Exercises 2.1

1. (a) yes; (b) no; (c) no
3. $f(x, y) = y^{1/5}$, so $f_y(t, y) = \frac{1}{5}y^{-4/5}$ is not continuous at $(0, 0)$ so uniqueness is not guaranteed. Solutions: $y = \left(\frac{4}{5}t\right)^{5/4}$, $y = 0$
5. $f(t, y) = 2\sqrt{|y|} = \begin{cases} 2\sqrt{y},\ y \geq 0 \\ 2\sqrt{-y},\ y < 0 \end{cases}$, so $f_y(t, y) = \begin{cases} y^{-1/2},\ y\ 0 \\ -(-y)^{1/2},\ y < 0 \end{cases}$ is not continuous at $(0, 0)$. Therefore, the hypotheses of the Existence and Uniqueness Theorem are not satisfied.

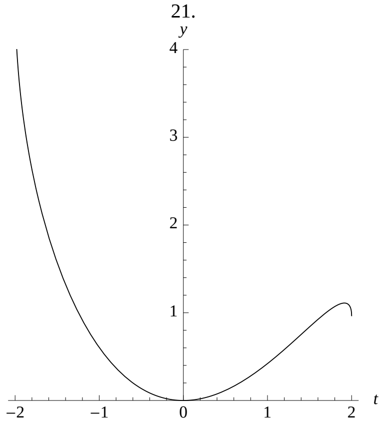
7. Yes. $\dfrac{dy}{dt} = y\sqrt{t} \Rightarrow \dfrac{1}{y}dy = t^{1/2}\,dt \Rightarrow ln|y| = \frac{2}{3}t^{3/2} + C_1 \Rightarrow y = Ce^{2t^{3/2}/3}$. Application of the initial conditions yields $y = \exp\left(\frac{2}{3}(t^{3/2} - 1)\right)$
9. Yes. $f(t, y) = \sin y - \cos t$ and $f_y(t, y) = \cos y$ are continuous on a region containing $(\pi, 0)$.
11. $y = \sec t$ so $y' = \sec t\, \tan t = y\, \tan t$ and $y(0) = 1$. $f_y(t, y) = t$ is continuous on $-\pi/2 < t < \pi/2$ and $f(t, y) = \sec t$ is continuous on $-\pi/2 < t < \pi/2$ so the largest interval on which the solution is valid is $-\pi/2 < t < \pi/2$.
13. $f(t, y) = \sqrt{y^2 - 1}$ and $f_y(t, y) = \frac{1}{2}(y^2 - 1)^{-1/2}$; unique solution guaranteed for (a) only.

15. $(0, \infty)$. Solution is $y = \frac{1}{4}t^{-1}(t^4 - 1)$.
17. $(0, \infty)$ because $y = \ln t$ has domain $t > 0$

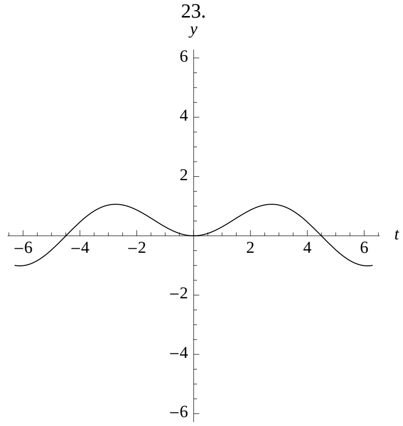
19. $(-\infty, 1)$ because $y = 1/(t-1)$ has domain $(-\infty, 1) \cup (1, \infty)$ and $y = 1/(t-3)$ has domain $(-\infty, 3) \cup (3, \infty)$

21. $(-2, 2)$
23. $t > 0$, $y = t^{-1}\sin t - \cos t$, $-\infty < t < \infty$

21.



23.

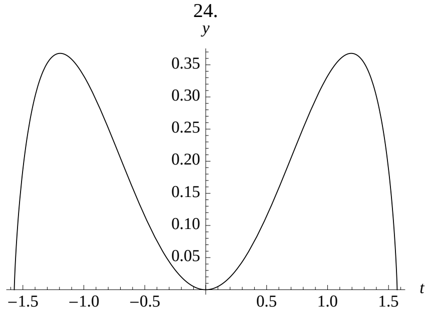


25. First, we solve the equation (see next section):

$$\begin{aligned} \frac{dy}{dt} &= y^2 \\ \frac{1}{y^2}dy &= dt \\ -\frac{1}{y} &= t + C \\ y &= \frac{-1}{t+C}. \end{aligned}$$

Applying the initial condition indicates that $-1/C = a \Rightarrow C = -1/a$ so $= a/(1-at)$. This solutions is defined for $t > 1/a$ or $t < 1/a$

24.



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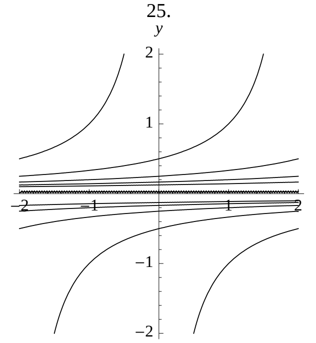
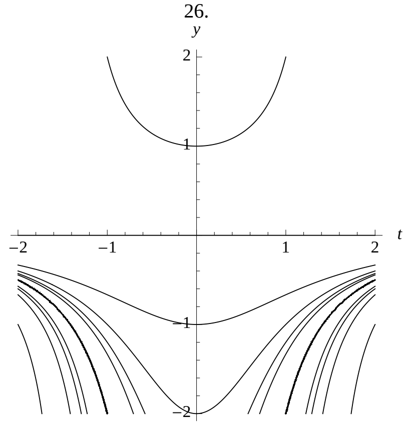
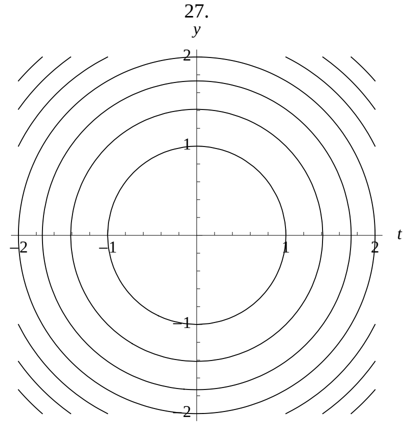


27. Separating variables (see next section) gives us $y\,dy = -t\,dt \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}t^2 + C \Rightarrow y = \pm\sqrt{C - t^2}$. Applying the initial conditions indicates that $C = a^2$ so $y = \sqrt{a^2 - t^2}$ (because $y(0) = a$ is positive). Thus, the interval of definition of the solution is $|t| < a$

26.



27.



Exercises 2.2

1. Separate variables and integrate:

$$\begin{aligned} y^2\,dy &= x\,dx \\ \frac{1}{3}y^3 &= \frac{1}{2}x^2 + C \\ y^3 &= \frac{3}{2}x^2 + C \\ y &= \left(\frac{3}{2}x^2 + C\right)^{1/3}. \end{aligned}$$

3. $y = \dfrac{1 - 2Cx + C^2x^2}{4x^2}$
5. $3t + \frac{1}{2}t^4 + \frac{5}{2}y - \frac{9}{14}y^{-7} = C$
7. $\sinh 3x - \frac{1}{2}\cosh 4y = C$
9. Separate variables and integrate:

$$\begin{aligned} \frac{1}{y+2}dy &= \frac{1}{2t+1}dt \\ \ln(2+y) &= \frac{1}{2}\ln(2t+1) + C \\ \ln(2+y) &= \ln\sqrt{2t+1} + C \\ 2+y &= Ce^{\ln\sqrt{2t+1}} \\ 2+y &= C\sqrt{2t+1} \\ y &= -2 + C\sqrt{2t+1}. \end{aligned}$$

11. $y = \sin^{-1}\left(-\frac{3}{4}\cos x + C\right)$
13. Separate variables and integrate:

$$\begin{aligned} \frac{dy}{dt} &= -ky \\ \frac{1}{y}\,dy &= -k\,dt \\ \ln|y| &= -kt + C \\ y &= Ce^{-kt} \end{aligned}$$

In the calculation above, remember that $e^{-kt+C} = e^{-kt}e^C$. C is arbitrary so e^C is positive and arbitrary.
15. $y = \cosh^{-1}\left(-\frac{1}{120}\sinh 6t - \frac{1}{16}\cosh 4t + C\right)$
17. $y = -\frac{1}{3}\ln\left(-\frac{3}{2}e^{2t} + C\right)$
19. $\frac{3}{4}\cos\theta - \frac{1}{12}\cos 3\theta + \frac{1}{16}\sin 4y - \sin y = C$
21. $y\sin 2x + 2xy + 2y^2 + 5 + 4Cy = 0$
23. $-\dfrac{1}{64}\,\dfrac{1}{\cos(2\,y) + 1}\Bigg(-64\,C\,\cos(2\,y\,(x)) - 64\,C + \cos(8\,x - 2\,y) + \cos(8\,x + 2\,y) +$

$2\,\cos\left(8\,x\right)+4\,\cos\left(4\,x-2\,y\right)+4\,\cos\left(4\,x+2\,y\right)+8\,\cos\left(4\,x\right)-128\Big)=0$

25. $-\frac{1}{4}x\sin 2x+\frac{1}{4}x^2-\frac{1}{8}\cos 2x-2\sin\sqrt{y}=C$

27. $\frac{1}{64}\,\frac{1}{-1+\sin\left(y\right)}\Big(58+6\,\sin\left(y\right)-2\,\cos\left(4\,x\right)+\sin\left(y+4\,x\right)-\sin\left(-y+4\,x\right)+$

$8\,\cos\left(2\,x\right)-4\,\sin\left(y+2\,x\right)+4\,\sin\left(-y+2\,x\right)-64\,C+64\,C\,\sin\left(y\right)\Big)=0$

29. $\frac{2}{3}(\ln x)^{3/2}+\frac{1}{3}e^{3/y}=C$

31. Factor first, then separate, use partial fractions and simplify.

$$\begin{aligned}
dy/dt &= (t^2+1)(y^2-1)\\
\frac{1}{y^2-1}\,dy &= (t^2+1)\,dt\\
\frac{1}{2}\left(\frac{1}{y-1}-\frac{1}{y+1}\right)dy &= (t^2+1)dt\\
\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| &= \frac{1}{3}t^2+t+C\\
\frac{y-1}{y+1} &= Ce^{\frac{2}{3}t^2+2t}\\
y &= \frac{1+Ce^{\frac{2}{3}t^2+2t}}{1-Ce^{\frac{2}{3}t^2+2t}}
\end{aligned}$$

33. $y=\dfrac{13+12C-4x(3C+1)}{4C(x-1)-3}$

35.

$$\begin{aligned}
\frac{1}{y^3+1}dy &= dt\\
\left(\frac{1}{3(y+1)}+\frac{2-y}{3(y^2-y+1)}\right)dy &= dt\\
-3t+\sqrt{3}\tan^{-1}\left(\frac{2y-1}{\sqrt{3}}\right)+3\ln\left|\frac{(y+1)^{1/3}}{(y^2-y+1)^{1/6}}\right| &= C
\end{aligned}$$

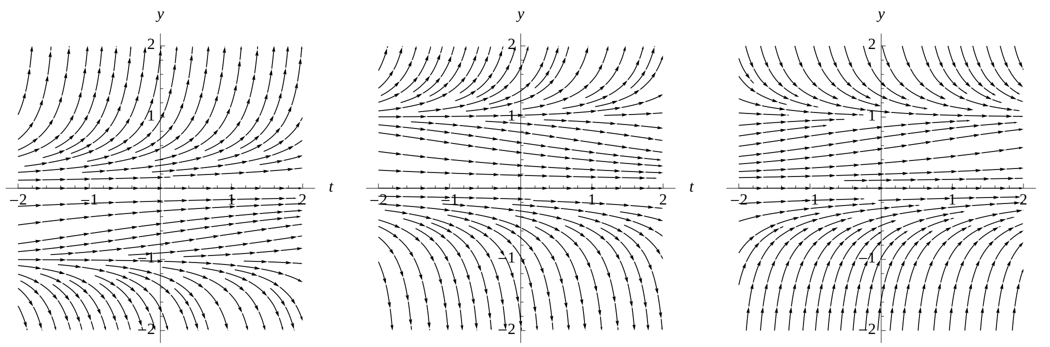


Figure 1: Direction fields for $dy/dt = y^3+y^2$, $dy/dt = y^3-y^2$, and $dy/dt = y^2-y^3$

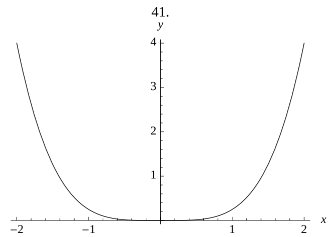
37.

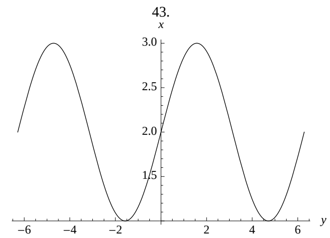
$$\begin{aligned}
\frac{1}{y^3+y}dy &= dt \\
\left(\frac{1}{y} - \frac{y}{y^2+1}\right) dy &= dt \\
\ln|y| - \frac{1}{2}\ln\left(y^2+1\right) &= t + C \\
\frac{y}{\sqrt{y^2+1}} &= Ce^t \\
y^2 &= \frac{Ce^{2t}}{1-Ce^{2t}} \\
y &= \pm\sqrt{\frac{Ce^{2t}}{1-Ce^{2t}}}
\end{aligned}$$

39.

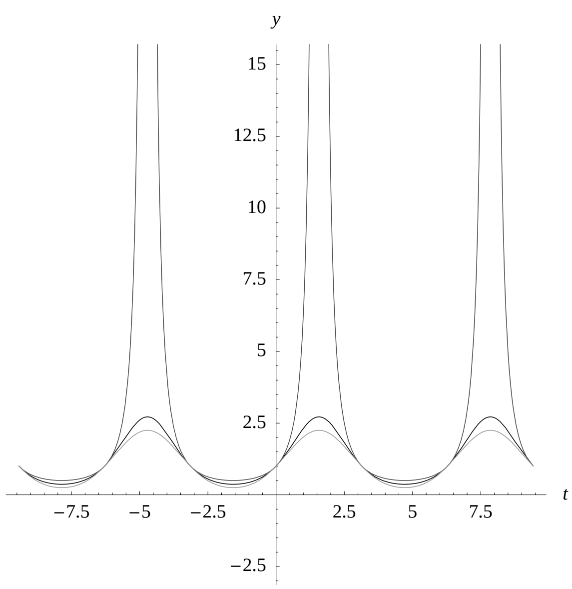
$$\begin{aligned}
\left(\frac{1}{2(y-1)} - \frac{1}{y} + \frac{1}{2(y+1)}\right) dy &= dt \\
\frac{1}{2}\ln|y-1| - \ln|y| + \frac{1}{2}\ln|y+1| &= t + C \\
\frac{\sqrt{y^2-1}}{y} &= Ce^t \\
y^2 &= \frac{1}{1-Ce^{2t}} \\
y &= \pm\sqrt{\frac{1}{1-Ce^{2t}}}
\end{aligned}$$

41. $dy = x^3\,dx \Rightarrow y = \frac{1}{4}x^4 + C$. $y(0) = \frac{1}{4} \cdot 0^4 + C = 0 \Rightarrow C = 0$ so $y(x) = 1/4\,x^4$
43. Integrating gives us $x = \sin y + C$ and applying the initial condition gives us $C = 2$ so $x(y) = \sin(y) + 2$





45. $y(t) = 2/3\,\sqrt{9 + 3\,t^{3/2}}$
47. $y(t) = -1 - \sqrt{-1 + 2\,e^t}$
49. $y(x) = e^{\sqrt{2}\sqrt{x}}$, $y(x) = e^{-\sqrt{2}\sqrt{x}}$
51. $y(x) = \arctan(x) + 1$
53. $y = -3 + 4(3x+1)^{1/3}$
55. $y = \ln\left(\frac{1}{2}e^{2x} - \frac{1}{2} + e\right)$
57. The solution for (a) is $y = e^{\sin t}$, for (b) it is $y = \dfrac{1}{1 - \sin t}$, and for (c) it is $y = \dfrac{1}{4}\left(4 + r\sin t + \sin^2 t\right)$.



59. $y = 2x(x-2)^{-1}$

67. Use partial fractions.

$$\begin{aligned}
\frac{1}{12+4y-y^2}\,dt &= dt \\
\left(\frac{1}{y+2}-\frac{1}{y-6}\right)\,dy &= 8\,dt \\
\ln\left|\frac{y+2}{y-6}\right| &= 8t+C \\
\frac{y+2}{y-6} &= Ce^{8t} \\
y &= 2\frac{3Ce^{8t}+1}{Ce^{8t}-1}.
\end{aligned}$$

71. $y = \exp\left(\frac{c}{t}(t-1)\right)$
73. $L(t) = L_\infty - e^{-r_B t}$

Exercises 2.3

1. The integrating factor is $\mu(t) = e^{-t}$. Multiplying through by the integrating factor, applying the theorem, integrating and solving for y gives us

$$\begin{aligned}
\frac{dy}{dt} - y &= 10 \\
e^{-t}\frac{dy}{dt} - ye^{-t} &= 10e^{-t} \\
\frac{d}{dt}\left(e^{-t}y\right) &= 10e^{-t} \\
e^{-t}y &= -10e^{-t} + C \\
y &= -10 + Ce^{t}.
\end{aligned}$$

The preferred solution is to use undetermined coefficients. A general solution of the corresponding homogeneous equation, $y' - y = 0$ is $y_h = Ce^t$. The forcing function is $f(t) = 10$. The associated set of functions for a constant is $F = \{1\}$. Because no multiple of 1 is a solution to the corresponding homogeneous equation, we assume that a particular solution takes the form $y_p = A \cdot 1 = A$. Differentiating $y_p' = 0$ and substituting into the *nonhomogeneous* equation gives us $y_p' - y_p = -A = 10$ so $A = -10$ and $y_p = -10$. Therefore, a general solution of the nonhomogeneous equation is $y = y_h + y_p = -10 + Ce^t$
3. The integrating factor is $\mu(t) = e^{-t}$. Multiplying through by the integrating

factor and integrating we have

$$\begin{aligned}\frac{dy}{dt} - y &= 2\cos t \\ e^{-t}\frac{dy}{dt} - e^{-t}y &= 2e^{-t}\cos t \\ \frac{d}{dt}\left(e^{-t}y\right) &= 2e^{-t}\cos t \\ e^{-t}y &= -e^{-t}\cos t + e^{-t}\sin t + C \\ y &= Ce^{t} - \cos t + \sin t.\end{aligned}$$

Observe that in integrating $e^{at}\cos bt\,dt$ and $e^{at}sinbt\,dt$ either required integration by parts twice or a table of integrals, $e^{at}\cos bt\,dt = \dfrac{1}{a^2+b^2}e^{at}\left(a\cos bt + b\sin bt\right)$ and $e^{at}\sin bt\,dt = \dfrac{1}{a^2+b^2}e^{at}\left(-b\cos bt + a\sin bt\right)$.

5. $y = Ce^{t} - 2te^{-t} - e^{-t}$

7. $y = \frac{1}{2}t + Ct^{-1}$

9. First write the equation in standard form $\dfrac{dy}{dx} + \dfrac{1}{x}y = \dfrac{1}{x}e^{-x}$ to see that $p(x) = 1/x$. Then, an integrating factor is $e^{\int 1/x\,dx} = x^{\ln x} = x$. Multiplying through by the integrating factor and integrating gives us

$$\begin{aligned}x\frac{dy}{dx} + y &= e^{-x} \\ \frac{d}{dx}(xy) &= e^{-x} \\ xy &= -e^{-x} + C \\ y &= Cx^{-1} - x^{-1}e^{-x}.\end{aligned}$$

11. $y = 4t^2 + 1 + C\sqrt{4t^2+1}$

13. An integrating factor is $\mu(t) = e^{\int \cot t\,dt} = e^{-\ln\csc t} = \sin t$. Multiplying through by the integrating factor, integrating the result, and solving for y results in

$$\begin{aligned}\sin t\frac{dy}{dt} + y\cos t &= \sin t\cos t \\ \frac{d}{dt}(y\sin t) &= \sin t\cos t \\ y\sin t &= \frac{1}{2}\sin^2 t + C \\ y &= \frac{1}{2}\sin t + C\csc t,\end{aligned}$$

which is equivalent to $y = -\frac{1}{2}\cos t\cot t + C\csc t$ because $\int \sin t\cos t\,dt = -\frac{1}{2}\cos^2 t + C$ when choosing $u = \cos t$ rather than $u = \sin t$ when calculating the integral.

15. An integrating factor is

$$\mu(t) = \exp\left(-\int \frac{4t}{4t^2-9}\,dt\right) = \exp\left(-\frac{1}{2}\ln(4t^2-9)\right) = \frac{1}{\sqrt{4t^2-9}}.$$

Multiplying through by the integrating factor, integrating the result, and solving for y results in

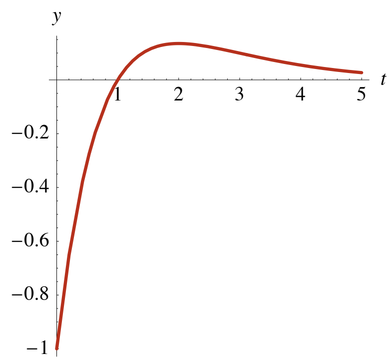
$$\begin{aligned}
\frac{1}{\sqrt{4t^2-9}}\frac{dy}{dt} - \frac{1}{\sqrt{4t^2-9}}\frac{4t}{4t^2-9}y &= \frac{t}{\sqrt{4t^2-9}} \\
\frac{d}{dt}\left(\frac{1}{\sqrt{4t^2-9}}y\right) &= \frac{t}{\sqrt{4t^2-9}} \\
\frac{1}{\sqrt{4t^2-9}}y &= \frac{1}{4}\sqrt{4t^2-9}+C \\
y &= \frac{1}{4}(4t^2-9) + C\sqrt{4t^2-9}.
\end{aligned}$$

17. An integrating factor is $\mu(t) = e^{2\int \cot x\,dx} = e^{-2\ln\csc x} = \sin^2 x$. Multiplying the equation by the integrating factor, integrating and solving for y yields

$$\begin{aligned}
\sin^2 x\frac{dy}{dx} + 2y\sin^2 x\cot x &= \sin^2 x\,\cos x \\
\frac{d}{dx}\left(\sin^2 x\,y\right) &= \sin^2 x\,\cos x \\
\sin^2 x\,y &= \frac{1}{3}\sin^3 x + C \\
y &= \frac{1}{3}\sin x + C\csc^2 x.
\end{aligned}$$

19. $\theta = -1 + Ce^{r^2/2}$
21. $x(y) = -1 - y + Ce^y$
23. $x(t) = C/(t^3-1)$
25. $v(s) = se^{-s} + Ce^{-s}$
27. Use undetermined coefficients to find a general solution of the equation. $y_h = Ce^{-t}$. The associated set of functions for the forcing function $f(t) = e^{-t}$ is $F = \{e^{-t}\}$. Because e^{-t} is a solution to the corresponding homogeneous equation, multiply F by t^n where n is the smallest integer so that no element of t^nF is a solution to the corresponding homogeneous equation. In this case, $tF = \{te^{-t}\}$ so we assume that a particular solution of the nonhomogeneous equation has the form $y_p = Ate^{-t}$. Differentiating y_p, $y_p' = -Ate^{-t} + Ae^{-t}$, and substituting into the nonhomogeneous equation yields $y_p' + y_p = -Ate^{-t} + Ae^{-t} + Ate^{-t} = Ae^{-t} = e^{-t}$ so $A = 1$ and $y_p = te^{-t}$. Therefore a general solution of the nonhomogeneous equation is $y = y_h + y_p = Ce^{-t} + te^{-t}$. Application of the initial condition yields

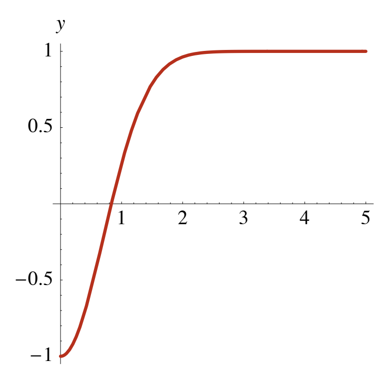
$y = e^{-t}(t-1)$.



29. An integrating factor is $\mu(t) = e^{\int 2t\,dt} = e^{t^2}$. Multiplying the equation by the integrating factor, integrating and solving for y yields

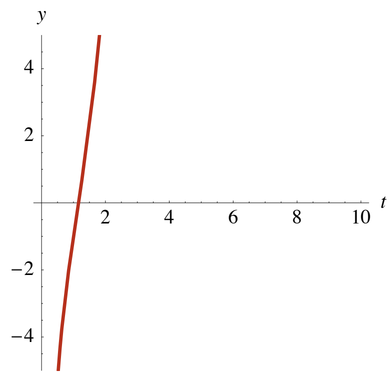
$$\begin{aligned} e^{t^2}\frac{dy}{dt} + 2te^{t^2}y &= 2te^{t^2} \\ \frac{d}{dt}\left(e^{t^2}y\right) &= 2te^{t^2} \\ e^{t^2}y &= e^{t^2} + C \\ y &= 1 + Ce^{-t^2}. \end{aligned}$$

Applying the initial condition yields $y = 1 - 2e^{-t^2}$.

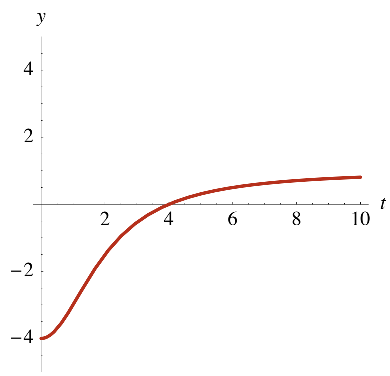


31. A general solution is $y = 2t^{-1}(t-1)e^t + Ct^{-1}$. Applying the initial conditions

yields $y = (2te^t - 2e^t - 1)/t$.



33. $y = \dfrac{t^2 - 16}{t^2 + 4}$

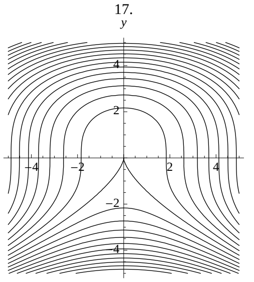


39. (a) $y = Ce^t + 1$; (b) $y = Ce^{-t} + t$; (c) $y = Ce^{-t} + \sin t$; (d) $y = Ce^t + e^{-t}$
43. $y' + y = t$ has solution $y = t - 1 + Ce^{-t}$. $y(0) = 1 \Rightarrow C = 2$ so $y = t - 1 + 2e^{-t}$ for $0 \le t < 1$. When $t = 1$, $y = 1 - 1 + 2e^{-1} = 2/e$. The solution to $y' + y = 0$, $y(1) = 2/e$ is $y = 2e^{-t}$. Thus, $y(t) = \begin{cases} t - 1 + 2e^{-t}, \ 0 \le t < 1 \\ 2e^{-t}, \ t \ge 1 \end{cases}$.
45. $y(t) = \begin{cases} e^{-2t}, \ 0 \le t < 1 \\ e^{2-4t}, \ t \ge 1 \end{cases}$
47. $y(t) = -2/5 \cos(2t) - 1/5 \sin(2t) + Ce^t$
49. $y(t) = (t + C)e^{-t}$
51. $y(t) = -1/25 - 1/5\, t + Ce^{5t}$
53. $y(t) = -2 \cos(t) + 4 \sin(t) + 4e^t + Ce^{1/2\, t}$
55. $y(t) = 2/11\, e^t + Ce^{-10t}$
57. $y(t) = (2t + C)e^t$
59. $y(t) = \cos(t) + \sin(t) + t - 1 + Ce^{-t}$
63. $y(t) = t - 1 + Ce^{-t}$, $y(t) = -1/2 \cos(t) + 1/2 \sin(t) + Ce^{-t}$, $y(t) = 1/2 \cos(t) + 1/2 \sin(t) + Ce^{-t}$, $y(t) = 1/2\, e^t + Ce^{-t}$
65. $y(t) = t - 1 - e^{-t}$, $y(t) = t - 1$, $y(t) = t - 1 + e^{-t}$, $y(t) = t - 1 + 2e^{-t}$, $y(t) = t - 1 + 3e^{-t}$

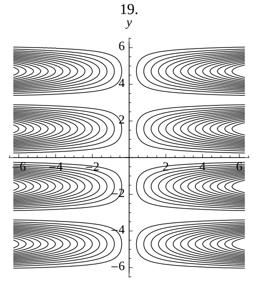
Exercises 2.4

1. $M_y(t,y) = 2y - \frac{1}{2}t^{-1/2} = N_t(t,y)$, exact
3. $M_y(t,y) = \cos ty - ty\sin ty = N_t(t,y)$, exact
5. The equations is exact because $\partial_t(sty^2) = 3y^2 = \partial_y(y^3)$
7. $M_y(t,y) = \sin 2t \neq 2\sin 2t = N_t(t,y)$, not exact
9. $M_y(t,y) = y^{-1} = N_t(t,y)$, exact
11. $y = C + t^3$
13. $y = 0$, $ty^2 = C$
15. Observe that the equation is exact because $\dfrac{\partial}{\partial y}(2t+y^3) = 3y^2 = \dfrac{\partial}{\partial t}(3ty^2+4)$. Let $F(t,y)$ have total derivative $(2t+y^3)\,dt + (3ty^2+4)\,dy$. Then, $F(t,y) = \int(2t+y^3)\,dt = t^2 + ty^3 + g(y)$. Differentiating F with respect to y, $F_y(t,y) = 3ty^2 + g'(y) = 3ty^2 + 4 \Rightarrow g'(y) = 4$ so $g(y) = 4y$ and $F(t,y) = t^2 + ty^3 + 4y$. A general solution is then $t^2 + ty^3 + 4y = C$ or $t^2 + ty^3 + 4\,y\,(t) = C$.
17. The equation is exact because $\dfrac{\partial}{\partial y}(2ty) = 2t = \dfrac{\partial}{\partial t}(t^2+y^2)$. Let $F(t,y)$ satisfy $F_t(t,y)dt + F_y(t,y)dy = 2ty\,dt + (t^2+y^2)\,dy$. Then, $F(t,y) = \int 2ty\,dt = t^2y + g'(y) = t^2 + y^2$ so $g'(y) = y^2 \Rightarrow g(y) = \frac{1}{3}y^3$. Therefore $F(t,y) = t^2 + \frac{1}{3}y^3$ and a general solution of the equation is $t^2 + \frac{1}{3}y^3 = C$. Observe that solving this as a homogeneous equation of degree 2 (see the next section) results in the following form of the solution: $-1/3\,\ln\left(\dfrac{y\left(3\,t^2+y^2\right)}{t^3}\right) - \ln(t) = C$.
19. The equation is exact because $\dfrac{\partial}{\partial y}(\sin^2 y) = 2\sin y\cos y = \sin 2y = \dfrac{\partial}{\partial t}(t\sin 2y)$. Let $F(t,y)$ satisfy $F_t(t,y)dt + F_y(t,y)dy = \sin^2 y\,dt + t\sin 2y\,dy$. Then, $F(t,y) = \int \sin^2 y\,dt = t\sin^2 y + g(y)$ so $g'(y) = 0 \Rightarrow g(y) = 0$ and $F(t,y) = t\sin^2 y$. A general solution is then $t\sin^2 y = C$ or $\ln t + 2\ln\sin y = C$.
21. The equation is exact because $\dfrac{\partial}{\partial y}(e^t\sin y) = e^t\cos y = \dfrac{\partial}{\partial t}(1 + e^t\cos y)$. Let $F(t,y)$ satisfy $F_t(t,y)dt + F_y(t,y)dy = e^t\sin y\,dt + (1+e^t\cos y)\,dy$. Then, $F(t,y) = \int e^t\sin y\,dt = e^t\sin y + g(y)$ so $g'(y) = 1 \Rightarrow g(y) = t$. Thus, $F(t,y) = e^t\sin y + t$ and a general solution of the equation is $e^t\sin y + y = C$.

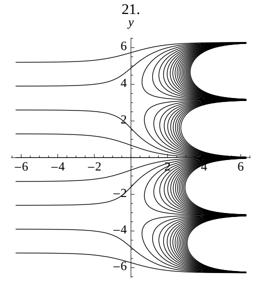
17.



19.



21.

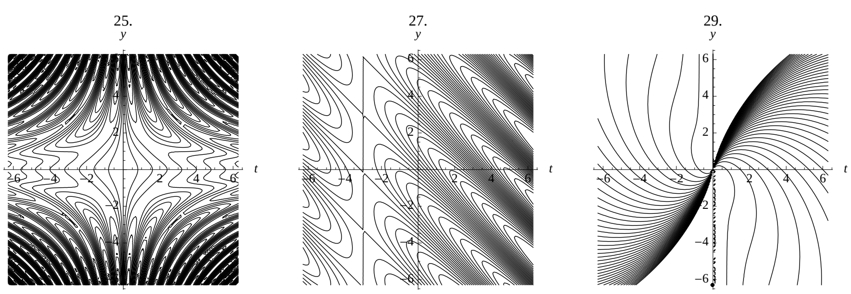


23. $y = 0$, $y = C\sqrt{\sec t^2 + \tan t^2}$
25. The equation is exact because $\dfrac{\partial}{\partial y}(1 + y^2\cos ty) = 2y\cos ty - ty^2\sin ty =$

$\dfrac{\partial}{\partial t}(\sin ty + ty\cos ty)$. Let $F(t,y)$ satisfy $F_t(t,y)dt + F_y(t,y)dy = (1+y^2\cos ty)\,dt + (\sin ty + ty\cos ty)\,dy$. Then, $F(t,y) = \int (1+y^2\cos ty)\,dt = t + y\sin ty + g(y)$ so $F_y(t,y) = \sin ty + ty\cos yt + g'(y)$ so $g'(y) = 0$ and $g(y) = 0$. Then $F(t,y) = t + y\sin(yt)$ and a general solution of the equation is $t + y\sin yt = C$.

27. The equation is exact because $\dfrac{\partial}{\partial y}((3+t)\cos(t+y) + \sin(t+y)) = \cos(t+y) - (3+t)\sin(t+y) = \dfrac{\partial}{\partial t}((3+t)\cos(t+y))$. Let $F(t,y)$ satisfy $F_t(t,y)dt + F_y(t,y)dy = ((3+t)\cos(t+y) + \sin(t+y))\,dt + (3+t)\cos(t+y)\,dy$. Then, $F(t,y) = \int ((3+t)\cos(t+y) + \sin(t+y))\,dt = (3+t)\sin(t+y) + g(y)$ so $F_y(t,y) = (3+t)\cos(t+y) + g'(y) = (3+t)\cos(t+y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = 0$ so $F(t,y) = 3\sin(t+y) + t\sin(t+y)$ and $3\sin(t+y) + t\sin(t+y) = C$.

29. The equation is exact because $\dfrac{\partial}{\partial y}\left(-t^{-2}y^2e^{y/t} + 1\right) = -t^{-3}ye^{y/t}(2t+y) = \dfrac{\partial}{\partial t}\left(e^{y/t}(1+y/t)\right)$. Let $F(t,y)$ satisfy $F_t(t,y)dt + F_y(t,y)dy = \left(-t^{-2}y^2e^{y/t} + 1\right)\,dt + e^{y/t}(1+y/t)\,dy$. Then, $F(t,y) = \left(-t^{-2}y^2e^{y/t} + 1\right)\,dt = t + ye^{y/t} + g(y)$. Next, $F_y(t,y) = e^{y/t}(1+y/t) + g'(y) = e^{y/t}(1+y/t)$ so $g'(y) = 0 \Rightarrow g(y) = 0$. Thus, $F(t,y) = ye^{y/t} + t$ and $ye^{y/t} + t = C$.



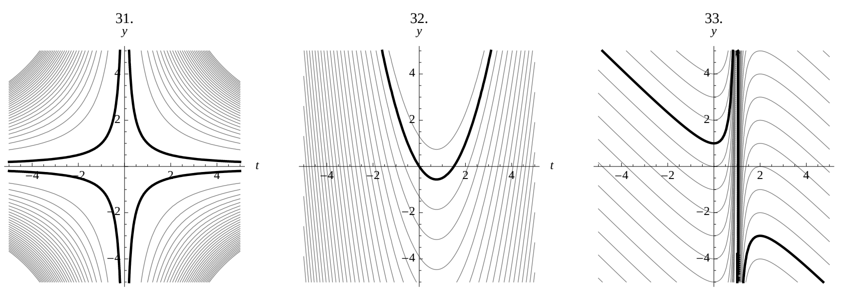
31. This equation is exact because $\dfrac{\partial}{\partial y}\left(2ty^2\right) = 4ty = \dfrac{\partial}{\partial t}\left(2t^2y\right)$. Let $F(t,y)$ satisfy $F_t(t,y)\,dt + F_y(t,y)\,dy = 2ty^2\,dt + 2t^2y\,dy$. Then, $F(t,y) = \int 2ty^2\,dt = t^2y^2 + g(y)$ so $F_y(t,y) = 2t^2y + g'(y) = 2t^2y$, which means that $g'(y) = 0 \Rightarrow g(y) = 0$. Therefore $F(t,y) = t^2y^2$ and a general solution (or, integral curves) of the differential equation are $t^2y^2 = C$. Application of the initial condition results in $y^2t^2 - 1 = 0$.

Observe that dividing the differential equation by $2ty$ yields $y\,dt + t\,dy = 0$, which is equivalent to $t\dfrac{dy}{dt} + y = 0$ or $\dfrac{dy}{dt} + \dfrac{1}{t}y = 0$. This is a first order linear homogeneous equation with integrating factor $\mu(t) = e^{\int 1/t\,dt} = t$. Multiplying through by the integrating factor, integrating and solving for y gives us

$$\begin{aligned} t\frac{dy}{dt} + y &= 0 \\ \frac{d}{dt}(ty) &= 0 \\ ty &= C \\ y &= Ct^{-1}. \end{aligned}$$

Observe that squaring both sides of the equation and solving for C gives us the same result as that obtained by solving the equation as an exact equation.

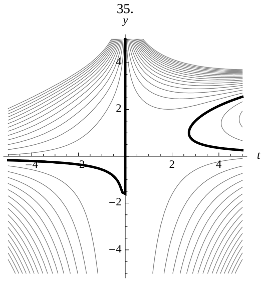
33. The equation is exact because $\frac{\partial}{\partial y}\left(2ty + 3t^2\right) = 2t = \frac{\partial}{\partial t}\left(t^2 - 1\right)$. Let $F(t,y)$ satisfy $F_t(t,y)\,dt + F_y(t,y)\,dy = \left(2ty + 3t^2\right)\,dt + \left(t^2 - 1\right)\,dy$. Then, $F(t,y) = \int \left(2ty + 3t^2\right)\,dt = t^2y + t^3 + g(y)$ so $F_y(t,y) = t^2 + g'(y) = t^2 - 1$ $\Rightarrow g'(y) = -1 \Rightarrow g(y) = -y$. Therefore, $F(t,y) = t^2y + t^3 - y$ and the integral curves are $t^2y + t^3 - y = C$. Applying the initial condition and solving for y gives us $y = \dfrac{-t^3 - 1}{(t-1)(t+1)}$.



35. The equation is exact because $\frac{\partial}{\partial y}\left(e^y - 2ty\right) = e^y - 2t = \frac{\partial}{\partial t}\left(te^y - t^2\right)$. Let $F(t,y)$ satisfy $F_t(t,y)\,dt + F_y(t,y)\,dy = \left(e^y - 2ty\right)\,dt + \left(te^y - t^2\right)\,dy$. Then, $F(t,y) = \int \left(e^y - 2ty\right)\,dt = te^y - t^2y + g(y)$. Differentiating with respect to y, $F_y(t,y) = te^y - t^2 + g'(y) = te^y - t^2 \Rightarrow g'(y) = 0 \Rightarrow g(y) = 0$ so $F(t,y) = te^y - t^2y$ and the integral curves are $te^y - t^2y = C$. Applying the initial condition results in $te^y - t^2y = 0$.

37. The equation is exact because $\frac{\partial}{\partial y}\left(y^2 - 2\sin 2t\right) = 2y = \frac{\partial}{\partial t}(1 + 2ty)$. Let $F(t,y)$ satisfy $F_t(t,y)\,dt + F(t,y)\,dy = \left(y^2 - 2\sin 2t\right)\,dt + (1 + 2ty)\,dy$. Then, $F(t,y) = \int \left(y^2 - 2\sin 2t\right)\,dt = ty^2 + \cos 2t + g(y)$. Differentiating with respect to y, $F_y(t,y) = 2ty + g'(y) = 1 + 2ty \Rightarrow g'(y) = 1 \Rightarrow g(y) = y$ so $F(t,y) = ty^2 + \cos 2t + y$ and the integral curves are $ty^2 + \cos 2t + y = C$. Applying the initial condition results in $ty^2 + \cos(2\,t) + y - 2 = 0$.

35.



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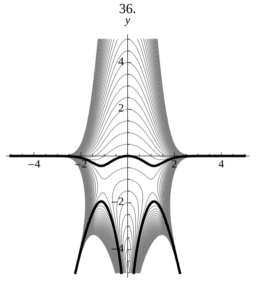

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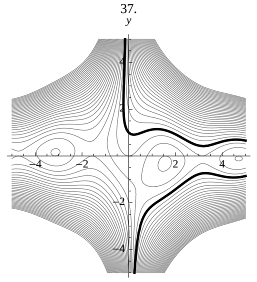
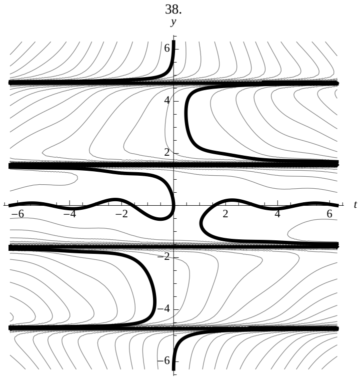
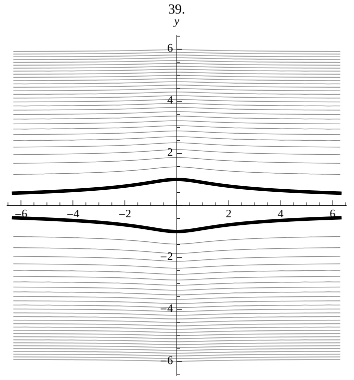


39. The equation is exact because $\dfrac{\partial}{\partial y}\left(\dfrac{1}{1+t^2}-y^2\right) = -2y = \dfrac{\partial}{\partial t}(-2ty)$. Let $F(t,y)$ satisfy $F_t(t,y)\,dt + F_y(t,y)\,dy = \left(\dfrac{1}{1+t^2}-y^2\right)dt - 2ty\,dy$. Then, $F(t,y) = \int\left(\dfrac{1}{1+t^2}-y^2\right)dt = \tan^{-1} t - ty^2 + g(y)$. Differentiating with respect to y, $F_y(t,y) = -2ty + g'(y) = -2ty \Rightarrow g'(y) = 0 \Rightarrow g(y) = 0$ so $F(t,y) = \tan^{-1} t - ty^2$ and the integral curves are $\tan^{-1} t - ty^2 = C$. Applying the initial condition results in $y^2 - \dfrac{\arctan(t)}{t} = 0$.

38.



39.



41. (a) The equation is exact because

$$\frac{\partial}{\partial y}\left(-2x - y\cos(xy)\right) = -\cos(xy) + xy\sin(xy) = \frac{\partial}{\partial x}\left(2y - x\cos(xy)\right).$$

The integral curves for the solution take the form $F(x,y) = C$, where

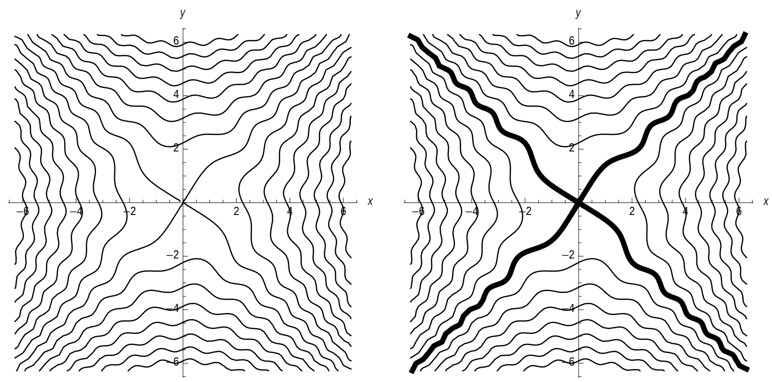
$$F(x,y) = \int \left(-2x - y\cos(xy)\right)\,dx = -x^2 - \sin(xy) + g(y).$$

Because

$$\frac{\partial}{\partial y}F(x,y) = \frac{\partial}{\partial y}\left(-x^2 - \sin(xy) + g(y)\right) = 2y - x\cos(xy) + g'(y),$$

$g'(y) = 2y$ so $g(y) = y^2$, $F(x,y) = y^2 - x^2 - \sin(xy)$, and the integral curves of the equation are given by $y^2 - x^2 - \sin(xy) = C$. Applying the initial condition

$y(0) = 0$ results in $0 = C$ so that the solution to the initial value problem is $y^2 = x^2 + \sin(xy)$.
Observe in the following figure that the initial condition $y(0) = 0$ does not result in a unique solution.



(b) Writing the differential equation in differential form gives us

$$\frac{dy}{dx} = \underbrace{\frac{2x + y\cos(xy)}{2y - x\cos(xy)}}_{f(x,y)}.$$

Using the notation in the theorem,

$$\frac{\partial f}{\partial y} = -\frac{x\left(4\left(x^2+y^2\right)\sin(xy) + \cos(2xy) + 9\right)}{2(x\cos(xy) - 2y)^2}.$$

The results do not contradict the Theorems because neither of these functions is continuous on a region containing the origin, $(0, 0)$.

43. Here, $\dfrac{dx}{dt} = \dfrac{2xy}{\sqrt{x^2+y^2}}$ and $\dfrac{dy}{dt} = -\dfrac{y^2 - x}{\sqrt{x^2+y^2}}$. Then,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y^2 - x}{2xy}$$

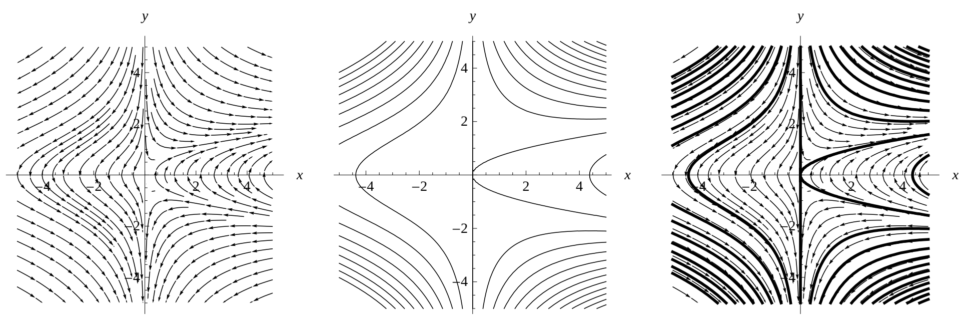
$$\left(-x + y^2\right)\,dx + 2xy\,dy = 0.$$

Using the notation in section, $M(x, y) = -x + y^2$ so $M_y(x, y) = 2y$ and $N(x, y) = 2xy$ so $N_x(x, y) = 2y$ so the equation $\left(-x + y^2\right)\,dx + 2xy\,dy = 0$ is exact. Let $F(x, y)$ be the potential function. Then,

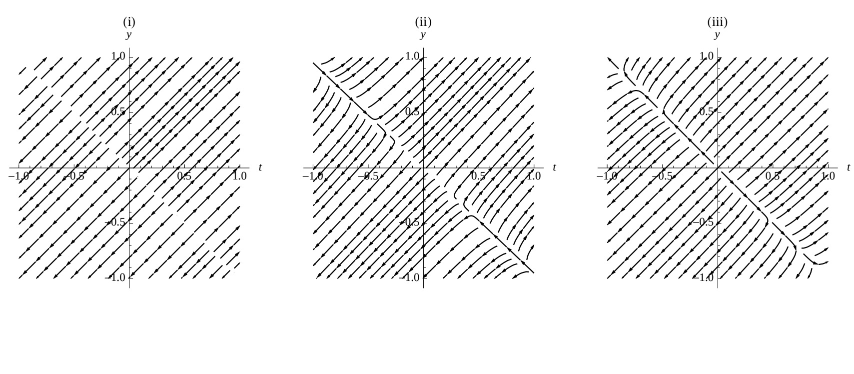
$$F(x, y) = \int \left(-x + y^2\right)\,dx$$

$$F(x, y) = -\frac{1}{2}x^2 + xy^2 + g(y).$$

Next, $F_y(x,y) = 2xy + g'(y) = 2xy$ so $g'(y) = 0$. We choose $g(y) = 0$ so that $F(x,y) = -\frac{1}{2}x^2 + xy^2$ and the integral curves are given by $-\frac{1}{2}x^2 + xy^2 = C$.



49. $y = \dfrac{C}{\left(e^t + t^2\right)^{2/3}}$
51. $y^{-1} - \dfrac{-t + C}{t^2} = 0$
53. $yt^5 + t^4\left(y\right)^2 + 1/4\, t^4 = C$
55. $\cos\left(y\left(t\right)\right)t^2 + \sin\left(y\right)t = C$
57. $t^2 + y\cos ty = C$
59. (a) (i) $y = -t$, $t^2 + 2ty + y^2 = C$ (ii) $y = \dfrac{1}{C}\left(-Ct \pm \frac{1}{10}\sqrt{10C^2t^2 + 10}\right)$ (iii) $y = \dfrac{1}{C}\left(-\frac{19}{20}Ct \pm \frac{1}{20}\sqrt{-39C^2t^2 + 40}\right)$



Exercises 2.5

1. This is Bernoulli with $n = -1$. Let $w = y^{1-(-1)} = y^2 \Rightarrow \dfrac{dw}{dt} = 2y\dfrac{dy}{dt}$ $\Rightarrow \dfrac{dy}{dt} = \dfrac{1}{2}y^{-1}\dfrac{dw}{dt}$. Then,

$$\begin{aligned}
\frac{dy}{dt} - \frac{1}{2}y &= ty^{-1} \\
\frac{1}{2}y^{-1}\frac{dw}{dt} - \frac{1}{2}y &= ty^{-1} \\
\frac{dw}{dt} - y^2 &= 2t \\
\frac{dw}{dt} - w &= 2t.
\end{aligned}$$

Use undetermined coefficients to solve for w. A general solution of the corresponding homogeneous equation is $w_h = Ce^t$. The associated set of functions for the forcing function $f(t) = 2t$ is $F = \{t, 1\}$. Because no element of F is a solution to the corresponding homogeneous equation we assume that a particular solution has the form $w_p = At + B \Rightarrow w_p' = A$. Substituting w_p into the nonhomogeneous equation yields $w_p' - w_p = -At + (A - B) = 2t$ so $A = -2$ and $B = -2$ so $w_p = -2t - 2$. A general solution is then $w = w_h + w_p = Ce^t - 2t - 2$. Because $w = y^2$, $y = \pm\sqrt{Ce^t - 2t - 2}$.

3. This is Bernoulli with $n = 3$ so we let $w = y^{1-3} = y^{-2}$. Then, $\dfrac{dw}{dt} = -2y^{-3}\dfrac{dy}{dt}$ so $-\dfrac{1}{2}y^3\dfrac{dw}{dt} = \dfrac{dy}{dt}$. Substituting into the equation gives us

$$\begin{aligned}
-ty^3\frac{dw}{dt} - y &= 2ty^3\cos t \\
\frac{dw}{dt} + \frac{1}{t}y^{-2} &= -2\cos t \\
\frac{dw}{dt} + \frac{1}{t}w &= -2\cos t \\
\frac{d}{dt}(tw) &= -2t\cos t \\
tw &= -2\cos t - 2t\sin t + C \\
w &= -2t^{-1}\cos t - 2\sin t + Ct^{-1} \\
\frac{1}{y^2} &= -2t^{-1}\cos t - 2\sin t + Ct^{-1} \\
y &= \pm\frac{1}{\sqrt{-2t^{-1}\cos t - 2\sin t + Ct^{-1}}}.
\end{aligned}$$

or $y = \pm\dfrac{\sqrt{-\left(2\,\cos(t) + 2\,t\sin(t) - C\right)t}}{2\,\cos(t) + 2\,t\sin(t) - C}$

5. $y^{3/2} + \frac{9}{20}\,\cos(t) - \frac{3}{20}\,\sin(t) - Ce^{3\,t} = 0$

7. $y = \dfrac{3t}{t^3 - 3C}$

9. This is a Bernoulli equation with $n = 2$ so we let $w = y^{1-2} = y^{-1} = 1/y$. Then, $\dfrac{dw}{dt} = -y^{-2}\dfrac{dy}{dt}$ so $\dfrac{dy}{dt} = -y^2\dfrac{dw}{dt}$. Then,

$$\begin{aligned}
\frac{dy}{dt} - \frac{1}{t}y &= \frac{y^2}{t} \\
-y^2\frac{dw}{dt} - \frac{1}{t}y &= \frac{y^2}{t} \\
\frac{dw}{dt} + \frac{1}{t}y^{-1} &= -\frac{1}{t} \\
\frac{dw}{dt} + \frac{1}{t}w &= -\frac{1}{t}.
\end{aligned}$$

The integrating factor for $\dfrac{dw}{dt} + \dfrac{1}{t}w = -\dfrac{1}{t}$ is $\mu(t) = e^{\int 1/t\,dt} = e^{\ln t} = t$, $t > 0$. Multiplying through by the integrating factor and solving for w gives us:

$$\begin{aligned}
\frac{dw}{dt} + \frac{1}{t}w &= -\frac{1}{t} \\
t\frac{dw}{dt} + w &= -1 \\
\frac{d}{dt}(tw) &= -1 \\
tw &= -t + C \\
w &= Ct^{-1} - 1.
\end{aligned}$$

Because $w = 1/y$, $y = 1/w$, so $1/y = Ct^{-1} - 1$ which means that $y = 1/(Ct^{-1} - 1)$ or $y = Ct/(1 - Ct)$.

11. Homogeneous of degree 0

13. Not homogeneous

15. Not homogeneous

17. The equation is homogeneous of degree 1. Let $t = vy$. Then, $dt = v\,dy + y\,dv$.

Substituting into the equation, separating and integrating yields

$$\begin{aligned}
2tdt + (y-3t)dy &= 0\\
2vy(vdy + ydv) + (y-3vy)dy &= 0\\
2v(vdy + ydv) + (1-3v)dy &= 0\\
(2v^2 - 3v + 1)dy &= -2vydv\\
\frac{1}{y}dy &= \frac{-2v}{2v^2-3v+1}dv\\
\frac{1}{y}dy &= 2\left(\frac{1}{2v-1} - \frac{1}{v-1}\right) dv\\
\ln y &= \ln(2v-1) - 2\ln(1-v) + C\\
y &= C\frac{2v-1}{(v-1)^2}\\
y &= C\frac{(2t-y)y}{(t-y)^2}\\
\frac{(t-y)^2}{2t-y} &= C.
\end{aligned}$$

Another form of the solution is$-2\,\ln\left(-\frac{-y+t}{t}\right) + \ln\left(\frac{y-2\,t}{t}\right) - \ln(t) = C.$

19. The equation is homogeneous of degree 2. Observe that either $t = vy$ or $y = ut$ results in an equivalent problem. We choose to use $y = ut \Rightarrow dy = udt + tdu$. Then,

$$\begin{aligned}
(ty - y^2)dt + t(t-3y)dy &= 0\\
(t^2u - t^2u^2)dt + t(t-3ut)(udt + tdu) &= 0\\
(u-u^2)dt + (1-3u)(udt + tdu) &= 0\\
2u(1-2u)dt + t(1-3u)du &= 0\\
2u(1-2u)dt &= t(3u-1)du\\
\frac{1}{t}dt &= \frac{3u-1}{2u(1-2u)}du\\
\frac{1}{t}dt &= -\frac{1}{2}\left(\frac{1}{u} + \frac{1}{2u-1}\right) du\\
\ln t &= -\frac{1}{2}\ln u - \frac{1}{4}\ln(2u-1) + C\\
-4\ln t &= 2\ln u + \ln(2u-1) + C\\
\frac{1}{t^4} &= Cu^2(2u-1)\\
\frac{1}{t^4} &= C\frac{y^2(t-2y)}{t^3}\\
ty^2(t-2y) &= C.
\end{aligned}$$

Another form of the solution is $-1/4\,\ln\left(\frac{-t+2\,y}{t}\right)-1/2\,\ln\left(\frac{y}{t}\right)-\ln(t)=C$.
21. $y^3-(3\,\ln(t)+C)\,t^3=0$
23. This is homogeneous of degree 1. (Also, observe that this is a first order linear equation in y.) Solving it as a homogeneous equation, we let $y=ut \Rightarrow dy=udt+tdu$. Then,

$$\begin{aligned}(t-y)dt+tdy&=0\\(t-ut)dt+t(udt+tdu)&=0\\(1-u)dt+udt+tdu&=0\\dt&=-tdu\\\frac{1}{t}dt&=-du\\\ln t&=-u+C\\\ln t&=-\frac{y}{t}+C\\y&=t(C-\ln t).\end{aligned}$$

25. $-2/3\,\ln\left(\frac{3\,y-2\,t}{t}\right)+1/2\,\ln\left(\frac{-t+2\,y}{t}\right)-\ln(t)=C$
27. The equation is homogeneous of degree 2. Let $t=vy \Rightarrow dt=vdy+ydv$. Then,

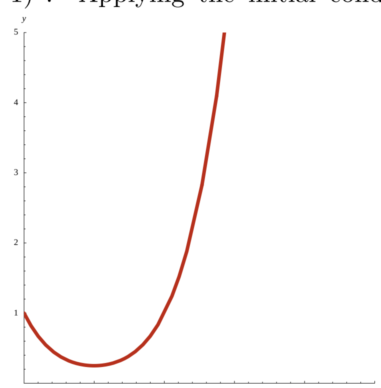
$$\begin{aligned}y^2dt&=(ty-4t^2)dy\\y^2(vdy+ydv)&=(vy^2-4v^2y^2)dy\\vdy+ydv&=(v-4v^2)dy\\ydv&=-4v^2dy\\-\frac{4}{y}dy&=\frac{1}{v^2}dv\\-4\ln y&=-\frac{1}{v}+C\\4\ln y&=\frac{y}{t}+C.\end{aligned}$$

Another form of the solution is $1/4\,\frac{y}{t}-\ln\left(\frac{y}{t}\right)-\ln(t)=C$
29. $y=-t$, $-1/2\,\ln\left(\frac{t^2-ty+y^2}{t^2}\right)+1/3\,\sqrt{3}\,\mathrm{arctan}\left(1/3\,\frac{(t-2\,y)\,\sqrt{3}}{t}\right)-\ln(t)=C$
31. $1/2\,(-y+t)\,y^{-1}\left(e^{-t/y}\right)^{-1}-\ln(y)=C$
33. $y'+2y=t^2y^{1/2}$ is Bernoulli with $n=1/2$. Let $w=y^{1-1/2}=y^{1/2}$. Then,

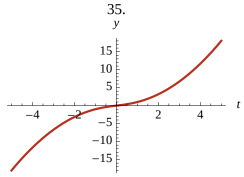
$\frac{dw}{dt} = \frac{1}{2} y^{-1/2} \frac{dy}{dt} \Rightarrow \frac{dy}{dt} = 2y^{1/2} \frac{dw}{dt}$. Substituting into the equation yields

$$\begin{aligned} \frac{dy}{dt} + 2y &= t^2 y^{1/2} \\ 2y^{1/2}\frac{dw}{dt} + 2y &= t^2 y^{1/2} \\ \frac{dw}{dt} + y^{1/2} &= \frac{1}{2}t^2 \\ \frac{dw}{dt} + w &= \frac{1}{2}t^2. \end{aligned}$$

Using undetermined coefficients, a general solution is $w = Ce^{-t} + \frac{1}{2}t^2 - t + 1$. Thus, $y = (Ce^{-t} + \frac{1}{2}t^2 - t + 1)^2$. Applying the initial condition results in $y = \frac{1}{4}(4 - 8\,t + 8\,t^2 - 4\,t^3 + t^4)$.



35. $y = 1/2\,\sqrt{2 + 2\,t^2}t$
37. $y = -1/2 + 1/2\,t^2$
39. $y = \sqrt[3]{-3\,\ln(t) + 27t}$

35.


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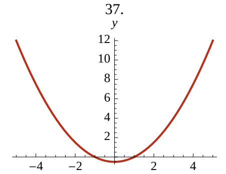

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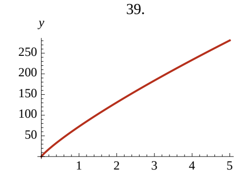


41. $y^4 dt + (t^4 - ty^3)dy = 0$ is homogeneous of degree 4. Let $t = vy \Rightarrow dt =$

$vdy + ydv$. Then,

$$\begin{aligned}
y^4 dt + (t^4 - ty^3)dy &= 0 \\
y^4(vdy + ydv) + (v^4y^4 - vy^4)dy &= 0 \\
(vdy + ydv) + (v^4 - v)dy &= 0 \\
v^4 dy &= -ydv \\
\frac{1}{y}dy &= -\frac{1}{v^4}dv \\
\ln y &= \frac{1}{3v^3} + C \\
3\ln y &= \frac{y^3}{t^3} + C.
\end{aligned}$$

Applying the initial condition results in $\dfrac{1}{3}\dfrac{y^3}{t^3} - \ln\left(\dfrac{y}{t}\right) - \ln t - 8/3 + \ln 2 = 0$.
43. We need to solve $dy/dt = y/t + t/y$ subject to $y(\sqrt{e}) = \sqrt{e}$. The equation is homogeneous of degree 2. To see so, we rewrite the equation:

$$\begin{aligned}
\frac{dy}{dt} &= \frac{y^2 + t^2}{yt} \\
yt\,dy &= (y^2 + t^2)dt.
\end{aligned}$$

Now, we let $y = ut \Rightarrow dy = udt + tdu$. Substituting then gives us

$$\begin{aligned}
ut^2(u\,dt + t\,du) &= (u^2t^2 + t^2)\,dt \\
u(u\,dt + t\,du) &= (u^2 + 1)\,dt \\
tu\,du &= dt \\
\frac{1}{t}dt &= u^2\,du \\
\ln t &= \frac{1}{3}u^3 + C \\
\ln t &= \frac{1}{3}\frac{y^3}{t^3} + C.
\end{aligned}$$

Now apply the initial condition and solve for y, $y = \sqrt{2}\,t\,\sqrt{\ln t}$.
45. Solve $y^2\,dx + (x^2 + y^2)\,dy = 0$; $y = \pm\sqrt{-x^2 \pm \sqrt{x^4 + C}}$
47. The general solution is $y^{-3} = (3\cot x + C)\sin^3 x$ or $y = 0$. So, the solution to the initial value problem is $y = 0$.
49. Because $M(t,y)dt + N(t,y)dy = 0$ is homogeneous, we can write the equation in the form $dy/dt = F(y,t)$. If $t = r\cos\theta$ and $y = r\sin\theta$, $dt = \cos\theta\,dy - r\sin\theta d\theta$

and $dy = \sin\theta\, dr + r\cos\theta\, d\theta$. Substituting into the equation gives us

$$\begin{aligned} dy/dt &= F(t,y) \\ \frac{\sin\theta\, dr + r\cos\theta\, d\theta}{\cos\theta\, dr - r\sin\theta d\theta} &= F\left(\frac{r\sin\theta}{r\cos\theta}\right) = F(\tan\theta) \\ \sin\theta\, dr + r\cos\theta\, d\theta &= F(\tan\theta)\cos\theta\, dr - F(\tan\theta) r\sin\theta d\theta \\ F(\tan\theta) r\sin\theta d\theta + r\cos\theta\, d\theta &= F(\tan\theta)\cos\theta\, dr - \sin\theta\, dr \\ \frac{F(\tan\theta)\sin\theta d\theta + \cos\theta}{F(\tan\theta)\cos\theta\ - \sin\theta} &= \frac{1}{r} dr. \end{aligned}$$

53. $f(t) = t-1$; $g(t) = t^2 - t$; General solution: $(tc - y) - 1 = c^2 - c \Rightarrow y = ct - 1 + c - c^2$; Singular solution: $\frac{d}{dt}(ty' - y - 1) = \frac{d}{dt}\left[(y')^2 - y'\right] \Rightarrow ty'' + y' = 2y'y'' - y'' \Rightarrow (t - 2y' + 1)y'' = 0 \Rightarrow y' = \frac{1}{2}(t+1) \Rightarrow y = \frac{1}{4}t^2 + \frac{1}{2}t - \frac{3}{4}$.
55. $f(t) = 1 - 2t$; $g(t) = t^{-2}$; General solution: $1 - 2(tc - y) = c^{-2} \Rightarrow y = \frac{1}{2}(c^{-2} + 2ct - 1)$; Singular solution: $y = \frac{1}{2}(3t^{2/3} - 1)$
59. We see that the equation is a Lagrange equation by rewriting it in the form $y = ty'^2 + (3y'^2 - 2y'^3)$ and identifying $f(y') = y'^2$ and $g(y) = 3y'^2 - 2y'^3$. Differentiating with respect to t yields the equation $y' = y'^2 + 2ty'y'' + 6y'y'' - 6y'^2y''$ and substituting $p = y'$ results in

$$\begin{aligned} p &= p^2 + 2tp\frac{dp}{dt} + 6p\frac{dp}{dt} - 6p^2\frac{dp}{dt} \\ p - p^2 &= \left(2xp + 6p - 6p^2\right)\frac{dp}{dx} \\ \frac{dx}{dp} &= \frac{2xp + 6p - 6p^2}{p - p^2} \\ \frac{dx}{dp} + \frac{2p}{p^2 - p}x &= \frac{6p - p^2}{p - p^2}. \end{aligned}$$

The solution of this linear equation is $x = \dfrac{2p^3 - 21p^2 + 36p + 6C}{6\left(p^2 - 2p + 1\right)}$ so

$$y = xp^2 + 3p^2 - 2p^3 = \frac{2p^3 - 21p^2 + 36p + 6C}{6\left(p^2 - 2p + 1\right)}p^2 + 3p^2 - 2p^3.$$

61. Differentiating the equation gives us $y' = (2 - y') + xy'' + 4y'y''$. Now, we let

$p = y'$ and solve for dx/dp:

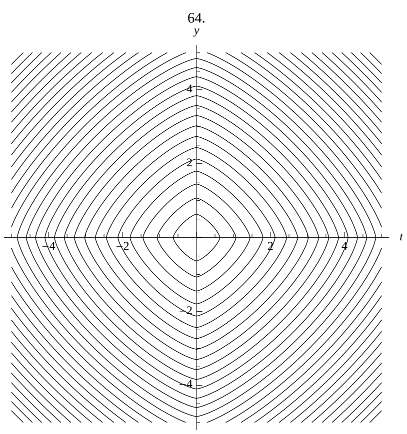
$$\begin{aligned} y' &= (2-y') + xy'' + 4y'y'' \\ p &= 2 - p + x\frac{dp}{dx} + 4p\frac{dp}{dx} \\ 2p-2 &= (x+4p)\frac{dp}{dx} \\ \frac{dx}{dp} &= \frac{x+4p}{2(p-1)} \\ \frac{dx}{dp} + \frac{1}{2-2p}x &= \frac{4p}{2(p-1)}. \end{aligned}$$

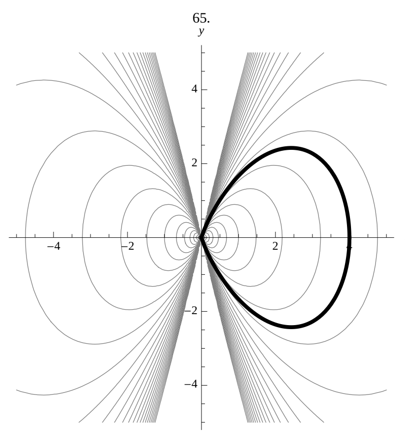
This linear equation has solution $x = -\frac{8}{3} + \frac{4}{3}p + \frac{4}{3}p^2 + C\sqrt{2-2p}$ so

$$y = x(2-o) + 2p^2 + 1 = \left(-\frac{8}{3} + \frac{4}{3}p + \frac{4}{3}p^2 + C\sqrt{2-2p}\right)(2-p) + 2p^2 + 1.$$

63. $\lim_{t\to\infty} y(t) = \dfrac{ky_0}{ay_0} = \dfrac{k}{a}$

65. General solution: $2t^2 \ln|t| + t^2 = Ct^2$. Initial value problem has two solutions: $y = \pm\sqrt{2}\sqrt{t^2(\ln 4 - \ln t)}$.





Exercises 2.6

1. 47.3742, 63.2572
3. 1.8857, 2.09847
5. 79.8458, 123.048
7. 1.95109, 1.95388
9. 83.6491, 88.6035
11. 2.37754, 2.41897
13. 185.34, 206.981

15. 1.95547, 1.95609
17. 90.6405, 90.6927
19. 216.582, 216.992
23. 1.95629, 1.95629
25-27. (a) $y(t) = e^{-t}$, $y(1) = 1/e \approx 0.367879$

	$h = 0.1$	$h = 0.005$	$h = 0.025$
25. (Euler's)	0.348678	0.358486	0.363232
26. (Improved)	0.368541	0.368039	0.367918
27. (4th-Order RK)	0.429069	0.414831	0.36788

29. $y(0.5) \approx 0.566144$, 1.12971, 1.68832, 2.23992, 2.78297

Chapter 2 Review Exercises

1. $y = 1/5\,\sqrt[3]{25\,t^6 + 125C}$

3. The equation $\dfrac{dy}{dt} - \dfrac{1}{t}y = \dfrac{1}{t}y^2$ is Bernoulli with $n = 2$. Let $w = y^{1-2} = y^{-1}$. Then, $dw/dt = -y^{-2}\,dy/dt$ so $-y^2\,dw/dt = dy/dt$. With this substitution we have,

$$\begin{aligned}\frac{dy}{dt} - \frac{1}{t}y &= \frac{1}{t}y^2\\ -y^2\frac{dw}{dt} - \frac{1}{t}y &= \frac{1}{t}y^2\\ \frac{dw}{dt} + \frac{1}{t}y^{-1} &= -\frac{1}{t} \qquad \text{(Divide by } -y^2.\text{)}\\ \frac{dw}{dt} + \frac{1}{t}w &= -\frac{1}{t}.\end{aligned}$$

An integrating factor for $\dfrac{dw}{dt} + \dfrac{1}{t}w = -\dfrac{1}{t}$, $t > 0$, is $\mu(t) = e^{\int 1/t\,dt} = e^{\ln t} = t$. Multiplying $\dfrac{dw}{dt} + \dfrac{1}{t}w = -\dfrac{1}{t}$ by the integrating factor and solving for w gives us

$$\begin{aligned}\frac{dw}{dt} + \frac{1}{t}w &= -\frac{1}{t}\\ t\frac{dw}{dt} + t\frac{1}{t}w &= -t\cdot\frac{1}{t}\\ \frac{d}{dt}(t\,w) &= -1\\ tw &= -t + C\\ w &= -1 + \frac{C}{t}\\ \frac{1}{y} &= \frac{-t + C}{t}\\ y &= \frac{t}{-t + C}.\end{aligned}$$

5. $y^4 dy = e^{5t} dt \Rightarrow \frac{1}{5}y^5 = \frac{1}{5}e^{5t} + C \Rightarrow y^5 = e^{5t} + C \rightarrow y = (e^{5t} + C)^{1/5}$

7. $y\,(t) = e^{\pm\sqrt{-e^{-2\,t}+C}}$

9. For this first-order linear equation, the preferred method of solution is using the method of undetermined coefficients to find a particular solution of the non homogeneous equation. The corresponding homogeneous equation is $y' + 3y = 0$ which is the first order linear homogeneous equation with constant coefficients, $y' + ky = 0$, which has general solution $y = Ce^{-kt}$ with $k = 3$ so $y' + 3y = 0$ has general solution $y_h = Ce^{-3t}$.

Now, look at the forcing function, $f(t) = -10\sin t$. The associated set for this forcing function is $F = \{\cos, \sin t\}$ and because neither of these is a solution of the corresponding homogeneous equation, we assume that y_p takes the form $y_p = A\cos t + B\sin t$ with derivative $y_p' = -A\sin t + B\cos t$. Substituting into the *non homogeneous* equation gives us

$$\begin{aligned} y_p' + 3y_p &= (3A + B\cos t + (-A + 3B\sin t \\ &= -10\sin t \end{aligned}$$

so

$$\begin{aligned} 3A + B &= 0 \\ -A + 3B &= -10, \end{aligned}$$

which has solution $A = 1$ and $B = -3$. Therefore, $y_p = \cos t - 3\sin t$ and $y = y_h + y_p = Ce^{-3t} + \cos t - 3\sin t$.

If you preferred to use the integrating factor approach, the integrating factor is $\mu(t) = e^{\int 3\,dt} = e^{3t}$. Now multiply through by the integrating factor and integrate the result. Observe that integrating the right hand side by hand involves integration by parts twice.

$$\begin{aligned} y' + 3y &= -10\sin t \\ e^{3t}y' + 3e^{3t}y &= -10e^{3t}\sin t \\ \frac{d}{dt}\left(e^{3t}y\right) &= -10e^{3t}\sin t \\ e^{3t}y &= e^{3t}\cos t - 3e^{3t}\sin t + C \\ y &= \cos t - 3\sin t + Ce^{-3t}. \end{aligned}$$

11. The equation $(y - t)\,dt + (t + y)\,dy = 0$ is homogeneous of degree 1. Either $y = ut$ or $t = vy$ result in an equivalent problem. We choose to use $y = ut \Rightarrow$

$dy = u\,dt + t\,du$. Then,

$$\begin{aligned}(y-t)\,dt + (t+y)\,dy &= 0\\ \left(u^2+2u-1\right)dt &= -t(u+1)du\\ \frac{1}{t}dt &= -\frac{u+1}{u^2+2u-1}du\\ \ln t &= -\frac{1}{2}\ln\left|u^2+2u-1\right| + C\\ t^{-2} &= C\left(u^2+2u-1\right)\\ t^{-2} &= Ct^{-2}\left(y^2+2ty-t^2\right)\\ y^2+2ty-t^2 &= C.\end{aligned}$$

$y = -t \pm \frac{1}{C}\sqrt{2C^2t^2+1}$

13. The equation $y^2\,dt + (ty+t^2)\,dy = 0$ is homogeneous of degree 2. We let $t = vy$ so $dt = v\,dy + y\,dv$. Then, substituting and separating variables gives us

$$\begin{aligned}y^2\,dt + (ty+t^2)\,dy &= 0\\ y^2\,(v\,dy + y\,dv) + (vy\cdot y + (vy)^2)\,dy &= 0\\ (v\,dy + y\,dv) + (v+v^2)\,dy &= 0 \qquad \text{(Divide by } y^2.)\\ (2v+v^2)\,dy &= -y\,dv\\ -\frac{1}{y}\,dy &= \frac{1}{2v+v^2}\,dv\\ -\frac{1}{y}\,dy &= \frac{1}{2}\left(\frac{1}{v}-\frac{1}{v+2}\right)dv\\ -\ln y &= \frac{1}{2}\left(\ln v - \ln(2+v)\right) + C\\ \ln y &= \frac{1}{2}\ln\left(\frac{2+v}{v}\right) + C\\ y &= C\sqrt{\frac{2+v}{v}}.\end{aligned}$$

Now replace v with t/y and solve for C.

$$\begin{aligned}y &= C\sqrt{\frac{2+v}{v}}\\ y &= C\sqrt{\frac{2+t/y}{t/y}}\\ y^2 &= C\frac{2+t/y}{t/y}\\ \frac{ty^2}{t+2y} &= C.\end{aligned}$$

Solving for y instead of C results in $y = \dfrac{C \pm \sqrt{C^2 + Ct^2}}{t}$.

15. $-\frac{5}{8}\,\ln\left(-\frac{-x+2\,t}{t}\right) - \frac{5}{8}\,\ln\left(\frac{x+2\,t}{t}\right) + \frac{1}{4}\,\ln\left(\frac{x}{t}\right) - \ln(t) = C$

17. $\frac{1}{3}\,t^3 y - \cos t - \sin y = C$

19. $\frac{1}{2}t^2 \ln y + y = C$

21. $y = 1 + Ce^{-t^2/2}$

23. $r = t^{-1}(\cos t + t \sin t + C)$

25. $y = t^{-1}(6e^t - 6te^t + 3t^2 e^t + C)^{1/3}$

27. $y = -2 + 2\ln(-2/t)$, $y = Ct + 2\ln C$

29. $y = -1/3\,t\sqrt{6\,t - 6\,\sqrt{t^2 - 12\,C}} + \frac{1}{54}\,\left(6\,t - 6\,\sqrt{t^2 - 12\,C}\right)^{3/2}$,
$y = 1/3\,t\sqrt{6\,t - 6\,\sqrt{t^2 - 12\,C}} - \frac{1}{54}\,\left(6\,t - 6\,\sqrt{t^2 - 12\,C}\right)^{3/2}$,
$y = -1/3\,t\sqrt{6\,t + 6\,\sqrt{t^2 - 12\,C}} + \frac{1}{54}\,\left(6\,t + 6\,\sqrt{t^2 - 12\,C}\right)^{3/2}$,
$y = 1/3\,t\sqrt{6\,t + 6\,\sqrt{t^2 - 12\,C}} - \frac{1}{54}\,\left(6\,t + 6\,\sqrt{t^2 - 12\,C}\right)^{3/2}$

31. $y + \sin(t - y) = \pi$

33. $t \sin y - y \sin t = 0$

35. $t \ln y + y \ln t = 0$

37.

n	x_n	y_n (Euler's)	y_n (Improved Euler's)	y_n (Runge-Kutta of order 4)
0	1	1	1	1
1	1.05	1	1.00559	1.00678
2	1.1	1.01118	1.0181	1.01921
3	1.15	1.02608	1.03383	1.03486
4	1.2	1.04368	1.052	1.05296
5	1.25	1.06345	1.07219	1.07309
6	1.3	1.08505	1.0941	1.09494
7	1.35	1.10823	1.11752	1.11831
8	1.4	1.13281	1.14228	1.14303
9	1.45	1.15866	1.16825	1.16896
10	1.5	1.18565	1.19533	1.19601
11	1.55	1.21368	1.22343	1.22407
12	1.6	1.24268	1.25247	1.25307
13	1.65	1.27256	1.28237	1.28295
14	1.7	1.30328	1.31309	1.31364
15	1.75	1.33478	1.34456	1.34509
16	1.8	1.36699	1.37675	1.37726
17	1.85	1.3999	1.40961	1.4101
18	1.9	1.43344	1.44311	1.44357
19	1.95	1.46759	1.4772	1.47764
20	2.	1.50232	1.51186	1.51229

39.

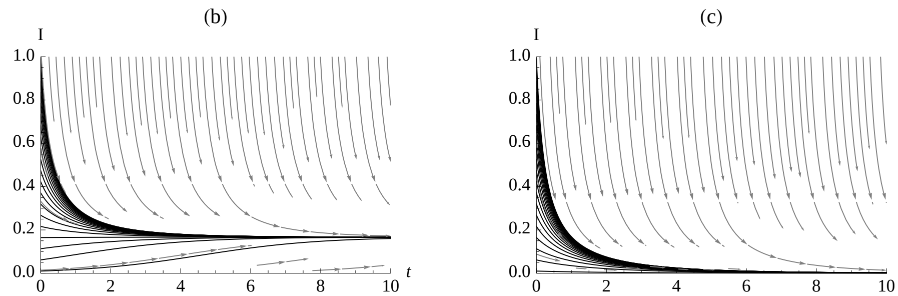
1. $V(x) = \int_0^x \pi y^2 dt$, $W(x) = \rho V(x) = \rho \int_0^x \pi y^2 dt$
2. $\sigma(x) = \frac{F(x)}{A(x)}$ where $F(x) = W(x) + L$ and $A(x) = \pi y^2$.
3. $y(x) = K \exp\left(\frac{\rho x}{2\sigma}\right)$. If $y(0) = 1$, then $K = 1$.

41. $\frac{dx}{dt} = kx^2$, $x(0) = 100$, $x(1) = 60$. $x(t) = \frac{-1}{kt+C}$ so that $C = -1/100$ and $k = -1/150$. When $t = 3$, $x(3) = \left(\frac{3}{150} + \frac{1}{100}\right)^{-1} = \frac{100}{3} \approx 33.33$ grams.

Differential Equations at Work

A. Modeling the Spread of a Disease

2.



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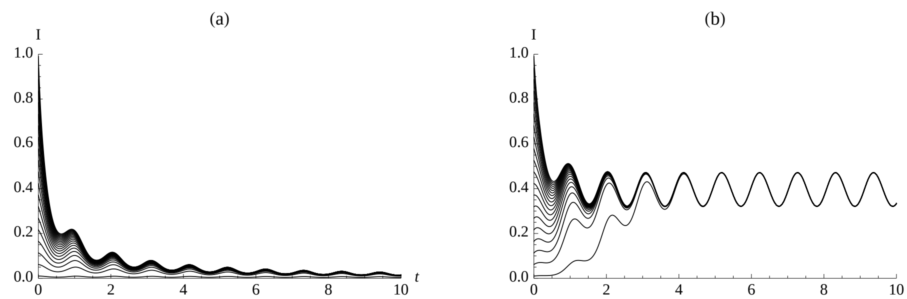
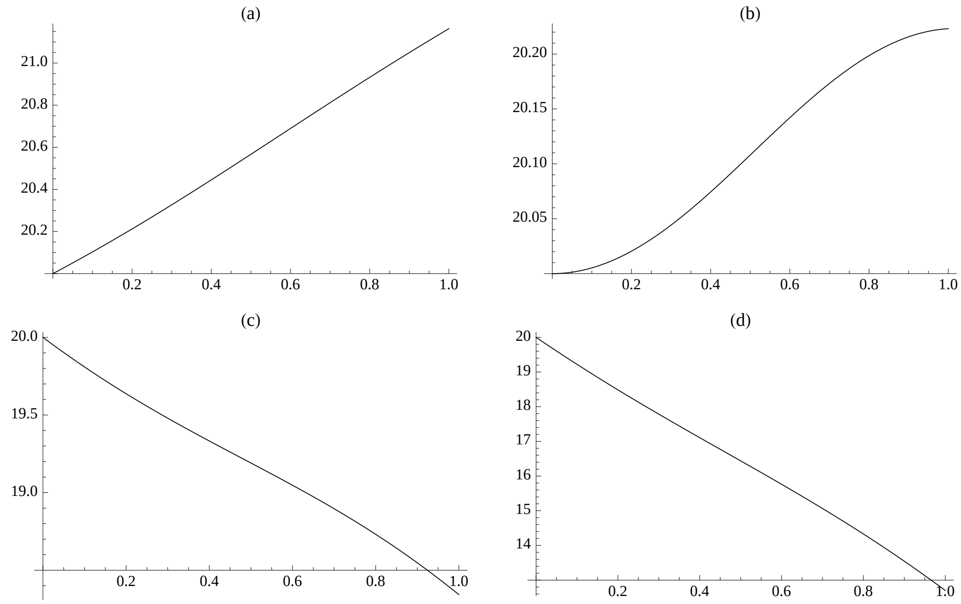


B. Linear Population Model with Harvesting

1. (a) $ay - h = 0$ has solution $y = h/a$. (c) $y = \dfrac{1}{a}\left((ay_0 - h)e^{at} + h\right)$ (d) $\lim_{t\to\infty} = \infty$ (e) $\lim_{t\to\infty} = h/a$ (f) $\lim_{t\to\infty} = -\infty$; $y(t) = 0$ when $t = -\dfrac{1}{a}\ln\left(1 - \dfrac{a}{h}y_0\right)$
2. (a) $y = 2$ (b) $y = (y_0 - 2)e^{t/2} + 2$ (c) ∞ (d) 2 (f) $-\infty$; $2\ln 4$
3. $y = \frac{1}{2}(2 - e^t) = 0$ when $t = \ln 2 \approx 0.693$; $y = \frac{1}{2}(4 - 3e^{t/2} = 0$ when $t = 2\ln(4/3) \approx 0.575$
4. $y = \frac{1}{2}(2 - e^{t/2} = 0$ when $t = 2\ln 2 \approx 1.386$
5. First solve $y' = \frac{1}{2}y - \frac{1}{2}$, $y(0) = 1/2$ to obtain $y = 1 - \frac{1}{2}e^{t/2}$. Then, $y(1) = 1 - \frac{1}{2}\sqrt{e} \approx 0.176$. Then, for year two, solve $y' = \frac{1}{2}y + \frac{1}{2}$, $y(1) = 1 - \frac{1}{2}\sqrt{e}$ to obtain $y = -1 + 2e^{(t-1)/2} - \frac{1}{2}e^{t/2}$ so $y(2) = -1 + 2\sqrt{e} - \frac{1}{2}e \approx 0.938$. $y' = \frac{1}{2}y + r$, $y(1) = 1 - \frac{1}{2}\sqrt{e}$ has solution $y = -\frac{1}{2}e^{t/2} - 2r + (2r+1)e^{(t-1)/2}$ so $y(2) = -\frac{1}{2}e - 2r + (2r+1)\sqrt{e} = \frac{1}{2} = y(0)$ when $r = \frac{1}{4}(\sqrt{e} - 1) \approx 0.162$.
6. Set $r = 0$ in the above. Then, $y(2) = \sqrt{e} - \frac{1}{2}e \approx 0.290$. $y(T) = e^{(t-1)/2} - \frac{1}{2}e^{t/2} = \frac{1}{2} = y(0)$ when $T = -2\ln\left(-1 + \dfrac{2}{\sqrt{e}}\right) \approx 3.092$
7.



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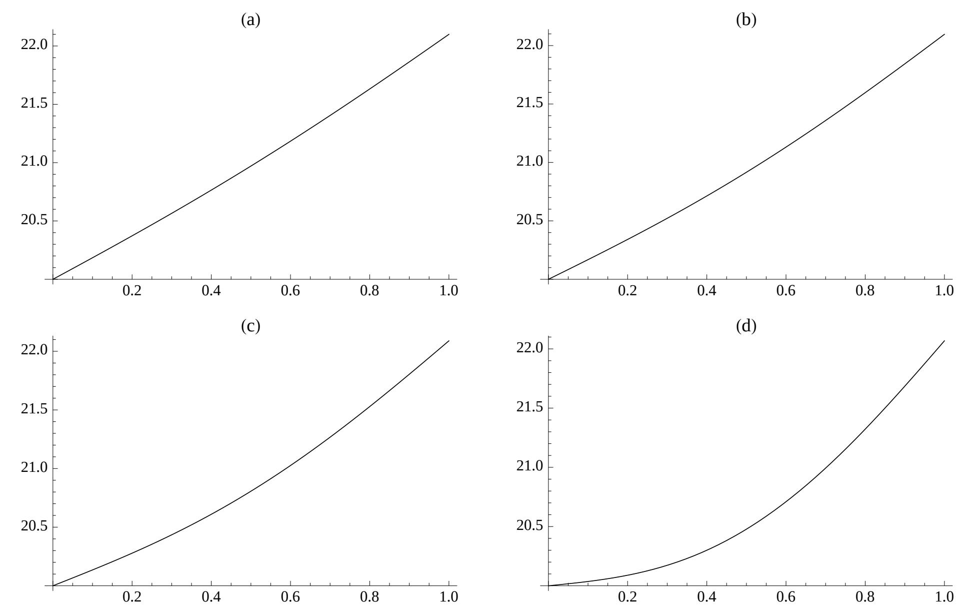
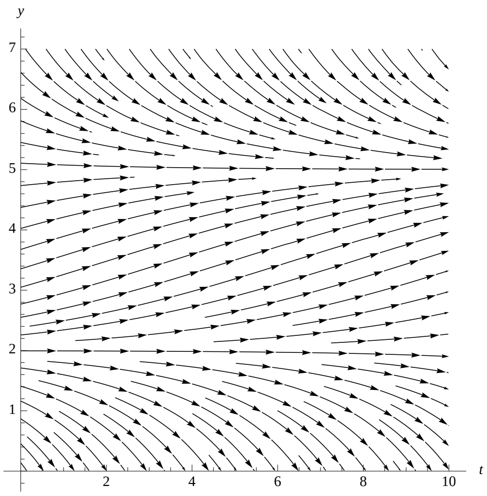
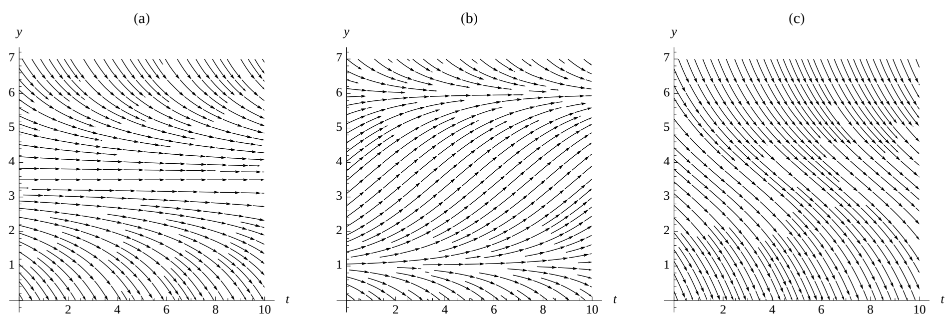


C. Logistic Model with Harvesting

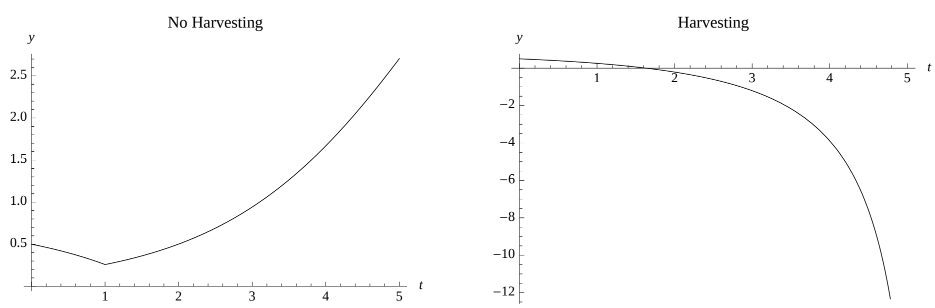
1. $y = \dfrac{1}{2c}\left(a \pm \sqrt{a^2 - 4ch}\right)$
2. (a) $y = 2$, $y = 5$ (b)



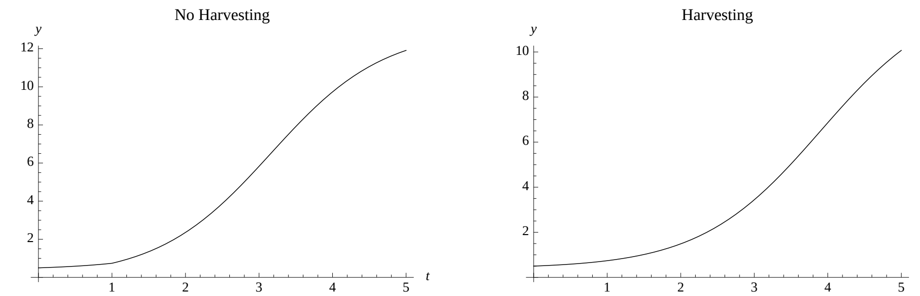
(c) 5 (d) 5 (e) 0 (extinction)
3. $49 - 40h = 0$ when $h = 49/40$



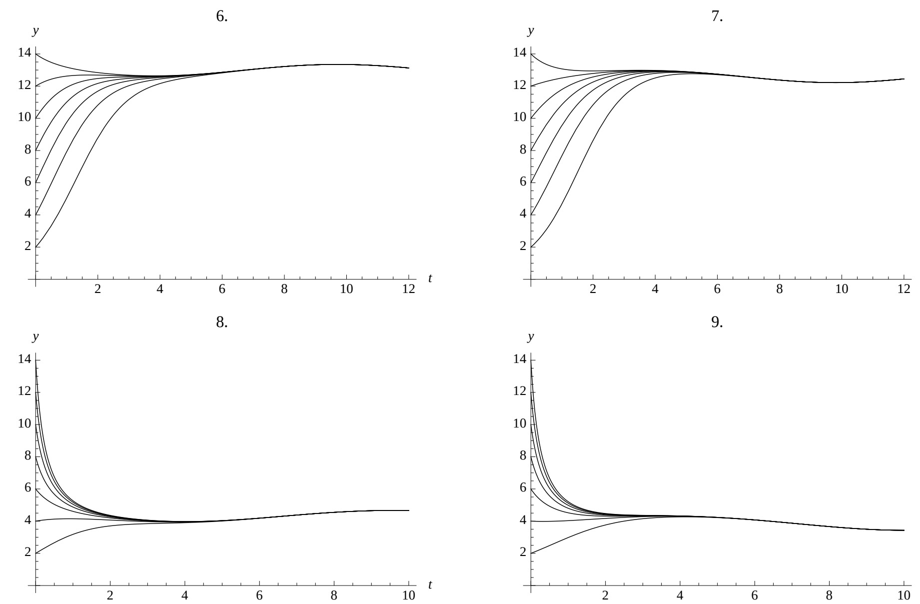
4.



5.

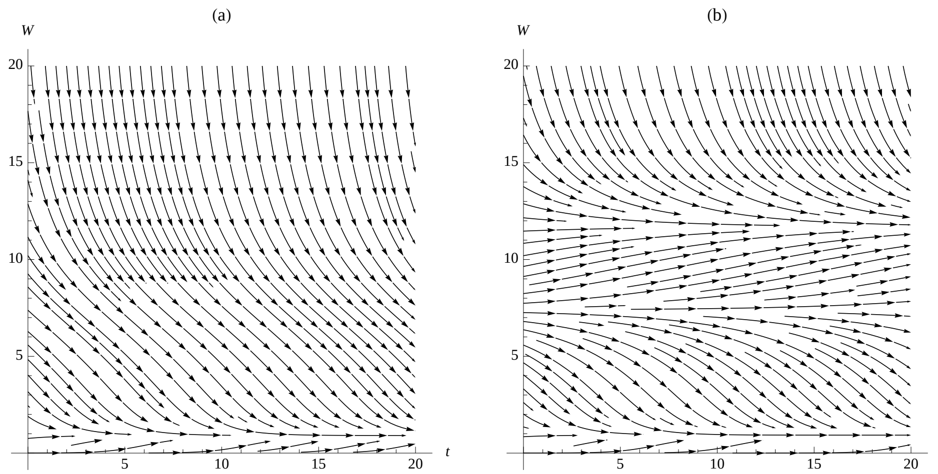


6.

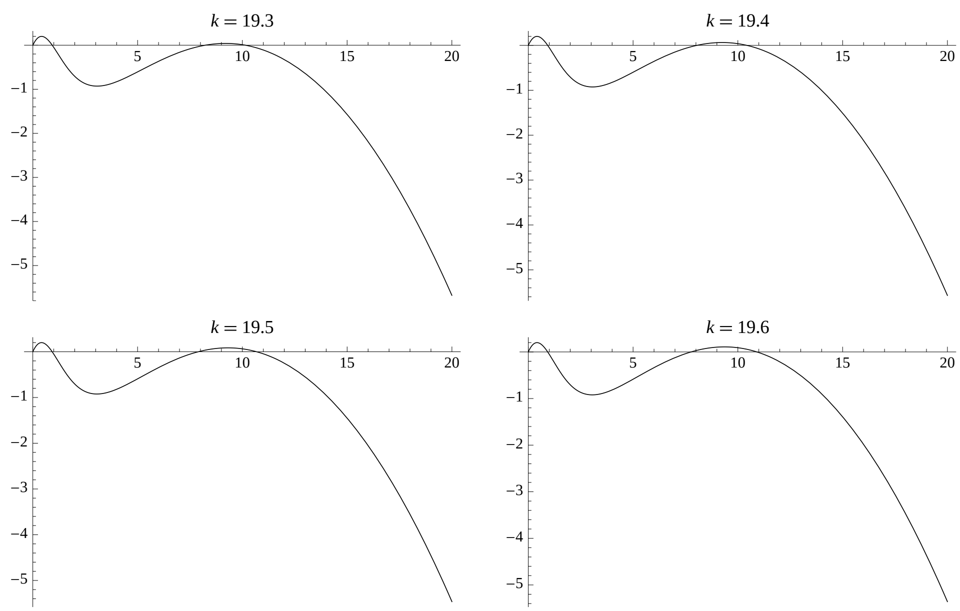


D. Logistic Model with Predation

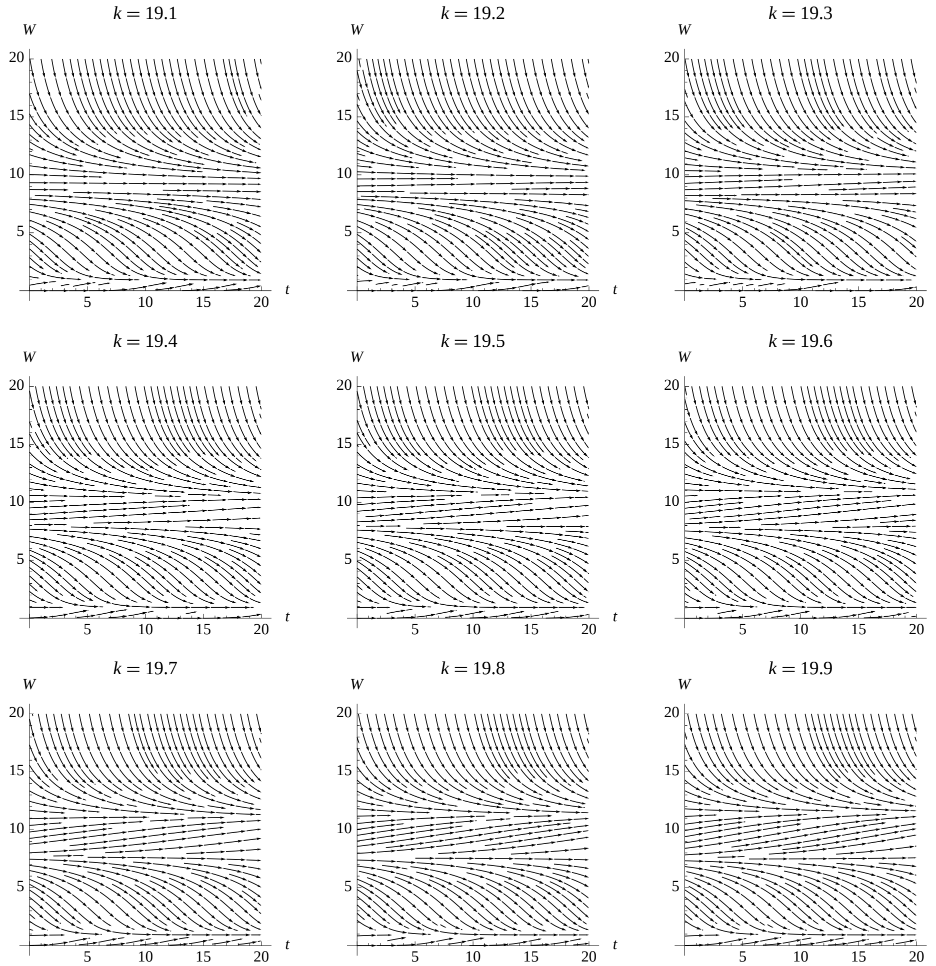
1. a 2.



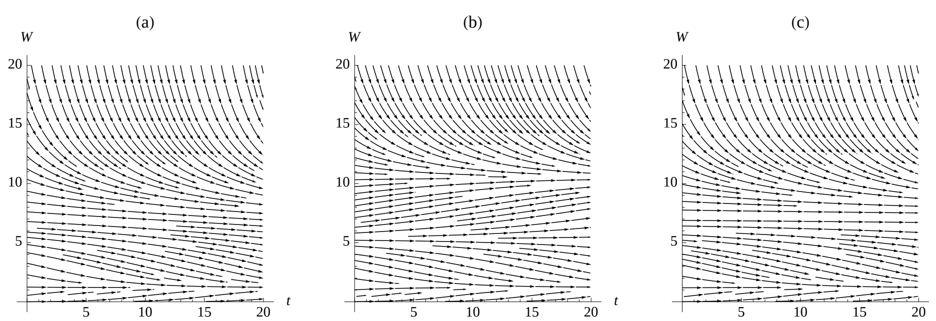
3. $k > 19.3$ will work



4.


k = 19.1
k = 19.2
k = 19.3
k = 19.4
k = 19.5
k = 19.6
k = 19.7
k = 19.8
k = 19.9
W
t
5
10
15
20

6.


(a)
(b)
(c)
W
t
5
10
15
20