

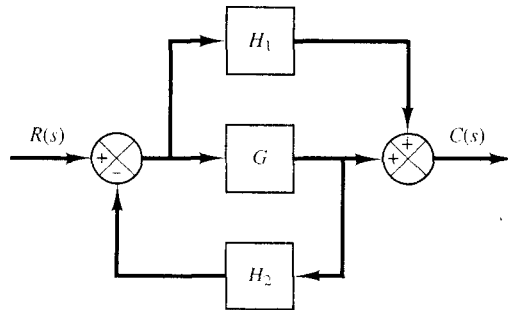
## EXAMPLE PROBLEMS AND SOLUTIONS

**A-3-1.** Simplify the block diagram shown in Figure 3-42.

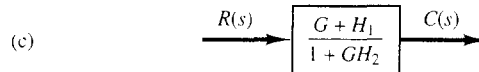
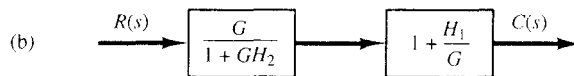
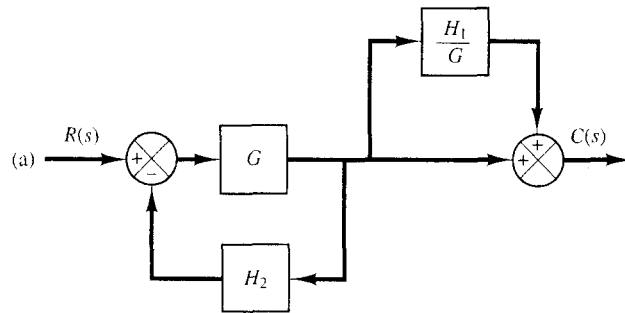
**Solution.** First, move the branch point of the path involving  $H_1$  outside the loop involving  $H_2$ , as shown in Figure 3-43(a). Then eliminating two loops results in Figure 3-43(b). Combining two blocks into one gives Figure 3-43(c).

**A-3-2.** Simplify the block diagram shown in Figure 3-44. Obtain the transfer function relating  $C(s)$  and  $R(s)$ .

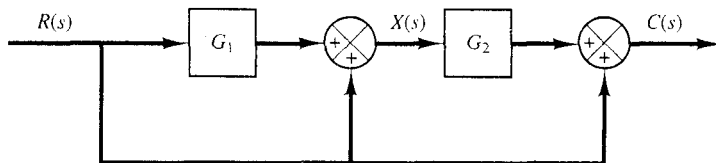
**Figure 3-42**  
Block diagram of a system.

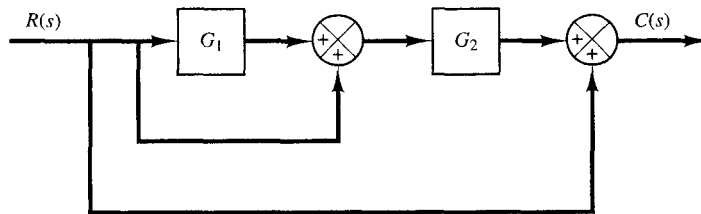


**Figure 3-43**  
Simplified block diagrams for the system shown in Figure 3-42.

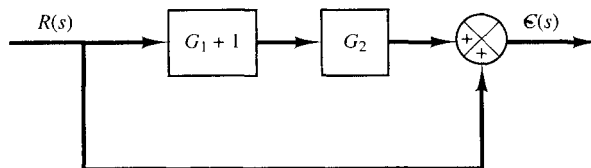


**Figure 3-44**  
Block diagram of a system.

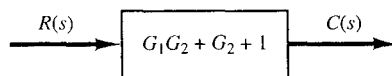




(a)



(b)



(c)

**Figure 3-45**  
Reduction of the  
block diagram shown  
in Figure 3-44.

**Solution.** The block diagram of Figure 3-44 can be modified to that shown in Figure 3-45(a). Eliminating the minor feedforward path, we obtain Figure 3-45(b), which can be simplified to that shown in Figure 3-45(c). The transfer function  $C(s)/R(s)$  is thus given by

$$\frac{C(s)}{R(s)} = G_1G_2 + G_2 + 1$$

The same result can also be obtained by proceeding as follows: Since signal  $X(s)$  is the sum of two signals  $G_1R(s)$  and  $R(s)$ , we have

$$X(s) = G_1R(s) + R(s)$$

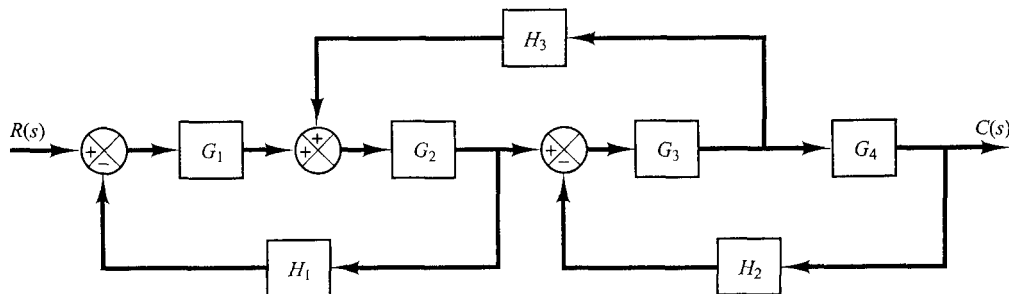
The output signal  $C(s)$  is the sum of  $G_2X(s)$  and  $R(s)$ . Hence

$$C(s) = G_2X(s) + R(s) = G_2[G_1R(s) + R(s)] + R(s)$$

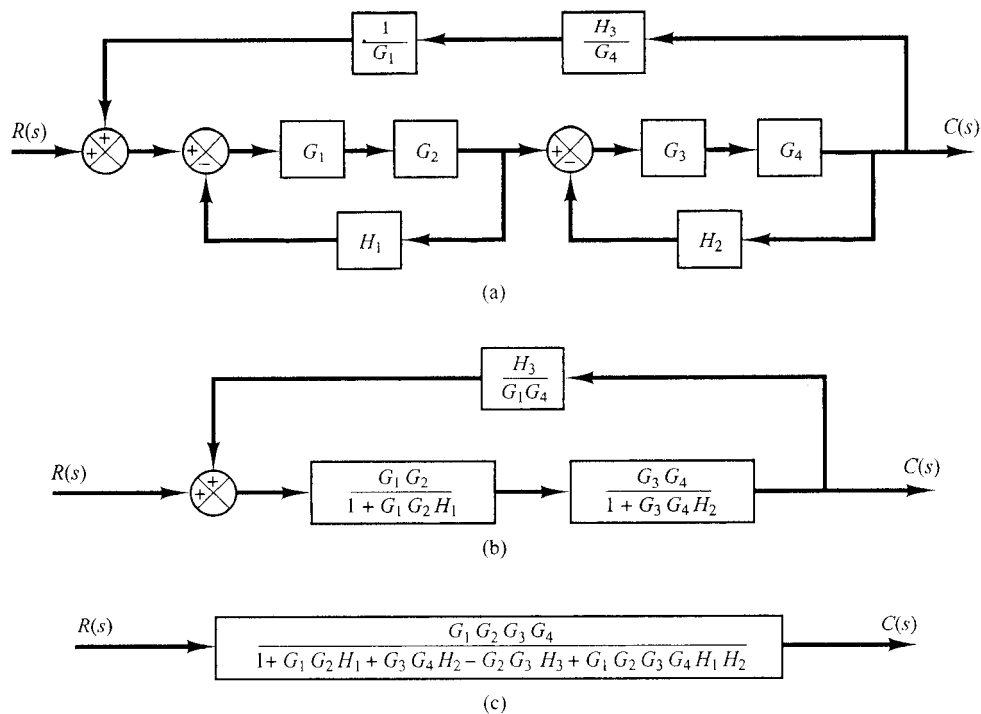
And so we have the same result as before:

$$\frac{C(s)}{R(s)} = G_1G_2 + G_2 + 1$$

**A-3-3.** Simplify the block diagram shown in Figure 3-46. Then, obtain the closed-loop transfer function  $C(s)/R(s)$ .



**Figure 3-46**  
Block diagram of a  
system.



**Figure 3-47**  
Successive  
reductions of the  
block diagram shown  
in Figure 3-46.

**Solution.** First move the branch point between  $G_3$  and  $G_4$  to the right-hand side of the loop containing  $G_3$ ,  $G_4$ , and  $H_2$ . Then move the summing point between  $G_1$  and  $G_2$  to the left-hand side of the first summing point. See Figure 3-47(a). By simplifying each loop, the block diagram can be modified as shown in Figure 3-47(b). Further simplification results in Figure 3-47(c), from which the closed-loop transfer function  $C(s)/R(s)$  is obtained as

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1 + G_1 G_2 H_1 + G_3 G_4 H_2 - G_2 G_3 H_3 + G_1 G_2 G_3 G_4 H_1 H_2}$$

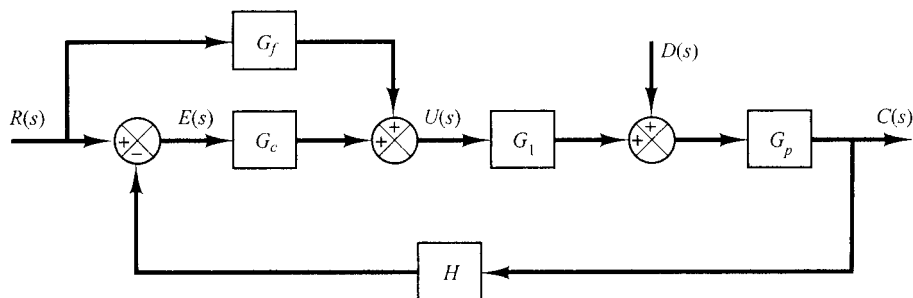
**A-3-4.** Obtain transfer functions  $C(s)/R(s)$  and  $C(s)/D(s)$  of the system shown in Figure 3-48.

**Solution.** From Figure 3-48 we have

$$U(s) = G_f R(s) + G_c E(s) \quad (3-88)$$

$$C(s) = G_p [D(s) + G_1 U(s)] \quad (3-89)$$

$$E(s) = R(s) - H C(s) \quad (3-90)$$



**Figure 3-48**  
Control system with  
reference input and  
disturbance input.

By substituting Equation (3-88) into Equation (3-89), we get

$$C(s) = G_p D(s) + G_1 G_p [G_f R(s) + G_c E(s)] \quad (3-91)$$

By substituting Equation (3-90) into Equation (3-91), we obtain

$$C(s) = G_p D(s) + G_1 G_p \{G_f R(s) + G_c [R(s) - HC(s)]\}$$

Solving this last equation for  $C(s)$ , we get

$$C(s) + G_1 G_p G_c HC(s) = G_p D(s) + G_1 G_p (G_f + G_c) R(s)$$

Hence

$$C(s) = \frac{G_p D(s) + G_1 G_p (G_f + G_c) R(s)}{1 + G_1 G_p G_c H} \quad (3-92)$$

Note that Equation (3-92) gives the response  $C(s)$  when both reference input  $R(s)$  and disturbance input  $D(s)$  are present.

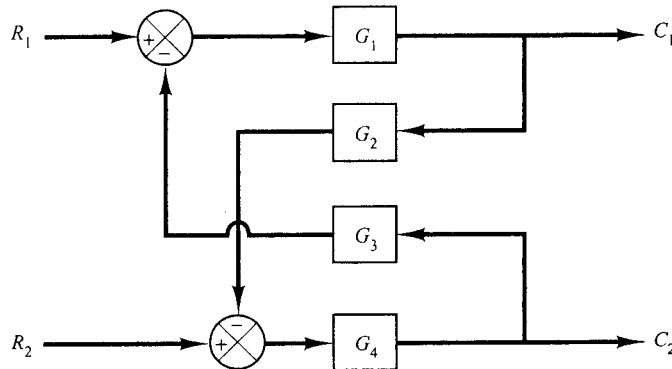
To find transfer function  $C(s)/R(s)$ , we let  $D(s) = 0$  in Equation (3-92). Then we obtain

$$\frac{C(s)}{R(s)} = \frac{G_1 G_p (G_f + G_c)}{1 + G_1 G_p G_c H}$$

Similarly, to obtain transfer function  $C(s)/D(s)$ , we let  $R(s) = 0$  in Equation (3-92). Then  $C(s)/D(s)$  can be given by

$$\frac{C(s)}{D(s)} = \frac{G_p}{1 + G_1 G_p G_c H}$$

**A-3-5.** Figure 3-49 shows a system with two inputs and two outputs. Derive  $C_1(s)/R_1(s)$ ,  $C_1(s)/R_2(s)$ ,  $C_2(s)/R_1(s)$ , and  $C_2(s)/R_2(s)$ . (In deriving outputs for  $R_1(s)$ , assume that  $R_2(s)$  is zero, and vice versa.)



**Figure 3-49**  
System with two  
inputs and two  
outputs.

**Solution.** From the figure, we obtain

$$C_1 = G_1(R_1 - G_3C_2) \quad (3-93)$$

$$C_2 = G_4(R_2 - G_2C_1) \quad (3-94)$$

By substituting Equation (3-94) into Equation (3-93), we obtain

$$C_1 = G_1[R_1 - G_3G_4(R_2 - G_2C_1)] \quad (3-95)$$

By substituting Equation (3-93) into Equation (3-94), we get

$$C_2 = G_4[R_2 - G_2G_1(R_1 - G_3C_2)] \quad (3-96)$$

Solving Equation (3-95) for  $C_1$ , we obtain

$$C_1 = \frac{G_1R_1 - G_1G_3G_4R_2}{1 - G_1G_2G_3G_4} \quad (3-97)$$

Solving Equation (3-96) for  $C_2$  gives

$$C_2 = \frac{-G_1G_2G_4R_1 + G_4R_2}{1 - G_1G_2G_3G_4} \quad (3-98)$$

Equations (3-97) and (3-98) can be combined in the form of the transfer matrix as follows:

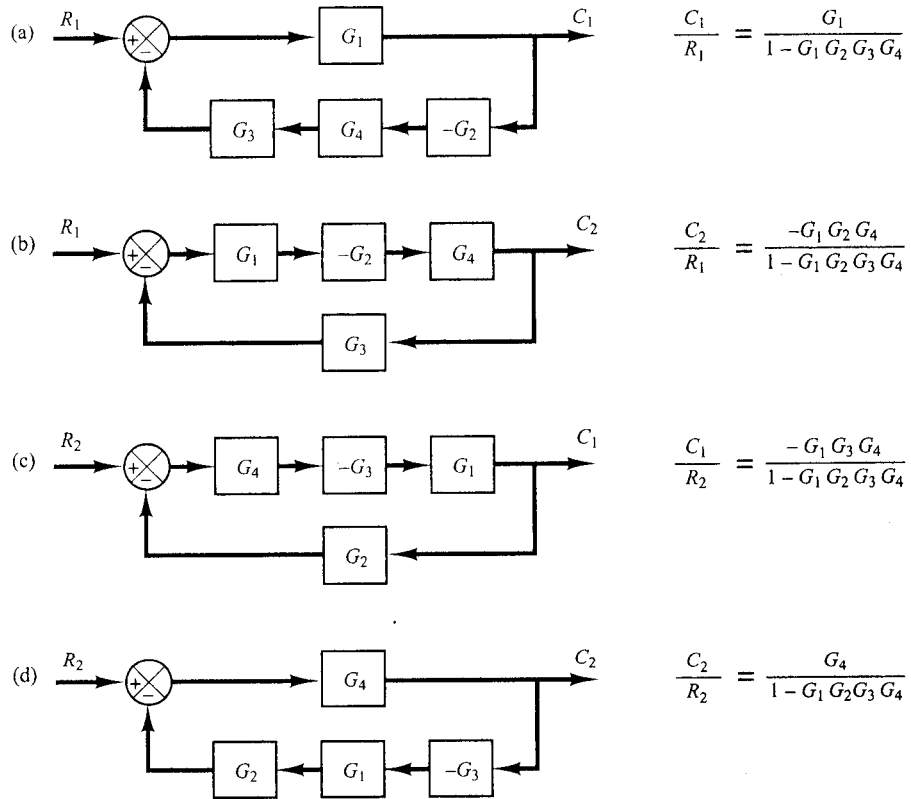
$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \frac{G_1}{1 - G_1G_2G_3G_4} & -\frac{G_1G_3G_4}{1 - G_1G_2G_3G_4} \\ -\frac{G_1G_2G_4}{1 - G_1G_2G_3G_4} & \frac{G_4}{1 - G_1G_2G_3G_4} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

Then the transfer functions  $C_1(s)/R_1(s)$ ,  $C_1(s)/R_2(s)$ ,  $C_2(s)/R_1(s)$  and  $C_2(s)/R_2(s)$  can be obtained as follows:

$$\begin{aligned} \frac{C_1(s)}{R_1(s)} &= \frac{G_1}{1 - G_1G_2G_3G_4}, & \frac{C_1(s)}{R_2(s)} &= -\frac{G_1G_3G_4}{1 - G_1G_2G_3G_4} \\ \frac{C_2(s)}{R_1(s)} &= -\frac{G_1G_2G_4}{1 - G_1G_2G_3G_4}, & \frac{C_2(s)}{R_2(s)} &= \frac{G_4}{1 - G_1G_2G_3G_4} \end{aligned}$$

Note that Equations (3-97) and (3-98) give responses  $C_1$  and  $C_2$ , respectively, when both inputs  $R_1$  and  $R_2$  are present.

Notice that when  $R_2(s) = 0$ , the original block diagram can be simplified to those shown in Figures 3-50(a) and (b). Similarly, when  $R_1(s) = 0$ , the original block diagram can be simplified to those shown in Figures 3-50(c) and (d). From these simplified block diagrams we can also obtain  $C_1(s)/R_1(s)$ ,  $C_2(s)/R_1(s)$ ,  $C_1(s)/R_2(s)$ , and  $C_2(s)/R_2(s)$ , as shown to the right of each corresponding block diagram.



**Figure 3-50**  
Simplified block  
diagrams and  
corresponding  
closed-loop transfer  
functions.

**A-3-6.** Show that for the differential equation system

$$\ddot{y} + a_1 \dot{y} + a_2 y = b_0 \ddot{u} + b_1 \dot{u} + b_2 \dot{u} + b_3 u \quad (3-99)$$

state and output equations can be given, respectively, by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u \quad (3-100)$$

and

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \beta_0 u \quad (3-101)$$

where state variables are defined by

$$x_1 = y - \beta_0 u$$

$$x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$

$$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$

and

$$\beta_0 = b_0$$

$$\beta_1 = b_1 - a_1\beta_0$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0$$

$$\beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0$$

**Solution.** From the definition of state variables  $x_2$  and  $x_3$ , we have

$$\dot{x}_1 = x_2 + \beta_1 u \quad (3-102)$$

$$\dot{x}_2 = x_3 + \beta_2 u \quad (3-103)$$

To derive the equation for  $\dot{x}_3$ , we first note from Equation (3-99) that

$$\ddot{y} = -a_1\dot{y} - a_2\dot{y} - a_3y + b_0\ddot{u} + b_1\dot{u} + b_2\dot{u} + b_3u$$

Since

$$x_3 = \dot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2u$$

we have

$$\begin{aligned} \dot{x}_3 &= \ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2\dot{u} \\ &= (-a_1\dot{y} - a_2\dot{y} - a_3y) + b_0\ddot{u} + b_1\dot{u} + b_2\dot{u} + b_3u - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2\dot{u} \\ &= -a_1(\dot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2u) - a_1\beta_0\ddot{u} - a_1\beta_1\dot{u} - a_1\beta_2u \\ &\quad - a_2(\dot{y} - \beta_0\ddot{u} - \beta_1\dot{u}) - a_2\beta_0\ddot{u} - a_2\beta_1\dot{u} - a_3(y - \beta_0u) - a_3\beta_0u \\ &\quad + b_0\ddot{u} + b_1\dot{u} + b_2\dot{u} + b_3u - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2\dot{u} \\ &= -a_1x_3 - a_2x_2 - a_3x_1 + (b_0 - \beta_0)\ddot{u} + (b_1 - \beta_1 - a_1\beta_0)\dot{u} \\ &\quad + (b_2 - \beta_2 - a_1\beta_1 - a_2\beta_0)\dot{u} + (b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0)u \\ &= -a_1x_3 - a_2x_2 - a_3x_1 + (b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0)u \\ &= -a_1x_3 - a_2x_2 - a_3x_1 + \beta_3u \end{aligned}$$

Hence, we get

$$\dot{x}_3 = -a_3x_1 - a_2x_2 - a_1x_3 + \beta_3u \quad (3-104)$$

Combining Equations (3-102), (3-103), and (3-104) into a vector-matrix equation, we obtain Equation (3-100). Also, from the definition of state variable  $x_1$ , we get the output equation given by Equation (3-101).

**A-3-7.** Obtain a state-space equation and output equation for the system defined by

$$\frac{Y(s)}{U(s)} = \frac{2s^3 + s^2 + s + 2}{s^3 + 4s^2 + 5s + 2}$$

**Solution.** From the given transfer function, the differential equation for the system is

$$\ddot{y} + 4\dot{y} + 5\dot{y} + 2y = 2\ddot{u} + \dot{u} + \dot{u} + 2u$$

Comparing this equation with the standard equation given by Equation (3-33), rewritten

$$\ddot{y} + a_1\dot{y} + a_2\dot{y} + a_3y = b_0\ddot{u} + b_1\dot{u} + b_2\dot{u} + b_3u$$

we find

$$\begin{aligned} a_1 &= 4, & a_2 &= 5, & a_3 &= 2 \\ b_0 &= 2, & b_1 &= 1, & b_2 &= 1, & b_3 &= 2 \end{aligned}$$

Referring to Equation (3-35), we have

$$\begin{aligned} \beta_0 &= b_0 = 2 \\ \beta_1 &= b_1 - a_1\beta_0 = 1 - 4 \times 2 = -7 \\ \beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 = 1 - 4 \times (-7) - 5 \times 2 = 19 \\ \beta_3 &= b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 \\ &= 2 - 4 \times 19 - 5 \times (-7) - 2 \times 2 = -43 \end{aligned}$$

Referring to Equation (3-34), we define

$$\begin{aligned} x_1 &= y - \beta_0 u = y - 2u \\ x_2 &= \dot{x}_1 - \beta_1 u = \dot{x}_1 + 7u \\ x_3 &= \dot{x}_2 - \beta_2 u = \dot{x}_2 - 19u \end{aligned}$$

Then referring to Equation (3-36),

$$\begin{aligned} \dot{x}_1 &= x_2 - 7u \\ \dot{x}_2 &= x_3 + 19u \\ \dot{x}_3 &= -a_3x_1 - a_2x_2 - a_1x_3 + \beta_3u \\ &= -2x_1 - 5x_2 - 4x_3 - 43u \end{aligned}$$

Hence, the state-space representation of the system is

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -7 \\ 19 \\ -43 \end{bmatrix} u \\ y &= [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2u \end{aligned}$$

This is one possible state-space representation of the system. There are many (infinitely many) others. If we use MATLAB, it produces the following state-space representation:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} -4 & -5 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \\ y &= [-7 \quad -9 \quad -2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2u \end{aligned}$$

See MATLAB Program 3-4. (Note that all state-space representations for the same system are equivalent.)



**MATLAB Program 3-4**

```

num = [2 1 1 2];
den = [1 4 5 2];
[A,B,C,D] = tf2ss(num, den)

```

A =

```

-4  -5  -2
 1   0   0
 0   1   0

```

B =

```

 1
 0
 0

```

C =

```

-7  -9  -2

```

D =

```

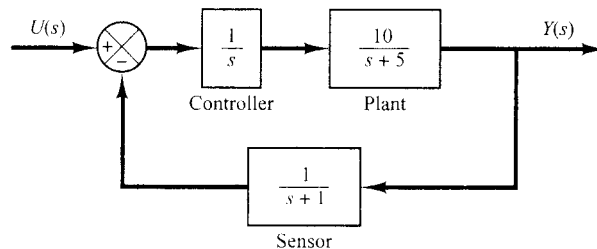
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**A-3-8.** Obtain a state-space model of the system shown in Figure 3-51.

**Solution.** The system involves one integrator and two delayed integrators. The output of each integrator or delayed integrator can be a state variable. Let us define the output of the plant as  $x_1$ , the output of the controller as  $x_2$ , and the output of the sensor as  $x_3$ . Then we obtain

$$\begin{aligned}\frac{X_1(s)}{X_2(s)} &= \frac{10}{s+5} \\ \frac{X_2(s)}{U(s) - X_3(s)} &= \frac{1}{s} \\ \frac{X_3(s)}{X_1(s)} &= \frac{1}{s+1} \\ Y(s) &= X_1(s)\end{aligned}$$



**Figure 3-51**  
Control system.

which can be rewritten as

$$sX_1(s) = -5X_1(s) + 10X_2(s)$$

$$sX_2(s) = -X_3(s) + U(s)$$

$$sX_3(s) = X_1(s) - X_3(s)$$

$$Y(s) = X_1(s)$$

By taking the inverse Laplace transforms of the preceding four equations, we obtain

$$\dot{x}_1 = -5x_1 + 10x_2$$

$$\dot{x}_2 = -x_3 + u$$

$$\dot{x}_3 = x_1 - x_3$$

$$y = x_1$$

Thus, a state-space model of the system in the standard form is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5 & 10 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

It is important to note that this is not the only state-space representation of the system. Many other state-space representations are possible. However, the number of state variables is the same in any state-space representation of the same system. In the present system, the number of state variables is three, regardless of what variables are chosen as state variables.

**A-3-9.** Obtain a state-space model for the system shown in Figure 3-52(a).

**Solution.** First, notice that  $(as + b)/s^2$  involves a derivative term. Such a derivative term may be avoided if we modify  $(as + b)/s^2$  as

$$\frac{as + b}{s^2} = \left( a + \frac{b}{s} \right) \frac{1}{s}$$

Using this modification, the block diagram of Figure 3-52(a) can be modified to that shown in Figure 3-52(b).

Define the outputs of the integrators as state variables, as shown in Figure 3-52(b). Then from Figure 3-52(b) we obtain

$$\frac{X_1(s)}{X_2(s) + a[U(s) - X_1(s)]} = \frac{1}{s}$$

$$\frac{X_2(s)}{U(s) - X_1(s)} = \frac{b}{s}$$

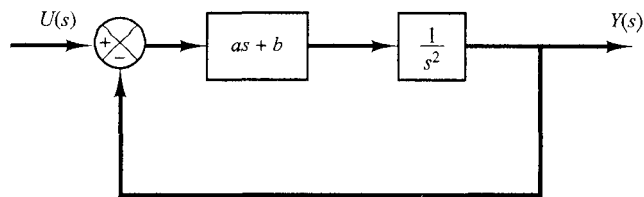
$$Y(s) = X_1(s)$$

which may be modified to

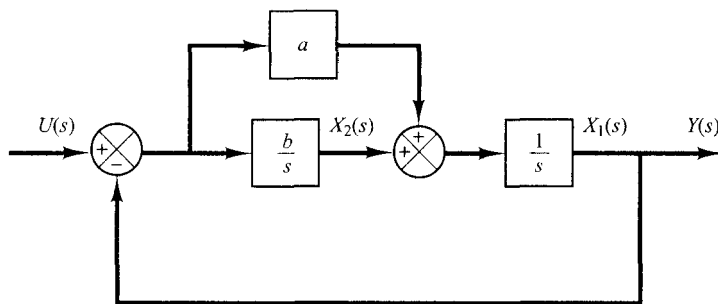
$$sX_1(s) = X_2(s) + a[U(s) - X_1(s)]$$

$$sX_2(s) = -bX_1(s) + bU(s)$$

$$Y(s) = X_1(s)$$



(a)



(b)

**Figure 3-52**

(a) Control system;  
(b) modified block  
diagram.

Taking the inverse Laplace transforms of the preceding three equations, we obtain

$$\dot{x}_1 = -ax_1 + x_2 + au$$

$$\dot{x}_2 = -bx_1 + bu$$

$$y = x_1$$

Rewriting the state and output equations in the standard vector-matrix form, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -a & 1 \\ -b & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

**A-3-10.** Obtain a state-space representation of the system shown in Figure 3-53(a).

**Solution.** In this problem, first expand  $(s + z)/(s + p)$  into partial fractions.

$$\frac{s + z}{s + p} = 1 + \frac{z - p}{s + p}$$

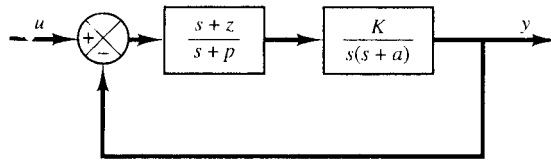
Next, convert  $K/[s(s + a)]$  into the product of  $K/s$  and  $1/(s + a)$ . Then redraw the block diagram, as shown in Figure 3-53(b). Defining a set of state variables, as shown in Figure 3-53(b), we obtain the following equations:

$$\dot{x}_1 = -ax_1 + x_2$$

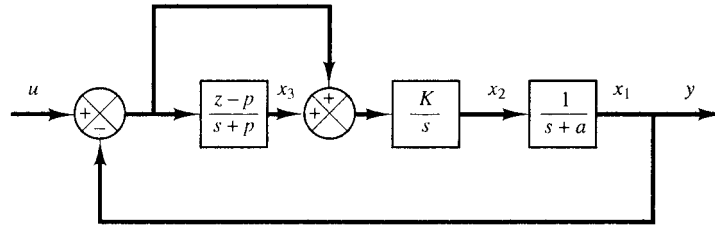
$$\dot{x}_2 = -Kx_1 + Kx_3 + Ku$$

$$\dot{x}_3 = -(z - p)x_1 - px_3 + (z - p)u$$

$$y = x_1$$



(a)



(b)

**Figure 3-53**

a) Control system;  
b) block diagram  
defining state  
variables for the  
system.

Rewriting gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a & 1 & 0 \\ -K & 0 & K \\ -(z-p) & 0 & -p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ K \\ z-p \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Notice that the output of the integrator and the outputs of the first-order delayed integrators  $[1/(s+a)$  and  $(z-p)/(s+p)]$  are chosen as state variables. It is important to remember that the output of the block  $(s+z)/(s+p)$  in Figure 3-53(a) cannot be a state variable, because this block involves a derivative term,  $s+z$ .

**A-3-11.** Obtain the transfer function of the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

**Solution.** Referring to Equation (3-29), the transfer function  $G(s)$  is given by

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

In this problem, matrices  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ , and  $D$  are

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad 0], \quad D = 0$$

Hence

$$\begin{aligned}
 G(s) &= [1 \quad 0 \quad 0] \begin{bmatrix} s+1 & -1 & 0 \\ 0 & s+1 & -1 \\ 0 & 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
 &= [1 \quad 0 \quad 0] \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2} & \frac{1}{(s+1)^2(s+2)} \\ 0 & \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
 &= \frac{1}{(s+1)^2(s+2)} = \frac{1}{s^3 + 4s^2 + 5s + 2}
 \end{aligned}$$

**A-3-12.** Obtain a state-space representation of the system shown in Figure 3-54.

**Solution.** The system equations are

$$m_1 \ddot{y}_1 + b \dot{y}_1 + k(y_1 - y_2) = 0$$

$$m_2 \ddot{y}_2 + k(y_2 - y_1) = u$$

The output variables for this system are  $y_1$  and  $y_2$ . Define state variables as

$$x_1 = y_1$$

$$x_2 = \dot{y}_1$$

$$x_3 = y_2$$

$$x_4 = \dot{y}_2$$

Then we obtain the following equations:

$$\dot{x}_1 = x_2$$

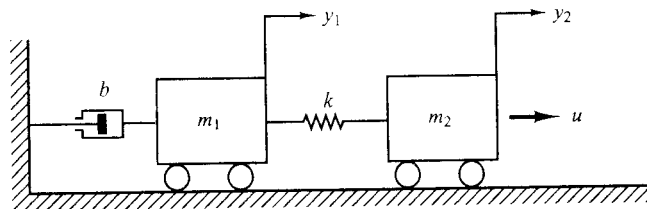
$$\dot{x}_2 = \frac{1}{m_1} [-b \dot{y}_1 - k(y_1 - y_2)] = -\frac{k}{m_1} x_1 - \frac{b}{m_1} x_2 + \frac{k}{m_1} x_3$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{1}{m_2} [-k(y_2 - y_1) + u] = \frac{k}{m_2} x_1 - \frac{k}{m_2} x_3 + \frac{1}{m_2} u$$

Hence, the state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & -\frac{b}{m_1} & \frac{k}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & 0 & -\frac{k}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} u$$



**Figure 3-54**  
Mechanical system.

and the output equation is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

- A-3-13.** Consider a system with multiple inputs and multiple outputs. When the system has more than one output, the command

$$[\text{NUM}, \text{den}] = \text{ss2tf}(\text{A}, \text{B}, \text{C}, \text{D}, \text{i}, \text{u})$$

produces transfer functions for all outputs to each input. (The numerator coefficients are returned to matrix NUM with as many rows as there are outputs.)

Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This system involves two inputs and two outputs. Four transfer functions are involved:  $Y_1(s)/U_1(s)$ ,  $Y_2(s)/U_1(s)$ ,  $Y_1(s)/U_2(s)$ , and  $Y_2(s)/U_2(s)$ . (When considering input  $u_1$ , we assume that input  $u_2$  is zero and vice versa.)

**Solution.** MATLAB Program 3-5 produces four transfer functions.

MATLAB Program 3-5			
A = [0 1;-25 -4];			
B = [1 1;0 1];			
C = [1 0;0 1];			
D = [0 0;0 0];			
[NUM,den] = ss2tf(A,B,C,D,1)			
NUM =			
	0	1	4
	0	0	-25
den =			
	1	4	25
[NUM,den] = ss2tf(A,B,C,D,2)			
NUM =			
	0	1.0000	5.0000
	0	1.0000	-25.0000
den =			
	1	4	25

This is the MATLAB representation of the following four transfer functions:

$$\begin{aligned}\frac{Y_1(s)}{U_1(s)} &= \frac{s + 4}{s^2 + 4s + 25}, & \frac{Y_2(s)}{U_1(s)} &= \frac{-25}{s^2 + 4s + 25} \\ \frac{Y_1(s)}{U_2(s)} &= \frac{s + 5}{s^2 + 4s + 25}, & \frac{Y_2(s)}{U_2(s)} &= \frac{s - 25}{s^2 + 4s + 25}\end{aligned}$$

- A-3-14.** Obtain the equivalent spring constants for the systems shown in Figures 3-55(a) and (b), respectively.

**Solution.** For the springs in parallel [Figure 3-55(a)] the equivalent spring constant  $k_{eq}$  is obtained from

$$k_1 x + k_2 x = F = k_{eq} x$$

or

$$k_{eq} = k_1 + k_2$$

For the springs in series [Figure-55(b)], the force in each spring is the same. Thus

$$k_1 y = F, \quad k_2(x - y) = F$$

Elimination of  $y$  from these two equations results in

$$k_2 \left( x - \frac{F}{k_1} \right) = F$$

or

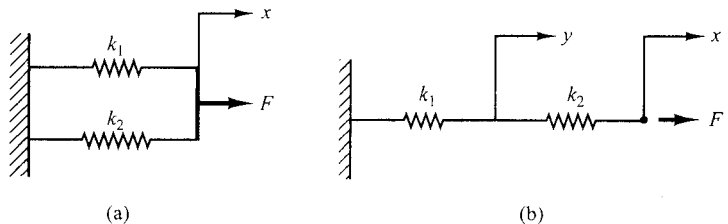
$$k_2 x = F + \frac{k_2}{k_1} F = \frac{k_1 + k_2}{k_1} F$$

The equivalent spring constant  $k_{eq}$  for this case is then found as

$$k_{eq} = \frac{F}{x} = \frac{k_1 k_2}{k_1 + k_2} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$$

**Figure 3-55**

(a) System consisting of two springs in parallel;  
(b) system consisting of two springs in series.



**A-3-15.** Obtain the equivalent viscous-friction coefficient  $b_{eq}$  for each of the systems shown in Figure 3-56(a) and (b).

**Solution.**

(a) The force  $f$  due to the dampers is

$$f = b_1(\dot{y} - \dot{x}) + b_2(\dot{y} - \dot{x}) = (b_1 + b_2)(\dot{y} - \dot{x})$$

In terms of the equivalent viscous friction coefficient  $b_{eq}$ , force  $f$  is given by

$$f = b_{eq}(\dot{y} - \dot{x})$$

Hence

$$b_{eq} = b_1 + b_2$$

(b) The force  $f$  due to the dampers is

$$f = b_1(\dot{z} - \dot{x}) = b_2(\dot{y} - \dot{z}) \quad (3-105)$$

where  $z$  is the displacement of a point between damper  $b_1$  and damper  $b_2$ . (Note that the same force is transmitted through the shaft.) From Equation (3-105), we have

$$(b_1 + b_2)\dot{z} = b_2\dot{y} + b_1\dot{x}$$

or

$$\dot{z} = \frac{1}{b_1 + b_2} (b_2\dot{y} + b_1\dot{x}) \quad (3-106)$$

In terms of the equivalent viscous friction coefficient  $b_{eq}$ , force  $f$  is given by

$$f = b_{eq}(\dot{y} - \dot{x})$$

By substituting Equation (3-106) into Equation (3-105), we have

$$\begin{aligned} f &= b_2(\dot{y} - \dot{z}) = b_2 \left[ \dot{y} - \frac{1}{b_1 + b_2} (b_2\dot{y} + b_1\dot{x}) \right] \\ &= \frac{b_1 b_2}{b_1 + b_2} (\dot{y} - \dot{x}) \end{aligned}$$

Thus,

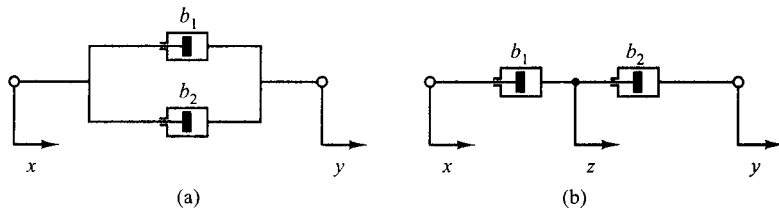
$$f = b_{eq}(\dot{y} - \dot{x}) = \frac{b_1 b_2}{b_1 + b_2} (\dot{y} - \dot{x})$$

Hence,

$$b_{eq} = \frac{b_1 b_2}{b_1 + b_2} = \frac{1}{\frac{1}{b_1} + \frac{1}{b_2}}$$

**Figure 3-56**

(a) Two dampers connected in parallel;  
(b) two dampers connected in series.





**A-3-16.**

Figure 3-57(a) shows a schematic diagram of an automobile suspension system. As the car moves along the road, the vertical displacements at the tires act as the motion excitation to the automobile suspension system. The motion of this system consists of a translational motion of the center of mass and a rotational motion about the center of mass. Mathematical modeling of the complete system is quite complicated.

A very simplified version of the suspension system is shown in Figure 3-57(b). Assuming that the motion  $x_i$  at point  $P$  is the input to the system and the vertical motion  $x_o$  of the body is the output, obtain the transfer function  $X_o(s)/X_i(s)$ . (Consider the motion of the body only in the vertical direction.) Displacement  $x_o$  is measured from the equilibrium position in the absence of input  $x_i$ .

**Solution.** The equation of motion for the system shown in Figure 3-57(b) is

$$m\ddot{x}_o + b(\dot{x}_o - \dot{x}_i) + k(x_o - x_i) = 0$$

or

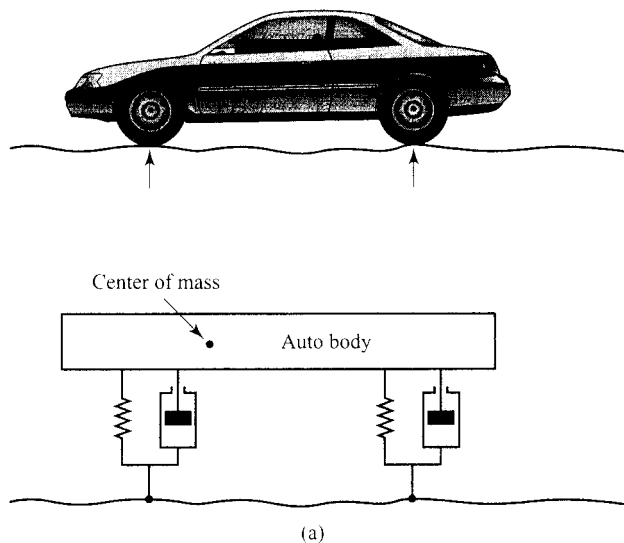
$$m\ddot{x}_o + b\dot{x}_o + kx_o = b\dot{x}_i + kx_i$$

Taking the Laplace transform of this last equation, assuming zero initial conditions, we obtain

$$(ms^2 + bs + k)X_o(s) = (bs + k)X_i(s)$$

Hence the transfer function  $X_o(s)/X_i(s)$  is given by

$$\frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms^2 + bs + k}$$



**Figure 3-57**  
(a) Automobile suspension system;  
(b) simplified suspension system.

**A-3-17.** Obtain the transfer function  $Y(s)/U(s)$  of the system shown in Figure 3-58. The input  $u$  is a displacement input. (Like the system of Problem A-3-16, this is also a simplified version of an automobile or motorcycle suspension system.)

**Solution.** Assume that displacements  $x$  and  $y$  are measured from respective steady-state positions in the absence of the input  $u$ . Applying the Newton's second law to this system, we obtain

$$m_1 \ddot{x} = k_2(y - x) + b(\dot{y} - \dot{x}) + k_1(u - x)$$

$$m_2 \ddot{y} = -k_2(y - x) - b(\dot{y} - \dot{x})$$

Hence, we have

$$m_1 \ddot{x} + b\dot{x} + (k_1 + k_2)x = b\dot{y} + k_2y + k_1u$$

$$m_2 \ddot{y} + b\dot{y} + k_2y = b\dot{x} + k_2x$$

Taking Laplace transforms of these two equations, assuming zero initial conditions, we obtain

$$[m_1 s^2 + bs + (k_1 + k_2)]X(s) = (bs + k_2)Y(s) + k_1U(s)$$

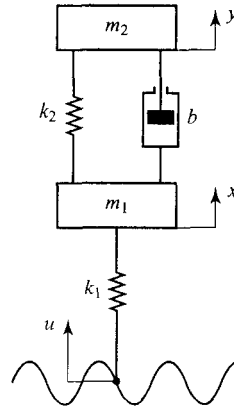
$$[m_2 s^2 + bs + k_2]Y(s) = (bs + k_2)X(s)$$

Eliminating  $X(s)$  from the last two equations, we have

$$(m_1 s^2 + bs + k_1 + k_2) \frac{m_2 s^2 + bs + k_2}{bs + k_2} Y(s) = (bs + k_2)Y(s) + k_1U(s)$$

which yields

$$\frac{Y(s)}{U(s)} = \frac{k_1(bs + k_2)}{m_1 m_2 s^4 + (m_1 + m_2)bs^3 + [k_1 m_2 + (m_1 + m_2)k_2]s^2 + k_1 bs + k_1 k_2}$$



**Figure 3-58**  
Suspension system.

**A-3-18.** Obtain the transfer function of the mechanical system shown in Figure 3-59(a). Also obtain the transfer function of the electrical system shown in Figure 3-59(b). Show that the transfer functions of the two systems are of identical form and thus they are analogous systems.

**Solution.** In Figure 3-59(a) we assume that displacements  $x_i$ ,  $x_o$ , and  $y$  are measured from their respective steady-state positions. Then the equations of motion for the mechanical system shown in Figure 3-59(a) are

$$b_1(\dot{x}_i - \dot{x}_o) + k_1(x_i - x_o) = b_2(\dot{x}_o - \dot{y})$$

$$b_2(\dot{x}_o - \dot{y}) = k_2 y$$

By taking the Laplace transforms of these two equations, assuming zero initial conditions, we have

$$b_1[sX_i(s) - sX_o(s)] + k_1[X_i(s) - X_o(s)] = b_2[sX_o(s) - sY(s)]$$

$$b_2[sX_o(s) - sY(s)] = k_2 Y(s)$$

If we eliminate  $Y(s)$  from the last two equations, then we obtain

$$b_1[sX_i(s) - sX_o(s)] + k_1[X_i(s) - X_o(s)] = b_2 s X_o(s) - b_2 s \frac{b_2 s X_o(s)}{b_2 s + k_2}$$

or

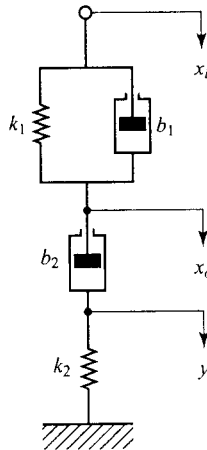
$$(b_1 s + k_1)X_i(s) = \left( b_1 s + k_1 + b_2 s - b_2 s \frac{b_2 s}{b_2 s + k_2} \right) X_o(s)$$

Hence the transfer function  $X_o(s)/X_i(s)$  can be obtained as

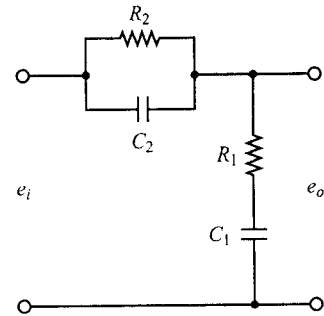
$$\frac{X_o(s)}{X_i(s)} = \frac{\left( \frac{b_1}{k_1} s + 1 \right) \left( \frac{b_2}{k_2} s + 1 \right)}{\left( \frac{b_1}{k_1} s + 1 \right) \left( \frac{b_2}{k_2} s + 1 \right) + \frac{b_2}{k_1} s}$$

For the electrical system shown in Figure 3-59(b), the transfer function  $E_o(s)/E_i(s)$  is found to be

$$\begin{aligned} \frac{E_o(s)}{E_i(s)} &= \frac{R_1 + \frac{1}{C_1 s}}{\frac{1}{(1/R_2) + C_2 s} + R_1 + \frac{1}{C_1 s}} \\ &= \frac{(R_1 C_1 s + 1)(R_2 C_2 s + 1)}{(R_1 C_1 s + 1)(R_2 C_2 s + 1) + R_2 C_1 s} \end{aligned}$$



(a)



(b)

**Figure 3-59**

(a) Mechanical system;  
(b) analogous electrical system.

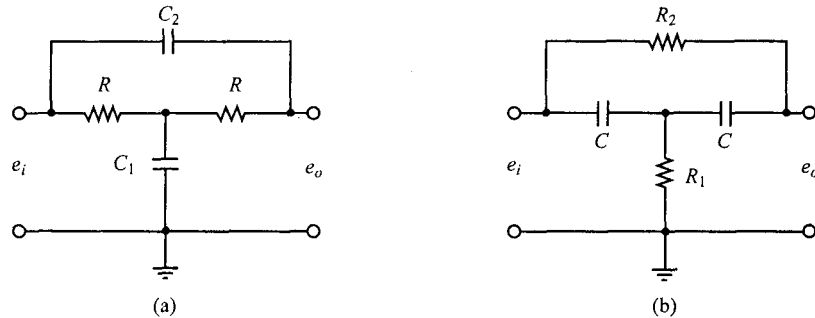
A comparison of the transfer functions shows that the systems shown in Figures 3-59(a) and (b) are analogous.

- A-3-19.** Obtain the transfer functions  $E_o(s)/E_i(s)$  of the bridged  $T$  networks shown in Figures 3-60(a) and (b).

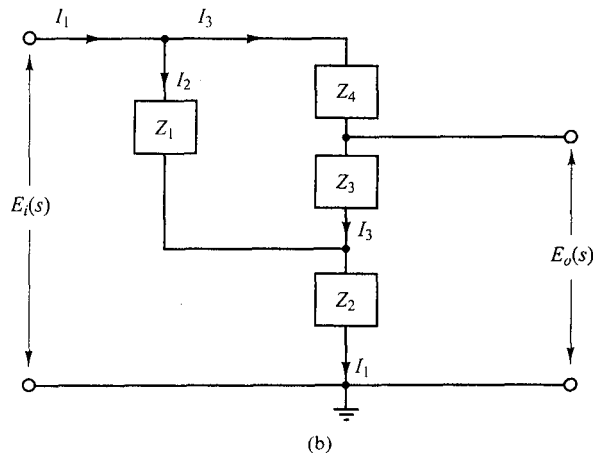
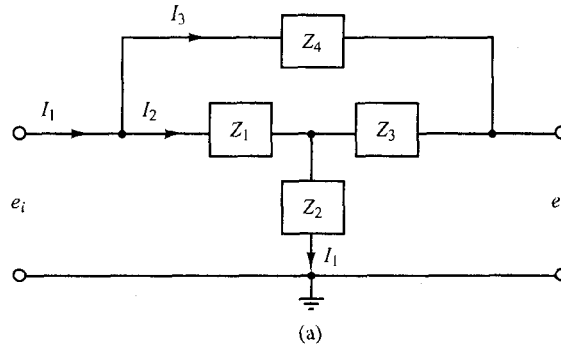
**Solution.** The bridged  $T$  networks shown can both be represented by the network of Figure 3-61(a), where we used complex impedances. This network may be modified to that, shown in Figure 3-61(b).

In Figure 3-61(b), note that

$$I_1 = I_2 + I_3, \quad I_2 Z_1 = (Z_3 + Z_4) I_3$$



**Figure 3-60**  
Bridged  $T$  networks.



**Figure 3-61**  
(a) Bridged  $T$  network in terms of complex impedances;  
(b) equivalent network.

Hence

$$I_2 = \frac{Z_3 + Z_4}{Z_1 + Z_3 + Z_4} I_1, \quad I_3 = \frac{Z_1}{Z_1 + Z_3 + Z_4} I_1$$

Then the voltages  $E_i(s)$  and  $E_o(s)$  can be obtained as

$$\begin{aligned} E_i(s) &= Z_1 I_2 + Z_2 I_1 \\ &= \left[ Z_2 + \frac{Z_1(Z_3 + Z_4)}{Z_1 + Z_3 + Z_4} \right] I_1 \\ &= \frac{Z_2(Z_1 + Z_3 + Z_4) + Z_1(Z_3 + Z_4)}{Z_1 + Z_3 + Z_4} I_1 \end{aligned}$$

$$\begin{aligned} E_o(s) &= Z_3 I_3 + Z_2 I_1 \\ &= \frac{Z_3 Z_1}{Z_1 + Z_3 + Z_4} I_1 + Z_2 I_1 \\ &= \frac{Z_3 Z_1 + Z_2(Z_1 + Z_3 + Z_4)}{Z_1 + Z_3 + Z_4} I_1 \end{aligned}$$

Hence, the transfer function  $E_o(s)/E_i(s)$  of the network shown in Figure 3–61(a) is obtained as

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_3 Z_1 + Z_2(Z_1 + Z_3 + Z_4)}{Z_2(Z_1 + Z_3 + Z_4) + Z_1 Z_3 + Z_1 Z_4} \quad (3-107)$$

For the bridged T network shown in Figure 3–60(a), substitute

$$Z_1 = R, \quad Z_2 = \frac{1}{C_1 s}, \quad Z_3 = R, \quad Z_4 = \frac{1}{C_2 s}$$

into Equation (3–107). Then, we obtain the transfer function  $E_o(s)/E_i(s)$  to be

$$\begin{aligned} \frac{E_o(s)}{E_i(s)} &= \frac{R^2 + \frac{1}{C_1 s} \left( R + R + \frac{1}{C_2 s} \right)}{\frac{1}{C_1 s} \left( R + R + \frac{1}{C_2 s} \right) + R^2 + R \frac{1}{C_2 s}} \\ &= \frac{RC_1 RC_2 s^2 + 2RC_2 s + 1}{RC_1 RC_2 s^2 + (2RC_2 + RC_1)s + 1} \end{aligned}$$

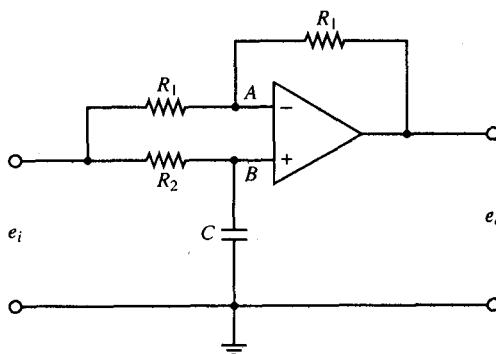
Similarly, for the bridged T network shown in Figure 3–60(b), we substitute

$$Z_1 = \frac{1}{Cs}, \quad Z_2 = R_1, \quad Z_3 = \frac{1}{Cs}, \quad Z_4 = R_2$$

into Equation (3–107). Then the transfer function  $E_o(s)/E_i(s)$  can be obtained as follows:

$$\begin{aligned} \frac{E_o(s)}{E_i(s)} &= \frac{\frac{1}{Cs} \frac{1}{Cs} + R_1 \left( \frac{1}{Cs} + \frac{1}{Cs} + R_2 \right)}{R_1 \left( \frac{1}{Cs} + \frac{1}{Cs} + R_2 \right) + \frac{1}{Cs} \frac{1}{Cs} + R_2 \frac{1}{Cs}} \\ &= \frac{R_1 C R_2 C s^2 + 2R_1 C s + 1}{R_1 C R_2 C s^2 + (2R_1 C + R_2 C)s + 1} \end{aligned}$$

**Figure 3-62**  
Operational-  
amplifier circuit.



**A-3-20.** Obtain the transfer function  $E_o(s)/E_i(s)$  of the op-amp circuit shown in Figure 3-62.

**Solution.** The voltage at point  $A$  is

$$e_A = \frac{1}{2}(e_i - e_o) + e_o$$

The Laplace-transformed version of this last equation is

$$E_A(s) = \frac{1}{2}[E_i(s) + E_o(s)]$$

The voltage at point  $B$  is

$$E_B(s) = \frac{\frac{1}{Cs}}{R_2 + \frac{1}{Cs}} E_i(s) = \frac{1}{R_2Cs + 1} E_i(s)$$

Since  $[E_B(s) - E_A(s)]K = E_o(s)$  and  $K \gg 1$ , we must have  $E_A(s) = E_B(s)$ . Thus

$$\frac{1}{2}[E_i(s) + E_o(s)] = \frac{1}{R_2Cs + 1} E_i(s)$$

Hence

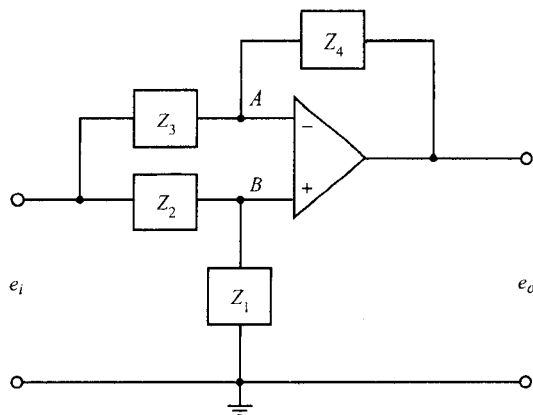
$$\frac{E_o(s)}{E_i(s)} = -\frac{R_2Cs - 1}{R_2Cs + 1} = -\frac{s - \frac{1}{R_2C}}{s + \frac{1}{R_2C}}$$

**A-3-21.** Obtain the transfer function  $E_o(s)/E_i(s)$  of the op-amp system shown in Figure 3-63 in terms of complex impedances  $Z_1, Z_2, Z_3$ , and  $Z_4$ . Using the equation derived, obtain the transfer function  $E_o(s)/E_i(s)$  of the op-amp system shown in Figure 3-62.

**Solution.** From Figure 3-63, we find

$$\frac{E_i(s) - E_A(s)}{Z_3} = \frac{E_A(s) - E_o(s)}{Z_4}$$

**Figure 3-63**  
Operational-  
amplifier circuit.



or

$$E_i(s) - \left(1 + \frac{Z_3}{Z_4}\right)E_A(s) = -\frac{Z_3}{Z_4}E_o(s) \quad (3-108)$$

Since

$$E_A(s) = E_B(s) = \frac{Z_1}{Z_1 + Z_2}E_i(s) \quad (3-109)$$

by substituting Equation (3-109) into Equation (3-108), we obtain

$$\left[ \frac{Z_4Z_1 + Z_4Z_2 - Z_4Z_1 - Z_3Z_1}{Z_4(Z_1 + Z_2)} \right] E_i(s) = -\frac{Z_3}{Z_4}E_o(s)$$

from which we get the transfer function  $E_o(s)/E_i(s)$  to be

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_4Z_2 - Z_3Z_1}{Z_3(Z_1 + Z_2)} \quad (3-110)$$

To find the transfer function  $E_o(s)/E_i(s)$  of the circuit shown in Figure 3-62, we substitute

$$Z_1 = \frac{1}{Cs}, \quad Z_2 = R_2, \quad Z_3 = R_1, \quad Z_4 = R_1$$

into Equation (3-110). The result is

$$\frac{E_o(s)}{E_i(s)} = -\frac{R_1R_2 - R_1\frac{1}{Cs}}{R_1\left(\frac{1}{Cs} + R_2\right)} = -\frac{R_2Cs - 1}{R_2Cs + 1}$$

which is, as a matter of course, the same as that obtained in Problem A-3-20.

**A-3-22.** Obtain the transfer function  $E_o(s)/E_i(s)$  of the operational-amplifier circuit shown in Figure 3-64.

**Solution.** We will first obtain currents  $i_1, i_2, i_3, i_4$ , and  $i_5$ . Then we will use node equations at nodes  $A$  and  $B$ .

$$i_1 = \frac{e_i - e_A}{R_1}; \quad i_2 = \frac{e_A - e_o}{R_3}, \quad i_3 = C_1 \frac{de_A}{dt}$$

$$i_4 = \frac{e_A}{R_2}, \quad i_5 = C_2 \frac{-de_o}{dt}$$

At node  $A$ , we have  $i_1 = i_2 + i_3 + i_4$ , or

$$\frac{e_i - e_A}{R_1} = \frac{e_A - e_o}{R_3} + C_1 \frac{de_A}{dt} + \frac{e_A}{R_2} \quad (3-111)$$

At node  $B$ , we get  $i_4 = i_5$ , or

$$\frac{e_A}{R_2} = C_2 \frac{-de_o}{dt} \quad (3-112)$$

By rewriting Equation (3-111), we have

$$C_1 \frac{de_A}{dt} + \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) e_A = \frac{e_i}{R_1} + \frac{e_o}{R_3} \quad (3-113)$$

From Equation (3-112), we get

$$e_A = -R_2 C_2 \frac{de_o}{dt} \quad (3-114)$$

By substituting Equation (3-114) into Equation (3-113), we obtain

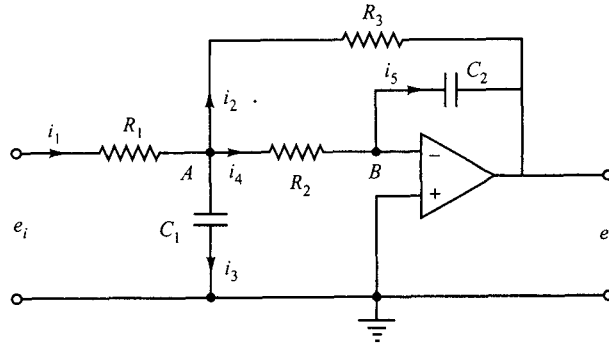
$$C_1 \left( -R_2 C_2 \frac{d^2 e_o}{dt^2} \right) + \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) (-R_2 C_2) \frac{de_o}{dt} = \frac{e_i}{R_1} + \frac{e_o}{R_3}$$

Taking the Laplace transform of this last equation, assuming zero initial conditions, we obtain

$$-C_1 C_2 R_2 s^2 E_o(s) + \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) (-R_2 C_2) s E_o(s) - \frac{1}{R_3} E_o(s) = \frac{E_i(s)}{R_1}$$

from which we get the transfer function  $E_o(s)/E_i(s)$  as follows:

$$\frac{E_o(s)}{E_i(s)} = - \frac{1}{R_1 C_1 R_2 C_2 s^2 + [R_2 C_2 + R_1 C_2 + (R_1/R_3) R_2 C_2] s + (R_1/R_3)}$$



**Figure 3-64**  
Operational-  
amplifier circuit.



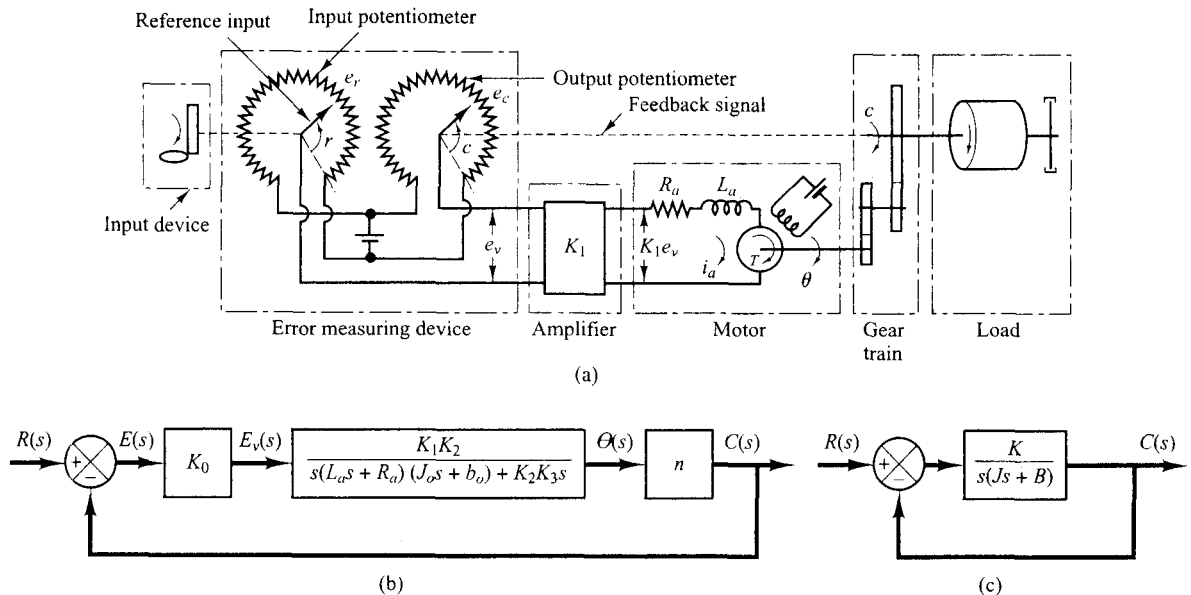
**A-3-23.** Consider the servo system shown in Figure 3-65(a). The motor shown is a servomotor, a dc motor designed specifically to be used in a control system. The operation of this system is as follows: A pair of potentiometers acts as an error-measuring device. They convert the input and output positions into proportional electric signals. The command input signal determines the angular position  $r$  of the wiper arm of the input potentiometer. The angular position  $r$  is the reference input to the system, and the electric potential of the arm is proportional to the angular position of the arm. The output shaft position determines the angular position  $c$  of the wiper arm of the output potentiometer. The difference between the input angular position  $r$  and the output angular position  $c$  is the error signal  $e$ , or

$$e = r - c$$

The potential difference  $e_r - e_c = e_v$  is the error voltage, where  $e_r$  is proportional to  $r$  and  $e_c$  is proportional to  $c$ ; that is,  $e_r = K_0 r$  and  $e_c = K_0 c$ , where  $K_0$  is a proportionality constant. The error voltage that appears at the potentiometer terminals is amplified by the amplifier whose gain constant is  $K_1$ . The output voltage of this amplifier is applied to the armature circuit of the dc motor. A fixed voltage is applied to the field winding. If an error exists, the motor develops a torque to rotate the output load in such a way as to reduce the error to zero. For constant field current, the torque developed by the motor is

$$T = K_2 i_a$$

where  $K_2$  is the motor torque constant and  $i_a$  is the armature current.



**Figure 3-65**

(a) Schematic diagram of servo system; (b) block diagram for the system; (c) simplified block diagram.

When the armature is rotating, a voltage proportional to the product of the flux and angular velocity is induced in the armature. For a constant flux, the induced voltage  $e_b$  is directly proportional to the angular velocity  $d\theta/dt$ , or

$$e_b = K_3 \frac{d\theta}{dt}$$

where  $e_b$  is the back emf,  $K_3$  is the back emf constant of the motor, and  $\theta$  is the angular displacement of the motor shaft.

Obtain the transfer function between the motor shaft angular displacement  $\theta$  and the error voltage  $e_v$ . Obtain also a block diagram for this system and a simplified block diagram when  $L_a$  is negligible.

**Solution.** The speed of an armature-controlled dc servomotor is controlled by the armature voltage  $e_a$ . (The armature voltage  $e_a = K_1 e_v$  is the output of the amplifier.) The differential equation for the armature circuit is

$$L_a \frac{di_a}{dt} + R_a i_a + e_b = e_a$$

or

$$L_a \frac{di_a}{dt} + R_a i_a + K_3 \frac{d\theta}{dt} = K_1 e_v \quad (3-115)$$

The equation for torque equilibrium is

$$J_0 \frac{d^2\theta}{dt^2} + b_0 \frac{d\theta}{dt} = T = K_2 i_a \quad (3-116)$$

where  $J_0$  is the inertia of the combination of the motor, load, and gear train referred to the motor shaft and  $b_0$  is the viscous-friction coefficient of the combination of the motor, load, and gear train referred to the motor shaft.

By eliminating  $i_a$  from Equations (3-115) and (3-116), we obtain

$$\frac{\Theta(s)}{E_v(s)} = \frac{K_1 K_2}{s(L_a s + R_a)(J_0 s + b_0) + K_2 K_3} \quad (3-117)$$

We assume that the gear ratio of the gear train is such that the output shaft rotates  $n$  times for each revolution of the motor shaft. Thus,

$$C(s) = n\Theta(s) \quad (3-118)$$

The relationship among  $E_v(s)$ ,  $R(s)$ , and  $C(s)$  is

$$E_v(s) = K_0[R(s) - C(s)] = K_0 E(s) \quad (3-119)$$

The block diagram of this system can be constructed from Equations (3-117), (3-118), and (3-119), as shown in Figure 3-65(b). The transfer function in the feedforward path of this system is

$$G(s) = \frac{C(s)}{\Theta(s)} \frac{\Theta(s)}{E_v(s)} \frac{E_v(s)}{E(s)} = \frac{K_0 K_1 K_2 n}{s[(L_a s + R_a)(J_0 s + b_0) + K_2 K_3]}$$

When  $L_a$  is small, it can be neglected, and the transfer function  $G(s)$  in the feedforward path becomes

$$\begin{aligned} G(s) &= \frac{K_0 K_1 K_2 n}{s[R_a(J_0 s + b_0) + K_2 K_3]} \\ &= \frac{K_0 K_1 K_2 n / R_a}{J_0 s^2 + \left(b_0 + \frac{K_2 K_3}{R_a}\right)s} \end{aligned} \quad (3-120)$$

The term  $[b_0 + (K_2 K_3 / R_a)]s$  indicates that the back emf of the motor effectively increases the viscous friction of the system. The inertia  $J_0$  and viscous friction coefficient  $b_0 + (K_2 K_3 / R_a)$  are referred to the motor shaft. When  $J_0$  and  $b_0 + (K_2 K_3 / R_a)$  are multiplied by  $1/n^2$ , the inertia and viscous-friction coefficient are expressed in terms of the output shaft. Introducing new parameters defined by

$$J = J_0 / n^2 = \text{moment of inertia referred to the output shaft}$$

$$B = [b_0 + (K_2 K_3 / R_a)] / n^2 = \text{viscous-friction coefficient referred to the output shaft}$$

$$K = K_0 K_1 K_2 / n R_a$$

the transfer function  $G(s)$  given by Equation (3-120) can be simplified, yielding

$$G(s) = \frac{K}{Js^2 + Bs}$$

or

$$G(s) = \frac{K_m}{s(T_m s + 1)}$$

where

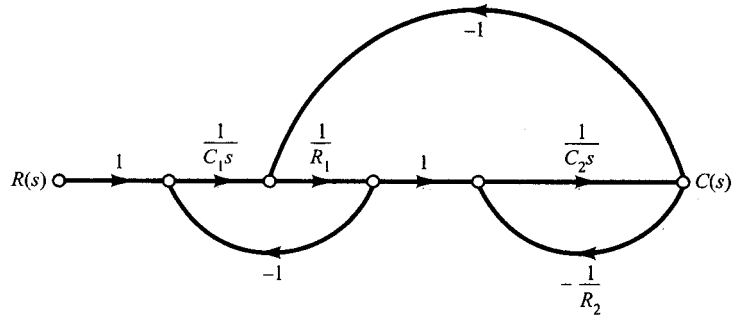
$$K_m = \frac{K}{B}, \quad T_m = \frac{J}{B} = \frac{R_a J_0}{R_a b_0 + K_2 K_3}$$

The block diagram of the system shown in Figure 3-65(b) can thus be simplified as shown in Figure 3-65(c).

**A-3-24.** Consider the system shown in Figure 3-66. Obtain the closed-loop transfer function  $C(s)/R(s)$ .

**Solution.** In this system there is only one forward path that connects the input  $R(s)$  and the Output  $C(s)$ . Thus,

$$P_1 = \frac{1}{C_1 s} \frac{1}{R_1} \frac{1}{C_2 s}$$



**Figure 3-66**  
Signal flow graph of  
a control system.

There are three individual loops. Thus,

$$L_1 = -\frac{1}{C_1 s} \frac{1}{R_1}$$

$$L_2 = -\frac{1}{C_2 s} \frac{1}{R_2}$$

$$L_3 = -\frac{1}{R_1} \frac{1}{C_2 s}$$

Loop  $L_1$  does not touch loop  $L_2$ . (Loop  $L_1$  touches loop  $L_3$ , and loop  $L_2$  touches loop  $L_3$ .) Hence the determinant  $\Delta$  is given by

$$\begin{aligned} \Delta &= 1 - (L_1 + L_2 + L_3) + (L_1 L_2) \\ &= 1 + \frac{1}{R_1 C_1 s} + \frac{1}{R_2 C_2 s} + \frac{1}{R_1 C_2 s} + \frac{1}{R_1 C_1 R_2 C_2 s^2} \end{aligned}$$

Since all three loops touch the forward path  $P_1$ , we remove  $L_1, L_2, L_3$ , and  $L_1 L_2$  from  $\Delta$  and evaluate the cofactor  $\Delta_1$  as follows:

$$\Delta_1 = 1$$

Thus we obtain the closed-loop transfer function to be

$$\begin{aligned} \frac{C(s)}{R(s)} &= \frac{P_1 \Delta_1}{\Delta} \\ &= \frac{1}{1 + \frac{1}{R_1 C_1 s} + \frac{1}{R_2 C_2 s} + \frac{1}{R_1 C_2 s} + \frac{1}{R_1 C_1 R_2 C_2 s^2}} \\ &= \frac{R_2}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_2 C_1)s + 1} \end{aligned}$$

**A-3-25.** Obtain the transfer function  $Y(s)/X(s)$  of the system shown in Figure 3-67.

**Solution.** The signal flow graph shown in Figure 3-67 can be successively simplified as shown in Figures 3-68 (a), (b), and (c). From Figure 3-68(c),  $X_3$  can be written as

$$X_3 = \frac{1}{s^2} X + \frac{a_1 s + a_2}{s^2} X_3$$

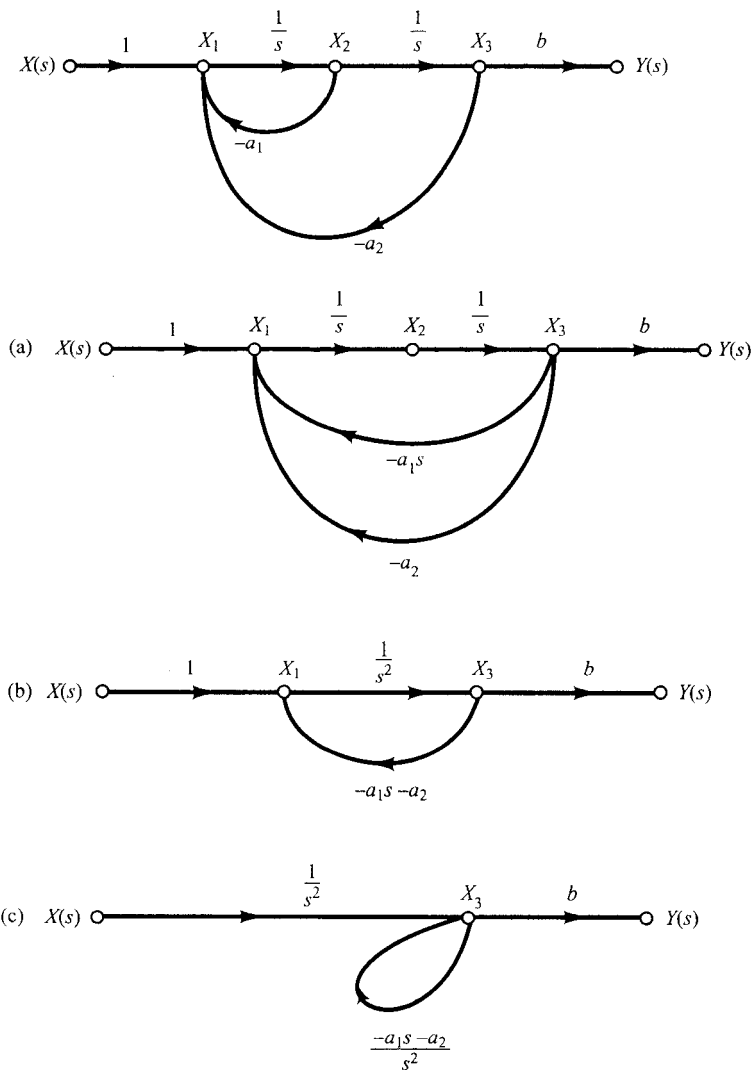
This last equation can be simplified as

$$(s^2 + a_1 s + a_2) X_3 = X$$

from which we obtain

$$\frac{Y(s)}{X(s)} = \frac{b X_3}{X} = \frac{b}{s^2 + a_1 s + a_2}$$

**Figure 3-67**  
Signal flow graph of  
a system.



**Figure 3-68**  
Successive  
simplifications of the  
signal flow graph of  
Figure 3-67.

**A-3-26.** Figure 3-69 is the block diagram of an engine-speed control system. The speed is measured by a set of flyweights. Draw a signal flow graph for this system.

**Solution.** Referring to Figure 3-36(e), a signal flow graph for

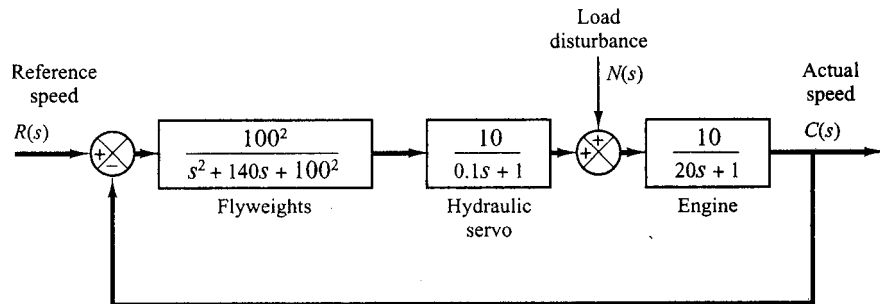
$$\frac{Y(s)}{X(s)} = \frac{1}{s + 140}$$

may be drawn as shown in Figure 3-70(a). Similarly, a signal flow graph for

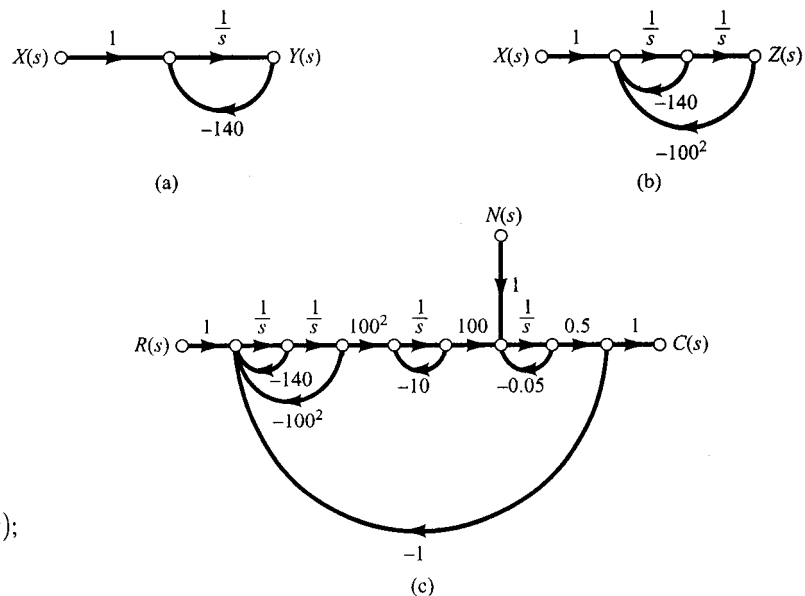
$$\frac{Z(s)}{X(s)} = \frac{1}{s^2 + 140s + 100^2} = \frac{\frac{1}{s + 140} \frac{1}{s}}{1 + \frac{100^2}{s + 140} \frac{1}{s}}$$

may be drawn as shown in Figure 3-70(b).

Drawing a signal flow graph for each of the system components and combining them together, a signal flow graph for the complete system may be obtained as shown in Figure 3-70(c).



**Figure 3-69**  
Block diagram of an engine-speed control system.



**Figure 3-70**  
(a) Signal flow graph for  $Y(s)/X(s) = 1/(s + 140)$ ;  
(b) signal flow graph for  $Z(s)/X(s) = 1/(s^2 + 140s + 100^2)$ ;  
(c) signal flow graph for the system shown in Fig. 3-69.

**A-3-27.** Linearize the nonlinear equation

$$z = x^2 + 4xy + 6y^2$$

in the region defined by  $8 \leq x \leq 10, 2 \leq y \leq 4$ .

**Solution.** Define

$$f(x, y) = z = x^2 + 4xy + 6y^2$$

Then

$$z = f(x, y) = f(\bar{x}, \bar{y}) + \left[ \frac{\partial f}{\partial x} (x - \bar{x}) + \frac{\partial f}{\partial y} (y - \bar{y}) \right]_{x=\bar{x}, y=\bar{y}} + \dots$$

where  $\bar{x} = 9, \bar{y} = 3$ .

Since the higher-order terms in the expanded equation are small, neglecting these higher-order terms, we obtain

$$z - \bar{z} = K_1(x - \bar{x}) + K_2(y - \bar{y})$$

where

$$K_1 = \left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}, y=\bar{y}} = 2\bar{x} + 4\bar{y} = 2 \times 9 + 4 \times 3 = 30$$

$$K_2 = \left. \frac{\partial f}{\partial y} \right|_{x=\bar{x}, y=\bar{y}} = 4\bar{x} + 12\bar{y} = 4 \times 9 + 12 \times 3 = 72$$

$$\bar{z} = \bar{x}^2 + 4\bar{x}\bar{y} + 6\bar{y}^2 = 9^2 + 4 \times 9 \times 3 + 6 \times 9 = 243$$

Thus

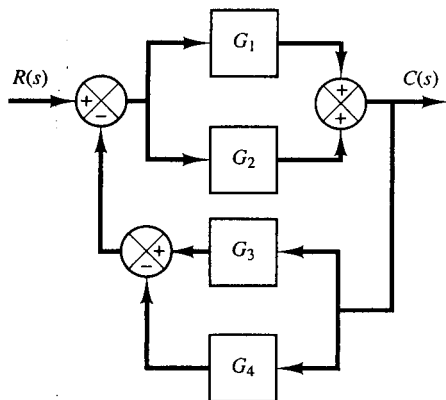
$$z - 243 = 30(x - 9) + 72(y - 3)$$

Hence a linear approximation of the given nonlinear equation near the operating point is

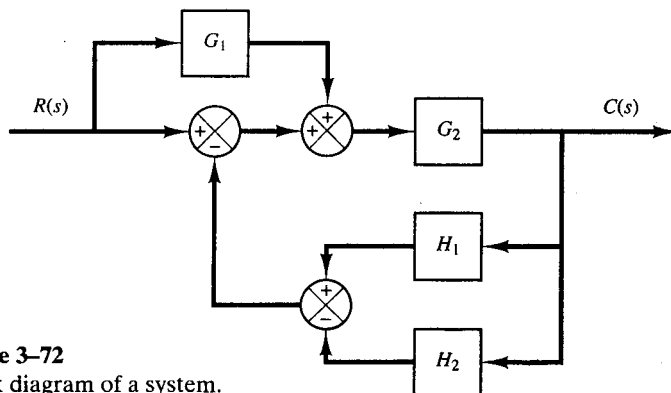
$$z - 30x - 72y + 243 = 0$$

## PROBLEMS

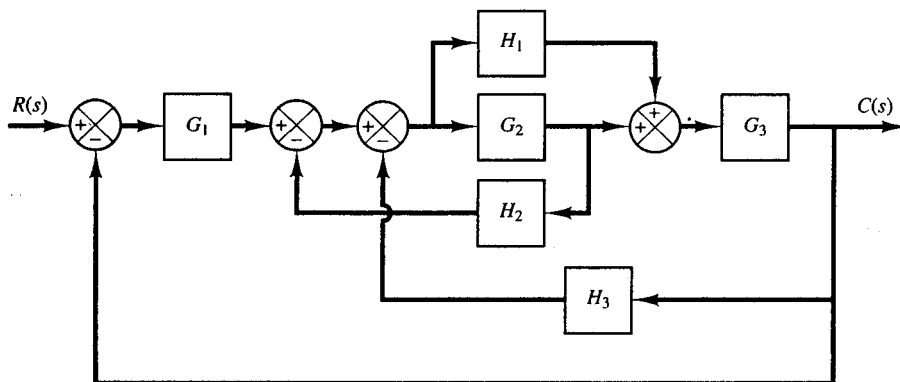
**B-3-1.** Simplify the block diagram shown in Figure 3-71 and obtain the closed-loop transfer function  $C(s)/R(s)$ .



**B-3-2.** Simplify the block diagram shown in Figure 3-72 and obtain the transfer function  $C(s)/R(s)$ .



**Figure 3-72**  
Block diagram of a system.



**Figure 3-73**  
Block diagram of a system.



**B-3-4.** Consider industrial automatic controllers whose control actions are proportional, integral, proportional-plus-integral, proportional-plus-derivative, and proportional-plus-integral-plus-derivative. The transfer functions of these controllers can be given, respectively, by

$$\frac{U(s)}{E(s)} = K_p$$

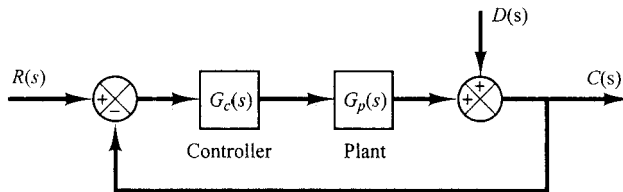
$$\frac{U(s)}{E(s)} = \frac{K_i}{s}$$

$$\frac{U(s)}{E(s)} = K_p \left( 1 + \frac{1}{T_i s} \right)$$

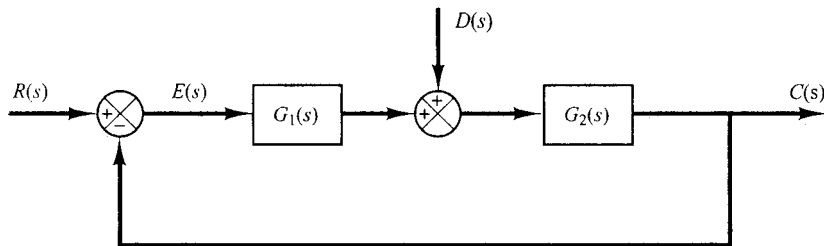
$$\frac{U(s)}{E(s)} = K_p (1 + T_d s)$$

$$\frac{U(s)}{E(s)} = K_p \left( 1 + \frac{1}{T_i s} + T_d s \right)$$

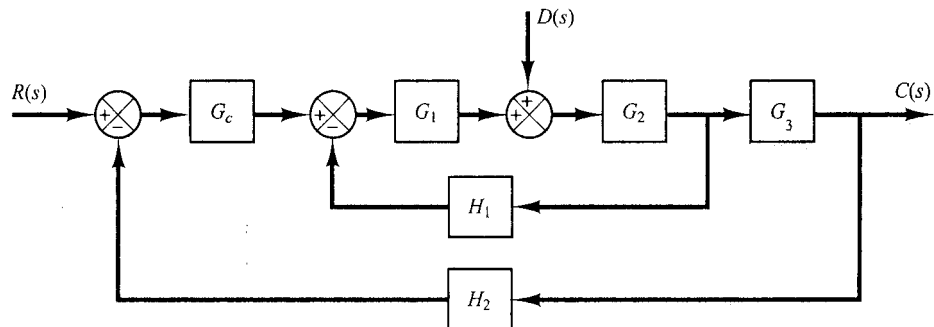
where  $U(s)$  is the Laplace transform of  $u(t)$ , the controller output, and  $E(s)$  the Laplace transform of  $e(t)$ , the actuating error signal.



**Figure 3-74**  
Closed-loop system.



**Figure 3-75**  
Control system.



**Figure 3-76**  
Control system.

ing error signal. Sketch  $u(t)$  versus  $t$  curves for each of the five types of controllers when the actuating error signal is

(a)  $e(t)$  = unit-step function

(b)  $e(t)$  = unit-ramp function

In sketching curves, assume that the numerical values of  $K_p$ ,  $K_i$ ,  $T_i$ , and  $T_d$  are given as

$K_p$  = proportional gain = 4

$K_i$  = integral gain = 2

$T_i$  = integral time = 2 sec

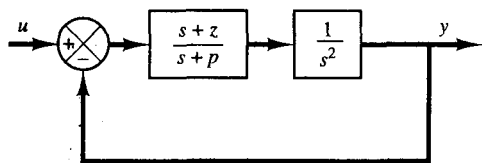
$T_d$  = derivative time = 0.8 sec

**B-3-5.** Figure 3-74 shows a closed-loop system with a reference input and disturbance input. Obtain the expression for the output  $C(s)$  when both the reference input and disturbance input are present.

**B-3-6.** Consider the system shown in Figure 3-75. Derive the expression for the steady-state error when both the reference input  $R(s)$  and disturbance input  $D(s)$  are present.

**B-3-7.** Obtain the transfer functions  $C(s)/R(s)$  and  $C(s)/D(s)$  of the system shown in Figure 3-76.

**B-3-8.** Obtain a state-space representation of the system shown in Figure 3-77.



**Figure 3-77**  
Control system.

**B-3-9.** Consider the system described by

$$\ddot{y} + 3\dot{y} + 2y = u$$

Derive a state-space representation of the system.

**B-3-10.** Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Obtain the transfer function of the system.

**B-3-11.** Consider a system defined by the following state-space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

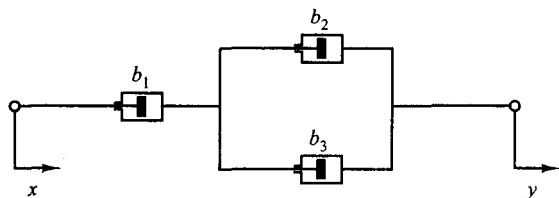
Obtain the transfer function  $G(s)$  of the system.

**B-3-12.** Obtain the transfer matrix of the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

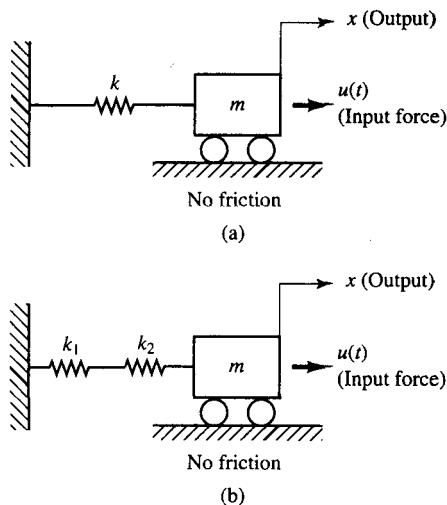
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

**B-3-13.** Obtain the equivalent viscous-friction coefficient  $b_{eq}$  of the system shown in Figure 3-78.



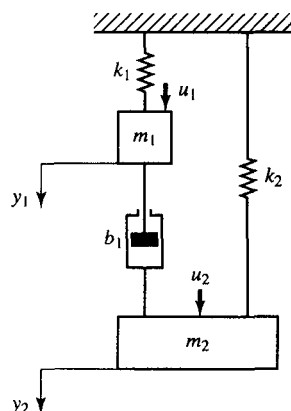
**Figure 3-78**  
Damper system.

**B-3-14.** Obtain mathematical models of the mechanical systems shown in Figure 3-79(a) and (b).



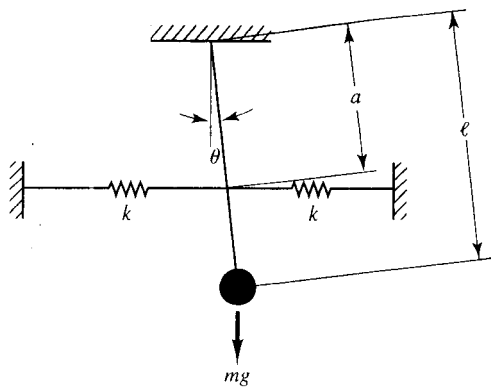
**Figure 3-79**  
Mechanical systems.

**B-3-15.** Obtain a state-space representation of the mechanical system shown in Figure 3-80, where  $u_1$  and  $u_2$  are the inputs and  $y_1$  and  $y_2$  are the outputs.



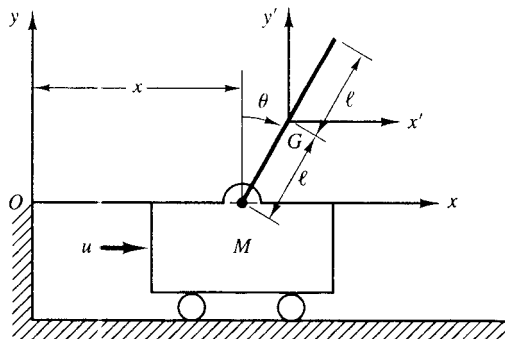
**Figure 3-80**  
Mechanical system.

**B-3-16.** Consider the spring-loaded pendulum system shown in Figure 3-81. Assume that the spring force acting on the pendulum is zero when the pendulum is vertical, or  $\theta = 0$ . Assume also that the friction involved is negligible and the angle of oscillation  $\theta$  is small. Obtain a mathematical model of the system.



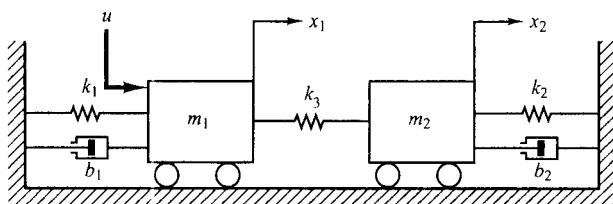
**Figure 3-81**  
Spring-loaded pendulum system.

**B-3-17.** Referring to Examples 3-8 and 3-9, consider the inverted pendulum system shown in Figure 3-82. Assume that the mass of the inverted pendulum is  $m$  and is evenly distributed along the length of the rod. (The center of gravity of the pendulum is located at the center of the rod.) Assuming that  $\theta$  is small, derive mathematical models for the system in the forms of differential equations, transfer functions, and state-space equations.



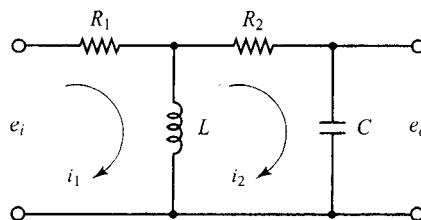
**Figure 3-82**  
Inverted pendulum system.

**B-3-18.** Obtain the transfer functions  $X_1(s)/U(s)$  and  $X_2(s)/U(s)$  of the mechanical system shown in Figure 3-83.



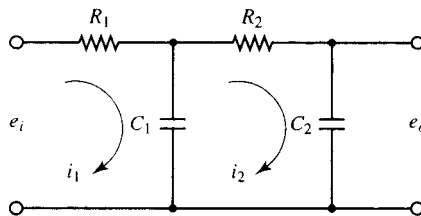
**Figure 3-83**  
Mechanical system.

**B-3-19.** Obtain the transfer function  $E_o(s)/E_i(s)$  of the electrical circuit shown in Figure 3-84.



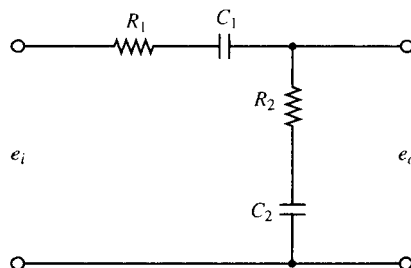
**Figure 3-84**  
Electrical circuit.

**B-3-20.** Consider the electrical circuit shown in Figure 3-85. Obtain the transfer function  $E_o(s)/E_i(s)$  by use of the block diagram approach.



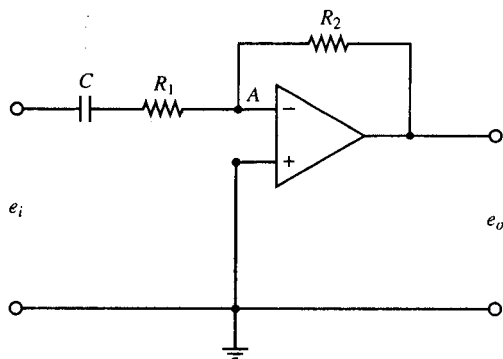
**Figure 3-85**  
Electrical circuit.

**B-3-21.** Derive the transfer function of the electrical circuit shown in Figure 3-86. Draw a schematic diagram of an analogous mechanical system.



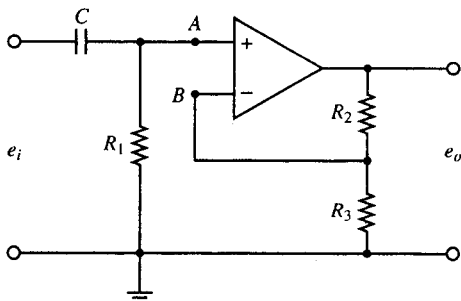
**Figure 3-86**  
Electrical circuit.

**B-3-22.** Obtain the transfer function  $E_o(s)/E_i(s)$  of the op-amp circuit shown in Figure 3-87.



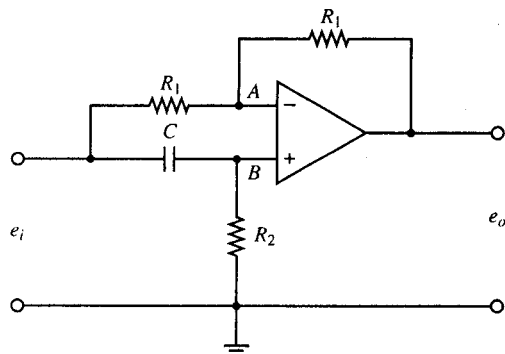
**Figure 3-87**  
Operational-amplifier circuit.

**B-3-23.** Obtain the transfer function  $E_o(s)/E_i(s)$  of the op-amp circuit shown in Figure 3-88.



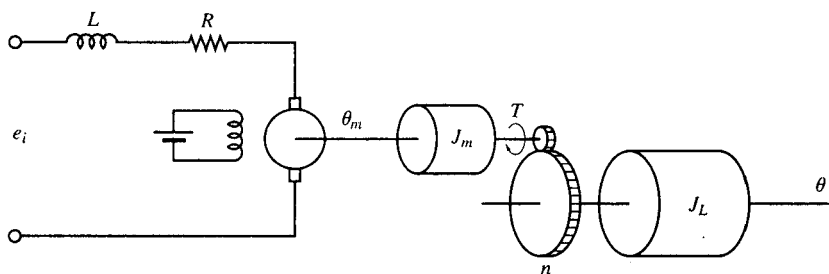
**Figure 3-88**  
Operational-amplifier circuit.

**B-3-24.** Using the impedance approach, obtain the transfer function  $E_o(s)/E_i(s)$  of the op-amp circuit shown in Figure 3-89.



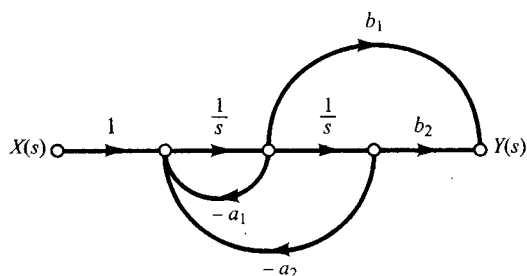
**Figure 3-89**  
Operational-amplifier circuit.

**B-3-25.** Consider the system shown in Figure 3-90. An armature-controlled dc servomotor drives a load consisting of the moment of inertia  $J_L$ . The torque developed by the motor is  $T$ . The moment of inertia of the motor rotor is  $J_m$ . The angular displacements of the motor rotor and the load element are  $\theta_m$  and  $\theta$ , respectively. The gear ratio is  $n = \theta/\theta_m$ . Obtain the transfer function  $\Theta(s)/E_i(s)$ .



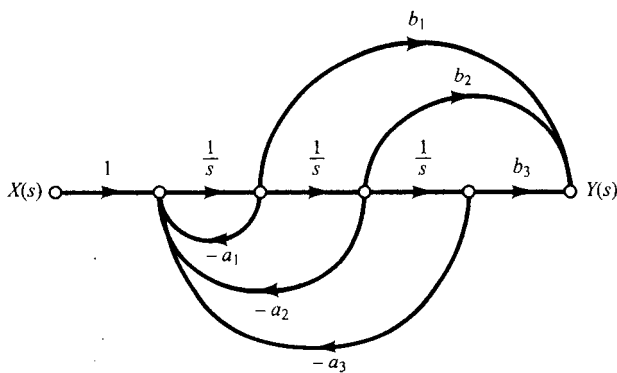
**Figure 3-90**  
Armature-controlled dc servomotor system.

**B-3-26.** Obtain the transfer function  $Y(s)/X(s)$  of the system shown, in Figure 3-91.



**Figure 3-91**  
Signal flow graph of a system.

**B-3-27.** Obtain the transfer function  $Y(s)/X(s)$  of the system shown in Figure 3-92.



**Figure 3-92**  
Signal flow graph of a system.

**B-3-28.** Linearize the nonlinear equation

$$z = x^2 + 8xy + 3y^2$$

in the region defined by  $2 \leq x \leq 4$ ,  $10 \leq y \leq 12$ .

**B-3-29.** Find a linearized equation for

$$y = 0.2x^3$$

about a point  $x = 2$ .