Exercise 2.12.1

(a) Player I has $3 \times 2 \times 2 = 12$ pure strategies. Player II has $3 \times 3 = 9$ pure strategies.

(b) I: $\{lll, llr, lrl, lrr, mll, mlr, mrl, mrr, rll, rlr, rrl, rrr\}$;
II: $\{LL, LM, LR, ML, MM, MR, RL, RM, RR\}$.

(c) $[rM]$.

(d) (rll, LR), (rrl, LR), (rll, MR), (rrl, MR), (rll, RR), (rrl, RR).

(e) See the table below.

	LL	LL	LR	ML	MM	MR	RL	RM	RR
lll	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
llr	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
lrl	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
lrr	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
mll	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
mlr	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
mrl	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
mrr	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
rll	$\mathcal{W}$	$\mathcal{D}$	$\mathcal{W}$	$\mathcal{W}$	$\mathcal{D}$	$\mathcal{W}$	$\mathcal{W}$	$\mathcal{D}$	$\mathcal{W}$
rlr	$\mathcal{W}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{W}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{W}$	$\mathcal{D}$	$\mathcal{D}$
rrl	$\mathcal{W}$	$\mathcal{D}$	$\mathcal{W}$	$\mathcal{W}$	$\mathcal{D}$	$\mathcal{W}$	$\mathcal{W}$	$\mathcal{D}$	$\mathcal{W}$
rrr	$\mathcal{W}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{W}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{W}$	$\mathcal{D}$	$\mathcal{D}$

(f) The saddle points are of the form (mxy, XM), where $x, y \in \{l, r\}$ and $X \in \{L, R, M\}$.

Exercise 2.12.2

See figure 2.12.2. below.

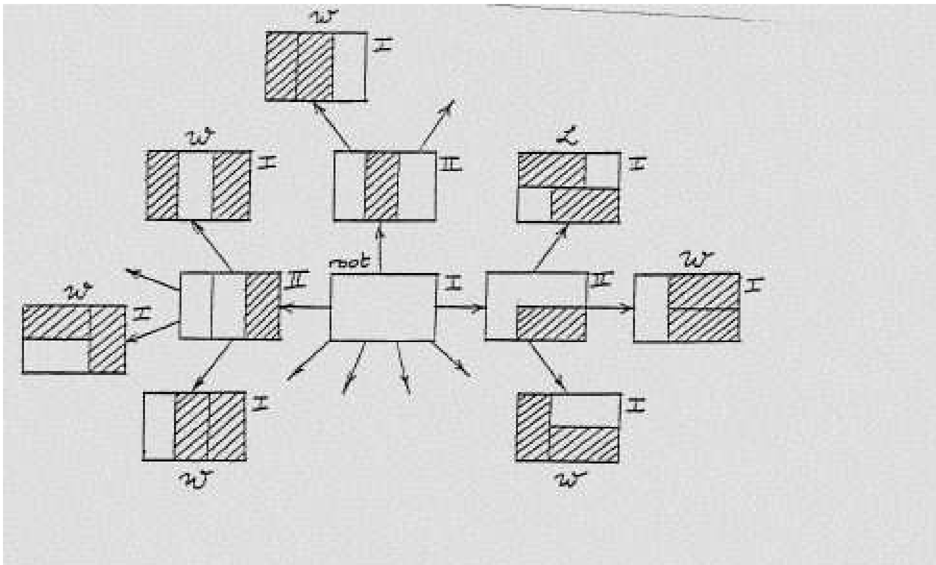


Figure 2.12.2

Exercise 2.12.3

See figure 2.12.3.

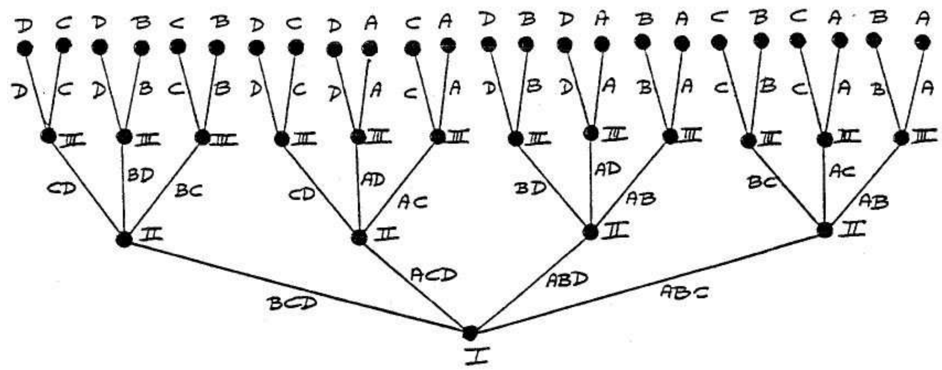


Figure 2.12.3

Exercise 2.12.4

See figure 2.12.4.

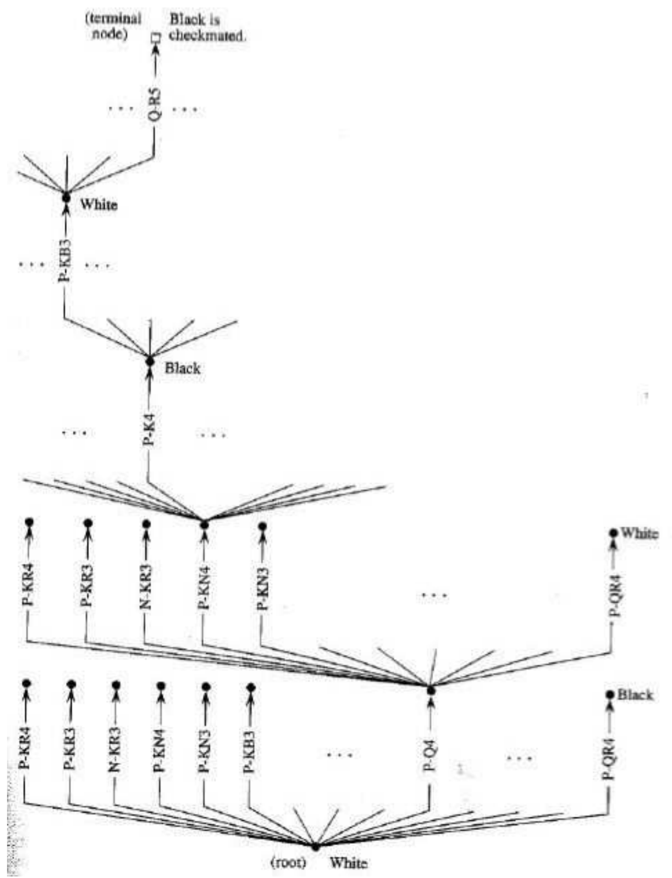


Figure 2.12.4

Exercise 2.12.5

See figure 2.12.5.

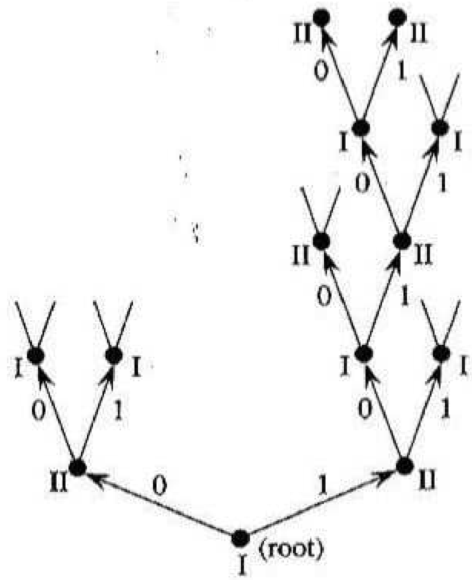


Figure 2.12.5

Exercise 2.12.6

$v(G) = \mathcal{D}, v(G_b) = \mathcal{L}, v(G_c) = \mathcal{D}$. The strategy rrr assures $\mathcal{D}$ or better because if player I chooses r at node a, all possible outcomes have a payoff of $\mathcal{W}$ or $\mathcal{D}$. This strategy is not selected by Zermelo's algorithm (backward induction) because if player I get to node d, it is not optimal to choose r.

Exercise 2.12.7

Let H be the game in exercise 2.12.2. The value of the game is $v(H) = \mathcal{W}$, and a winning strategy for I is to place the domino so as to completely cover one of the three columns. That is, choose one of the moves shown in figure 2.12.7.

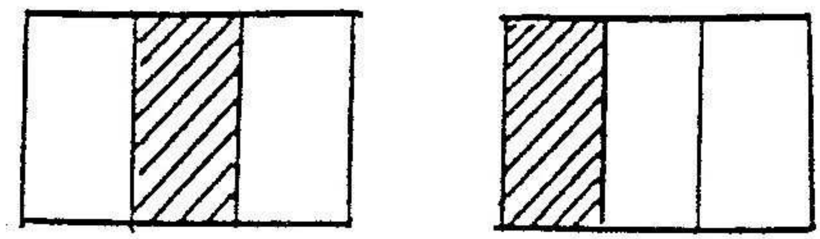


Figure 2.12.7

Exercise 2.12.8

Player I always has a winning strategy because the game is unbalanced. With $n = 3$, player I begins by taking a match from the largest pile. Play then continues as in Section 2.6. of the book.

With k-th pile containing k matches Player I always has a winning strategy because the game is balanced.

Exercise 2.12.9

(a) Player II has a winning strategy in this game by mirroring player I's moves.

(b) Player I has a winning strategy. As an example, apply backward induction to exercise 2.12.2.

(c) Player II has a winning strategy by always putting his domino immediately next to player I's on his first move.

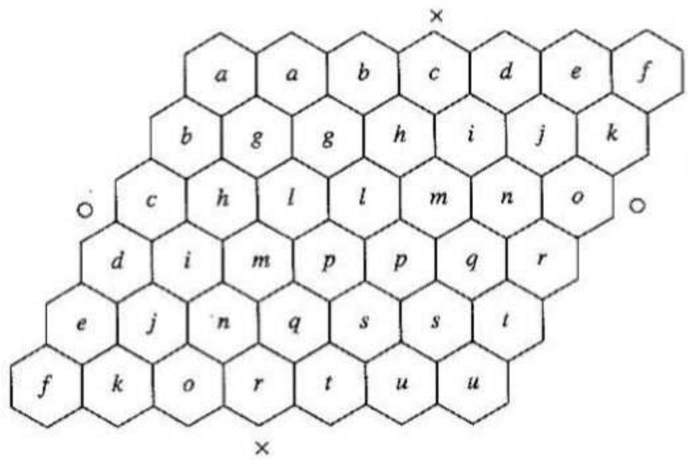
Exercise 2.12.10

In 3×3 and 5×5 Hex, Circle should open by playing in the central hexagon. In 4×4 Hex, with the board oriented as in figure 2.14(a), he should play in the topmost hexagon not on the boundary.

Exercise 2.12.11

Cross should always reply to Circle by occupying the hexagon labeled with the same letter in figure 2.12.11. as that just occupied by Circle. Such a blocking strategy makes it impossible for Circle to win and so Cross must win.

This solution is from Martin Gardner's puzzle column in the Scientific American, now long gone.



Exercise 2.12.12

The argument only proves that such strategy would be a winning strategy if player II had a winning strategy. However, player II doesn't have a winning strategy, which was proven by contradiction. In particular, as the next argument shows, if player I happens to pick the first hex in an acute corner, he will not have a winning strategy any more. To prove that in Beck's Hex, player II doesn't have a winning strategy proceed by induction on n^2, the size of the board. It is easy to check that the statement is true for $n = 2$. Next, if the statement is true for a board of size n^2, we need to show that it is true for the board of size $(n+1)^2$. So suppose that the board is as in figure 2.14(a) and suppose that the first circle was in the left acute corner, label it h_0. We prove by contradiction that player II can't have a winning strategy. So assume that II had a winning strategy on a board of size $n + 1$. Then the connecting path would necessarily have to incorporate the circled hex in the acute corner (since in the ordinary Hex game II doesn't have a winning strategy). Therefore, the path has to incorporate either the hex to the right and down from the corner (label it h_1) or the hex to the right an up (h_2). If it incorporated only h_2 this would be a contradiction with the fact that in the original hex player II doesn't have a winning strategy, since then the circled hex in the corner would be irrelevant. If it incorporated only h_1 then this would imply that II had a winning strategy in the board where the upper left line of hexes is deleted, as well as the line in the other dimension (take the one where player I placed the first cross). But such board has size n^2 and player II starts with a circle in an acute corner. By induction hypothesis we know that player II doesn't have a winning strategy on that board. This contradicts the path incorporating h_1. Thus the path doesn't incorporate either h_1 or h_2 hence it cannot incorporate h_0. But then player II can't have a winning strategy.

Exercise 2.12.13

This game is equivalent to Hex because there is a one-to-one correspondence between the components of 7×7 Hex and the components of this game. For example, each intersection corresponds to a hexagon, and the streets correspond to one of the six sides of the hexagon. The objectives of both games are obviously the same.

Exercise 2.12.14

(a) Figure 2.12.14(a) shows a Bridgit board on which no further linkages can be made without violating the rules. The result is a sort of maze. Someone who entered the maze at the bottom-left would finally exit at the bottom-right, having kept White's linkages always on his left. Thus White must have won.

(b) Since the outcome $\mathcal{D}$ is impossible, the value of Bridgit must be either $\mathcal{W}$ or $\mathcal{L}$.

(c) A strategy-stealing argument just like that given for Hex applies.

(d) If Black moves first, he should begin by moving as shown in figure 2.12.14(b). It is pointless for either player to link two nodes that lie on the edge of the board. If White avoids such pointless moves, her linkages will always touch an end of one of the broken lines shown in figure 2.12.14(a). Black should always reply by making a linkage that touches the other end of this broken line.

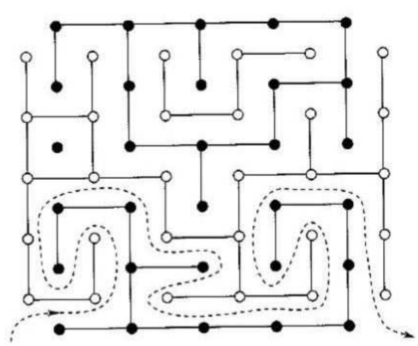


Figure 2.12.14(a)

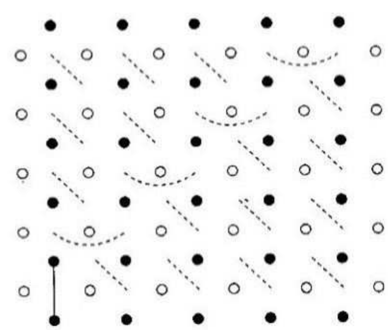


Figure 2.12.14(b)

Exercise 2.12.15

If there exists a set E, then player II can always win. The reasoning is as follows: Since each node is an end point of one of the edges in E by hypothesis, player I's choice will leave player II with either one or two choices. If she has two choices, she should choose the one which will leave her opponent no choices, thus winning. If she has one choice, she will take it, thus forcing her opponent to choose a node which gives her the win. These possibilities are shown in figure 2.12.15(a). A winning strategy for player I with the graph $\mathcal{G}$ in figure 2.22, is shown in figure 2.12.15(b). Player I will pick the nodes identified with Arabic numerals, and player II will choose those identified with Roman numerals. Player II will have only one node at which to make a choice, and player I can win regardless of her choice.

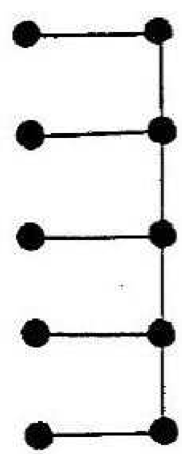


Figure 2.12.15(a)

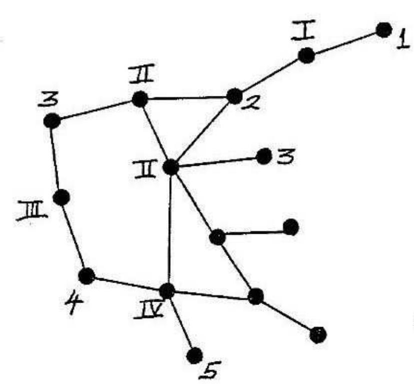


Figure 2.12.15(b)

Exercise 2.12.16

If the second player has a winning strategy, then the first one does as well, thus the second one cannot have a winning strategy. To guaranteee a draw, the second player must occupy a corner. The value of tic-tac-toe is D.

Exercise 2.12.17

Suppose that Black had a winning strategy in Chess. Then White might seek to steal it by planning to make whatever move Black would make in every position if the colors of the pieces were reversed. But this will not help White at the beginning of the game because the positions he will be facing then *cannot* be reached if the colors of the pieces are reversed. Nor will it help White to pretend tat he did not make certain moves. This ploy worked in Hex only because of the very simple structure of the game.

Exercise 2.12.18

Player I can ensure victory when $E = \{x : x > \frac{1}{2}\}$ by choosing 1 at both the first and the second opportunity. To win when $E = \{x : x \geq \frac{2}{3}\}$, player I must choose 1 at *every* opportunity. Player II can ensure victory when $E = \{x : x > \frac{2}{3}\}$ by choosing 0 at every opportunity. The decimal expansion of a rational number is one that is eventually periodic, and the same is true for binary expansions. To win when E is the set of all rational numbers, player II therefore merely needs to destroy any periodicity that looks as though it might be getting established. If she is satisfied to win with probability 1, she can do this simply by playing at random.

Exercise 2.12.19

Since (s, t) is a saddle point, $v(a, t) \succeq_1 v(s, t) \succeq_1 v(s, b)$, for all a in S and all b in T. Similarly, $v(c, t') \succeq_1 v(s', t') \succeq_1 v(s', d)$, for all c in S and all d in T. To prove that (s, t') is a saddle point, take $a = s'$ and $d = t$. Then $v(c, t') \succeq_1 v(s', t') \succeq_1 v(s', t) \succeq_1 v(s, t) \succeq_1 v(s, b)$.

Exercise 2.12.20

Any strategy combination of the form (yxx, XM) with $y \in \{m, r\}, x \in \{l, r\}$ and $X \in \{L, R, M\}$ forms a Nash equilibrium. The set of subgame-perfect equilibria is $\{(mll, RM), (rll, RM)\}$.

Exercise 2.12.21

The subgame-perfect equilibria are shown in figure 2.12.21(a) with double edges. There are many Nash equilibria that are not subgame-perfect. One is shown in figure 2.12.21(b).

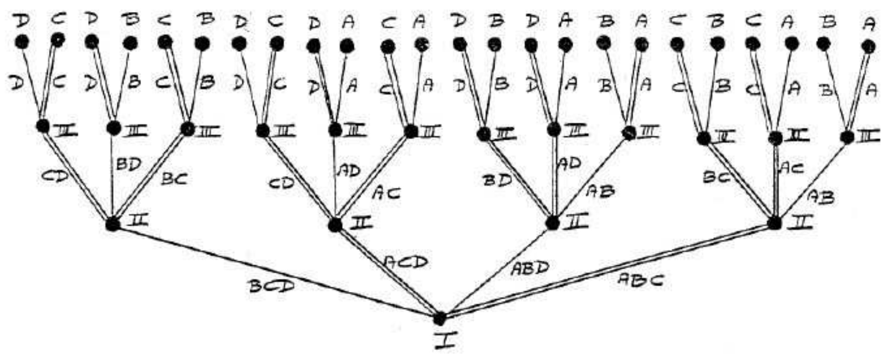


Figure 2.12.21(a)

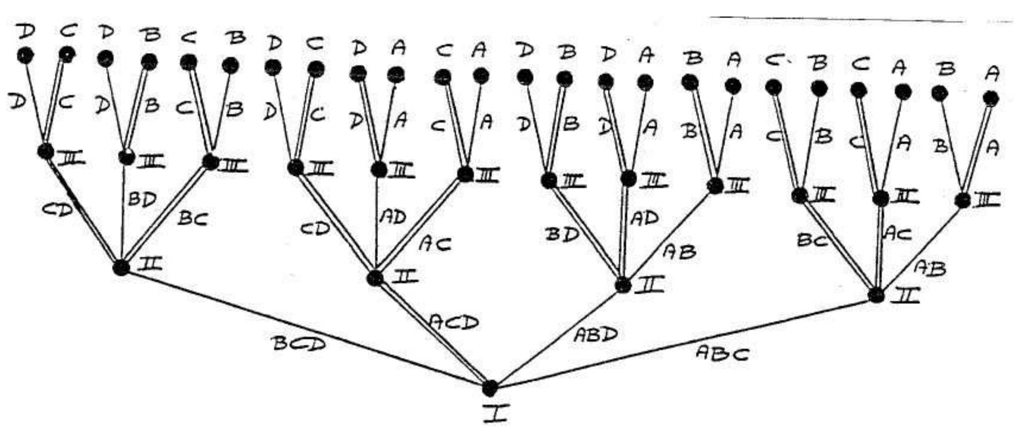


Figure 2.12.21(b)

Exercise 2.12.22

The payoff table is

Agency

Firm		M	Tu	W	Th	F
	M	L	W	W	W	W
	Tu	W	L	W	W	W
	W	W	W	L	W	W
	Th	W	W	W	L	W
	F	W	W	W	W	L

If the Agency puts probability $\frac{1}{5}$ on every day of the week, and if the Firm uses a pure strategy X, where X is a given day of the week then the Agency will win precisely one time out of five, that is when it will monitor exactly on day X. Now suppose that the Firm plays a mixed strategy $(p_M, p_{Tu}, ..., p_F)$ where p_X is the probability that the Firm puts on day X. Then the expected number of times the Agency will win is equal to

$$\frac{1}{5}p_M + \frac{1}{5}p_{Tu} + ... + \frac{1}{5}p_F = \frac{1}{5}\left(p_M + p_{Tu} + ... + p_F\right) = \frac{1}{5},$$

since the probabilities in the Firm's strategy have to add up to 1. If both, the Agency and the Firm put equal probabilities on all days, then they are both indifferent between any strategy. In particular, the strategy of the Firm is its best reply to the strategy of the Agency, and vice versa. Hence, these strategies constitute a Nash equilibrium.

Exercise 2.12.23

In the figure, W, denotes a win for Adam, and L a win for Eve. First, Adam guesses whether the test will be today or tomorrow, then Eve either gives the test today, or if not, she surely has to give the test tomorrow. Thus if there was no test today, Adam can "guess" that the test will be tomorrow. Then Eve gives the test tomorrow. Eve, when deciding whether to give a test or not, doesn't know what Adam's guess was. The result of the backward induction is the strategy (t, T) for Adam and (t, T) for Eve, resulting in Eve giving the test today and Adam guessing it right.

Exercise 2.12.24

The strategic form of the game is

A \ E	tT	TT
tT	W	W
TT	L	W

If we eliminate weakly dominated strategies we obtain either both Nash equilibria, or just the Subgame Perfect Nash equilibrium, depending on the order of elimination.

Exercise 2.12.25

Adam should then know for sure that the surprise test will be on the only possible day.

Exercise 2.12.26

If everyone votes according to their rankings Alice wins: she wins over Bob in the first stage and over Nobody in the second stage. If Horace solves the game using backward induction he should vote for Bob in the first stage so that Bob wins over Nobody in the second stage. If everyone votes strategically then in the last stage Alice wins over Nobody and Nobody wins over Bob, hence in the first stage Alice wins over Bob (this is true by backward induction, because Alice wins over Nobody). Thus, the outcome is that Alice wins.