Excursions in Modern Mathematics
Sixth Edition

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Chapter 3
Fair Division

The Mathematics of Sharing

Fair Division
Outline/learning Objectives

- State the fair-division problem and identify assumptions used in developing solution methods.
- Recognize the differences between continuous and discrete fair-division problems.
- Apply the divider-chooser, lone-divider, lone-chooser, and last diminisher methods to continuous fair-division problems
- Apply the method of sealed bids and the method of markers to a discrete fair-division problem

Fair Division

3.1 Fair Division Games

Fair Division- Underlying Elements

- ***The goods (or booty)*.**

  This is the informal name we will give to the item(s) being divided and is denoted by ***S***.

- ***The players*.**

  They are the players in the game.

- ***The value systems*.**

  Each player has an internalized *value system*.

Fair Division Assumptions

- Rationality
- Cooperation
- Privacy
- Symmetry

Fair Division

Fair Share

Suppose that *s* denotes a share of the booty *S* and *P* is one of the players in a fair division game with *N* players. We will say that *S* is a **fair share to player** *P* if *S* is worth *at least* 1/*N*th of the total value of *S* in the opinion of *P*.

Fair Division-Division Methods

- ***Continuous***

  The set *S* is divisible.

- ***Discrete***

  The set *S* is indivisible.

- ***Mixed***

  Some are continuous and some discrete.

Fair Division

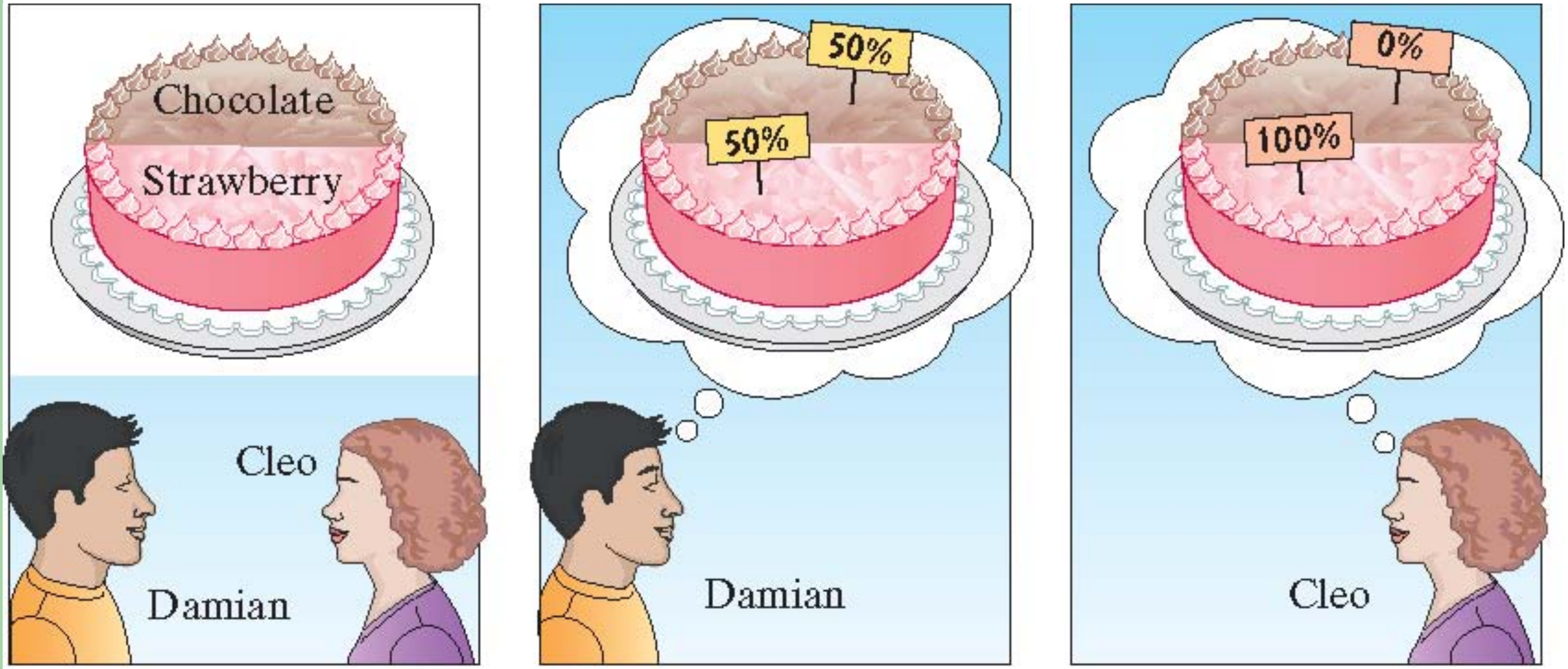
3.2 Two Players: The Divider-Chooser Method

The Divider-Chooser Method

- The best known of all continuous fair-division methods.
- This method can be used anytime it involves two players and a continuous set *S*.
- Also known as "you cut– I choose" method.

The Divider-Chooser Method

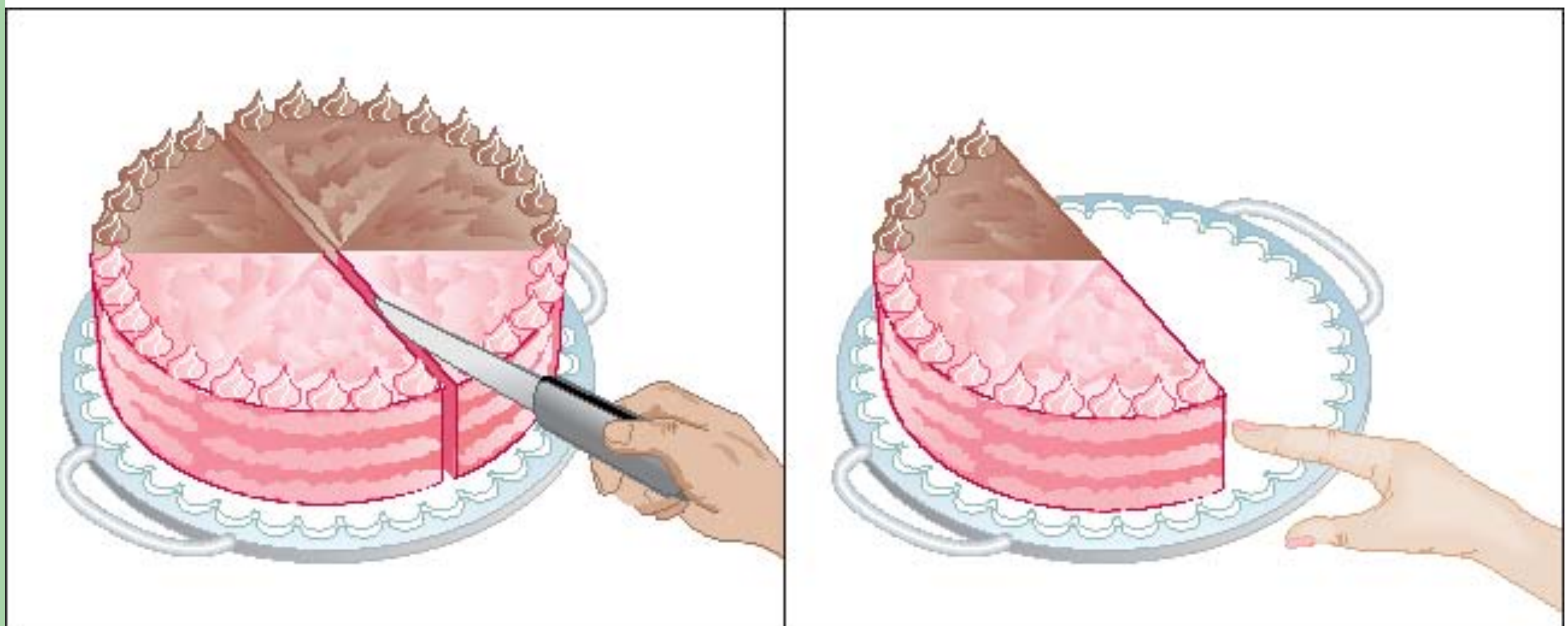
Two Players: The Divider-Chooser Method



(a) (b) (c)

The Divider-Chooser Method

Two Players: The Divider-Chooser Method



(a) (b)

Fair Division

3.3 The Lone-Divider Method

Lone-Divider Method

– **The Lone-Divider Method for Three Players**

- **Preliminaries**. One of the three players will be the divider; the other two players will be choosers. We'll call the divider *D* and the choosers C_1 and C_2 . N = 3

Lone-Divider Method

– **The Lone-Divider Method for Three Players**

- **Step 1 (Division)**. The divider *D* divides the cake into three pieces (s_1 , s_2 and s_3 .) *D* will get one of these pieces, but at this point does not know which one. (Not knowing which of the pieces will be his share is critical– it forces *D* to divide the cake equally)

Lone-Divider Method

– **The Lone-Divider Method for Three Players**

- **Step 2 (Bidding)**. C_1 declares (usually by writing on a slip of paper) which of the three pieces are fair shares to her. Independently, C_2 does the same. These are the chooser' *bid lists*. A choosers bid list should include *every piece that he or she values to be a fair share*.

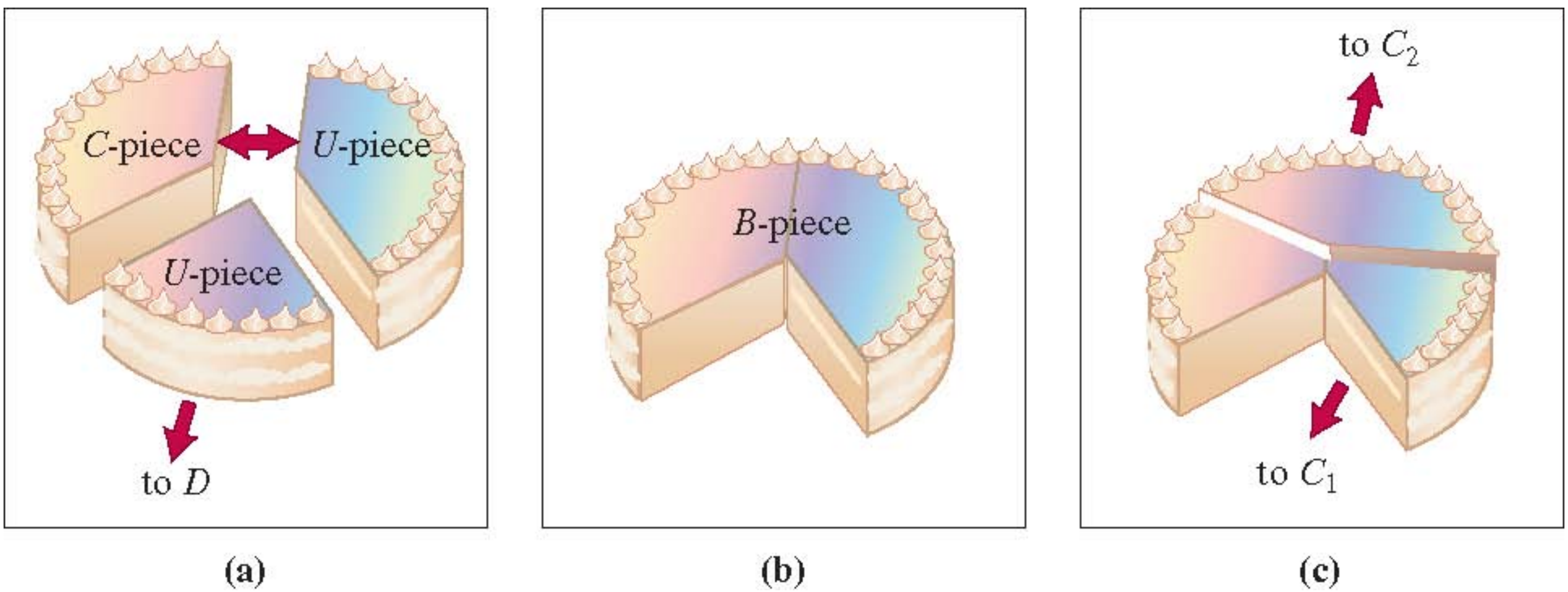
Lone-Divider Method

– **The Lone-Divider Method for Three Players**

- **Step 3 (Distribution)**. Who gets the piece? The answer depends on the bid lists. For convenience, we will separate the pieces into two groups: *chosen* pieces (let's call them *C*-pieces), and *unwanted* pieces (let's call them *U*- pieces).
- Note: Swapping pieces after the distribution is perfectly fine.

Lone-Divider Method

The Lone-Divider Method for Three Players



Lone-Divider Method

	S_1	S_2	S_3
*Dale	33.3%	33.3%	33.3%
Cindy	35%	10%	55%
Cher	40%	25%	35%

Dale is the divider.

Lone-Divider Method

	S_1	S_2	S_3
*Dale	33.3%	33.3%	33.3%
Cindy	30%	40%	30%
Cher	60%	15%	25%

Dale is the divider.

Lone-Divider Method

	S_1	S_2	S_3
*Dale	33.3%	33.3%	33.3%
Cindy	20%	30%	50%
Cher	10%	20%	70%

Dale is the divider.

Lone-Divider Method

- **The Lone-Divider Method for More Than Three Players**

- **Preliminaries**. One of the players will be the divider *D*; and the remaining **N – 1** players are going to be all choosers. As always, it's better to be a chooser than a divider.

Lone-Divider Method

- **The Lone-Divider Method for More Than Three Players**

- **Step 1 (Division)**. The divider *D* divides the set *S* into *N* shares S_1, S_2, S_3, … $S_{N.}$ *D* is guaranteed of getting one of these shares, but doesn't know which one.

Lone-Divider Method

– **The Lone-Divider Method for More Than Three Players**

- **Step 2 (Bidding)**. Each of the **N - 1** choosers independently submits a bid list consisting of every share that he or she considers to be a fair share (1/*N*th or more of S).

Lone-Divider Method

– **The Lone-Divider Method for More Than Three Players**

- **Step 3 (Distribution)**. The bid lists are opened.

Lone-Divider Method

	S_1	S_2	S_3	S_4
*Demi	25%	25%	25%	25%
Chan	30%	20%	35%	15%
Chloe	20%	20%	40%	20%
Chris	25%	20%	20%	35%

Demi is the divider.

Lone-Divider Method

	S_1	S_2	S_3	S_4
*Demi	25%	25%	25%	25%
Chan	20%	20%	20%	40%
Chloe	15%	35%	30%	20%
Chris	22%	23%	20%	35%

Demi is the divider.

Fair Division

3.4 The Lone-Chooser Method

The Lone-Chooser Method

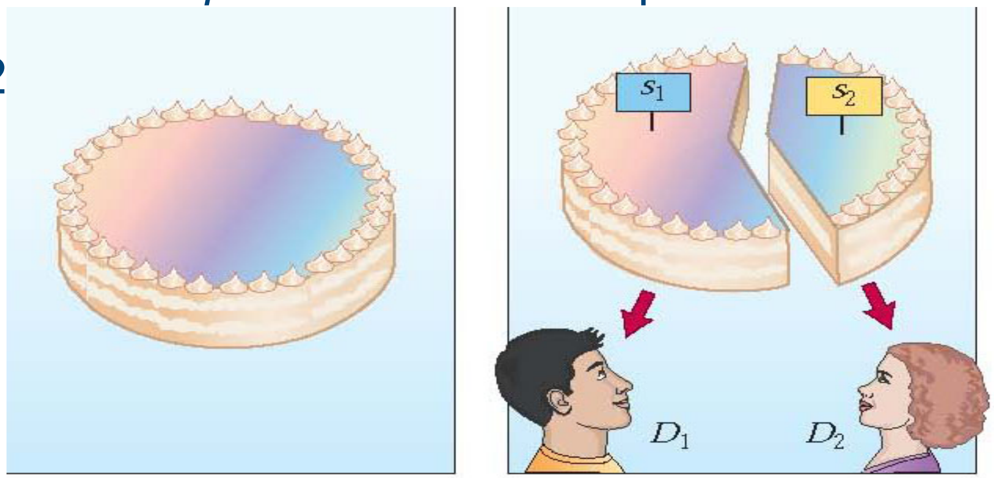
– **The Lone-Chooser Method for Three Players**

- **Preliminaries**. We have one chooser and two dividers. Let's call the chooser *C and the dividers* D_1 and D_2. As usual, we decide who is what by a random draw.

The Lone-Chooser Method

– **The Lone-Chooser Method for Three Players**

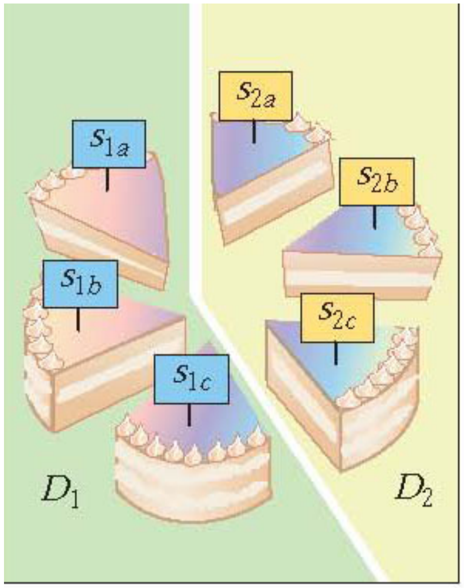
- **Step 1 (Division)**. D_1 and D_2 divide S between themselves into two fair shares. To do this, they use the divider-chooser method. Let's say that D_1 ends with S_1 and D_2 ends with S_2.



The Lone-Chooser Method

– **The Lone-Chooser Method for Three Players**

- **Step 2 (Subdivision)**. Each divider divides his or her share into three subshares. Thus D_1 divides S_1 into three subshares, which we will call S_{1a}, S_{1b} and S_{1c} . Likewise, D_2 divides S_2 into three subshares, which we will call S_{2a}, S_{2b} and S_{2c} .

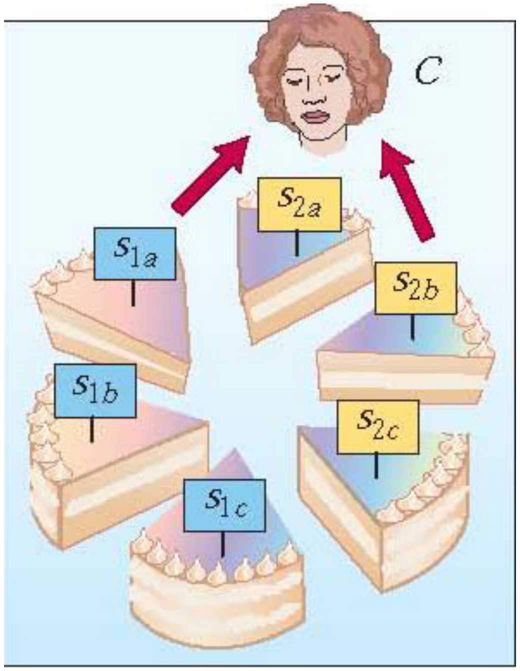


(c)

The Lone-Chooser Method

– **The Lone-Chooser Method for Three Players**

- **Step 3 (Selection)**. The chooser *C* now selects one of D_1 ‘s three subshares and one of D_2 ‘s three subshares. These two subshares make up *C*’s final share. D_1 then keeps the remaining two subshares from S_1 , and D_2 keeps the remaining two subshares from S_2 .

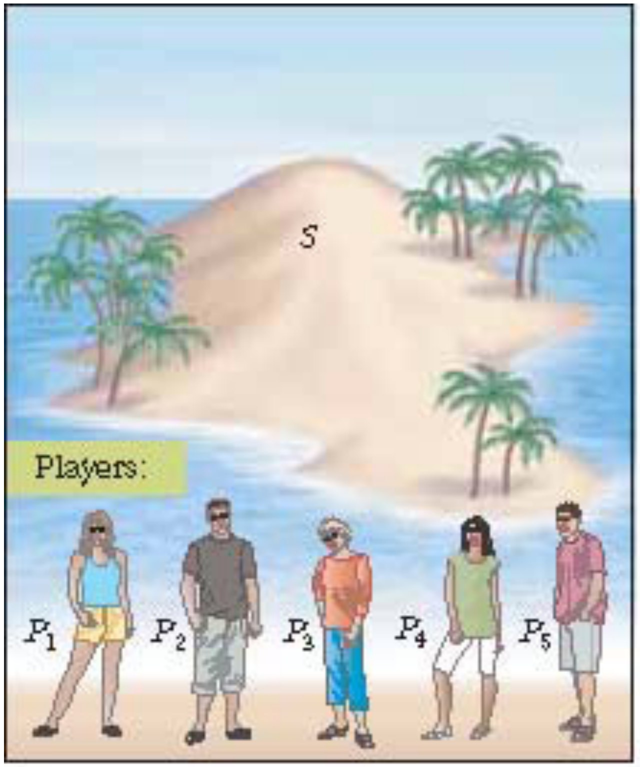


Fair Division

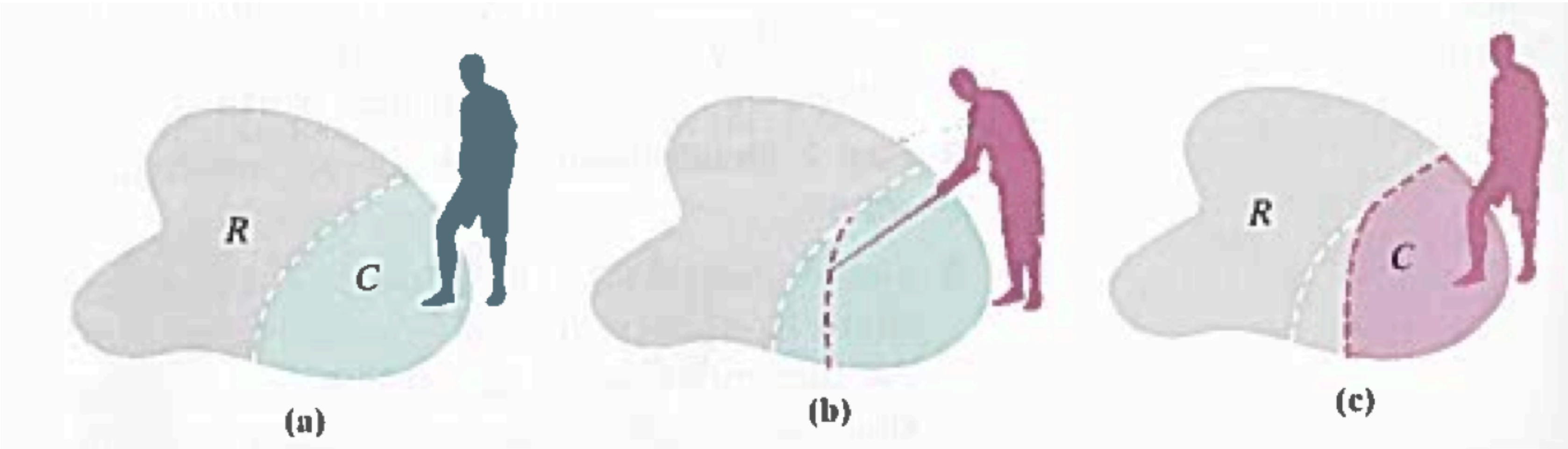
3.5 The Last-Diminisher Method

The Last-Diminisher Method

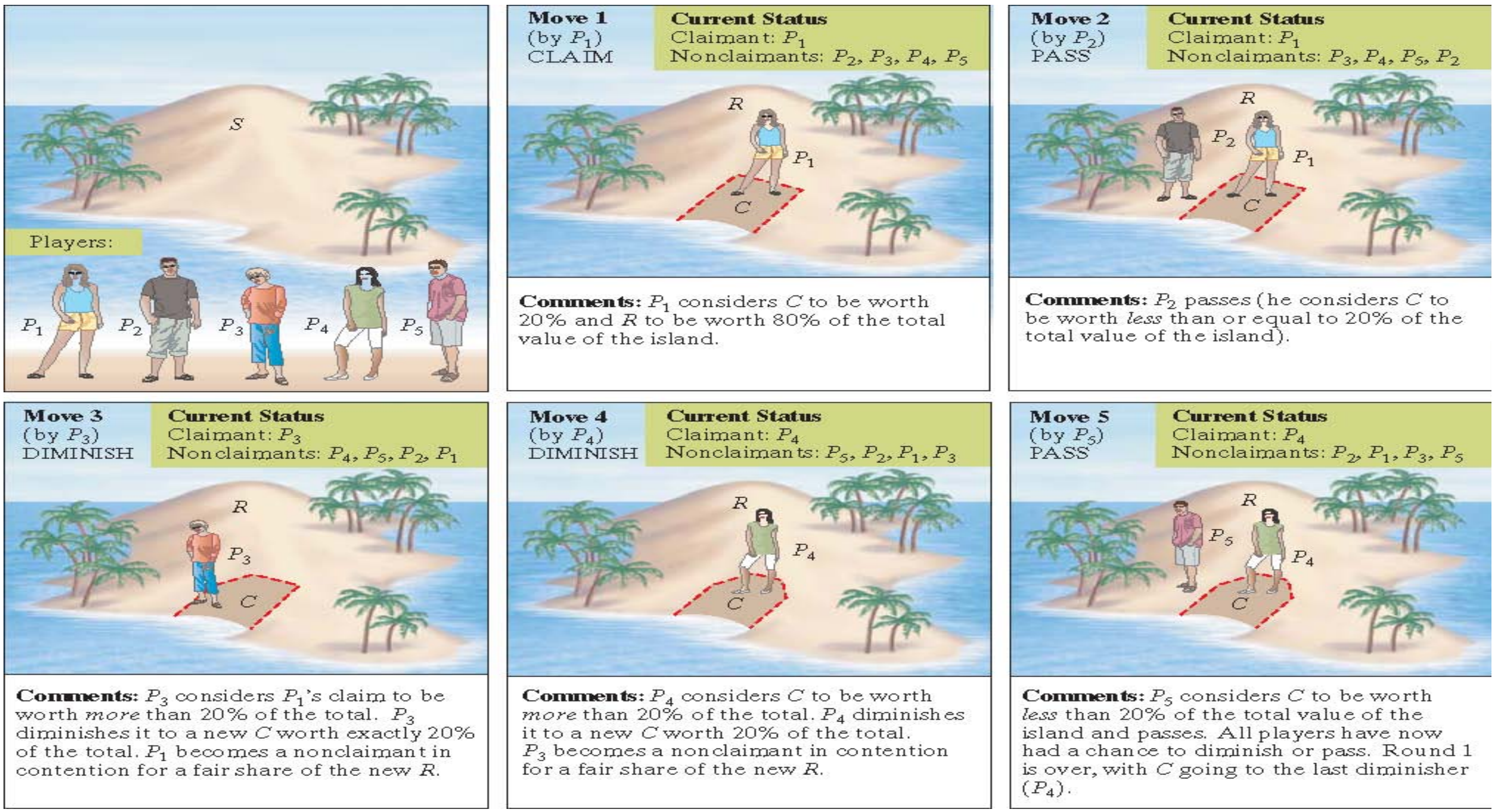
- **Preliminaries**. Before the game starts the players are randomly assigned an order of play. The game is played in rounds, and at the end of the each round there is one fewer player and a smaller *S* to be divided.



The Last-Diminisher Method



The Last-Diminisher Method



The Last-Diminisher Method

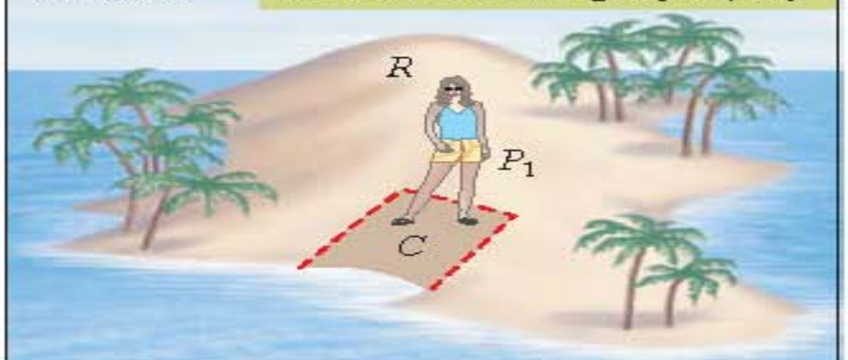
- **Round 1**. P_1 kicks the off by "cutting" for herself a 1/Nth share of S. This will be the current C-piece, and P_1 is its claimant. P_1 does not know whether or not she will end up with this share.

  P_2 comes next and has a choice: *pass* or *diminish*

The Last-Diminisher Method

Move 1
(by P_1)
CLAIM

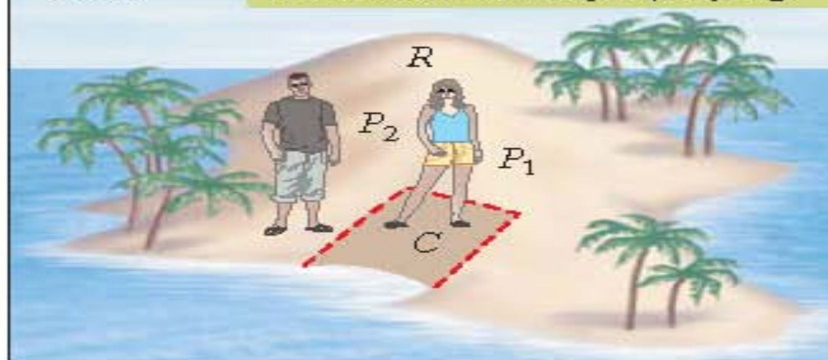
Current Status
Claimant: P_1
Nonclaimants: P_2, P_3, P_4, P_5



Comments: P_1 considers C to be worth 20% and R to be worth 80% of the total value of the island.

Move 2
(by P_2)
PASS

Current Status
Claimant: P_1
Nonclaimants: P_3, P_4, P_5, P_2



Comments: P_2 passes (he considers C to be worth *less* than or equal to 20% of the total value of the island).

The Last-Diminisher Method

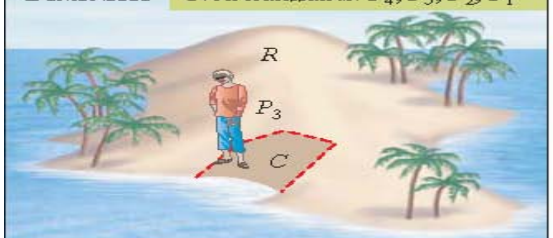
- **(Round 1 continued)**. P_3 comes next and has the same opportunity as P_2 : *Pass* or *diminish* the current *C*-piece.

  The round continues this way, each player in turn having an opportunity to *pass* or *diminish*.

The Last-Diminisher Method

Move 3
(by P_3)
DIMINISH

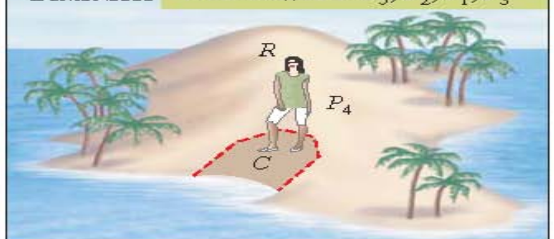
Current Status
Claimant: P_3
Nonclaimants: P_4, P_5, P_2, P_1



Comments: P_3 considers P_1's claim to be worth *more* than 20% of the total. P_3 diminishes it to a new C worth exactly 20% of the total. P_1 becomes a nonclaimant in contention for a fair share of the new R.

Move 4
(by P_4)
DIMINISH

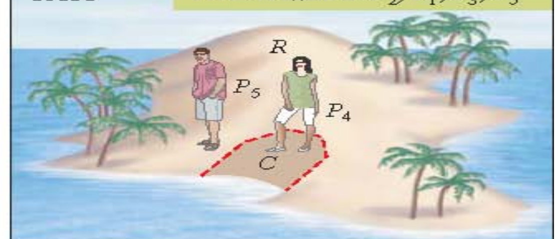
Current Status
Claimant: P_4
Nonclaimants: P_5, P_2, P_1, P_3



Comments: P_4 considers C to be worth *more* than 20% of the total. P_4 diminishes it to a new C worth 20% of the total. P_3 becomes a nonclaimant in contention for a fair share of the new R.

Move 5
(by P_5)
PASS

Current Status
Claimant: P_4
Nonclaimants: P_2, P_1, P_3, P_5

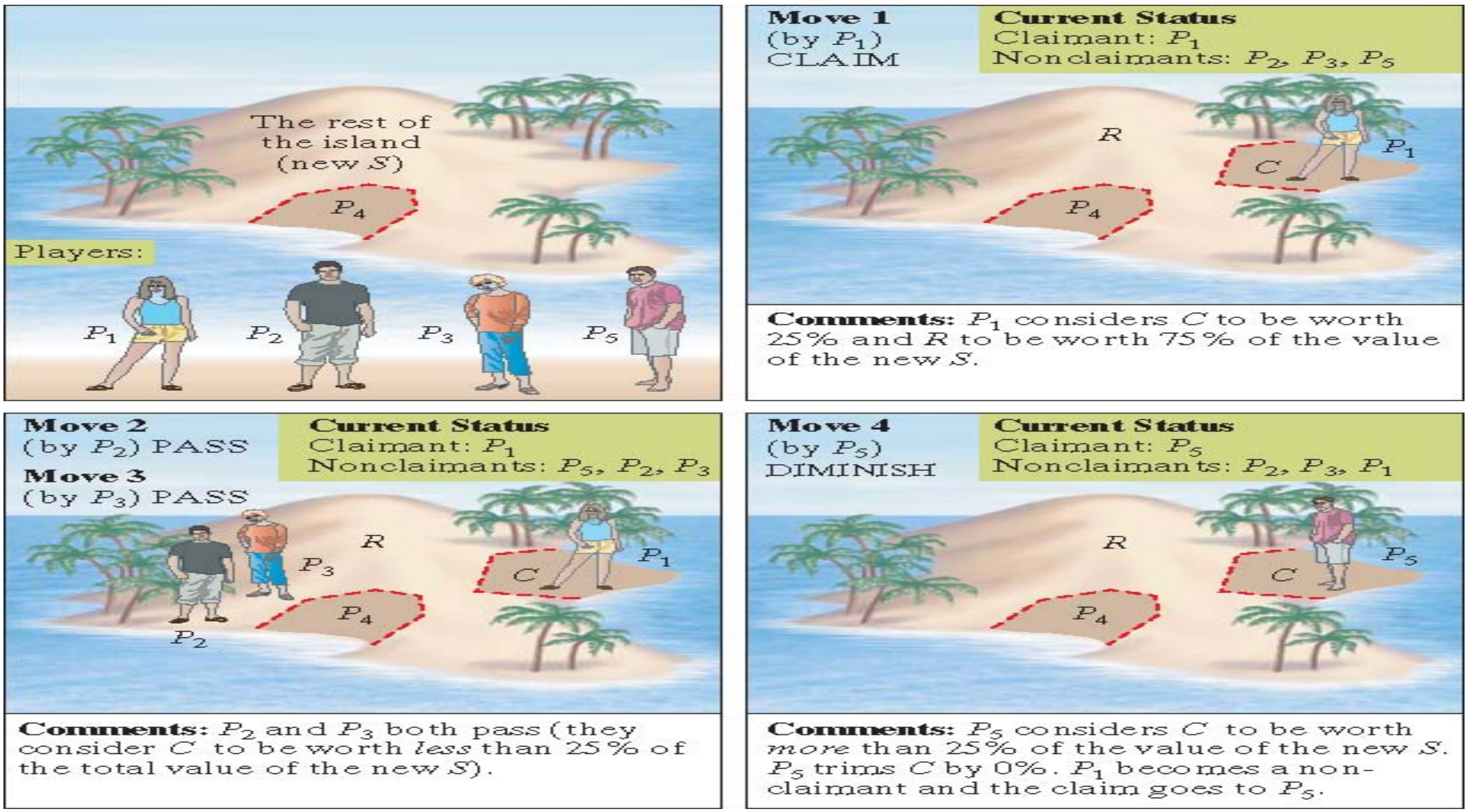


Comments: P_5 considers C to be worth *less* than 20% of the total value of the island and passes. All players have now had a chance to diminish or pass. Round 1 is over, with C going to the last diminisher (P_4).

The Last-Diminisher Method

- **Round 2**. The *R*- piece becomes the new *S* and a new version of the game is played with the new *S* and the N – 1 remaining players. At the end of this round, the last diminisher gets to keep the current *C*-piece and is out of the game.

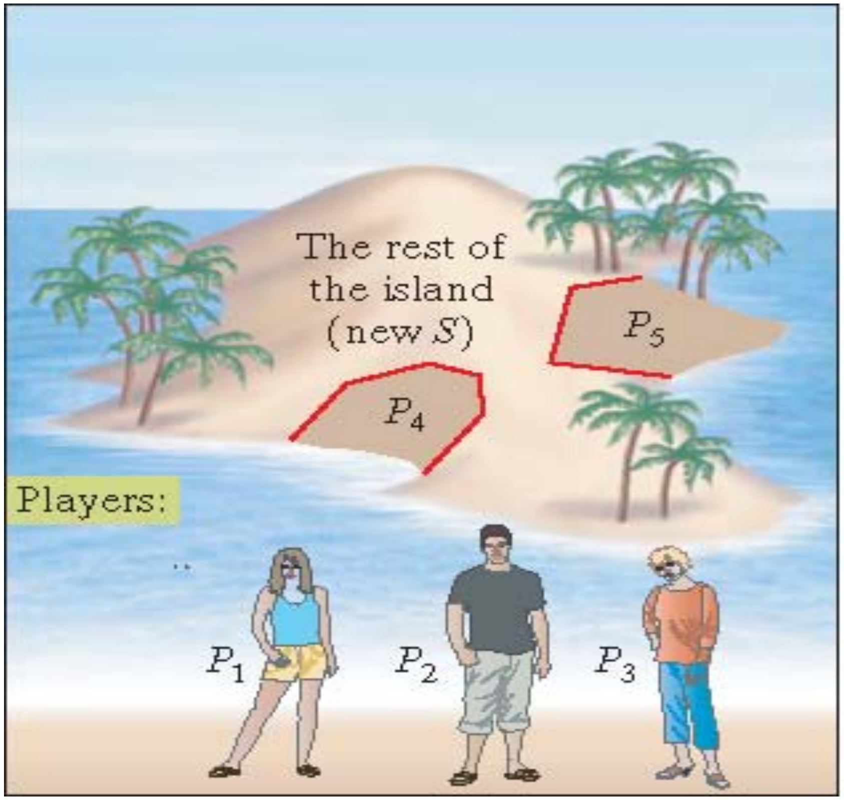
The Last-Diminisher Method

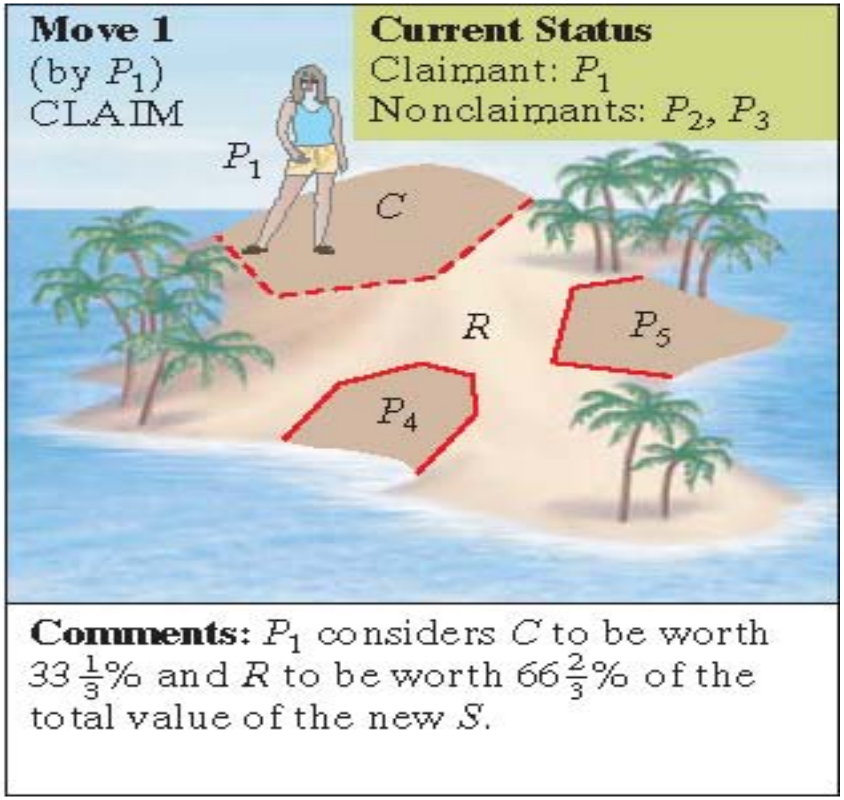


The Last-Diminisher Method

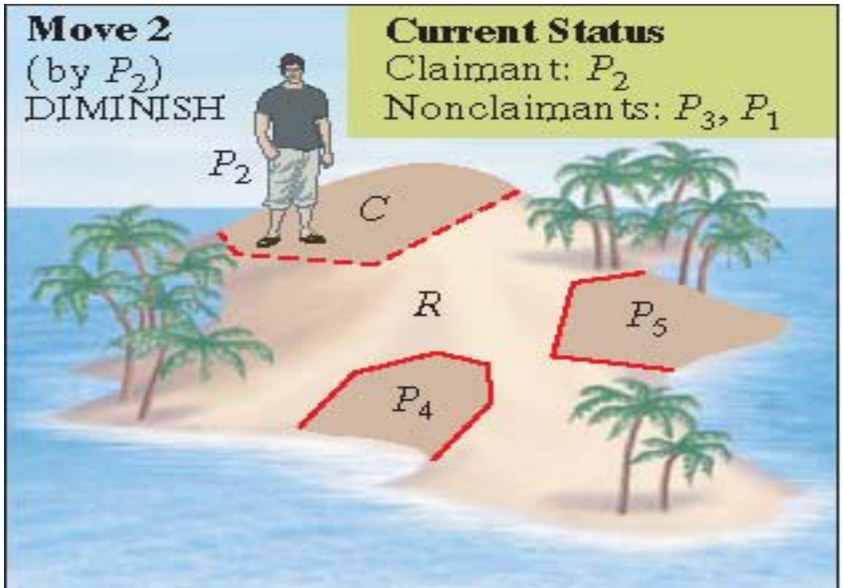
- **Round 3, 4, etc**. Repeat the process, each time with one fewer player and a smaller S, until there are just two players left. At this point, divide the remaining piece between the final two players using the *divider-chooser* method.

The Last-Diminisher Method

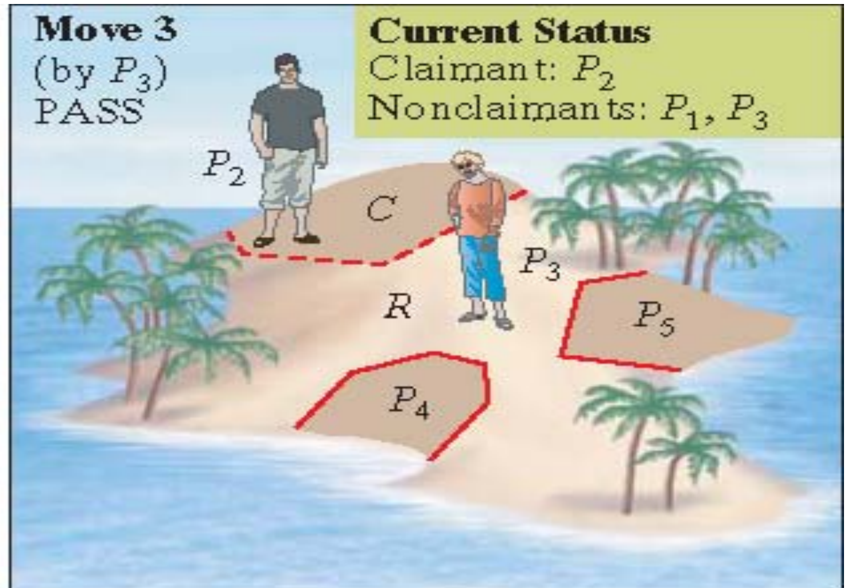




The Last-Diminisher Method

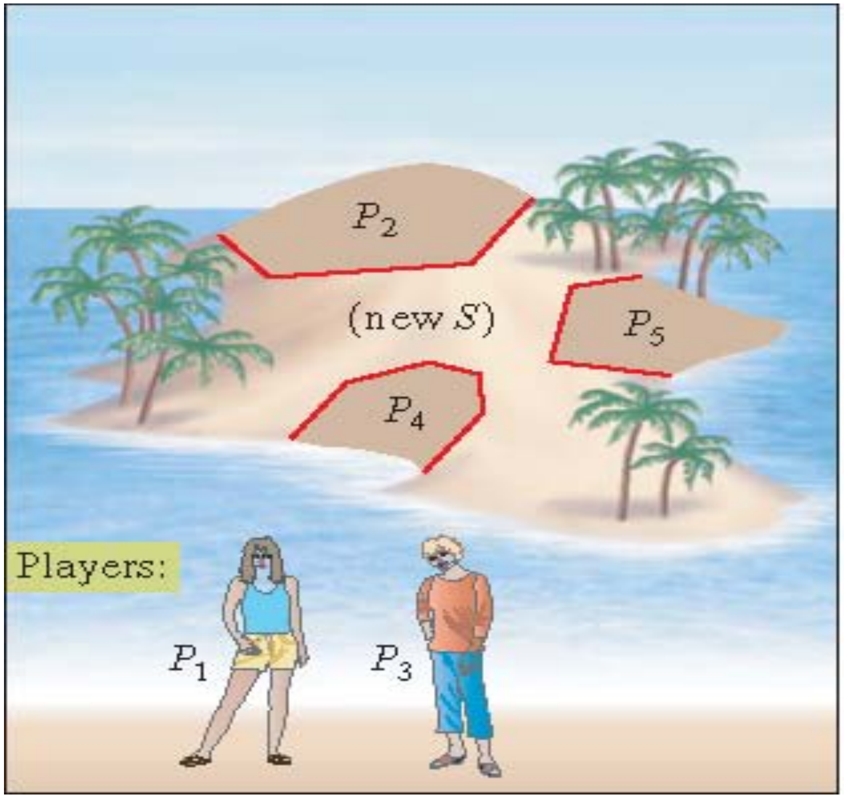


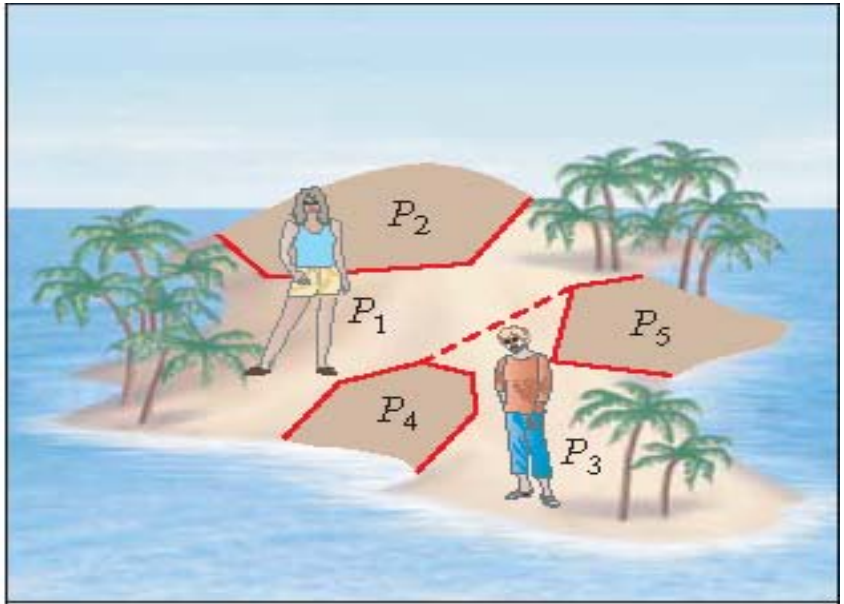
Comments: P_2 considers C to be worth *more* than $33\frac{1}{3}\%$ of the value of S. P_2 diminishes it to a new C worth exactly $33\frac{1}{3}\%$ of the value of S. P_1 goes back to being a nonclaimant.



Comments: P_3 passes (she considers C to be worth *less* than $33\frac{1}{3}\%$ of the value of S). The claim C goes to P_2.

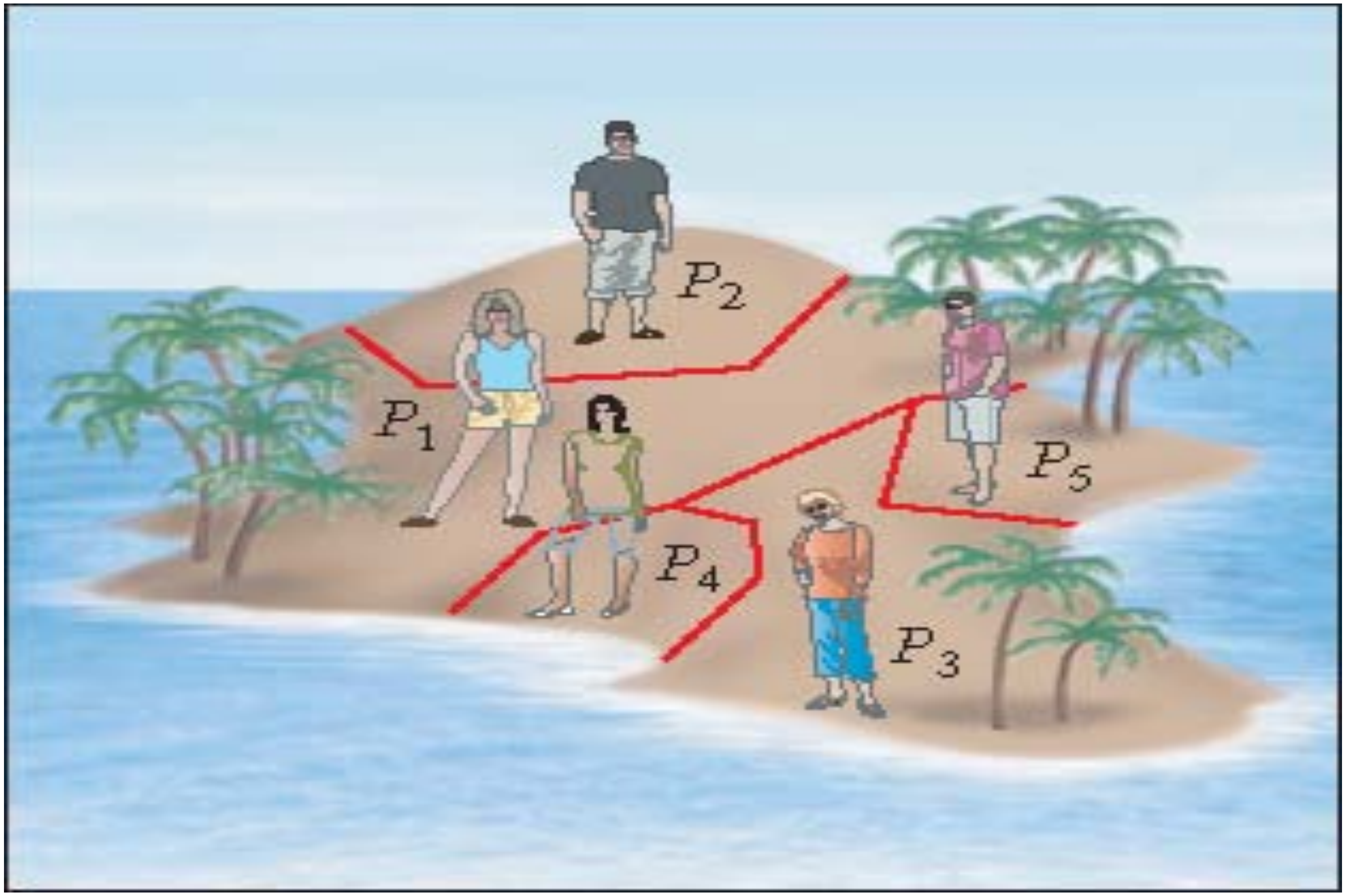
The Last-Diminisher Method





Comments: P_1 and P_3 divide S using the divider-chooser method. Since P_1 goes first, P_1 is the divider and P_3 is the chooser.

The Last-Diminisher Method



Fair Division

3.6 The Method of Sealed Bids

The Method of Sealed Bids

- **Step 1 (Bidding)**. Each of the players makes a bid (in dollars) for each of the items in the estate, giving his or her honest assessment of the actual value of each item. Each player submits their own bid in a sealed envelope.

The Method of Sealed Bids

- **Step 2 (Allocation)**. Each item will go to the highest bidder for that item. (If there is a tie, the tie can be broken with a coin flip.)

The Method of Sealed Bids

- **Step 3 (First Settlement)**. Depending on what items (if any) a player gets in Step 2, he or she will owe money to or be owed money by the estate. To determine how much a player owes or is owed, we first calculate each player's fair-dollar share of the estate.

The Method of Sealed Bids

- **Step 4 (Division of the Surplus)**. The surplus is common money that belongs to the estate, and thus to be divided equally among the players.

The Method of Sealed Bids

- **Step 5 (Final Settlement)**. The final settlement is obtained by adding the surplus money to the first settlement obtained in Step 3.

Fair Division

3.7 The Method of Markers

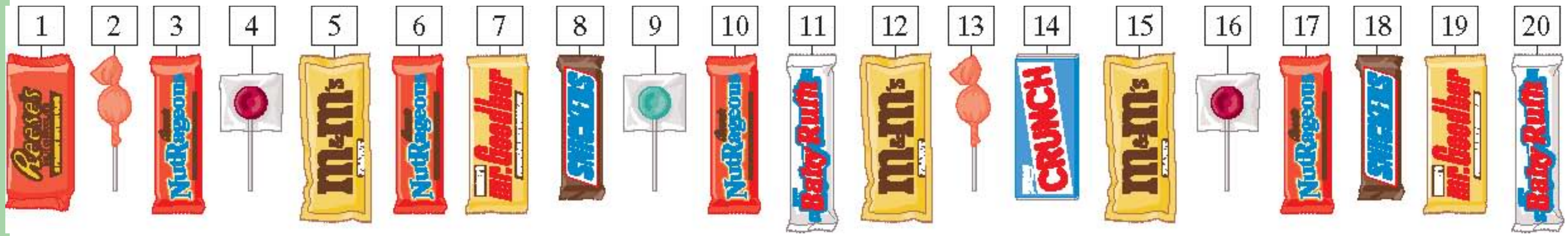
The Method of Markers

The Method of Markers

- No money up front.
- Must have more items than players.
- Items must be close in value.

The Method of Markers

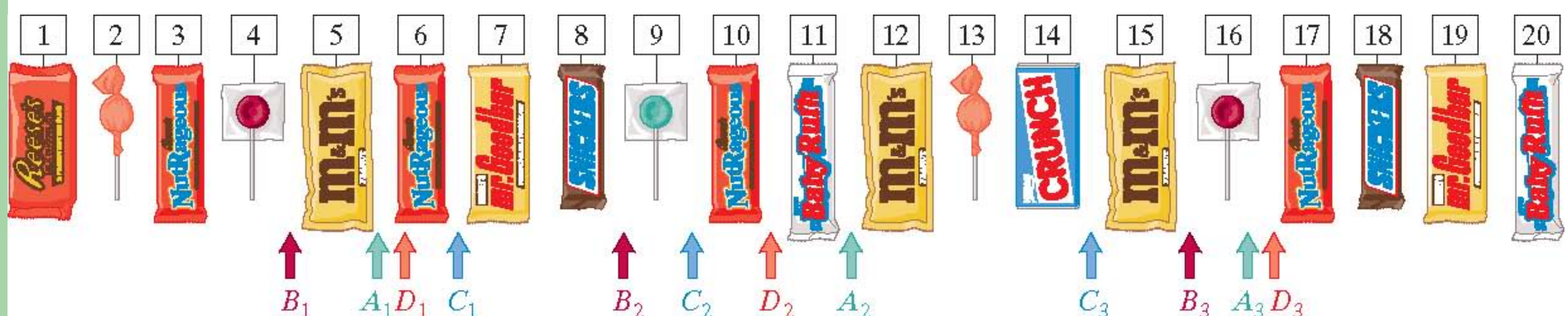
- **Preliminaries**. The items are arranged randomly into an ***array***.



Array – a set of numbers or objects that follow a specific pattern. Arrays are usually orderly arranged in rows, columns or a matrix.

The Method of Markers

- **Step 1 (Bidding)**. Each player independently divides the array into N segments by placing markers along the array.

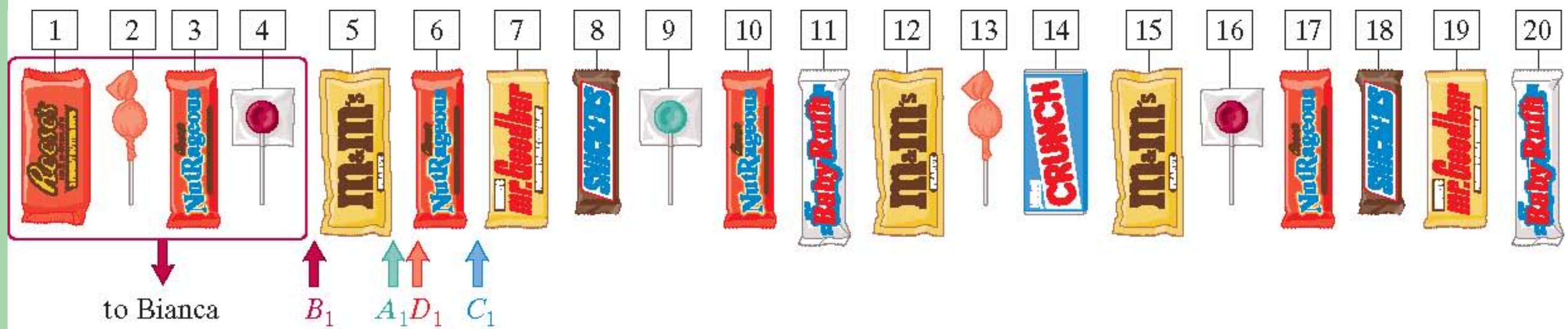


The Method of Markers

- **Step 2 (Allocations)**. Scan the array from left to right until the first *first marker* is located. The player owning that marker goes first, and gets the first segment in his bid. That players markers are removed, and we continue scanning left to right, looking for the first ***second marker***.

The Method of Markers

The Method of Markers - Step 2

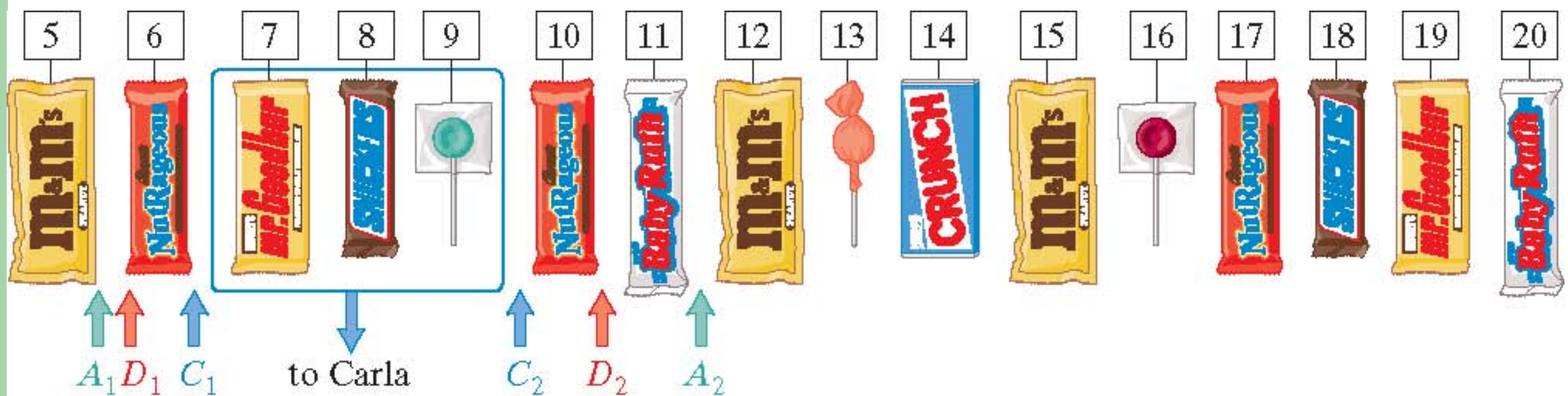


The Method of Markers

- **Step 2 (Allocations continued)**. The player owning that marker goes second and gets the second segment in her bid. Continue this process, assigning to each player in turn one of the segments in her bid. The last player gets the last segment in her bid.

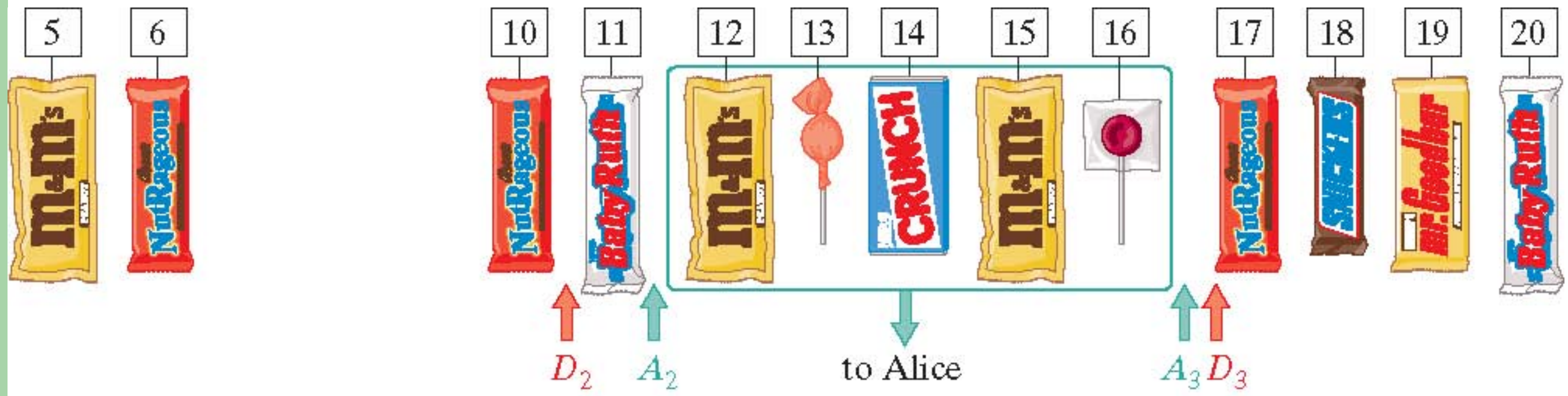
The Method of Markers

The Method of Markers - Step 2



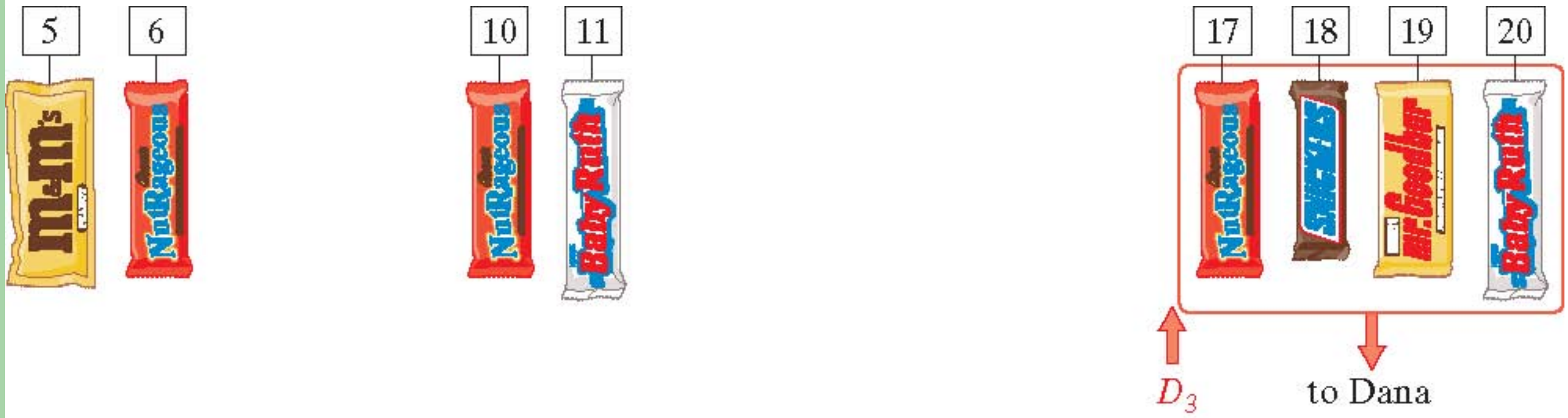
The Method of Markers

The Method of Markers - Step 2



The Method of Markers

The Method of Markers - Step 2



The Method of Markers

- **Step 3 (Dividing Leftovers)**. The leftover items can be divided among the players by some form of lottery, and, in the rare case that there are many more leftover items than players, the method of markers could be used again.

The Method of Markers

The Method of Markers - Step 3

Fair Division Conclusion

- Fair Division from a Mathematical perspective
- Developed different methods for solving fair-division problems
- Classified fair-division problems into continuous and discrete
- Overview of how to get humans to share in a reasonable and fair way.