A-5-1. In the system of Figure 5–52, $x(t)$ is the input displacement and $\theta(t)$ is the output angular displacement. Assume that the masses involved are negligibly small and that all motions are restricted to be small; therefore, the system can be considered linear. The initial conditions for x and θ are zeros, or $x(0-) = 0$ and $\theta(0-) = 0$. Show that this system is a differentiating element. Then obtain the response $\theta(t)$ when $x(t)$ is a unit-step input.

Solution. The equation for the system is

$$b(\dot{x} - L\dot{\theta}) = kL\theta$$

or

$$L\dot{\theta} + \frac{k}{b}L\theta = \dot{x}$$

The Laplace transform of this last equation, using zero initial conditions, gives

$$\left(Ls + \frac{k}{b}L\right)\Theta(s) = sX(s)$$

And so

$$\frac{\Theta(s)}{X(s)} = \frac{1}{L}\frac{s}{s + (k/b)}$$

Thus the system is a differentiating system.

For the unit-step input $X(s) = 1/s$, the output $\Theta(s)$ becomes

$$\Theta(s) = \frac{1}{L}\frac{1}{s + (k/b)}$$

The inverse Laplace transform of $\Theta(s)$ gives
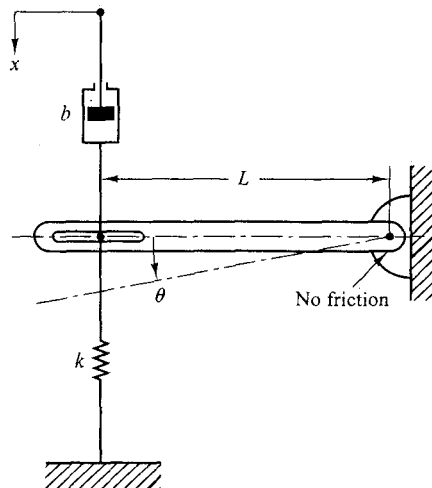
$$\theta(t) = \frac{1}{L}e^{-(k/b)t}$$
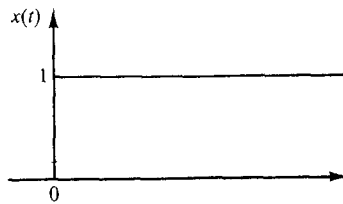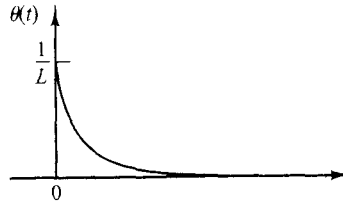

Figure 5–52
Mechanical system.

Figure 5–53
Unit-step input and
the response of the
mechanical system
shown in Figure
5–52.

Note that if the value of k/b is large the response $\theta(t)$ approaches a pulse signal as shown in Figure 5–53.

A–5–2. Consider the mechanical system shown in Figure 5–54. Suppose that the system is at rest initially $[x(0) = 0, \dot{x}(0) = 0]$, and at $t = 0$ it is set into motion by a unit-impulse force. Obtain a mathematical model for the system. Then find the motion of the system.

Solution. The system is excited by a unit-impulse input. Hence

$$m\ddot{x} + kx = \delta(t)$$

This is a mathematical model for the system.

Taking the Laplace transform of both sides of this last equation gives

$$m\left[s^2 X(s) - sx(0) - \dot{x}(0)\right] + kX(s) = 1$$

By substituting the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ into this last equation and solving for $X(s)$, we obtain

$$X(s) = \frac{1}{ms^2 + k}$$

The inverse Laplace transform of $X(s)$ becomes

$$x(t) = \frac{1}{\sqrt{mk}} \sin\sqrt{\frac{k}{m}}\, t$$

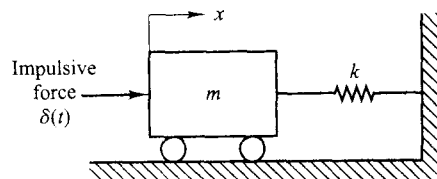The oscillation is a simple harmonic motion. The amplitude of the oscillation is $1/\sqrt{mk}$.

Figure 5–54
Mechanical system.

Example Problems and Solutions

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A–5–3. Gear trains are often used in servo systems to reduce speed, to magnify torque, or to obtain the most efficient power transfer by matching the driving member to the given load.

Consider the gear train system shown in Figure 5–55. In this system, a load is driven by a motor through the gear train. Assuming that the stiffness of the shafts of the gear train is infinite (there is neither backlash nor elastic deformation) and that the number of teeth on each gear is proportional to the radius of the gear, obtain the equivalent moment of inertia and equivalent viscous-friction coefficient referred to the motor shaft and referred to the load shaft.

In Figure 5–55 the numbers of teeth on gears 1, 2, 3, and 4 are N_1, N_2, N_3, and N_4, respectively. The angular displacements of shafts, 1, 2, and 3 are θ_1, θ_2, and θ_3, respectively. Thus, $\theta_2/\theta_1 = N_1/N_2$ and $\theta_3/\theta_2 = N_3/N_4$. The moment of inertia and viscous-fraction coefficient of each gear train component are denoted by $J_1, b_1; J_2, b_2$; and J_3, b_3; respectively. (J_3 and b_3 include the moment of inertia and friction of the load.)

Solution. For this gear train system, we can obtain the following equations: For shaft 1,

$$J_1 \ddot{\theta}_1 + b_1 \dot{\theta}_1 + T_1 = T_m \tag{5-63}$$

where T_m is the torque developed by the motor and T_1 is the load torque on gear 1 due to the rest of the gear train. For shaft 2,

$$J_2 \ddot{\theta}_2 + b_2 \dot{\theta}_2 + T_3 = T_2 \tag{5-64}$$

where T_2 is the torque transmitted to gear 2 and T_3 is the load torque on gear 3 due to the rest of the gear train. Since the work done by gear 1 is equal to that of gear 2,

$$T_1 \theta_1 = T_2 \theta_2 \qquad \text{or} \qquad T_2 = T_1 \frac{N_2}{N_1}$$

If $N_1/N_2 < 1$, the gear ratio reduces the speed as well as magnifies the torque. For shaft 3,

$$J_3 \ddot{\theta}_3 + b_3 \dot{\theta}_3 + T_L = T_4 \tag{5-65}$$

where T_L is the load torque and T_4 is the torque transmitted to gear 4. T_3 and T_4 are related by

$$T_4 = T_3 \frac{N_4}{N_3}$$

and θ_3 and θ_1 are related by

$$\theta_3 = \theta_2 \frac{N_3}{N_4} = \theta_1 \frac{N_1}{N_2} \frac{N_3}{N_4}$$
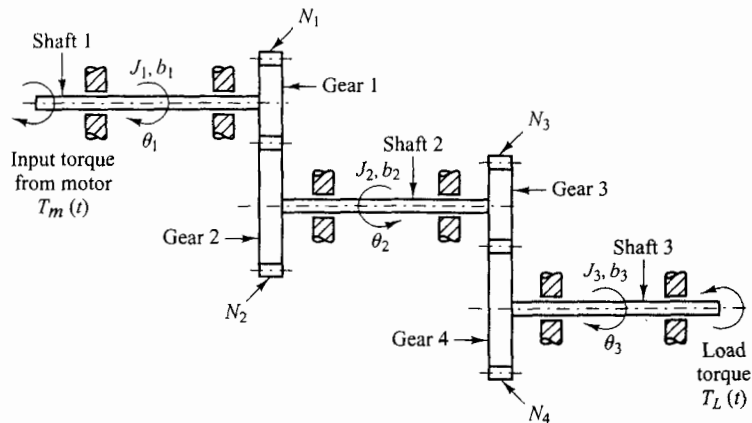

Figure 5–55
Gear train system.

Eliminating T_1, T_2, T_3, and T_4 from Equations (5–63), (5–64), and (5–65) yields

$$J_1\ddot{\theta}_1 + b_1\dot{\theta}_1 + \frac{N_1}{N_2}(J_2\ddot{\theta}_2 + b_2\dot{\theta}_2) + \frac{N_1 N_3}{N_2 N_4}(J_3\ddot{\theta}_3 + b_3\dot{\theta}_3 + T_L) = T_m$$

Eliminating θ_2 and θ_3 from this last equation and writing the resulting equation in terms of θ_1 and its time derivatives, we obtain

$$\left[J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2 + \left(\frac{N_1}{N_2}\right)^2\left(\frac{N_3}{N_4}\right)^2 J_3 \right]\ddot{\theta}_1$$

$$+ \left[b_1 + \left(\frac{N_1}{N_2}\right)^2 b_2 + \left(\frac{N_1}{N_2}\right)^2\left(\frac{N_3}{N_4}\right)^2 b_3 \right]\dot{\theta}_1 + \left(\frac{N_1}{N_2}\right)\left(\frac{N_3}{N_4}\right)T_L = T_m \qquad (5\text{–}66)$$

Thus, the equivalent moment of inertia and viscous-friction coefficient of the gear train referred to shaft 1 are given, respectively, by

$$J_{1eq} = J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2 + \left(\frac{N_1}{N_2}\right)^2\left(\frac{N_3}{N_4}\right)^2 J_3$$

$$b_{1eq} = b_1 + \left(\frac{N_1}{N_2}\right)^2 b_2 + \left(\frac{N_1}{N_2}\right)^2\left(\frac{N_3}{N_4}\right)^2 b_3$$

Similarly, the equivalent moment of inertia and viscous-friction coefficient of the gear train referred to the load shaft (shaft 3) are given, respectively, by

$$J_{3eq} = J_3 + \left(\frac{N_4}{N_3}\right)^2 J_2 + \left(\frac{N_2}{N_1}\right)^2\left(\frac{N_4}{N_3}\right)^2 J_1$$

$$b_{3eq} = b_3 + \left(\frac{N_4}{N_3}\right)^2 b_2 + \left(\frac{N_2}{N_1}\right)^2\left(\frac{N_4}{N_3}\right)^2 b_1$$

The relationship between J_{1eq} and J_{3eq} is thus

$$J_{1eq} = \left(\frac{N_1}{N_2}\right)^2\left(\frac{N_3}{N_4}\right)^2 J_{3eq}$$

and that between b_{1eq} and b_{3eq} is

$$b_{1eq} = \left(\frac{N_1}{N_2}\right)^2\left(\frac{N_3}{N_4}\right)^2 b_{3eq}$$

The effect of J_2 and J_3 on an equivalent moment of inertia is determined by the gear ratios N_1/N_2 and N_3/N_4. For speed-reducing gear trains, the ratios, N_1/N_2 and N_3/N_4 are usually less than unity. If $N_1/N_2 \ll 1$ and $N_3/N_4 \ll 1$, then the effect of J_2 and J_3 on the equivalent moment of inertia J_{1eq} is negligible. Similar comments apply to the equivalent viscous-friction coefficient b_{1eq} of the gear train. In terms of the equivalent moment of inertia J_{1eq} and equivalent viscous-friction coefficient b_{1eq}, Equation (5–66) can be simplified to give

$$J_{1eq}\ddot{\theta}_1 + b_{1eq}\dot{\theta}_1 + nT_L = T_m$$

where

$$n = \frac{N_1 N_3}{N_2 N_4}$$

A–5–4. When the system shown in Figure 5–56(a) is subjected to a unit-step input, the system output responds as shown in Figure 5–56(b). Determine the values of K and T from the response curve.

Solution. The maximum overshoot of 25.4% corresponds to $\zeta = 0.4$. From the response curve we have

$$t_p = 3$$

Consequently,

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1 - \zeta^2}} = \frac{\pi}{\omega_n\sqrt{1 - 0.4^2}} = 3$$

It follows that

$$\omega_n = 1.14$$

From the block diagram we have
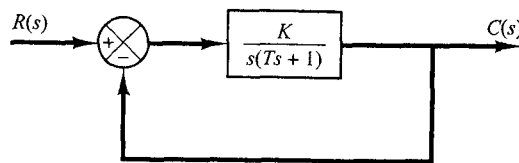
$$\frac{C(s)}{R(s)} = \frac{K}{Ts^2 + s + K}$$

from which

$$\omega_n = \sqrt{\frac{K}{T}}, \qquad 2\zeta\omega_n = \frac{1}{T}$$
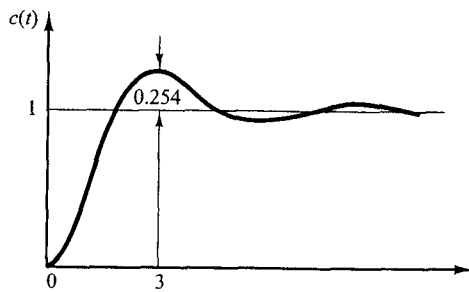
Therefore, the values of T and K are determined as

$$T = \frac{1}{2\zeta\omega_n} = \frac{1}{2 \times 0.4 \times 1.14} = 1.09$$

$$K = \omega_n^2 T = 1.14^2 \times 1.09 = 1.42$$

(a)

(b)

Figure 5–56
(a) Closed-loop system; (b) unit-step response curve.

A-5-5. Determine the values, of K and k of the closed-loop system shown in Figure 5–57 so that the maximum overshoot in unit-step response is 25% and the peak time is 2 sec. Assume that $J = 1$ kg-m^2.

Solution. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Kks + K}$$

By substituting $J = 1$ kg-m^2 into this last equation, we have

$$\frac{C(s)}{R(s)} = \frac{K}{s^2 + Kks + K}$$

Note that in this problem

$$\omega_n = \sqrt{K}, \qquad 2\zeta\omega_n = Kk$$

The maximum overshoot M_p is

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

which is specified as 25%. Hence

$$e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.25$$

from which

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 1.386$$

or

$$\zeta = 0.404$$

The peak time t_p is specified as 2 sec. And so

$$t_p = \frac{\pi}{\omega_d} = 2$$

or

$$\omega_d = 1.57$$

Then the undamped natural frequency ω_n is

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{1.57}{\sqrt{1-0.404^2}} = 1.72$$

Therefore, we obtain

$$K = \omega_n^2 = 1.72^2 = 2.95 \text{ N-m}$$

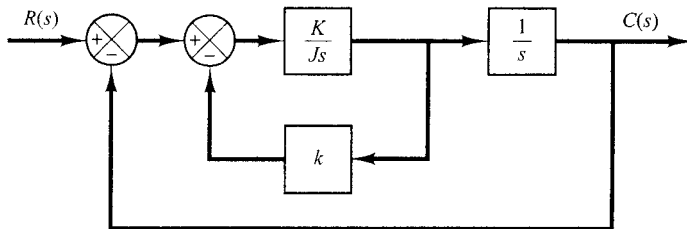$$k = \frac{2\zeta\omega_n}{K} = \frac{2 \times 0.404 \times 1.72}{2.95} = 0.471 \text{ sec}$$

Figure 5–57
Closed-loop system.

A-5-6. Figures 5–58(a) shows a mechanical vibratory system. When 2 lb of force (step input) is applied to the system, the mass oscillates, as shown in Figure 5–58(b). Determine m, b, and k of the system from this response curve. The displacement x is measured from the equilibrium position.

Solution. The transfer function of this system is

$$\frac{X(s)}{P(s)} = \frac{1}{ms^2 + bs + k}$$

Since

$$P(s) = \frac{2}{s}$$

we obtain

$$X(s) = \frac{2}{s(ms^2 + bs + k)}$$

It follows that the steady-state value of x is

$$x(\infty) = \lim_{s \to 0} sX(s) = \frac{2}{k} = 0.1 \text{ ft}$$

Hence

$$k = 20 \text{ lb}_f/\text{ft}$$

Note that $M_p = 9.5\%$ corresponds to $\zeta = 0.6$. The peak time t_p is given by

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = \frac{\pi}{0.8\omega_n}$$

The experimental curve shows that $t_p = 2$ sec. Therefore,

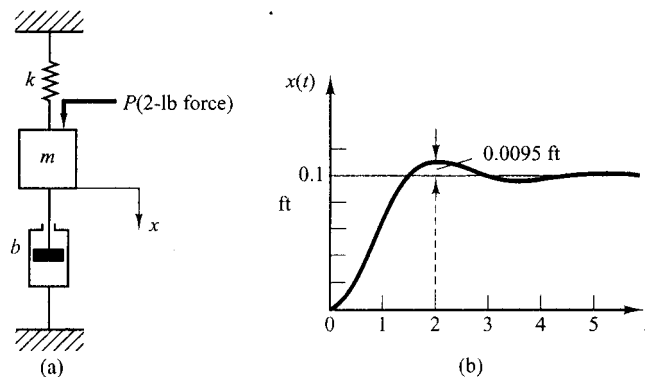$$\omega_n = \frac{3.14}{2 \times 0.8} = 1.96 \text{ rad/sec}$$

Figure 5–58
(a) Mechanical vibratory system;
(b) step-response curve.

Chapter 5 / Transient and Steady-State Response Analyses

Since $\omega^2{}_n = k/m = 20/m$, we obtain

$$m = \frac{20}{\omega_n^2} = \frac{20}{1.96^2} = 5.2 \text{ slugs} = 166 \text{ lb}$$

(Note that 1 slug = 1 lb_f-sec^2/ft.) Then b is determined from

$$2\zeta\omega_n = \frac{b}{m}$$

or

$$b = 2\zeta\omega_n m = 2 \times 0.6 \times 1.96 \times 5.2 = 12.2 \text{ lb}_f/\text{ft}/\text{sec}$$

A-5-7. Consider the unit-step response of the second-order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The amplitude of the exponentially·damped sinusoid changes as a geometric series. At time $t = t_p = \pi/\omega_d$, the amplitude is equal to $e^{-(\sigma/\omega_d)\pi}$. After one oscillation, or at $t = t_p + 2\pi/\omega_d = 3\pi/\omega_d$, the amplitude is equal to $e^{-(\sigma/\omega_d)3\pi}$; after another cycle of oscillation, the amplitude is $e^{-(\sigma/\omega_d)5\pi}$. The logarithm of the ratio of successive amplitudes is called the *logarithmic decrement*. Determine the logarithmic decrement for this second-order system. Describe a method for experimental determination of the damping ratio from the rate of decay of the oscillation.

Solution. Let us define the amplitude of the output oscillation at $t = t_i$ to be x_i, where $t_i = t_p + (i - 1)T$ (T = period of oscillation). The amplitude ratio per one period of damped oscillation is

$$\frac{x_1}{x_2} = \frac{e^{-(\sigma/\omega_d)\pi}}{e^{-(\sigma/\omega_d)3\pi}} = e^{2(\sigma/\omega_d)\pi} = e^{2\zeta\pi/\sqrt{1-\zeta^2}}$$

Thus, the logarithmic decrement δ is

$$\delta = \ln\frac{x_1}{x_2} = \frac{2\zeta\pi}{\sqrt{1 - \zeta^2}}$$

It is a function only of the damping ratio ζ. Thus, the damping ratio ζ can be determined by use of the logarithmic. decrement.

In the experimental determination of the damping ratio ζ from the rate of decay of the oscillation, we measure the amplitude x_1 at $t = t_p$ and amplitude x_n at $t = t_p + (n - 1)T$. Note that it is necessary to choose n large enough so that the ratio o x_1/x_n is not near unity. Then

$$\frac{x_1}{x_n} = e^{(n-1)2\zeta\pi/\sqrt{1-\zeta^2}}$$

or

$$\ln\frac{x_1}{x_n} = (n - 1)\frac{2\zeta\pi}{\sqrt{1 - \zeta^2}}$$

Hence

$$\zeta = \frac{\dfrac{1}{n - 1}\left(\ln\dfrac{x_1}{x_n}\right)}{\sqrt{4\pi^2 + \left[\dfrac{1}{n - 1}\left(\ln\dfrac{x_1}{x_n}\right)\right]^2}}$$
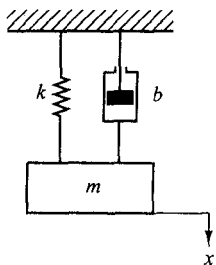
A-5-8.

Figure 5–59 · Spring-mass-damper system.

In the system shown in Figure 5–59, the numerical values of m, b, and k are given as $m = 1$ kg, $b = 2$ N-sec/m, and $k = 100$ N/m. The mass is displaced 0.05 m and released without initial velocity. Find the frequency observed in the vibration. In addition, find the amplitude four cycles later. The displacement x is measured from the equilibrium position.

Solution. The equation of motion for the system is

$$m\ddot{x} + b\dot{x} + kx = 0$$

Substituting the numerical values for m, b, and k into this equation gives

$$\ddot{x} + 2\dot{x} + 100x = 0$$

where the initial conditions are $x(0) = 0.05$ and $\dot{x}(0) = 0$. From this last equation the undamped natural frequency ω_n and the damping ratio ζ are found to be

$$\omega_n = 10, \qquad \zeta = 0.1$$

The frequency actually observed in the vibration is the damped natural frequency ω_d.

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 10\sqrt{1 - 0.01} = 9.95 \text{ rad/sec}$$

In the present analysis, $\dot{x}(0)$ is given as zero. Thus, solution $x(t)$ can be written as

$$x(t) = x(0)e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}}\sin\omega_d t\right)$$

It follows that at $t = nT$, where $T = 2\pi/\omega_d$,

$$x(nT) = x(0)e^{-\zeta\omega_n nT}$$

Consequently, the amplitude four cycles later becomes

$$x(4T) = x(0)e^{-\zeta\omega_n 4T} = x(0)e^{-(0.1)(10)(4)(0.6315)}$$

$$= 0.05e^{-2.526} = 0.05 \times 0.07998 = 0.004 \text{ m}$$

A-5-9. Obtain both analytically and computationally the unit-step response of the following higher-order system:

$$\frac{C(s)}{R(s)} = \frac{3s^3 + 25s^2 + 72s + 80}{s^4 + 8s^3 + 40s^2 + 96s + 80}$$

[Obtain the partial-fraction expansion of $C(s)$ with MATLAB when $R(s)$ is a unit-step function.]

Solution. MATLAB Program 5–19 yields the unit-step response curve shown in Figure 5–60. It also yields the partial-fraction expansion of $C(s)$ as follows:

$$C(s) = \frac{3s^3 + 25s^2 + 72s + 80}{s^4 + 8s^3 + 40s^2 + 96s + 80}\frac{1}{s}$$

$$= \frac{-0.2813 - j0.1719}{s + 2 - j4} + \frac{-0.2813 + j0.1719}{s + 2 + j4}$$

$$+ \frac{-0.4375}{s + 2} + \frac{-0.375}{(s + 2)^2} + \frac{1}{s}$$

$$= \frac{-0.5626(s + 2)}{(s + 2)^2 + 4^2} + \frac{(0.3438) \times 4}{(s + 2)^2 + 4^2}$$

$$- \frac{0.4375}{s + 2} - \frac{0.375}{(s + 2)^2} + \frac{1}{s}$$

MATLAB Program 5–19

% ------- Unit-Step Response of C(s)/R(s) and Partial-Fraction Expansion of C(s) -------

num = [0 3 25 72 80];
den = [1 8 40 96 80];
step(num,den);
v = [0 3 0 1.2]; axis(v), grid

% To obtain the partial-fraction expansion of C(s), enter commands

%   num1 = [0 0 3 25 72 80];
%   den1 = [1 8 40 96 80 0];
%   [r,p,k] = residue(num1,den1)

num1 = [0 0 3 25 72 80];
den1 = [1 8 40 96 80 0];
[r,p,k] = residue(num1,den1)

r =

   -0.2813- 0.1719i
   -0.2813+ 0.1719i
   -0.4375
   -0.3750
    1.0000

p =

   -2.0000+ 4.0000i
   -2.0000- 4.0000i
   -2.0000
   -2.0000
    0

k =

    []

Hence, the time response $c(t)$ can be given by

$$c(t) = -0.5626e^{-2t}\cos 4t + 0.3438e^{-2t}\sin 4t$$

$$- 0.4375e^{-2t} - 0.375te^{-2t} + 1$$

The fact that the response curve is an exponential curve superimposed by damped sinusoidal curves can be seen from Figure 5–60.
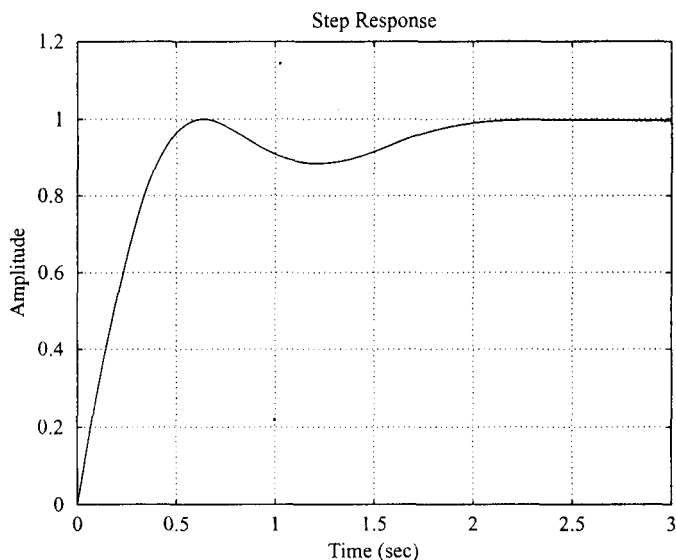
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Figure 5–60
Unit-step response
curve.

A–5–10. Obtain both analytical and computational solutions of the unit-step response of a unity-feedback system whose open-loop transfer function is

$$G(s) = \frac{5(s + 20)}{s(s + 4.59)(s^2 + 3.41s + 16.35)}$$

Solution. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{5(s + 20)}{s(s + 4.59)(s^2 + 3.41s + 16.35) + 5(s + 20)}$$

$$= \frac{5s + 100}{s^4 + 8s^3 + 32s^2 + 80s + 100}$$

$$= \frac{5(s + 20)}{(s^2 + 2s + 10)(s^2 + 6s + 10)}$$

The unit-step response of this system is then

$$C(s) = \frac{5(s + 20)}{s(s^2 + 2s + 10)(s^2 + 6s + 10)}$$

$$= \frac{1}{s} + \frac{\dfrac{3}{8}(s + 1) - \dfrac{17}{8}}{(s + 1)^2 + 3^2} + \frac{-\dfrac{11}{8}(s + 3) - \dfrac{13}{8}}{(s + 3)^2 + 1^2}$$

The time response $c(t)$ can be found by taking the inverse Laplace transform of $C(s)$ as follows:

$$c(t) = 1 + \frac{3}{8} e^{-t} \cos 3t - \frac{17}{24} e^{-t} \sin 3t - \frac{11}{8} e^{-3t} \cos t - \frac{13}{8} e^{-3t} \sin t, \quad \text{for } t \geq 0$$

A MATLAB program to obtain the unit-step response of this system is shown in MATLAB Program 5–20. The resulting unit-step response curve is shown in Figure 5–61.

```
MATLAB Program 5-20

% -------------- Unit-step-response --------------

num = [0  0  0  5  100];
den = [1  8  32  80  100];
step(num,den)
grid
title('Unit-Step Response of C(s)/R(s) = (5s + 100)/(s^4 + 8s^3 + 32s^2 + 80s + 100)')
```
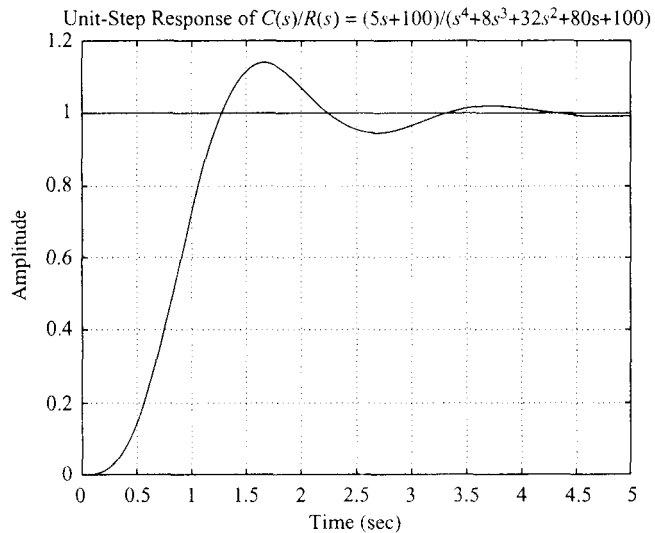

Unit-Step Response of $C(s)/R(s) = (5s+100)/(s^4+8s^3+32s^2+80s+100)$

Figure 5-61
Unit-step response
curve.

A-5-11. When the closed-loop system involves a numerator dynamics, the unit-step response curve may exhibit a large overshoot. Obtain the unit-step response of the following system with MATLAB:

$$\frac{C(s)}{R(s)} = \frac{10s + 4}{s^2 + 4s + 4}$$

Obtain also the unit-ramp response with MATLAB.

Solution. MATLAB Program 5-21 produces the unit-step response as well as the unit-ramp response of the system. The unit-step response curve and unit-ramp response curve, together with the unit-ramp input, are shown in Figures 5-62(a) and (b), respectively.

Notice that the unit-step response curve exhibits over 215% of overshoot. The unit-ramp response curve leads the input curve. These phenomena occurred because of the presence of a large derivative term in the numerator.

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```
MATLAB Program 5–21

num = [0  10  4];
den = [1  4  4];
t = 0:0.02:10;
y = step(num,den,t);
plot(t,y)
grid
title('Unit-Step Response')
xlabel('t (sec)')
ylabel('Output')

num1 = [0  0  10  4];
den1 = [1  4  4  0];
y1 = step(num1,den1,t);
plot(t,t,'--',t,y1)
v = [0  10  0  10]; axis(v);
grid
title('Unit-Ramp Response')
xlabel('t (sec)')
ylabel('Unit-Ramp Input and Output')
text(6.1,5.0,'Unit-Ramp Input')
text(3.5,7.1,'Output')
```
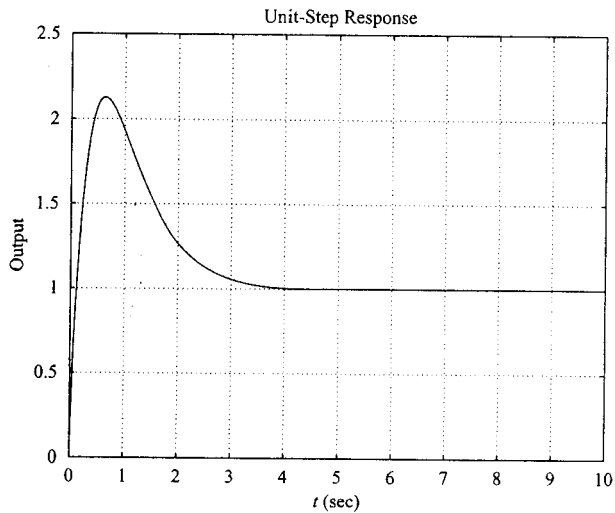

(a)
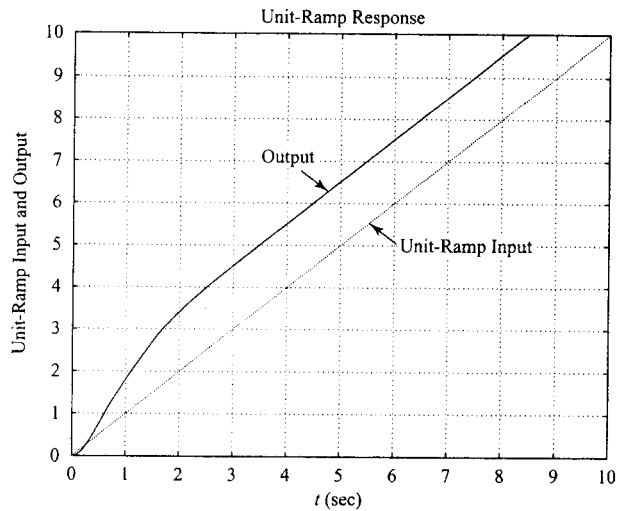
(b)

Figure 5–62

(a) Unit-step response curve; (b) unit-ramp response curve plotted with unit-ramp input.

A–5–12. Consider a higher-order system defined by

$$\frac{C(s)}{R(s)} = \frac{6.3223s^2 + 18s + 12.811}{s^4 + 6s^3 + 11.3223s^2 + 18s + 12.811}$$

Using MATLAB, plot the unit-step response curve of this system. Using MATLAB, obtain the rise time, peak time, maximum overshoot, and settling time.

Solution. MATLAB Program 5–22 plots the unit-step response curve as well as to give the rise time, peak time, maximum overshoot, and settling time. The unit-step response curve is shown in Figure 5–63.

MATLAB Program 5–22

```
% ------- This program is to plot the unit-step response curve, as well as to
% find the rise time, peak time, maximum overshoot, and settling time.
% In this program the rise time is calculated as the time required for the
% response to rise from 10% to 90% of its final value. -------

num = [0  0  6.3223  18  12.811];
den = [1  6  11.3223  18  12.811];
t = 0:0.02:20;
[y,x,t] = step(num,den,t);
plot(t,y)
grid
title('Unit-Step Response')
xlabel('t (sec)')
ylabel('Output y(t)')

r1 = 1; while y(r1) < 0.1, r1. = r1+1; end;
r2 = 1; while y(r2) < 0.9, r2 = r2+1; end;
rise_time = (r2-r1)*0. 02

rise_time =

   0.5800

[ymax,tp] = max(y);
peak_time = (tp-1)*0.02

peak_time =

   1.6600

max_overshoot = ymax-1

max_overshoot =

   0.6182

s = 1001; while y(s) > 0.98 & y(s) < 1.02; s = s-1; end;
settling_time = (s-1)*0.02

settling_time =

   10.0200
```
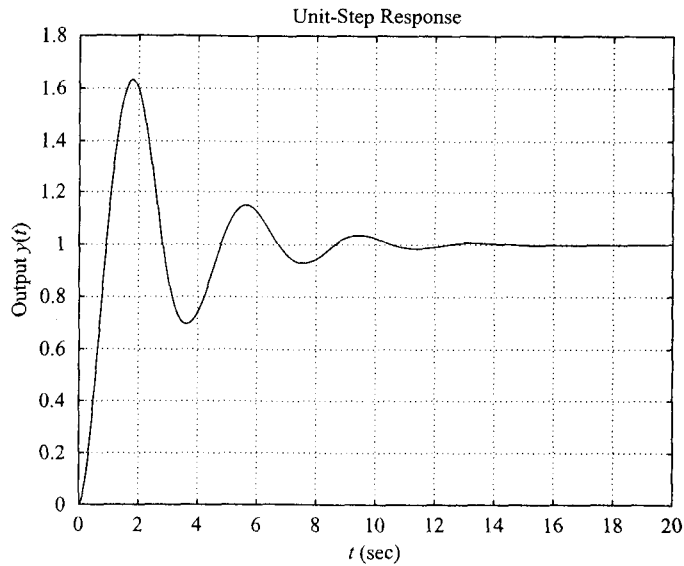

Figure 5–63
Unit-step response
curve.

A–5–13. Consider the closed-loop system defined by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Using a "for loop," write a MATLAB program to obtain unit-step response of this system for the following four cases:

Case 1: $\zeta = 0.3,$ $\omega_n = 1$

Case 2: $\zeta = 0.5,$ $\omega_n = 2$

Case 3: $\zeta = 0.7,$ $\omega_n = 4$

Case 4: $\zeta = 0.8,$ $\omega_n = 6$

Solution. Define $\omega_n^2 = a$ and $2\zeta\omega_n = b$. Then, a and b each have four elements as follows:

$$a = \begin{bmatrix} 1 & 4 & 16 & 36 \end{bmatrix}$$

$$b = \begin{bmatrix} 0.6 & 2 & 5.6 & 9.6 \end{bmatrix}$$

Using vectors a and b, MATLAB Program 5–23 will produce the unit-step response curves as shown in Figure 5–64.

MATLAB Program 5–23

```
a = [1  4  16  36];
b = [0.6  2  5.6  9.6];
t = 0:0.1:8;
y = zeros(81,4);
    for i = 1:4;
    num = [0  0  a(i)];
    den = [1  b(i)  a(i)];
    y(:,i) = step(num,den,t);
    end
plot(t,y(:,1),'o',t,y(:,2),'x',t,y(:,3),'-',t,y(:,4),'-.')
grid
title('Unit-Step Response Curves for Four Cases')
xlabel('t Sec')
ylabel('Outputs')
gtext('1')
gtext('2')
gtext('3')
gtext('4')
```
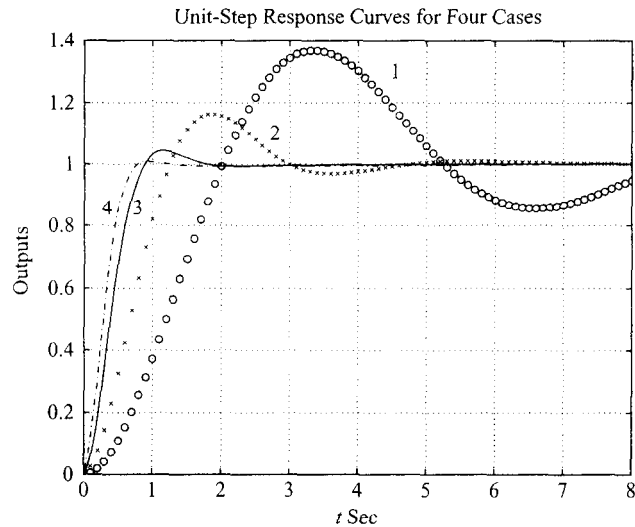
Figure 5–64
Unit-step response
curves for four cases.

Unit-Step Response Curves for Four Cases

A–5–14. Using MATLAB, obtain the unit-ramp response of the closed-loop control system whose closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{s + 10}{s^3 + 6s^2 + 9s + 10}$$

Also, obtain the response of this system when the input is given by

$$r = e^{-0.5t}$$

Solution. MATLAB Program 5–24 produces the unit-ramp response and the response to the exponential input $r = e^{-0.5t}$. The resulting response curves are shown in Figures 5–65(a) and (b), respectively.

MATLAB Program 5–24

```
% --------- Unit-Ramp Response ---------

num = [0  0  1  10];
den = [1  6  9  10];
t = 0:0.1:10;
r = t;
y = lsim(num,den,r,t);
plot(t,r,'-',t,y,'o')
grid
title('Unit-Ramp Response by Use of Command "lsim"')
xlabel('t Sec')
ylabel('Output')
text(3.2,6.5,'Unit-Ramp Input')
text(6.0,3.1,'Output')

% --------- Response to Input r1 = exp(-0.5t). ---------

num = [0  0  1  10];
den = [1  6  9  10];
t = 0:0.1:12;
r1 = exp(-0.5*t);
y1 = lsim(num,den,r1,t);
plot(t,r1,'-',t,y1,'o')
grid
title('Response to Input r1 = exp(-0.5t)')
xlabel('t Sec')
ylabel('Input and Output')
text(1.4,0.75,'Input r1 = exp(-0.5t)')
text(6.2,0.34,'Output')
```
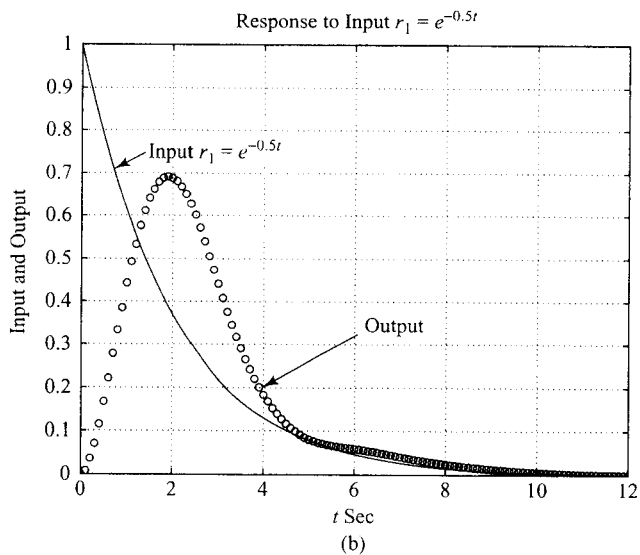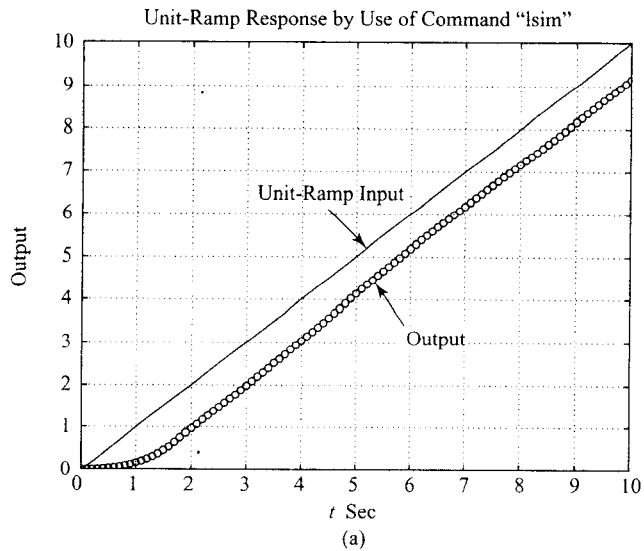
Figure 5–65
(a) Unit-ramp
response curve;
(b) response to
exponential input
$r = e^{-0.5t}$.

Unit-Ramp Response by Use of Command "lsim"

Response to Input $r_1 = e^{-0.5t}$

A–5–15. Obtain the response of the closed-loop system defined by

$$\frac{C(s)}{R(s)} = \frac{5}{s^2 + s + 5}$$

when the input $r(t)$ is given by

$$r(t) = 2 + t$$

[The input $r(t)$ is a step input of magnitude 2 plus unit-ramp input.]

Example Problems and Solutions

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Solution. A possible MATLAB program is shown in MATLAB Program 5–25. The resulting response curve, together with a plot of the input function, is shown in Figure 5–66.

MATLAB Program 5–25

```
num = [0  0  5];
den = [1  1  5];
t = 0:0.05:10;
r = 2+t;
c = lsim(num,den,r,t);
plot(t,r,'-',t,c,'o')
grid
title('Response to Input r(t) = 2 + t')
xlabel('t Sec')
ylabel('Output c(t) and Input r(t) = 2 + t')
```
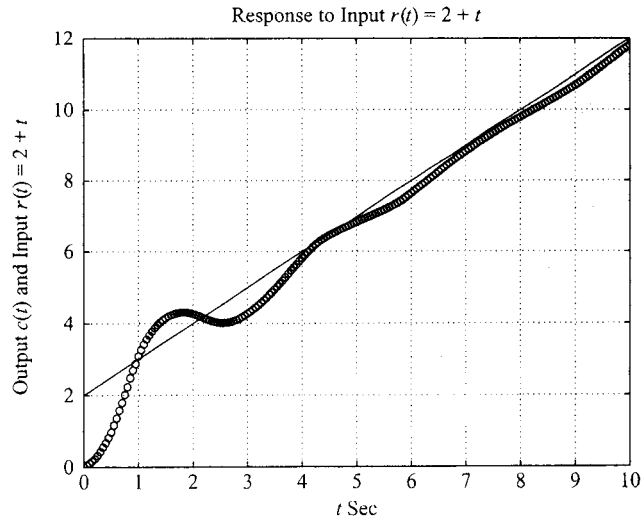

Figure 5–66
Response to input
$r(t) = 2 + t$.

A–5–16. Obtain the response of the system shown in Figure 5–67 when the input $r(t)$ is given by

$$r(t) = \frac{1}{2}t^2$$

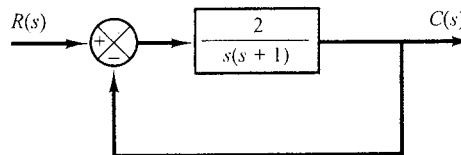[The input $r(t)$ is the unit-acceleration input.]

Figure 5–67
Control system.

Solution. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{2}{s^2 + s + 2}$$

MATLAB Program 5–26 produces the unit-accelelation response. The resulting response, together with the unit-acceleration input, is shown in Figure 5–68.

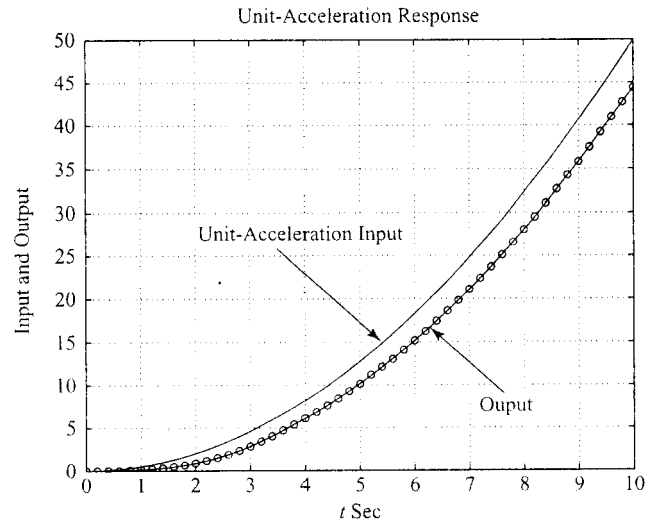| MATLAB Program 5–26 |
|---|
| num = [0 0 2];
den = [1 1 2];
t = 0:0.2:10;
r = 0.5*t.^2;
y = lsim(num,den,r,t);
plot(t,r,'-',t,y,'o',t,y,'-')
grid
title('Unit-Acceleration Response')
xlabel('t Sec')
ylabel('Input and Output')
text(2.1,27.5,'Unit-Acceleration Input')
text(7.2,7.5,'Output') |

Figure 5–68
Response to unit-acceleration input.

A–5–17. Consider the system defined by

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + 2\zeta s + 1}$$

where $\zeta = 0$, 0.2, 0.4, 0.6, 0.8, and 1.0. Write a MATLAB program using a "for loop" to obtain the two-dimensional and three-dimensional plots of the system output. The input is the unit-step function.

Solution. MATLAB Program 5–27 is a possible program to obtain two-dimensional and three-dimensional plots. Figure 5–69(a) is the two-dimensional plot of the unit-step response curves for various values of ζ. Figure 5–69(b) is the three-dimensional plot obtained by use of the command "mesh(y)" and Figure 5–69(c) is obtained by use of the command "mesh(y')". (These two three-dimensional plots are basically the same. The only difference is that x axis and y axis are interchanged.)

MATLAB Program 5–27

```
t = 0:0.2:12;
     for n = 1:6;
     num = [0  0  1];
     den = [1  2*(n-1)*0.2  1];
     [y(1:61,n),x,t] = step(num,den,t);
     end
plot(t,y)
grid
title('Unit-Step Response Curves')
xlabel('t Sec')
ylabel('Outputs')
gtext('\zeta = 0'),
gtext('0.2')
gtext('0.4')
gtext('0.6')
gtext('0.8')
gtext('1.0')

% To draw a three-dimensional plot, enter the following command: mesh(y) or mesh(y').
% We shall show two three-dimensional plots, one using "mesh(y)" and the other using
% "mesh(y')". These two plots are the same, except that the x axis and y axis are
% interchanged.

mesh(y)
title('Three-Dimensional Plot of Unit-Step Response Curves using Command "mesh(y)"')
xlabel('n, where n = 1,2,3,4,5,6')
ylabel('Computation Time Points')
zlahel('Outputs')

mesh(y')
title('Three-Dimensional Plot of Unit-Step Response Curves using Command "mesh(y transpose)"')
xlabel('Computation Time Points')
ylabel('n, where n = 1,2,3,4,5,6')
zlabel('Outputs')
```
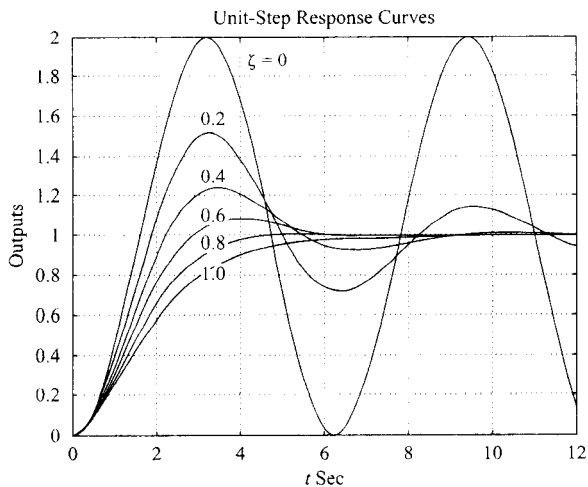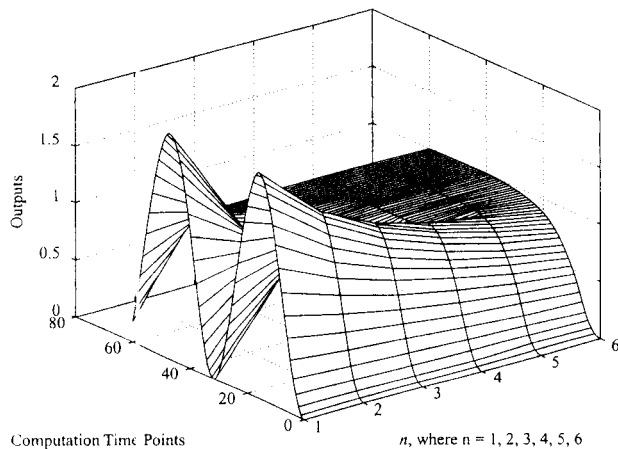

Figure 5–69
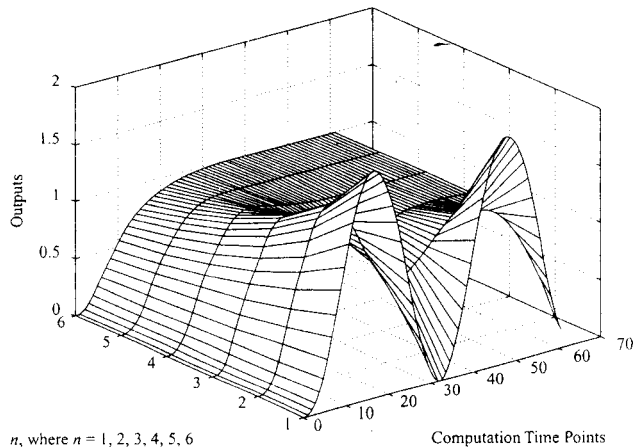(a) Two-dimensional
plot of unit-step
response curves;
(b) three-dimensional
plot of unit-step
response curves
usign command
"mesh(y)";
(c) three-dimensional
plot of unit-step
response curves
using command
"mesh(y′)".

(a)

(b)

(c)

A–5–18. Consider the following characteristic equation:

$$s^4 + Ks^3 + s^2 + s + 1 = 0$$

Determine the range of K for stability.

Solution. The Routh array of coefficients is

$$
\begin{array}{cccc}
s^4 & 1 & 1 & 1 \\
s^3 & K & 1 & 0 \\
s^2 & \dfrac{K-1}{K} & 1 & 0 \\
s^1 & 1 - \dfrac{K^2}{K-1} & 0 & \\
s^0 & 1 & &
\end{array}
$$

For stability, we require that

$$K > 0$$

$$\frac{K - 1}{K} > 0$$

$$1 - \frac{K^2}{K - 1} > 0$$

From the first and second conditions, K must be greater than 1. For $K > 1$, notice that the term $1 - \left[K^2/(K - 1)\right]$ is always negative, since

$$\frac{K - 1 - K^2 \cdot}{K - 1} = \frac{-1 + K(1 - K)}{K - 1} < 0$$

Thus, the three conditions cannot be fulfilled simultaneously. Therefore, there is no value of K that allows stability of the system.

A–5–19. Consider the characteristic equation given by

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0 \tag{5–67}$$

The Hurwitz stability criterion, given next, gives conditions for all the roots to have negative real parts in terms of the coefficients of the polynomial. As stated in the discussions of Routh's stability criterion in Section 5–7, for all the roots to have negative real parts, all the coefficients a's must be positive. This is a necessary condition but not a sufficient condition. If this condition is not satisfied, it indicates that some of the roots have positive real parts or are imaginary or zero. A sufficient condition for all the roots to have negative real parts is given in the following Hurwitz stability criterion: If all the coefficients of the polynomial are positive, arrange these coefficients in the following determinant:

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & \cdots & \cdot & \cdot & \cdot \\ 0 & a_1 & a_3 & \cdots & a_n & 0 & 0 \\ 0 & a_0 & a_2 & \cdots & a_{n-1} & 0 & 0 \\ \cdot & \cdot & \cdot & & a_{n-2} & a_n & 0 \\ \cdot & \cdot & \cdot & & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & a_{n-4} & a_{n-2} & a_n \end{vmatrix}$$

where we substituted zero for a_s if $s > n$. For all the roots to have negative real parts, it is necessary and sufficient that successive principal minors of Δ_n be positive. The successive principal minors are the following determinants:

$$\Delta_i = \begin{vmatrix} a_1 & a_3 & \cdots & a_{2i-1} \\ a_0 & a_2 & \cdots & a_{2i-2} \\ 0 & a_1 & \cdots & a_{2i-3} \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & a_i \end{vmatrix} \qquad (i = 1, 2, \ldots, n - 1)$$

where $a_s = 0$ if $s > n$. (It is noted that some of the conditions for the lower-order determinants are included in the conditions for the higher-order determinants.) If all these determinants are positive, and $a_0 > 0$ as already assumed, the equilibrium state of the system whose characteristic

equation is given by Equation (5–67) is asymptotically stable. Note that exact values of determinants are not needed; instead, only signs of these determinants are needed for the stability criterion.

Now consider the following characteristic equation:

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0$$

Obtain the conditions for stability using the Hurwitz stability criterion.

Solution. The conditions for stability are that all the a's be positive and that

$$\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = a_1 a_2 - a_0 a_3 > 0$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}$$

$$= a_1(a_2 a_3 - a_1 a_4) - a_0 a_3^2$$

$$= a_3(a_1 a_2 - a_0 a_3) - a_1^2 a_4 > 0$$

It is clear that, if all the a's are positive and if the condition $\Delta_3 > 0$ is satisfied, the condition $\Delta_2 > 0$ is also satisfied. Therefore, for all the roots of the given characteristic equation to have negative real parts, it is necessary and sufficient that all the coefficients a's are positive and $\Delta_3 > 0$.

A–5–20. Show that the first column of the Routh array of

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

is given by

$$1, \quad \Delta_1, \quad \frac{\Delta_2}{\Delta_1}, \quad \frac{\Delta_3}{\Delta_2}, \cdots, \frac{\Delta_n}{\Delta_{n-1}}$$

where

$$\Delta_r = \begin{vmatrix} a_1 & 1 & 0 & 0 & \cdot & 0 \\ a_3 & a_2 & a_1 & 1 & \cdot & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ a_{2r-1} & \cdot & \cdot & \cdot & \cdot & a_r \end{vmatrix}, \quad (n \geq r \geq 1)$$

$$a_k = 0 \quad \text{if } k > n$$

Solution. The Routh array of coefficients has the form

$$
\begin{array}{cccccc}
1 & a_2 & a_4 & a_6 & \cdots & a_n \\
a_1 & a_3 & a_5 & \cdots & & \\
b_1 & b_2 & b_3 & \cdots & & \\
c_1 & c_2 & \cdot & & & \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & &
\end{array}
$$

The first term in the first column of the Routh array is 1. The next term in the first column is a_1, which is equal to Δ_1. The next term is b_1, which is equal to

$$\frac{a_1 a_2 - a_3}{a_1} = \frac{\Delta_2}{\Delta_1}$$

The next term in the first column is c_1, which is equal to

$$\frac{b_1 a_3 - a_1 b_2}{b_1} = \frac{\left[\dfrac{a_1 a_2 - a_3}{a_1}\right] a_3 - a_1 \left[\dfrac{a_1 a_4 - a_5}{a_1}\right]}{\left[\dfrac{a_1 a_2 - a_3}{a_1}\right]}$$

$$= \frac{a_1 a_2 a_3 - a_3^2 - a_1^2 a_4 + a_1 a_5}{a_1 a_2 - a_3}$$

$$= \frac{\Delta_3}{\Delta_2}.$$

In a similar manner the remaining terms in the first column of the Routh array can be found.

The Routh array has the property that the last nonzero terms of any columns are the same; that is, if the array is given by

$$\begin{array}{cccc}
a_0 & a_2 & a_4 & a_6 \\
a_1 & a_3 & a_5 & a_7 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3 \\
d_1 & d_2 \\
e_1 & e_2 \\
f_1 \\
g_1
\end{array}$$

then

$$a_7 = c_3 = e_2 = g_1$$

and if the array is given by

$$\begin{array}{cccc}
a_0 & a_2 & a_4 & a_6 \\
a_1 & a_3 & a_5 & 0 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & 0 \\
d_1 & d_2 \\
e_1 & 0 \\
f_1
\end{array}$$

then

$$a_6 = b_3 = d_2 = f_1$$

In any case, the last term of the first column is equal to a_n, or

$$a_n = \frac{\Delta_{n-1} a_n}{\Delta_{n-1}} = \frac{\Delta_n}{\Delta_{n-1}}$$

For example, if $n = 4$, then

$$\Delta_4 = \begin{vmatrix} a_1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 \\ a_5 & a_4 & a_3 & a_2 \\ a_7 & a_6 & a_5 & a_4 \end{vmatrix} = \begin{vmatrix} a_1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{vmatrix} = \Delta_3 a_4$$

Thus it has been shown that the first column of the Routh array is given by

$$1, \quad \Delta_1, \quad \frac{\Delta_2}{\Delta_1}, \quad \frac{\Delta_3}{\Delta_2}, \quad \cdots, \quad \frac{\Delta_n}{\Delta_{n-1}}$$

A–5–21. Show that the Routh's stability criterion and Hurwitz stability criterion are equivalent.

Solution. If we write Hurwitz determinants in the triangular form

$$\Delta_i = \begin{vmatrix} a_{11} & & & & * \\ & a_{22} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & a_{ii} \end{vmatrix}, \qquad (i = 1, 2, \ldots, n)$$

where the elements below the diagonal line are all zeros and the elements above the diagonal line any numbers, then the Hurwitz conditions for asymptotic stability become

$$\Delta_i = a_{11} a_{22} \cdots a_{ii} > 0, \qquad (i = 1, 2, \ldots, n)$$

which are equivalent to the conditions

$$a_{11} > 0, \qquad a_{22} > 0, \qquad \ldots, \qquad a_{nn} > 0$$

We shall show that these conditions are equivalent to

$$a_1 > 0, \qquad b_1 > 0, \qquad c_1 > 0, \qquad \ldots$$

where $a_1, b_1, c_1, \ldots,$ are the elements of the first column in the Routh array.

Consider, for example, the following Hurwitz determinant, which corresponds to $i = 4$:

$$\Delta_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$

The determinant is unchanged if we subtract from the ith row k times the jth row. By subtracting from the second row a_0/a_1 times the first row, we obtain

$$\Delta_4 = \begin{vmatrix} a_{11} & a_3 & a_5 & a_7 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$

where

$$a_{11} = a_1$$

$$a_{22} = a_2 - \frac{a_0}{a_1} a_3$$

$$a_{23} = a_4 - \frac{a_0}{a_1} a_5$$

$$a_{24} = a_6 - \frac{a_0}{a_1} a_7$$

Similarly, subtracting from the fourth row a_0/a_1 times the third row yields

$$\Delta_4 = \begin{vmatrix} a_{11} & a_3 & a_5 & a_7 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_1 & a_3 & a_5 \\ 0 & 0 & \hat{a}_{43} & \hat{a}_{44} \end{vmatrix}$$

where

$$\hat{a}_{43} = a_2 - \frac{a_0}{a_1} a_3$$

$$\hat{a}_{44} = a_4 - \frac{a_0}{a_1} a_5$$

Next, subtracting from the third row a_1/a_{22} times the second row yields

$$\Delta_4 = \begin{vmatrix} a_{11} & a_3 & a_5 & a_7 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & \hat{a}_{43} & \hat{a}_{44} \end{vmatrix}$$

where

$$a_{33} = a_3 - \frac{a_1}{a_{22}} a_{23}$$

$$a_{34} = a_5 - \frac{a_1}{a_{22}} a_{24}$$

Finally, subtracting from the last row $\hat{a}_{43}/a_{33}$ times the third row yields

$$\Delta_4 = \begin{vmatrix} a_{11} & a_3 & a_5 & a_7 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{vmatrix}$$

where

$$a_{44} = \hat{a}_{44} - \frac{\hat{a}_{43}}{a_{33}} a_{34}$$

From this analysis, we see that

$$\Delta_4 = a_{11}a_{22}a_{33}a_{44}$$

$$\Delta_3 = a_{11}a_{22}a_{33}$$

$$\Delta_2 = a_{11}a_{22}$$

$$\Delta_1 = a_{11}$$

The Hurwitz conditions for asymptotic stability

$$\Delta_1 > 0, \qquad \Delta_2 > 0, \qquad \Delta_3 > 0, \qquad \Delta_4 > 0, \qquad \cdots$$

reduce to the conditions

$$a_{11} > 0, \qquad a_{22} > 0, \qquad a_{33} > 0, \qquad a_{44} > 0, \qquad \cdots$$

The Routh array for the polynomial

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0$$

where $a_0 > 0$ and $n = 4$, is given by

$$\begin{array}{ccc}
a_0 & a_2 & a_4 \\
a_1 & a_3 & \\
b_1 & b_2 & \\
c_1 & & \\
d_1 & &
\end{array}$$

From this Routh array, we see that

$$a_{11} = a_1$$

$$a_{22} = a_2 - \frac{a_0}{a_1}a_3 = b_1$$

$$a_{33} = a_3 - \frac{a_1}{a_{22}}a_{23} = \frac{a_3 b_1 - a_1 b_2}{b_1} = c_1$$

$$a_{44} = \hat{a}_{44} - \frac{\hat{a}_{43}}{a_{33}}a_{34} = a_4 = d_1$$

(The last equation is obtained using the fact that $a_{34} = 0$, $\hat{a}_{44} = a_4$, and $a_4 = b_2 = d_1$.) Hence the Hurwitz conditions for asymptotic stability become

$$a_1 > 0, \qquad b_1 > 0, \qquad c_1 > 0, \qquad d_1 > 0$$

Thus we have demonstrated that Hurwitz conditions for asymptotic stability can be reduced to Routh's conditions for asymptotic stability. The same argument can be extended to Hurwitz determinants of any order, and the equivalence of Routh's stability criterion and Hurwitz stability criterion can be established.

A–5–22. Consider the characteristic equation

$$s^4 + 2s^3 + (4 + K)s^2 + 9s + 25 = 0$$

Using the Hurwitz stability criterion, determine the range of K for stability.

Solution. Comparing the given characteristic equation

$$s^4 + 2s^3 + (4 + K)s^2 + 9s + 25 = 0$$

with the following standard fourth-order characteristic equation:

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0$$

we find

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 4 + K, \quad a_3 = 9, \quad a_4 = 25$$

The Hurwitz stability criterion states that Δ_4 is given by

$$\Delta_4 = \begin{vmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$

For all the roots to have negative real parts, it is necessary and sufficient that succesive principal minors of Δ_4 be positive. The successive principal minors are

$$\Delta_1 = |a_1| = 2$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} 2 & 9 \\ 1 & 4 + K \end{vmatrix} = 2K - 1$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} 2 & 9 & 0 \\ 1 & 4 + K & 25 \\ 0 & 2 & 9 \end{vmatrix} = 18K - 109$$

For all principal minors to be positive, we require that $\Delta_i (i = 1, 2, 3)$ be positive. Thus, we require

$$2K - 1 > 0$$

$$18K - 109 > 0$$

from which we obtain the region of K for stability to be

$$K > \frac{109}{18}$$

A–5–23. Explain why the proportional control of a plant that does not possess an integrating property (which means that the plant transfer function does not include the factor $1/s$) suffers offset in response to step inputs.

Solution. Consider, for example, the system shown in Figure 5–70. At steady state, if c were equal to a nonzero constant r, then $e = 0$ and $u = Ke = 0$, resulting in $c = 0$, which contradicts the assumption that $c = r =$ nonzero constant.

A nonzero offset must exist for proper operation of such a control system. In other words, at steady state, if e were equal to $r/(1 + K)$, then $u = Kr/(1 + K)$ and $c = Kr/(1 + K)$, which results in the assumed error signal $e = r/(1 + K)$. Thus the offset of $r/(1 + K)$ must exist in such a system.
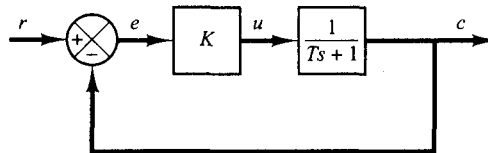
Figure 5–70
Control system.

A-5-24. Consider the system shown in Figure 5–71. Show that the steady-state error in following the unit-ramp input is B/K. This error can be made smaller by choosing B small and/or K large. However, making B small and/or K large would have the effect of making the damping ratio small, which is normally not desirable. Describe a method or methods to make B/K small and yet make the damping ratio have reasonable value ($0.5 < \zeta < 0.7$).

Solution. From Figure 5–71 we obtain

$$E(s) = R(s) - C(s) = \frac{Js^2 + Bs}{Js^2 + Bs + K} R(s)$$

The steady-state error for the unit-ramp response can be obtained as follows: For the unit-ramp input, the steady-state error e_{ss} is

$$e_{ss} = \lim_{s \to 0} sE(s)$$

$$= \lim_{s \to 0} s \frac{Js^2 + Bs}{Js^2 + Bs + K} \frac{1}{s^2}$$

$$= \frac{B}{K}$$

$$= \frac{2\zeta}{\omega_n}$$

where

$$\zeta = \frac{B}{2\sqrt{KJ}}, \qquad \omega_n = \sqrt{\frac{K}{J}}$$

To assure acceptable transient response and acceptable steady-state error in following a ramp input, ζ must not be too small and ω_n must be sufficiently large. It is possible to make the steady-state error e_{ss} small by making the value of the gain K large. (A large value of K has an additional advantage of suppressing undesirable effects caused by dead zone, backlash, coulomb friction, and the like.) A large value of K would, however, make the value of ζ small and increase the maximum overshoot, which is undesirable.

It is therefore necessary to compromise between the magnitude of the steady-state error to a ramp input and the maximum overshoot to a unit-step input. In the system shown in Figure 5–71, a reasonable compromise may not be reached easily. It is then desirable to consider other types of control action that may improve both the transient-response and steady-state behavior. Two schemes to improve both the transient-response and steady-state behavior are available. One scheme is to use a proportional-plus-derivative controller and the other is to use tachometer feedback.
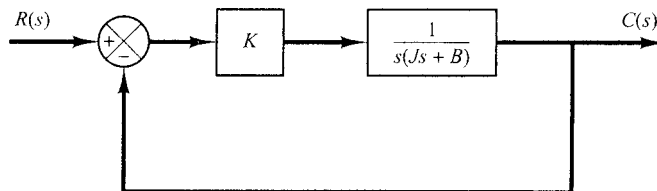

Figure 5–71
Control system.

A–5–25. The block diagram of Figure 5–72 shows a speed control system in which the output member of the system is subject to a torque disturbance. In the diagram, $\Omega_r(s)$, $\Omega(s)$, $T(s)$, and $D(s)$ are the Laplace transforms of the reference speed, output speed, driving torque, and disturbance torque, respectively. In the absence of a disturbance torque, the output speed is equal to the reference speed.
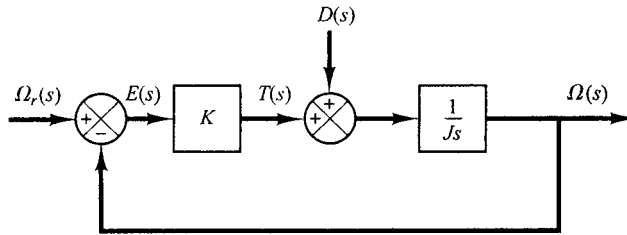

Figure 5–72
Block diagram of a speed control system.

Investigate the response of this system to a unit-step disturbance torque. Assume that the reference input is zero, or $\Omega_r(s) = 0$.

Solution. Figure 5–73 is a modified block diagram convenient for the present analysis. The closed-loop transfer function is

$$\frac{\Omega_D(s)}{D(s)} = \frac{1}{Js + K}$$

where $\Omega_D(s)$ is the Laplace transform of the output speed due to the disturbance torque. For a unit-step disturbance torque, the steady-state output velocity is

$$\omega_D(\infty) = \lim_{s \to 0} s\Omega_D(s)$$

$$= \lim_{s \to 0} \frac{s}{Js + K} \frac{1}{s}$$

$$= \frac{1}{K}$$

From this analysis, we conclude that, if a step disturbance torque is applied to the output member of the system, an error speed will result so that the ensuing motor torque will exactly cancel the disturbance torque. To develop this motor torque, it is necessary that there be an error in speed so that nonzero torque will result.
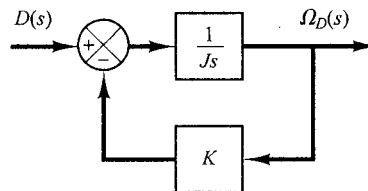

Figure 5–73
Block diagram of the speed control system of Figure 5–72 when $\Omega_r(s) = 0$.

A-5-26. In the system considered in Problem **A-5-25**, it is desired to eliminate as much as possible the speed errors due to torque disturbances.

Is it possible to cancel the effect of a disturbance torque at steady state so that a constant disturbance torque applied to the output member will cause no speed change at steady state?

Solution. Suppose that we choose a suitable controller whose transfer function is $G_c(s)$, as shown in Figure 5-74. Then in the absence of the reference input the closed-loop transfer function between the output velocity $\Omega_D(s)$ and the disturbance torque $D(s)$ is

$$\frac{\Omega_D(s)}{D(s)} = \frac{\dfrac{1}{Js}}{1 + \dfrac{1}{Js}G_c(s)}$$

$$= \frac{1}{Js + G_c(s)}$$

The steady-state output speed due to a unit-step disturbance torque is

$$\omega_D(\infty) = \lim_{s \to 0} s\Omega_D(s)$$

$$= \lim_{s \to 0} \frac{s}{Js + G_c(s)} \frac{1}{s}$$

$$= \frac{1}{G_c(0)}$$

To satisfy the requirement that

$$\omega_D(\infty) = 0$$

we must choose $G_c(0) = \infty$. This can be realized if we choose

$$G_c(s) = \frac{K}{s}$$

Integral control action will continue to correct until the error is zero. This controller, however, presents a stability problem because the characteristic equation will have two imaginary roots.

One method of stabilizing such a system is to add a proportional mode to the controller or choose
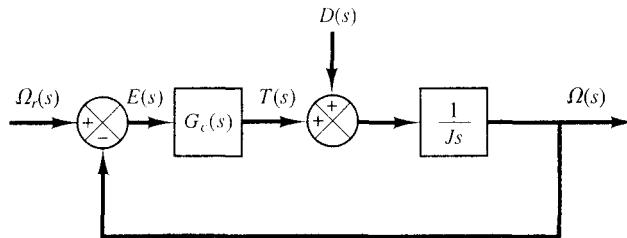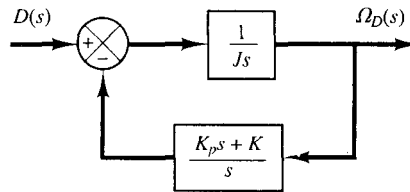
$$G_c(s) = K_p + \frac{K}{s}$$

Figure 5-74
Block diagram of a
speed control system.

Figure 5–75
Block diagram of the
speed control system
of Figure 5–74 when
$G_c(s) = K_p + (K/s)$
and $\Omega_r(s) = 0$.

With this controller, the block diagram of Figure 5–74 in the absence of the reference input can be modified to that of Figure 5–75. The closed-loop transfer function $\Omega_D(s)/D(s)$ becomes

$$\frac{\Omega_D(s)}{D(s)} = \frac{s}{Js^2 + K_p s + K}$$

For a unit-step disturbance torque, the steady-state output speed is

$$\omega_D(\infty) = \lim_{s \to 0} s\Omega_D(s) = \lim_{s \to 0} \frac{s^2}{Js^2 + K_p s + K}\frac{1}{s} = 0$$

Thus, we see that the proportional-plus-integral controller eliminates speed error at steady state.

The use of integral control action has increased the order of the system by 1. (This tends to produce an oscillatory response.)

In the present system, a step disturbance torque will cause a transient error in the output speed, but the error will become zero at steady state. The integrator provides a nonzero output with zero error. (The nonzero output of the integrator produces a motor torque that exactly cancels the disturbance torque.)

Note that the integrator in the transfer function of the plant does not eliminate the steady-state error due to a step disturbance torque. To eliminate this, we must have an integrator before the point where the disturbance torque enters.

A–5–27. Consider the system shown in Figure 5–76(a). The steady-state error to a unit-ramp input is $e_{ss} = 2\zeta/\omega_n$. Show that the steady-state error for following a ramp input may be eliminated if the input is introduced to the system through a proportional-plus-derivative filter, as shown in Figure 5–76(b), and the value of k is properly set. Note that the error $e(t)$ is given by $r(t) - c(t)$.

Solution. The closed-loop transfer function of the system shown in Figure 5–76(b) is

$$\frac{C(s)}{R(s)} = \frac{(1 + ks)\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Then

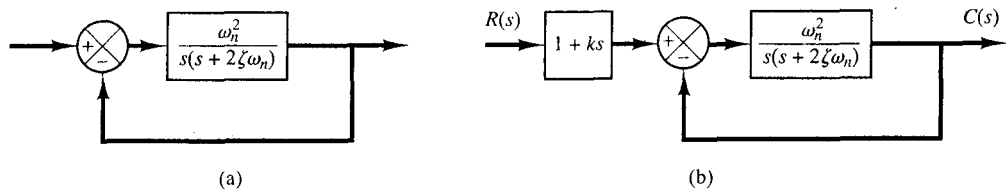$$R(s) - C(s) = \left(\frac{s^2 + 2\zeta\omega_n s - \omega_n^2 ks}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)R(s)$$

Figure 5–76
(a) Control system;
(b) control system
with input filter.

(a)   (b)

If the input is a unit ramp, then the steady-state error is

$$e(\infty) = r(\infty) - c(\infty)$$

$$= \lim_{s \to 0} s \left(\frac{s^2 + 2\zeta\omega_n s - \omega_n^2 ks}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) \frac{1}{s^2}$$

$$= \frac{2\zeta\omega_n - \omega_n^2 k}{\omega_n^2}$$

Therefore, if k is chosen as

$$k = \frac{2\zeta}{\omega_n}$$

then the steady-state error for following a ramp input can be made equal to zero. Note that, if there are any variations in the values of ζ and/or ω_n due to environmental changes or aging, then a nonzero steady-state error for a ramp response may result.

A–5–28. Consider the servo system with tachometer feedback shown in Figure 5–77. Obtain the error signal $E(s)$ when both the reference input $R(s)$ and disturbance input $D(s)$ are present. Obtain also the steady-state error when the system is subjected to a reference input (unit-ramp input) and disturbance input (step input of magnitude d).
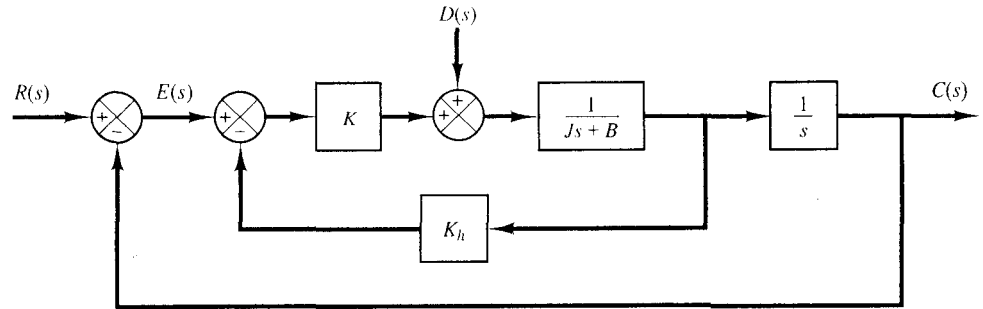
Figure 5–77
Servo system with tachometer feedback.

Solution. When we consider the reference input $R(s)$ we can assume that the disturbance input $D(s)$ is zero, and vice versa. Then, a block diagram that relates the reference input $R(s)$ and the output $C(s)$ may be drawn as shown in Figure 5–78(a). Similarly, Figure 5–78(b) relates the disturbance input $D(s)$ and the output $C(s)$.

The closed-loop transfer function $C(s)/R(s)$ can be obtained from Figure 5–78(a) as follows:

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K}$$

Similarly. the closed-loop transfer function $C(s)/D(s)$ can be obtained from Figure 5–78(b) as

$$\frac{C(s)}{D(s)} = \frac{1}{Js^2 + (B + KK_h)s + K}$$

If both $R(s)$ and $D(s)$ are present, then

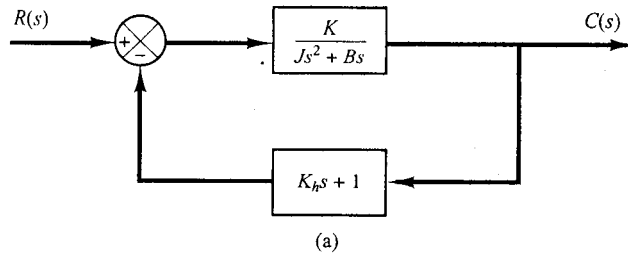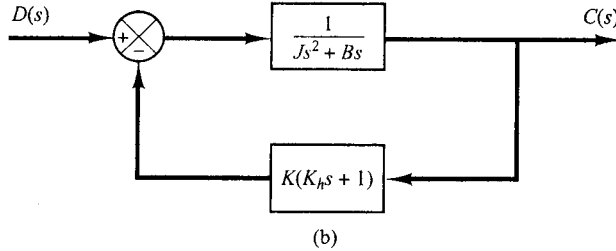$$C(s) = \frac{1}{Js^2 + (B + KK_h)s + K} \left[KR(s) + D(s) \right]$$

Figure 5–78
(a) Block diagram
that relates reference
input $R(s)$ and
output $C(s)$;
(b) block diagram
that relates
disturbance input
$D(s)$ and output
$C(s)$.

(a)

(b)

Since

$$E(s) = R(s) - C(s)$$

$$= R(s) - \frac{KR(s) + D(s)}{Js^2 + (B + KK_h)s + K}$$

we obtain

$$E(s) = \frac{1}{Js^2 + (B + KK_h)s + K} \left\{ [Js^2 + (B + KK_h)s]R(s) - D(s) \right\}$$

Hence the steady-state error can be obtained as follows:

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

$$= \lim_{s \to 0} \frac{s}{Js^2 + (B + KK_h)s + K} \left[s(Js + B + KK_h)R(s) - D(s) \right]$$

$$= \lim_{s \to 0} \left[\frac{B + KK_h}{K} s^2 R(s) - \frac{1}{K} sD(s) \right]$$

Since $R(s) = 1/s^2$ (unit-ramp input) and $D(s) = d/s$ (step input of magnitude d) the steady-state error is

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} \left[\frac{B + KK_h}{K} \frac{s^2}{s^2} - \frac{1}{K} \frac{sd}{s} \right]$$

$$= \frac{B + KK_h}{K} - \frac{d}{K}$$

A–5–29. Consider the stable unity-feedback control system with feedforward transfer function $G(s)$. Suppose that the closed-loop transfer function can be written

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{(T_1 s + 1)(T_2 s + 1)\cdots(T_n s + 1)} \qquad (m \le n)$$

Show that

$$\int_0^\infty e(t)\, dt = (T_1 + T_2 + \cdots + T_n) - (T_a + T_b + \cdots + T_m)$$

where $e(t)$ is the error in the unit-step response. Show also that

$$\frac{1}{K_v} = \frac{1}{\lim\limits_{s \to 0} sG(s)} = (T_1 + T_2 + \cdots + T_n) - (T_a + T_b + \cdots + T_m)$$

Solution. Let us define

$$(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1) = P(s)$$

and

$$(T_1 s + 1)(T_2 s + 1)\cdots(T_n s + 1) = Q(s)$$

Then

$$\frac{C(s)}{R(s)} = \frac{P(s)}{Q(s)}$$

and

$$E(s) = \frac{Q(s) - P(s)}{Q(s)} R(s)$$

For a unit-step input, $R(s) = 1/s$ and

$$E(s) = \frac{Q(s) - P(s)}{sQ(s)}$$

Since the system is stable, $\int_0^\infty e(t)\, dt$ converges to a constant value. Referring to Table 2–2 (item 9), we have

$$\int_0^\infty e(t)\, dt = \lim_{s \to 0} s\, \frac{E(s)}{s} = \lim_{s \to 0} E(s)$$

Hence

$$\int_0^\infty e(t)\, dt = \lim_{s \to 0} \frac{Q(s) - P(s)}{sQ(s)}$$

$$= \lim_{s \to 0} \frac{Q'(s) - P'(s)}{Q(s) + sQ'(s)}$$

$$= \lim_{s \to 0} \left[Q'(s) - P'(s) \right]$$

Since

$$\lim_{s\to 0} P'(s) = T_a + T_b + \cdots + T_m$$

$$\lim_{s\to 0} Q'(s) = T_1 + T_2 + \cdots + T_n$$

we have

$$\int_0^\infty e(t)\,dt = (T_1 + T_2 + \cdots + T_n) - (T_a + T_b + \cdots + T_m)$$

For a unit-step input $r(t)$, since

$$\int_0^\infty e(t)\,dt = \lim_{s\to 0} E(s) = \lim_{s\to 0} \frac{1}{1+G(s)} R(s) = \lim_{s\to 0} \frac{1}{1+G(s)} \frac{1}{s} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{K_v}$$

we have

$$\frac{1}{K_v} = \frac{1}{\lim_{s\to 0} sG(s)} = (T_1 + T_2 + \cdots + T_n) - (T_a + T_b + \cdots + T_m)$$

Note that zeros in the left half-plane (that is, positive $T_a, T_b, \ldots, T_m$) will improve K_v. Poles close to the origin cause low velocity-error constants unless there are zeros nearby.

PROBLEMS

B–5–1. A thermometer requires 1 min to indicate 98% of the response to a step input. Assuming the thermometer to be a first-order system, find the time constant.

  If the thermometer is placed in a bath, the temperature of which is changing linearly at a rate of 10°/min, how much error does the thermometer show?

B–5–2. Consider the unit-step response of a unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{1}{s(s+1)}$$

Obtain the rise time, peak time, maximum overshoot, and settling time.

B–5–3. Consider the closed-loop system given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Determine the values of ζ and ω_n so that the system responds to a step input with approximately 5% overshoot and with a settling time of 2 sec. (Use the 2% criterion.)

B–5–4. Figure 5–79 is a block diagram of a space-vehicle attitude-control system. Assuming the time constant T of the controller to be 3 sec and the ratio K/J to be $\frac{2}{9}$ rad^2/sec^2, find the damping ratio of the system.
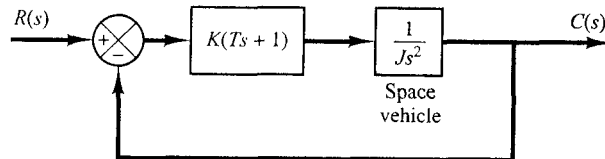

Figure 5–79
Space-vehicle attitude-control system.

B–5–5. Consider the system shown in Figure 5–80. The system is initially at rest. Suppose that the cart is set into motion by an impulsive force whose strength is unity. Can it be stopped by another such impulsive force?
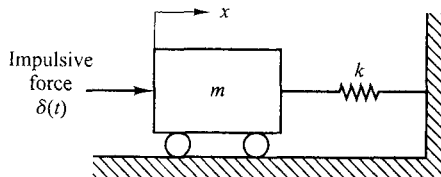

Figure 5–80
Mechanical system.

B-5-6. Obtain the unit-impulse response and the unit-step response of a unity-feedback system whose open-loop transfer function is

$$G(s) = \frac{2s + 1}{s^2}$$

B-5-7. Consider the system shown in Figure 5-81. Show that the transfer function $Y(s)/X(s)$ has a zero in the right-half s plane. Then obtain $y(t)$ when $x(t)$ is a unit step. Plot $y(t)$ versus t.

B-5-8. An oscillatory system is known to have a transfer function of the following form:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Assume that a record of a damped oscillation is available as shown in Figure 5-82. Determine the damping ratio ζ of the system from the graph.

B-5-9. Consider the system shown in Figure 5-83(a). The damping ratio of this system is 0.158 and the undamped natural frequency is 3.16 rad/sec. To improve the relative stability, we employ tachometer feedback. Figure 5-83(b) shows such a tachometer-feedback system.

Determine the value of K_h so that the damping ratio of the system is 0.5. Draw unit-step response curves of both the original and tachometer-feedback systems. Also draw the error-versus-time curves for the unit-ramp response of both systems.
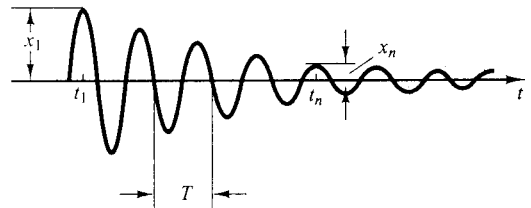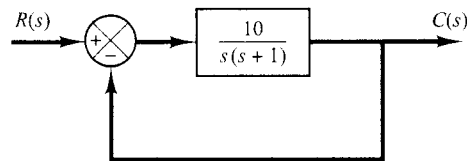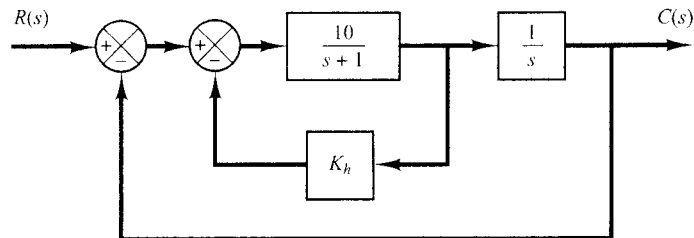

Figure 5-81
System with zero in the right-half s plane.

Figure 5-82
Decaying oscillation.

(a)

(b)

Figure 5-83
(a) Control system; (b) control system with tachometer feedback.

B–5–10. Referring to the system shown in Figure 5–84, determine the values of K and k such that the system has a damping ratio ζ of 0.7 and an undamped natural frequency ω_n of 4 rad/sec.

B–5–11. Consider the system shown in Figure 5–85. Determine the value of k such that the damping ratio ζ is 0.5. Then obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time t_s in the unit-step response.

B–5–12. Using MATLAB, obtain the unit-step response, unit-ramp response, and unit-impulse response of the following system:

$$\frac{C(s)}{R(s)} = \frac{10}{s^2 + 2s + 10}$$

where $R(s)$ and $C(s)$ are Laplace transforms of the input $r(t)$ and output $c(t)$, respectively.

B–5–13. Using MATLAB, obtain the unit-step response, unit-ramp response, and unit-impulse response of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where u is the input and y is the output.

B–5–14. Obtain both analytically and computationally the rise time, peak time, maximum overshoot, and settling time in the unit-step response of a closed-loop system given by
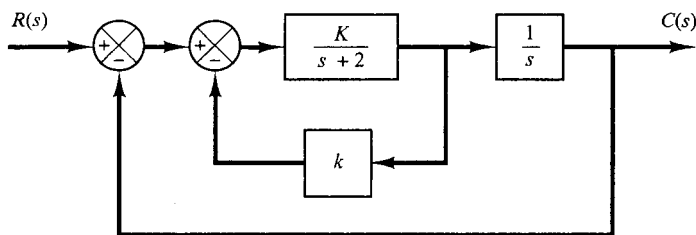
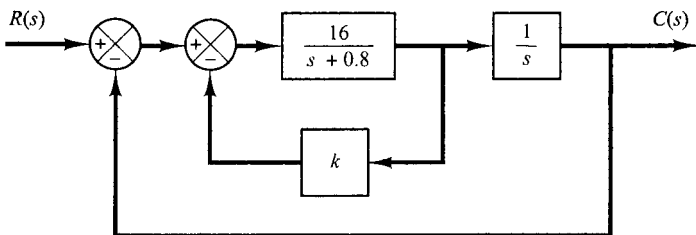$$\frac{C(s)}{R(s)} = \frac{36}{s^2 + 2s + 36}$$

Figure 5–84
Closed-loop system.

Figure 5–85
Block diagram of a system.

B–5–15. Figure 5–86 shows three systems. System I is a positional servo system. System II is a positional servo system with PD control action. System III is a positional servo system with velocity feedback. Compare the unit-step, unit-impulse, and unit-ramp responses of the three systems. Which system is best with respect to the speed of response and maximum overshoot in the step response?

B–5–16. Consider the position control system shown in Figure 5–87. Write a MATLAB program to obtain a unit-step response and a unit-ramp response of the system. Plot curves $x_1(t)$ versus t, $x_2(t)$ versus t, $x_3(t)$ versus t, and $e(t)$ versus t [where $e(t) = r(t) - x_1(t)$] for both the unit-step response and the unit-ramp response.
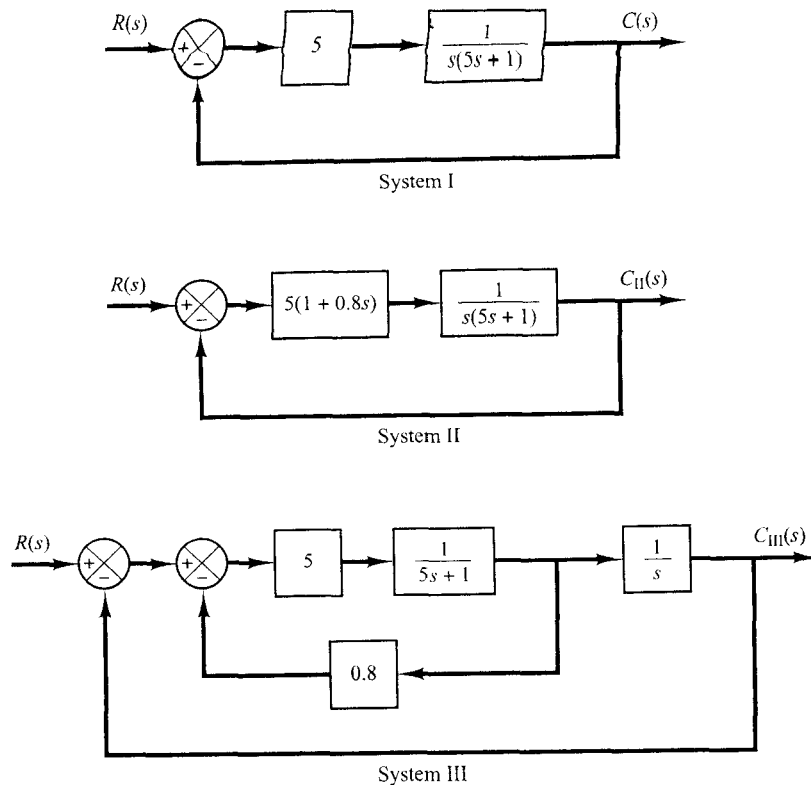

Figure 5–86
Positional servo system (System I), positional servo system with PD control action (System II), and positional servo system with velocity feedback (System III).
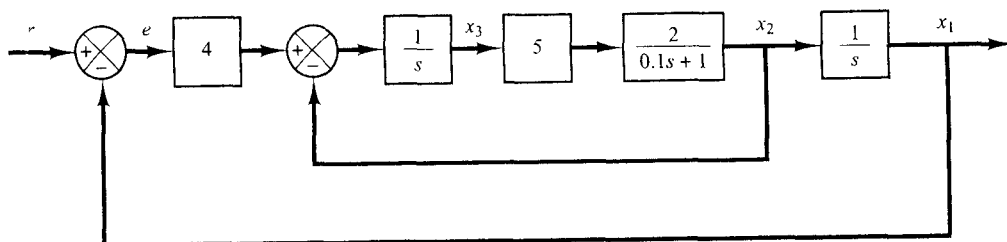

Figure 5–87
Position control system.

B–5–17. Using MATLAB, obtain the unit-step response curve for the unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{10}{s(s + 2)(s + 4)}$$

Using MATLAB, obtain also the rise time, peak time, maximum overshoot, and settling time in the unit-step response curve.

B–5–18. Consider the closed-loop system defined by

$$\frac{C(s)}{R(s)} = \frac{2\zeta s + 1}{s^2 + 2\zeta s + 1}$$

where $\zeta = 0.2, 0.4, 0.6, 0.8,$ and 1.0. Using MATLAB, plot a two-dimensional diagram of unit-impulse response curves. Also plot a three-dimensional plot of the response curves.

B–5–19. Consider the second-order system defined by

$$\frac{C(s)}{R(s)} = \frac{s + 1}{s^2 + 2\zeta s + 1}$$

where $\zeta = 0.2, 0.4, 0.6, 0.8, 1.0$. Plot a three-dimensional diagram of the unit-step response curves.

B–5–20. Obtain the unit-ramp response of the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where u is the unit-ramp input. Use lsim command to obtain the response.

B–5–21. Using MATLAB obtain the unit acceleration response curve of the unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{10(s + 1)}{s^2(s + 4)}$$

The unit acceleration input is defined by

$$r(t) = \frac{1}{2} t^2 \qquad (t \geq 0)$$

B–5–22. Consider the differential equation system given by

$$\ddot{y} + 3\dot{y} + 2y = 0, \qquad y(0) = 0.1, \qquad \dot{y}(0) = 0.05$$

Obtain the response $y(t)$, subject to the given initial condition.

B–5–23. Determine the range of K for stability of a unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{K}{s(s + 1)(s + 2)}$$

B–5–24. Consider the unity-feedback control system with the following open-loop transfer function:

$$G(s) = \frac{10}{s(s - 1)(2s + 3)}$$

Is this system stable?

B–5–25. Consider the following characteristic equation:

$$s^4 + 2s^3 + (4 + K)s^2 + 9s + 25 = 0$$

Using Routh stability criterion, determine the range of K for stability.

B–5–26. Consider the closed-loop system shown in Figure 5–88. Determine the range of K for stability. Assume that $K > 0$.
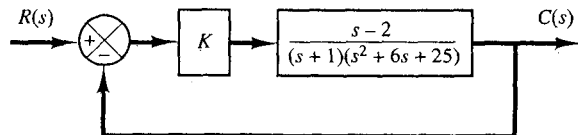

Figure 5–88
Closed-loop system.

B–5–27. Consider the satellite attitude control system shown in Figure 5–89(a). The output of this system exhibits continued oscillations and is not desirable. This system can be stabilized by use of tachometer feedback, as shown in Figure 5–89(b). If $K/J = 4$, what value of K_h will yield the damping ratio to be 0.6?

B–5–28. Consider the servo system with tachometer feedback shown in Figure 5–90. Determine the ranges of stability for K and K_h. (Note that K_h must be positive.)
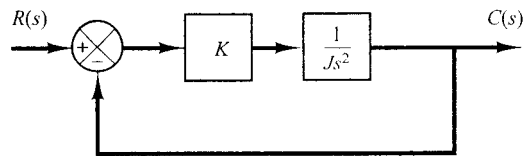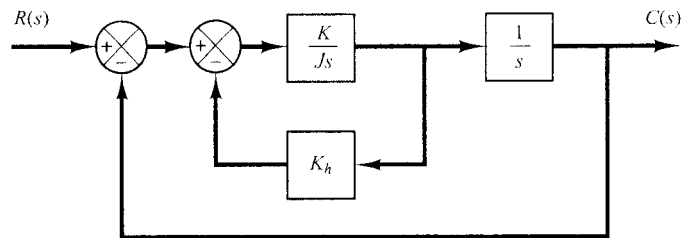
B–5–29. Consider the system

$$\dot{x} = Ax$$

where matrix A is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -b_3 & 0 & 1 \\ 0 & -b_2 & -b_1 \end{bmatrix}$$

(A is called Schwarz matrix.) Show that the first column of the Routh's array of the characteristic equation $|sI - A| = 0$ consists of $1, b_1, b_2$, and $b_1 b_3$.

(a)

(b)

Figure 5–89
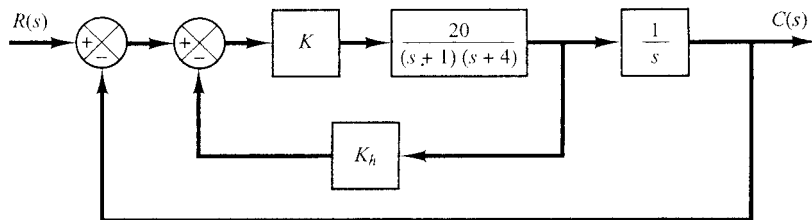(a) Unstable satellite attitude control system; (b) stabilized system.

Figure 5–90
Servo system with tachometer feedback.

B–5–30. Consider a unity-feedback control system with the closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{Ks + b}{s^2 + as + b}$$

Determine the open-loop transfer function $G(s)$.

Show that the steady-state error in the unit-ramp response is given by

$$e_{ss} = \frac{1}{K_v} = \frac{a - K}{b}$$

B–5–31. Consider a unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{K}{s(Js + B)}$$

Discuss the effects that varying the values of K and B has on the steady-state error in unit-ramp response. Sketch typical unit-ramp response curves for a small value, medium value, and large value of K, assuming that B is constant.

B–5–32. If the feedforward path of a control system contains at least one integrating element, then the output continues to change as long as an error is present. The output stops when the error is precisely zero. If an external disturbance enters the system, it is desirable to have an integrating element between the error-measuring element and the point where the disturbance enters so that the effect of the external disturbance may be made zero at steady state.

Show that, if the disturbance is a ramp function, then the steady-state error due to this ramp disturbance may be eliminated only if two integrators precede the point where the disturbance enters.