) Two-parameter Pareto with $\alpha = 2$, $\theta = 40$

34. A reinsurance company offers a stop-loss reinsurance contract that pays the excess of annual aggregate losses over 3.

You are given:

(i) Loss counts follow a binomial distribution with $m = 3$ and $q = 0.2$.
(ii) Loss sizes have the following distribution:

Size	Probability
1	0.6
2	0.2
3	0.1
4	0.1

Calculate the expected annual payment under the stop-loss reinsurance contract.

(A) 0.09 (B) 0.10 (C) 0.11 (D) 0.12 (E) 0.13

Exam questions continue on the next page . . .

35. Annual claim frequency follows a Poisson distribution. Loss sizes follow a Weibull distribution with $\tau = 0.5$.

Full credibility for aggregate loss experience is granted if the probability that aggregate losses differ from expected by less than 6% is 95%.

Determine the number of expected claims needed for full credibility.

(A) 6403 (B) 6755 (C) 7102 (D) 7470 (E) 7808

Solutions to the above questions begin on page 1159.

Appendix A. Solutions to the Practice Exams

Answer Key for Practice Exam 1

1	C	11	B	21	D	31	A
2	B	12	C	22	C	32	B
3	B	13	A	23	C	33	B
4	C	14	D	24	A	34	E
5	D	15	A	25	D	35	A
6	E	16	C	26	B		
7	D	17	D	27	E		
8	A	18	C	28	A		
9	E	19	D	29	E		
10	A	20	A	30	C		

Practice Exam 1

1. [**Section 4.1**] You may either use the conditional variance formula (equation (4.2)), or compute first and second moments.

With the conditional variance formula, let I be the indicator of whether the policyholder is healthy or in bad health. Let X be annual claim counts then

$$\begin{aligned}\operatorname{Var}(X) &= \operatorname{Var}_I\big(\mathbf{E}_X[X \mid I]\big) + \mathbf{E}_I\big[\operatorname{Var}_X(X \mid I)\big]\\ &= \operatorname{Var}_I(2000, 10{,}000) + \mathbf{E}_I[10{,}000{,}000, 50{,}000{,}000]\end{aligned}$$

where $\operatorname{Var}_I(2{,}000, 10{,}000)$ means the variance of a random variable that is 2,000 with probability 0.75 and 10,000 with probability 0.25. By the Bernoulli shortcut, the variance of such a random variable is

$$(0.75)(0.25)(10{,}000 - 2{,}000)^2 = 12{,}000{,}000$$

$\mathbf{E}_I[10{,}000{,}000, 50{,}000{,}000]$ means the expected value of a random variable that is 10,000,000 with probability 0.75 and 50,000,000 with probability 0.25. The expected value of such a random variable is

$$0.75(10{,}000{,}000) + 0.25(50{,}000{,}000) = 20{,}000{,}000$$

Adding up the variance of the mean and the mean of the variances, we get $\operatorname{Var}(X) = 12{,}000{,}000 + 20{,}000{,}000 =$ **32,000,000**. **(C)**

With first and second moments, the overall first moment of annual claim counts is

$$0.75(2{,}000) + 0.25(10{,}000) = 4{,}000$$

The overall second moment of annual claim counts is the weighted average of the individual second moments, and the second moment for each type of driver is the variance plus the mean squared.

$$0.75(10{,}000{,}000 + 2{,}000^2) + 0.25(50{,}000{,}000 + 10{,}000^2) = 48{,}000{,}000$$

The overall variance of claim counts is $48{,}000{,}000 - 4{,}000^2 =$ **32,000,000**. **(C)**

2. [**Lesson 5**] Big Insurance Company pays Jane 7000 and receives the scrap metal, for a net loss of 6500. That is the amount that it gets upon subrogration. The remaining 1000 of the subrogation is paid to Jane. **(B)**

3. **[Lesson 15]** The layers are 100–200, 200–400, and 400–800. The amount of the catastrophic loss in each of those layers is 100, 200, and 250 respectively. The reinsurance pays 0.8(100) + 0.85(200) + 0.9(250) = **475** million. **(B)**

4. **[Lesson 21]** The number of employees plays no role in the solution.
The negative binomial parameters are

$$\begin{aligned} r\beta &= 27 \\ r\beta(1+\beta) &= 67.5 \\ \beta &= 1.5 \\ r &= \frac{27}{1.5} = 18 \end{aligned}$$

We need to modify the β parameter by multiplying by the probability that a claim X is greater than 500.

$$\Pr(X > 500) = 1 - \left(\frac{x}{x+\theta}\right)^{\tau} = 1 - \left(\frac{500}{1500}\right)^2 = \frac{8}{9}$$

So the modified β is 1.5(8/9) = 4/3. The variance of the modified negative binomial distribution is

$$18(4/3)(7/3) = \boxed{\mathbf{56}}. \qquad \textbf{(C)}$$

5. **[Section 10.1]** The policies will be sold over the period 10/1/CY6 through 9/30/CY8. The average sale date is 10/1/CY7. The policies are six-month policies. The average accident date is 3 months after policy issue, or 1/1/CY8. The average accident in AY1 occurred on 6/30/CY1. So the average accident on the policies for which rates are filed occurs 6.5 years after the average accident in our experience. The trend factor is $1.065^{6.5}$.

Accidents occur in AY1 and data has been developed through CY4, which is development year 3. (Remember, CY1 is development year 0, so CY2 is development year 1, CY3 is development year 2, and CY4 is development year 3.) Future development is from year 3 to ultimate, so we multiply the development factors for 4/3 and ∞/4 to obtain the future development for our losses: (1.05)(1.05) = 1.1025. That is the development factor.

Trended and developed losses are 3,500,000(1.1025)($1.065^{6.5}$) = **5,810,575**. **(D)**

6. **[Section 7.2]** Projected losses based on the loss ratio are

$$0.6(120{,}000 + 140{,}000 + 150{,}000) = 246{,}000$$

Paid to date is 48,000 + 45,000 + 33,000 = 126,000. The reserve is 246,000 − 126,000 = **120,000**. **(E)**

7. **[Lesson 29]** The likelihood function, ignoring the multiplicative constant $1/\prod x_i^2$, is

$$L(\theta) = \theta^5 e^{-\theta \sum 1/x_i}$$

and logging and differentiating,

$$\begin{aligned} l(\theta) &= 5\ln\theta - \theta \sum \frac{1}{x_i} \\ \frac{\mathrm{d}l}{\mathrm{d}\theta} &= \frac{5}{\theta} - \sum \frac{1}{x_i} = 0 \\ \hat{\theta} &= \frac{5}{\sum 1/x_i} = 7.5604 \end{aligned}$$

The mode is $\hat{\theta}/2$ = **3.7802**. **(D)**

8. **[Lesson 41]** The expected number of claims given a is $0.2a + 2(0.9 - 1.2a) + 3(0.1) = 2.1 - 2.2a$. This is 1 for $a = 0.5$ and 0.78 for $a = 0.6$. So the Bayesian estimate of expected value given 0 claims in a year is

$$\frac{(0.25)(0.5)(1) + (0.75)(0.6)(0.78)}{(0.25)(0.5) + (0.75)(0.6)} = \boxed{\mathbf{0.8278}} \qquad \textbf{(A)}$$

9. **[Lesson 20]** The distribution of λ is gamma with $\alpha = 2$ and $\theta = 3$, so the mixed distribution of number of sick days is negative binomial with $r = 2$ and $\beta = 3$. Then the probability of more than 2 sick days is

$$1 - p_0 - p_1 - p_2 = 1 - \left(\frac{1}{4}\right)^2 - 2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right) - \left(\frac{2 \cdot 3}{2!}\right)\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2 = \boxed{\mathbf{0.738281}} \qquad \textbf{(E)}$$

10. **[Section 14.2]** We can ignore losses below 500. The total losses paid in the interval 501–1000 with a deductible of 500 are 674(285) = 192,090. Similar calculations for the other intervals yield:

Loss size	Number of losses	Average loss size	Total payments 500 deductible	Total payments 2500 deductible
501–1,000	674	785	192,090	
1,001–2,500	1,055	1,600	1,160,500	
Over 2,500	933	4,800	4,011,900	2,145,900
Total			5,364,490	2,145,900

The indicated deductible relativity for a deductible of 2,500 is 2,145,900/5,364,490 = $\boxed{\mathbf{0.4000}}$. **(A)**

11. **[Lesson 42]** The likelihood function, ignoring the constant denominators x_i^2, is

$$\theta^3 e^{-\theta/10-\theta/20-\theta/40} = \theta^3 e^{-0.175\theta}$$

The product of the likelihood and the prior (ignoring the constant factor 2 of the prior) is $e^{-0.175\theta}$. The integral of this is

$$\int_1^\infty e^{-0.175\theta}\,\mathrm{d}\theta = \frac{e^{-0.175}}{0.175}$$

So the posterior function is $0.175e^{-0.175\theta}e^{0.175}$. To calculate the expected value, we could integrate this times θ from 1 to ∞, but instead we'll be clever and identify this function as a shifted exponential with parameter 1/0.175 and shift 1. In other words,

$$\pi(\theta \mid x_i) = 0.175e^{-0.175(\theta-1)}$$

The mean of a shifted exponential is the parameter plus the shift, or $1/0.175 + 1 = \boxed{\mathbf{6.7143}}$. **(B)**

12. **[Lesson 13]** $\mathbf{E}[X] = \mathbf{E}[X \wedge \infty] = 2500$, and after inflation this doubles to 5000. After inflation, X becomes $2X$ and $\mathbf{E}[2X \wedge 2000] = 2\,\mathbf{E}[X \wedge 1000] = 2(400) = 800$. So the revised LER is $\frac{800}{5000} = \boxed{\mathbf{0.16}}$. **(C)**

13. **[Section 31.1]**

$$L(a) = (a+1)^4 \prod x_i^a$$
$$l(a) = 4\ln(a+1) + a\ln\prod x_i$$
$$\frac{\mathrm{d}l}{\mathrm{d}a} = \frac{4}{a+1} + \ln\prod x_i = 0$$

$$a = -\frac{4}{\ln\prod x_i} - 1 = -0.203296$$
$$\frac{\mathrm{d}^2 l}{\mathrm{d}a^2} = -\frac{4}{(a+1)^2}$$

The asymptotic variance is $(a+1)^2/4$, estimated as $(-0.203296+1)^2/4 = \boxed{0.1587}$. **(A)**

14. **[Lesson 50]** EHM $= \mathbf{E}[\Theta] = 3$. VHM $= \mathrm{Var}(\Theta) = 9$. The expected value of the process variance is

$$\mathrm{EPV} = \int e^{0.3\theta} f_\Theta(\theta)\mathrm{d}\theta = \int_0^\infty e^{0.3\theta}\left(\frac{e^{-\theta/3}}{3}\right)\mathrm{d}\theta = \frac{\int_0^\infty e^{-\theta/30}\mathrm{d}\theta}{3} = 10$$

So $k = \frac{10}{9}$, $Z = \frac{75}{75+\frac{10}{9}} = \frac{675}{685}$. There are 75 exposures and 270 aggregate losses, so $\bar{x} = \frac{270}{75} = \frac{54}{15}$ and the credibility premium per exposure is

$$\frac{675(54/15) + 10(3)}{685} = 3.5912$$

The credibility premium for the group is $35(3.5912) = \boxed{125.69}$. **(D)**

15. **[Lesson 29]** The likelihoods of the 5 observations of (i) are $f(x_i)$. The likelihoods of the 5 observations of (ii) are $1 - F(10{,}000)$. The likelihoods of the 5 observations of (iii) are $f(y_i)/\bigl(1 - F(10{,}000)\bigr)$. Multiplying everything together, we get **(A)**.

16. **[Lessons 30 and 35]** The fitted θ of the exponential is

$$\begin{aligned} L(\theta) &= (e^{-1000/\theta})^{220+162(2)+84(3)+94(4)}(1-e^{-1000/\theta})^{380+220+162+84} \\ &= u^{1172}(1-u)^{846} \end{aligned}$$

where $u = e^{-1000/\theta}$. As indicated in Table 30.2, with this likelihood function form, the maximum occurs at $u = 1172/(1172+846) = 0.580773$. This is enough to calculate the expected number of observations in each interval:

Interval	Expected
(0,1000)	$940(1-u) = 394.07$
[1000,2000)	$940u(1-u) = 228.87$
[2000,3000)	$940u^2(1-u) = 132.92$
[3000,4000)	$940u^3(1-u) = 77.20$
4000 and over	$940u^4 = 106.94$

The chi-square statistic, using the alternative formula (equation (35.2)), is

$$Q = \frac{380^2}{394.07} + \frac{220^2}{228.86} + \frac{162^2}{132.92} + \frac{84^2}{77.20} + \frac{94^2}{106.94} - 940 = 9.374$$

Since one parameter was fitted, the number of degrees of freedom is 3. At 3 degrees of freedom, 9.374 is between the 97.5$^{\text{th}}$ percentile and the 99$^{\text{th}}$ percentile, so the answer is **(C)**.

17. **[Section 31.2]** The inverse of a diagonal matrix is the matrix of reciprocals of elements of the diagonal. So the variance of $\hat{\mu}$ is $1/200$ and the variance of $\hat{\sigma}$ is $1/400$.

For a lognormal X, the mean is $g(\mu,\sigma) = \mathbf{E}[X] = e^{\mu+\sigma^2/2}$. Then

$$\frac{\partial g}{\partial \mu} = e^{\mu+\sigma^2/2} = e^{3+0.5^2/2} = 22.7599$$
$$\frac{\partial g}{\partial \sigma} = \sigma e^{\mu+\sigma^2/2} = 0.5e^{3+0.5^2/2} = 11.3799$$

By formula (31.7) for the delta method, taking into account that the covariance is 0, the asymptotic variance of the estimate of the mean is

$$\frac{1}{200}(22.7599^2) + \frac{1}{400}(11.3799^2) = \boxed{\mathbf{2.9138}} \qquad \textbf{(D)}$$

18. [**Subsection 30.4.2**] Using the formula for the MLE of a single-parameter Pareto in Table 30.1,

$$\begin{aligned}-K &= 4(\ln 500 - \ln 400) + 4(\ln 1000 - \ln 500) + 3(\ln 2{,}000 - \ln 500)\\ &\quad + 2(\ln 5{,}000 - \ln 400) + 5(\ln 10{,}000 - \ln 400)\\ &= 28.9699\end{aligned}$$

There are 13 uncensored observations, so $\hat{\alpha} = 13/28.9699 = \boxed{\mathbf{0.44874}}$. **(C)**

19. [**Section 7.2**] The link ratio from development year 0 to development year 1 is

$$\frac{8{,}900 + 9{,}800 + 9{,}500 + 10{,}000}{7{,}800 + 9{,}100 + 8{,}600 + 9{,}500} = 1.09143$$

The link ratio from development year 1 to development year 2 is

$$\frac{9{,}500 + 10{,}500 + 10{,}100}{8{,}900 + 9{,}800 + 9{,}500} = 1.06738$$

The link ratio from development year 2 to development year 3 is

$$\frac{11{,}000 + 10{,}800}{9{,}500 + 10{,}500} = 1.09$$

Cumulative factors from age i to ultimate, f_i, are

$$\begin{aligned}f_3 &= 1.09\\ f_2 &= 1.09(1.06738) = 1.16344\\ f_1 &= 1.16344(1.09143) = 1.26981\end{aligned}$$

The reserve is

$$0.7\left(23{,}100\left(1 - \frac{1}{1.09}\right) + 24{,}000\left(1 - \frac{1}{1.16344}\right) + 25{,}000\left(1 - \frac{1}{1.26981}\right)\right) = \boxed{\mathbf{7{,}408}} \qquad \textbf{(D)}$$

20. [**Lesson 22**] Use the compound variance formula, equation (22.4). For a two-parameter Pareto random variable X:

$$\begin{aligned}\mathbf{E}[X] &= \frac{\theta}{\alpha - 1} = \frac{10}{3-1} = 5\\ \mathbf{E}[X^2] &= \frac{2\theta^2}{(\alpha-1)(\alpha-2)} = \frac{2\cdot 10^2}{(3-1)(3-2)} = 100\\ \mathrm{Var}(X) &= 100 - 5^2 = 75\end{aligned}$$

The variance of annual aggregate claim costs is

$$\mathrm{Var}(S) = 0.2(75) + 0.3(5^2) = \boxed{\mathbf{22.5}} \qquad \textbf{(A)}$$

21. **[Section 10.1]** Existing relativity is 900/600 = 1.5.

$$1.5\left(\frac{\text{LR}_B}{\text{LR}_A}\right) = 1.8$$
$$1.5(0.8) = 1.8\text{LR}_A$$
$$\text{LR}_A = \boxed{\mathbf{\frac{2}{3}}} \qquad \textbf{(D)}$$

22. **[Section 53.2]**

$$\bar{x}_1 = \frac{420}{60} = 7$$
$$\bar{x}_2 = \frac{250}{20} = 12.5$$
$$\bar{x} = \frac{420+250}{80} = 8.375$$
$$\widehat{\text{EPV}} = \frac{15(10-7)^2 + 20(5-7)^2 + 25(6.8-7)^2 + 5(10-12.5)^2 + 15(200/15-12.5)^2}{(3-1)+(2-1)} = 85.8889$$
$$\widehat{\text{VHM}} = \frac{60(7-8.375)^2 + 20(12.5-8.375)^2 - 85.8889}{80-(60^2+20^2)/80} = 12.2620$$
$$\hat{K} = \frac{85.8889}{12.2620} = 7.0045$$
$$Z_A = \frac{60}{60+7.0045} = \boxed{\mathbf{0.8955}} \qquad \textbf{(C)}$$

23. **[Lesson 49]** The hypothetical mean is $\theta/3$ and the process variance is

$$\frac{\theta^2}{3} - \left(\frac{\theta}{3}\right)^2 = \frac{2\theta^2}{9}$$

The prior distribution is an inverse gamma with $\alpha = 3$, $\theta = 1000$. (The θ parameter of the inverse gamma is not the θ of this question.) Hence

$$\mathbf{E}[\theta^2] = \frac{1000^2}{(2)(1)} = 500{,}000$$
$$\text{Var}(\theta) = 500{,}000 - \left(\frac{1000}{2}\right)^2 = 250{,}000$$
$$\text{EPV} = \mathbf{E}[2\theta^2/9] = 1{,}000{,}000/9$$
$$\text{VHM} = \text{Var}(\theta/3) = 250{,}000/9$$
$$K = \frac{\text{EPV}}{\text{VHM}} = 4$$
$$Z = \frac{4}{4+4} = \boxed{\mathbf{0.5}} \qquad \textbf{(C)}$$

24. **[Lesson 54]**

$$\bar{x} = \frac{105 + 25(2) + 5(3)}{500} = 0.34$$

$$\begin{aligned}\sum x_i^2 &= 105 + 25(4) + 5(9) = 250\\ s^2 &= \left(\frac{500}{499}\right)\left(\frac{250}{500} - 0.34^2\right) = 0.385170\\ \hat{\mu} &= \hat{v} = 0.34\\ \hat{a} &= 0.385170 - 0.34 = 0.045170\\ \hat{Z} &= \frac{0.045170}{0.385170} = 0.117274\end{aligned}$$

The credibility estimate for an individual with 2 claims is $(0.117274)(2) + (1 - 0.117274)(0.34) =$ **0.5347**. **(A)**

25. [**Section 8.3**] We must calculate the year-by-year projected payments. Projected cumulative losses are 8,000(1.25) = 10,000 in CY5, 10,000(1.11) = 11,100 in CY6, and 11,100(1.05) = 11,655 in CY7. Incremental losses are 10,000 − 8,000 = 2,000 in CY5, 11,100 − 10,000 = 1,100 in CY6, and 11,655 − 11,100 = 555 in CY7. Discounting these payments, the reserve is

$$\frac{2{,}000}{1.05^{0.5}} + \frac{1{,}100}{1.05^{1.5}} + \frac{555}{1.05^{2.5}} = \boxed{3{,}465} \qquad \textbf{(D)}$$

26. [**Lesson 34**] The Pareto cumulative distribution function is $1 - \big(\theta/(\theta + x)\big)^\alpha = 1 - \big(1600/(1600 + x)\big)^2$. So our table is

y_i	$F_5(y_i^-)$	$F_5(y_i)$	$F^*(y_i)$	Max dif
200	0	0.4	0.2099	0.2099
500	0.4	0.6	0.4195	0.1805
2500	0.6	0.8	0.8477	0.2477
5000	0.8	1.0	0.9412	0.1412

The largest difference is **0.2477**. **(B)**

27. [**Lessons 25 and 26**] First, the expectations:

$$\begin{aligned}\mathbf{E}[N] &= (0.3)(10) = 3\\ \mathbf{E}[X] &= 0.4(10) + 0.3(20) + 0.3(35) = 20.5\\ \mathbf{E}[S] &= (3)(20.5) = 61.5\end{aligned}$$

Now we calculate the aggregate probabilities of 0, 10, 20.

$$\begin{aligned}g_0 &= 0.7^{10} = 0.028248\\ g_{10} &= \left(\binom{10}{1}(0.7^9)(0.3)\right)(0.4) = 10(0.7^9)(0.3)(0.4) = 0.048424\\ g_{20} &= \left(\binom{10}{2}(0.7^8)(0.3^2)\right)(0.4^2) + \left(\binom{10}{1}(0.7^9)(0.3)\right)(0.3)\\ &= 45(0.7^8)(0.3^2)(0.4^2) + 10(0.7^9)(0.3)(0.3) = 0.073674\\ S_S(0) &= 1 - 0.028248 = 0.971752\\ S_S(10) &= 0.971752 - 0.048424 = 0.923328\\ S_S(20) &= 0.951576 - 0.073674 = 0.849654\\ \mathbf{E}[S \wedge 30] &= 10(0.971752 + 0.923328 + 0.849654) = 27.44734\end{aligned}$$

Alternatively, you can compute $\mathbf{E}[S \wedge 30]$ as

$$\mathbf{E}[S \wedge 30] = 10g_{10} + 20g_{20} + 30(1 - g_0 - g_{10} - g_{20})$$

$$= 10(0.048424) + 20(0.073674) + 30(1 - 0.028248 - 0.048424 - 0.073674) = 27.44734$$

The answer is then

$$\mathbf{E}[(S-30)_+] = 61.5 - 27.44734 = \boxed{\mathbf{34.0526}} \qquad \textbf{(E)}$$

28. **[Lesson 17]** Annual payments are $0.6(X \wedge 10{,}000 - X \wedge 1{,}000)$, and we use the tables to evaluate the expected value of this.

$$\mathbf{E}[X \wedge 1{,}000] = \frac{\theta}{\alpha-1}\left(1 - \left(\frac{\theta}{\theta+1000}\right)^{\alpha-1}\right) = \frac{8000}{3}\left(1 - \left(\frac{8000}{9000}\right)^3\right) = 793.78$$

$$\mathbf{E}[X \wedge 10{,}000] = \frac{8000}{3}\left(1 - \left(\frac{8000}{18000}\right)^3\right) = 2432.56$$

The answer is $0.6(2432.56 - 793.78) = \boxed{\mathbf{983.26}}$. **(A)**

29. **[Section 16.3]** Let q be the 95^{th} percentile. Let's calculate q.

$$\begin{aligned}
1 - 0.6e^{-q/10} - 0.4e^{-q/20} &= 0.95 \\
0.6x^2 + 0.4x = 0.05 &\quad \text{where } x = e^{-q/20} \\
x = \frac{-0.4 + \sqrt{0.4^2 + 4(0.6)(0.05)}}{1.2} &= 0.107625 \\
q = -20 \ln x &= 44.5820
\end{aligned}$$

$\mathrm{TVaR}_{0.95}(X) = q + e(q)$. We'll calculate $e(q)$ using

$$e(q) = \frac{\int_q^\infty S(x)\,\mathrm{d}x}{0.05}$$

The numerator is

$$\int_q^\infty (0.6e^{-x/10} + 0.4e^{-x/20})\mathrm{d}x = -10(0.6)e^{-x/10} - 20(0.4)e^{-x/20}\Big|_q^\infty = 6e^{-q/10} + 8e^{-q/20} = 0.930501$$

So

$$\mathrm{TVaR}_{0.95}(X) = 44.5820 + \frac{0.930501}{0.05} = \boxed{\mathbf{63.19}} \qquad \textbf{(E)}$$

30. **[Lesson 19]** We use the $(a, b, 0)$ recursion for probabilities.

$$\begin{aligned}
\frac{p_3}{p_2} &= \frac{0.08192}{0.0768} = 1.066667 = a + \frac{b}{3} \\
\frac{p_4}{p_3} &= 1 = a + \frac{b}{4} \\
\frac{b}{12} &= 0.066667 \\
b &= 0.8 \\
a &= 1.066667 - \frac{0.8}{3} = 0.8
\end{aligned}$$

Then $a + b = 1.6$ and $a + b/2 = 1.2$, so $p_0 = p_2/(1.6 \cdot 1.2) = \boxed{\mathbf{0.04}}$. **(C)** It is not necessary to back out the distribution that has these probabilities, but the underlying distribution is negative binomial with $r = 2$ and $\beta = 4$.

31. [**Section 10.1**] The ratio of the old differentials to the new ones is

$$\frac{0.6(1)+0.3(1.5)+0.1(2)}{0.6(1)+0.3(1.8)+0.1(2.2)}=0.919118$$

so that is the balance-back factor. The indicated premium for Territory I is

$$500(1.05)(0.919118)=\boxed{\mathbf{482.54}}\qquad\textbf{(A)}$$

32. [**Lesson 13**] Losses follow a beta distribution with $\theta=100$, $a=1$, $b=3$. The mean is $\theta a/(a+b)=100/4=25$. The expression for $\mathbf{E}[X\wedge 15]$ in the tables isn't so easy to use because it has an incomplete beta, so let's calculate $\mathbf{E}[X\wedge 15]$ directly by integrating the survival function, which is

$$S(t)=\int_t^{100}f(x)\,\mathrm{d}x=\int_t^{100}\frac{3}{100^3}(100-x)^2\mathrm{d}x=-\frac{(100-x)^3}{100^3}\bigg|_t^{100}=\left(\frac{100-t}{100}\right)^3$$

Integrating this from 0 to 15,

$$\int_0^{15}\left(\frac{100-t}{100}\right)^3=-\frac{(100-t)^4}{4(100^3)}\bigg|_0^{15}=25-\frac{85^4}{4(100^3)}=11.94984$$

The loss elimination ratio is $11.94984/25=\boxed{\mathbf{0.4780}}$. **(B)**

33. [**Section 33.2**] All of these distributions have low fitted values for $F(1)$, so it's hard to eliminate them using 1; however, (A) can be eliminated since for a uniform distribution on $[1,45]$, $F^*(1)=0$ and the point with x-coordinate 0.1 does not have $y=0$. To distinguish the distributions, let's instead look at $F(45)$, whose fitted value should be about 0.9 according to the plot since the point with x-coordinate 0.9 has y-coordinate approximately 0.9. We have

$$\begin{array}{ll}\text{(B)} & F^*(45)=1-e^{-45/20}=0.8946\\ \text{(C)} & F^*(45)=\Phi(2.5)=0.9938\\ \text{(D)} & F^*(45)=\Phi\big((\ln 45-2.4)/1.4\big)=\Phi(1.00)=0.8413\\ \text{(E)} & F^*(45)=1-\left(\frac{40}{85}\right)^2=0.7785\end{array}$$

Only the exponential fit has $F^*(45)\approx 0.9$, so the answer is **(B)**.

34. [**Lesson 26**] The loss count distribution through p_2 is

$$\begin{aligned}p_0&=0.8^3=0.512\\ p_1&=3(0.8^2)(0.2)=0.384\\ p_2&=3(0.8)(0.2^2)=0.096\end{aligned}$$

Aggregate probabilities are

$$\begin{aligned}g_0&=0.512\\ g_1&=(0.384)(0.6)=0.2304\\ g_2&=(0.384)(0.2)+(0.096)(0.6^2)=0.11136\end{aligned}$$

The expected value of each loss X is

$$\mathbf{E}[X]=0.6+0.2(2)+0.1(3+4)=1.7$$

The expected value of annual aggregate losses S is $\mathbf{E}[S] = 1.7mq = 1.7(3)(0.2) = 1.02$.

The limited expected value of annual aggregate losses at 3 is

$$\mathbf{E}[S \wedge 3] = 0.2304 + 0.11136(2) + (1 - 0.512 - 0.2304 - 0.11136)(3) = 0.89184$$

The expected annual payment under the reinsurance contract is $1.02 - 0.89184 = \boxed{0.12816}$. **(E)**

35. **[Lesson 38]** 1 plus the coefficient of variation squared can be expressed as

$$1 + \mathrm{CV}_X^2 = 1 + \frac{\mathrm{Var}(X)}{\mathbf{E}[X]^2} = \frac{\mathbf{E}[X]^2 + \mathrm{Var}(X)}{\mathbf{E}[X]^2} = \frac{\mathbf{E}[X^2]}{\mathbf{E}[X]^2}$$

For a Weibull distribution with $\tau = 0.5$,

$$\begin{aligned}
\mathbf{E}[X] &= \theta\Gamma(1 + 1/\tau) = \theta\Gamma(3) = 2\theta \\
\mathbf{E}[X^2] &= -\theta^2\Gamma(1 + 2/\tau) = \theta^2\Gamma(5) = 24\theta^2 \\
\frac{\mathbf{E}[X^2]}{\mathbf{E}[X]^2} &= \frac{24}{2^2} = 6
\end{aligned}$$

The standard for full credibility is

$$\lambda^F = \left(\frac{1.96}{0.06}\right)^2 (6) = \boxed{6403} \qquad \textbf{(A)}$$