Exercises on structural mechanics

Ice-breaking material for the course Advanced Structural Mechanics

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A short overview on the required background to successfully attend the course in Advanced Structural Mechanics (ASM).

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Civil Engineering course program

Bologna

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Preface

1.1 Outlook

The International Master program in Civil Engineering attracts students with a considerable variety of backgrounds. In particular, regarding the structural mechanics, students have been generally trained on different topics, considering different methods and using different syntax and nomenclatures.

As instructors of the course Advanced Structural Mechanics (ASM), delivered in the first semester of the first year, it is our interest to establish a common level of understanding on some structural mechanics problems at the outset of the course.

To this purpose, these handouts list a series of topics in structural mechanics whose knowledge is recommended to successfully attend the course Advanced Structural Mechanics. Each section presents one or more solved examples. As a self evaluation of the required standards, you are encouraged to attempt their solution.

1.2 Supporting textbook

The exercises considered in this handout have been taken from the textbook "Statics and Mechanics of Structures" by S. Krenk and J. Høgsberg, Springer Netherlands, 2013. The textbook can be used as a reference for those students who needs to strengthen their knowledge on the fundamentals of structural mechanics.

1.3 Course notes

Once enrolled in the Master Program you will have access to the repository of ASM (https://iol.unibo.it/) in which you can find the slides of the course, some suggested readings as well as some solved midterm and final exams. The slides are only meant to point the main topics that will be covered during the course and do not pretend to be comprehensive.

1.4 Course bibliography

Below you can find the textbooks useful for students looking to do further reading on the topics of ASM and more in general on structural mechanics and mechanics of materials.

- Matrix Structural Analysis, W. McGuire, R.H. Gallagher, R.D. Ziemian, John Wiley Sons, 2000.
- Fundamentals of Structural Stability, G.J. Simitses, D.H. Hodges, Elsevier, 2006.
- Advanced Structural Mechanics. A. Carpinteri, Taylor Francis, 1997.

2

Truss structures

2.1 Fundamentals

- Notion of force, moment of a force.
- Definition of support condition.
- Notion of statically determined vs. statically indetermined structure.
- Equilibrium via method of joints and via method of sections.
- Notion of axial force, stress and strain.

2.2 Example E1: V-truss structure

2.2.1 Analysis of the structure

Let us analyse the equilibrium of a simply supported V-truss structure [1], loaded by a concentrated force at node B (see Fig. 2.1). The truss structure is planar and can deform in the $x-y$ plane only. It consists of seven truss elements which count for a total of 21 degrees of freedoms (DoF), $DoF = 7 \times 3 = 21$, connected via five *internal hinges* in A, B, C, D, and E.

Each internal hinge (h) provides a number of internal degrees of restraints (DoR) equal to $DoR^h_{int} = 2 \times (n-1)$, where n is the number of truss elements joining at the hinge. For instance the internal hinge at node E provides $DoR^E_{int} = 2 \times (3-1) = 4$. It follows that the total number of degree of restraints of the internal hinges is $DoR_{int} = 4+4+6+2+2 = 18$.

The support at node A, termed *external hinge*, restrains both the horizontal and the vertical translations of node A. Conversely, the support in C, termed *slider*, restrains the vertical translation of node C only. As a result, the support in A can express both

[1] Adapted from the Example 2.2 in Ref. [1]

vertical, V_A, and horizontal, H_A, reaction components, while the support in C can express a vertical reaction V_C only. Hence, the number of external degrees of restraints, namely those between the structure and the external reference, is equal $DoR_{ext} = 3$, and the overall degree of restraint is $DoR = DoR_{int} + DoR_{ext} = 21$. As the number of DoR pairs the number of DoF, the necessary condition for having a statically determinate problem is satisfied.

In addition, since the restraints in A and C have no common centers of rotation and the internal structure is formed by simple triangles connected together, no kinematic mechanisms are allowed, i.e., there are $DoR_{ext} = 21$ which are effective. As the number of effective restrains pairs the number of degrees of freedom, the structure is *statically determined*. Hence, its reactions forces in A and C, as well as the internal forces in the seven truss elements, can be determined from equilibrium equations of forces and/or moments only.

2.2.2 Internal forces and external reactions via method of joints.

For instance, by considering the global equilibrium in terms of horizontal forces we get:

$$F_H : \quad H_A = 0.$$

The equilibrium of moments in C (clockwise direction) yields:

$$M_C : \quad V_A 4L - P2L = 0 \rightarrow V_A = \frac{1}{2}P.$$

Finally, imposing the global vertical force equilibrium:

$$F_V : \quad V_A + V_C - P = 0 \rightarrow V_C = \frac{1}{2}P.$$

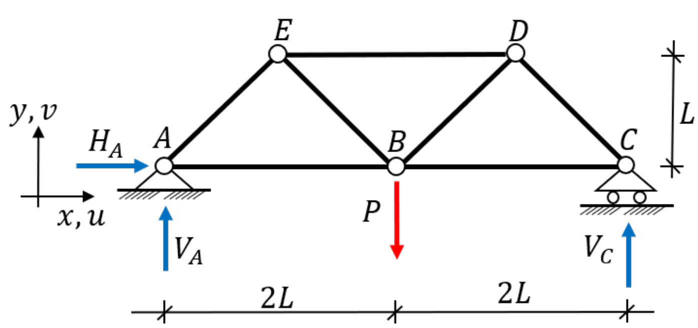


Fig. 2.1: V-truss. Global equilibrium.

Note that the same result could be quickly obtained by observing that the structure and the loading are symmetric with respect to a vertical line through B. Since the horizontal reaction component H_A is null, also the reactions forces in A and C must be symmetric.

Due to symmetry, also the internal axial forces (N_i) in the truss elements present the same symmetry, for instance the axial force in the element AB must be equal the one in the element BC, i.e., $N_{AB} = N_{BC}$.
For this reason, only the right half of the structure, namely nodes B, C and D need to be considered to determine the axial force in all the truss elements of the structure. These forces can be calculated by applying the equilibrium at the truss joints (see Fig. 2.2).
Let us consider the equilibrium of joint C. The node equilibrium contains two unknown forces, the truss forces N_{BC} and N_{CD}[2].

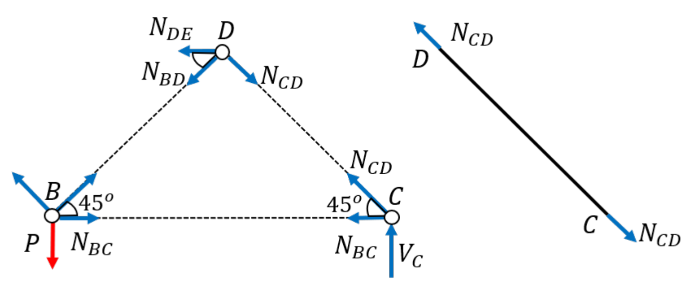


Fig. 2.2: V-truss. Equilibrium of nodes.

Imposing the equilibrium of node C along the vertical (V) and horizontal (H) direction yields, respectively[3]:

$$F_V: \quad N_{CD}\sin(45^o) + V_C = 0 \rightarrow N_{CD}\sin(45^o) + \frac{1}{2}P = 0 \rightarrow N_{CD} = -\frac{1}{2}\sqrt{2}P.$$

$$F_H: \quad N_{BC} + N_{CD}\cos(45^o) = 0 \rightarrow N_{BC} = \frac{1}{2}P$$

Similarly, for the joint D, the equilibrium along the vertical and horizontal direction yields:

[2]We suggest to take the unknown axial forces in the truss elements always as tensile forces. For instance, with reference to the Fig. 2.2, the unknown axial force in the truss element CD has been taken as a tensile force N_{CD}. Alike the axial forces in the others truss elements BD and BC have been taken as tensile forces.

[3]As you are writing an equilibrium equation, there is not an absolute reference system for positive and negative forces. The only requirement is to be consistent within the same equation.

$$F_V: \quad N_{BD}\sin(45^o) + N_{CD}\sin(45^o) = 0 \rightarrow N_{BD} = \frac{1}{2}\sqrt{2}P.$$

$$F_H: \quad N_{ED} + N_{BD}\cos(45^o) - N_{CD}\cos(45^o) = 0 \rightarrow N_{ED} = -P$$

The rest of the bar forces are defined due to symmetry, e.g., N_{BE} = N_{BD} and N_{AB} = N_{BC}. The diagram of the axial forces in the bar is provided in Fig. 2.3 and the values of the internal forces N_i are collected in Table 2.1.

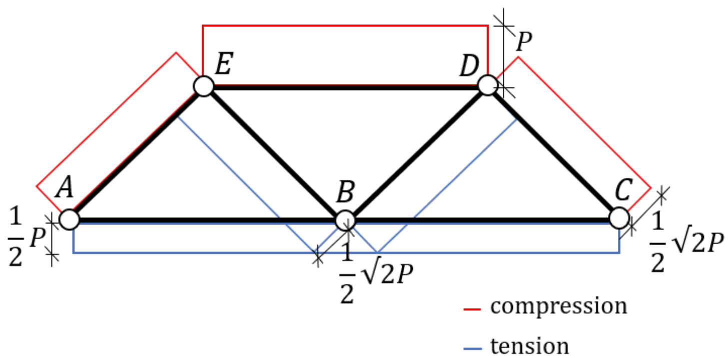


Fig. 2.3: V-truss. Axial force diagram.

2.2.3 Computation of displacements by Principle of Virtual Work.

Let us recall some fundamental notions related to the Principle of Virtual Work (PVW). By definition, the *virtual work* δW is the work done by a distribution of real forces $\mathbf{F}$ acting through a system of *virtual displacements* $\delta\mathbf{u}$ or a distribution of *virtual forces* $\delta\mathbf{F}$ acting through a system of *real displacements* $\mathbf{u}$. A *virtual displacement* is any displacement consistent with the constraints of the structure, i.e., that satisfy the boundary conditions at the supports. A *virtual force* is any system of forces in equilibrium.

The PVW states that for any deformable structure in equilibrium the external work δW_{ext} equates the internal one δW_{int}, i.e., $\delta W_{ext} = \delta W_{int}$, namely:

$$\delta\mathbf{F}\mathbf{u} = \int_V \delta\sigma\epsilon dV$$

where $\delta\sigma$ is the stress distribution in equilibrium with $\delta\mathbf{F}$, ϵ is the strain distribution compatible with the displacement field $\mathbf{u}$, and V is the volume of the structure. For a truss structure loaded at the nodes only, the equality between external and internal virtual work can be written as:

$$\sum_{j=1}^{n} \delta F_j u_j = \sum_{i=1}^{m} \delta N_i \epsilon_i$$

where δF_j and u_j are, respectively, the virtual force and the real displacement at node $j = 1, \ldots, n$, while δN_i and ϵ_i are the internal virtual force and real strain at the $i = 1, \ldots, m$ truss element, with:

$$\epsilon_i = \frac{N_i L_i}{E_i A_i}$$

being E_i, L_i, A_i and N_i the Young modulus, the length, the cross section area and real internal force of the $i-th$ truss element, respectively. The above equation is used in the *unit load method* to find redundant forces or reactions, as well as to find real structural displacements, as will be shown in the following.

Let us calculate the vertical displacement of node B, here labelled as v_B. To such purpose, we consider an auxiliary truss structure, also called unitary load configuration '1', identical to one of the given problem, but now loaded with a single unitary load acting at the node of interest along the direction of the sought displacement, in this case a unitary vertical force applied at node B, i.e. $\delta F_B = P^1 = 1$ (see Fig. 2.4).

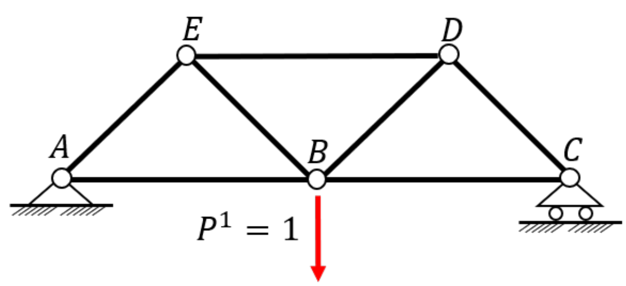


Fig. 2.4: V-truss subjected to a unitary vertical load at node B.

The concentrated load $\delta F_B = P^1 = 1$ yields to a distribution of virtual internal forces in the structure $\delta N_i = N_i^1$ (see Table 2.1). Again, N_i and u_j are the internal forces and displacements associated to the real load configuration (Fig. 2.1).

Due to the assumed unitary load consisting of a single force, the external virtual work counts only one term $\delta W_{ext} = P^1 v_B = 1 v_B$. As a result, an explicit formula for the displacement v_B in the direction of the unit test force is obtained:

$$1 v_B = \sum_{i=1}^{m} N_i^1 \frac{N_i L_i}{E_i A_i}$$

In table 2.1 are reported the values of the internal forces N_i, N_i^1 associated to the actual and unitary load configurations, respectively.

truss element	N_i	N_i^1
AB	$\frac{P}{2}$	$\frac{1}{2}$
BC	$\frac{P}{2}$	$\frac{1}{2}$
AE	$-\frac{\sqrt{2}P}{2}$	$-\frac{\sqrt{2}}{2}$
CD	$-\frac{\sqrt{2}P}{2}$	$-\frac{\sqrt{2}}{2}$
BE	$\frac{\sqrt{2}P}{2}$	$\frac{\sqrt{2}}{2}$
BD	$\frac{\sqrt{2}P}{2}$	$\frac{\sqrt{2}}{2}$
ED	$-P$	-1

Table 2.1: Bar internal forces associated to the real load configuration (N_i) and to the unitary vertical load (N_i^1) applied at node B.

Substitution of the values of the bar forces yields a vertical displacement $v_B = (3 + 2\sqrt{2})\frac{PL}{EA}$ [4].

The reader can replicate the procedure for any node of the truss and verify, for example, that the horizontal displacement of node C is $u_C = 2\frac{PL}{EA}$. For the calculation of this displacement, a unitary load $P^1 = 1$ should be applied at node C along the horizontal direction (see Fig. 2.5). This load generates the set of bar forces N_i^1 collected in table 2.2.

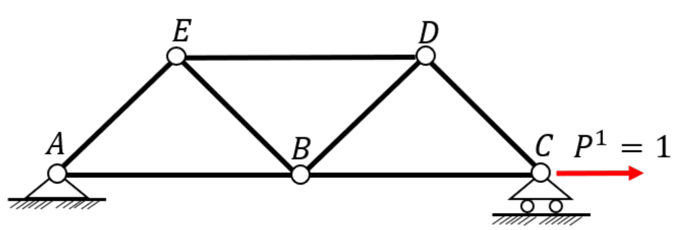


Fig. 2.5: V-truss subjected to a unitary horizontal load at node C.

[4] Following the principle of virtual work, a positive displacement is directed towards the same direction of the applied P^1 force, regardless the positive directions of the axes of the assumed reference system.

truss element	N_i^1
AB	1
BC	1
AE	0
CD	0
BE	0
BD	0
ED	0

Table 2.2: Bar internal forces associated to the unitary vertical load at node C.

The truss deformed shape under the actual load configuration is shown in Fig. 2.6.

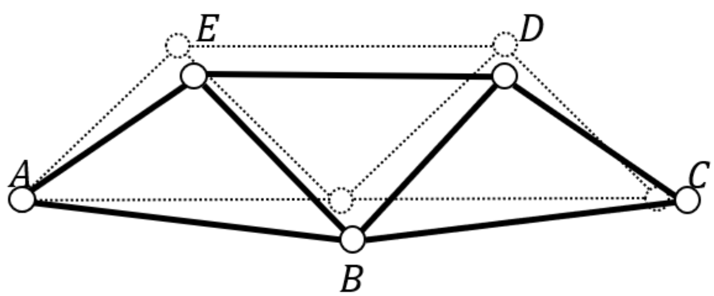


Fig. 2.6: V-truss deformed configuration under actual load.

3

Statics of Beams and Frames

3.1 Fundamentals

- Statically determined beams and frames: solution via equilibrium equations and by virtual work.
- Notion and calculation of internal forces and moments.
- Deformation of beams and frames. Euler Bernoulli beam model.
- Statically indetermined beams and frames: solution via the force method.

3.2 Example E1: Two beams with an internal hinge

3.2.1 Analysis of the structure

The system of interest[1] consists of two beams AD and DB, connected at node D by an *internal hinge* (see Fig. 3.1). The structure is restrained through a fixed external hinge in C and two sliders in A and B. The external hinge expresses vertical, V_C, and horizontal, H_C, reactions, while the sliders in A and B allow for horizontal translation and exert only vertical reactions, V_A and V_B, respectively. The structure is subjected to a concentrated force P acting at cross-section E placed at the mid-span of the beam DB.

Being the system formed by two elements the total number of degrees of freedom is $DoF = 6$. The sum of the external and internal degrees of restraints, $DoR_{ext} = 4$ and $DoR_{int} = 2$ respectively, is $DoR = DoR_{ext} + DoR_{int} = 6$, which pairs the number of DoF. As the restraints in A and C do not share a common center of rotation, the beam AD has no allowed kinematic mechanisms. It follows that the node D does not move due to kinematic mechanisms and the beam DB, restrained in B, cannot move as well. Therefore, the $DoR = 6$ are effective, and the structure is *statically determined*.

[1] Adapted from the example 1.5 in Ref. [1]

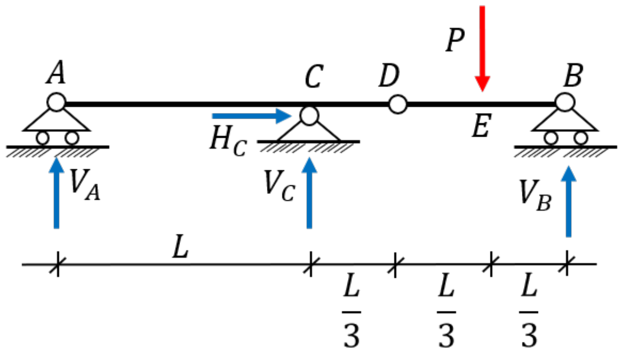


Fig. 3.1: Beam with an hinge. Global equilibrium.

3.2.2 Equilibrium equations

First, we can observed that the bending moment vanishes at the internal hinge in D while both vertical and horizontal forces can be transmitted. In addition, as the hinge is unloaded, the internal forces on its left hand side must be equal in modulus and opposite in sign w.r.t. those on its right hand side as illustrated in Fig. 3.2.

Let us determine the reactions on the beam DB. From the horizontal force equilibrium:

$$F_H : \quad H_D = 0.$$

Thus, the hinge at D only transmits a vertical force V_D. The vertical reaction in D is obtained by equilibrium of moments in B (clockwise):

$$M_B : \quad \frac{2L}{3}V_D - \frac{L}{3}P = 0 \rightarrow V_D = \frac{1}{2}P.$$

From the vertical force equilibrium of the beam BD:

$$F_V : \quad V_D + V_B - P = 0 \rightarrow V_B = \frac{1}{2}P.$$

The same results by observing the symmetry of the beam DB that carries a concentrated force P at its center. Due to symmetry, the two reactions V_D and V_B are equal, so that V_D = V_B = $\frac{1}{2}P$.

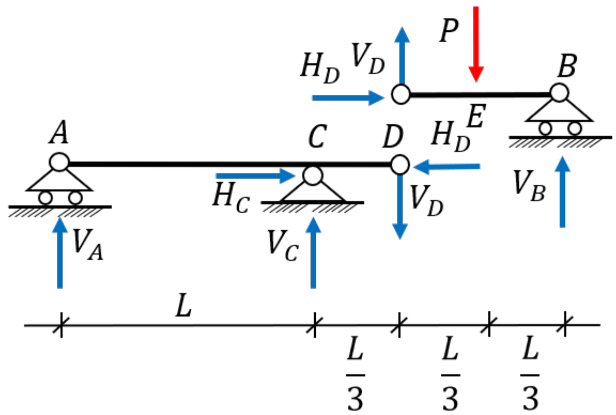


Fig. 3.2: Beam with an hinge. Global equilibrium. Reactions and hinge forces.

The remaining reactions are determined by considering the equilibrium of the beam AD , simply supported at A and C and loaded by $V_D = \frac{1}{2}P$, as shown in Fig. 3.2. Horizontal force equilibrium yields:

$$F_H : \quad H_C + H_D = 0 \rightarrow H_C = 0.$$

Counterclockwise moment about A gives:

$$M_A : \quad LV_C - \frac{4L}{3}V_D = 0 \rightarrow V_C = \frac{2}{3}P.$$

while clockwise moment about C yields:

$$M_C : \quad LV_A + \frac{L}{3}V_D = 0 \rightarrow V_A = -\frac{1}{6}P.$$

As a proof, one can verify that the vertical equilibrium equation of the whole structure is satisfied:

$$F_V : \quad V_A + V_B + V_C - P = 0$$

Internal forces

The internal forces (or internal reactions) for a given beam section are the elementary internal reactions transmitted by the section itself. For a plane beam with load acting in the plane (as in Fig. 3.3), there are three internal forces which fully characterize the equilibrium of the section, namely axial force N, shear force Q and bending moment M. The internal forces are related to the external loads via the indefinite equations of equilibrium, which for a plane beam read (see Fig.3.3):

$$\frac{dN}{dx} + p(x) = 0$$

$$\frac{dQ}{dx} + q(x) = 0$$
$$\frac{dM}{dx} + Q(x) = 0$$

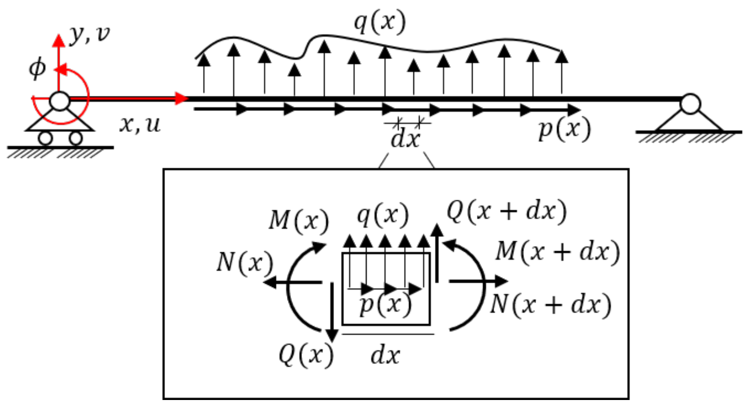


Fig. 3.3: Plane beam loaded with axial p and transverse q distributed loads in the plane. A detail on the cross-section equilibrium.

In the equilibrium equations, the assumed conventions regarding the positive signs of the internal forces are:

- the axial force is positive when giving tension in the beam;
- the shear force is positive when it tends to rotate the infinitesimal portion of beam where it acts in a counter-clockwise direction;
- the bending moment is positive when it stretches the lower beam fibres and compresses the upper beam fibres (lower and upper are related to the positive direction defined by the y-axis).

Given the above sign convention, the internal forces in the example of interest (Fig. 3.1) are determined by imposing the equilibrium across suitable sections along the beam axis. First, one can note that the axial force at any section of the beam is $N = 0$, since no horizontal load and reactions are exerted.

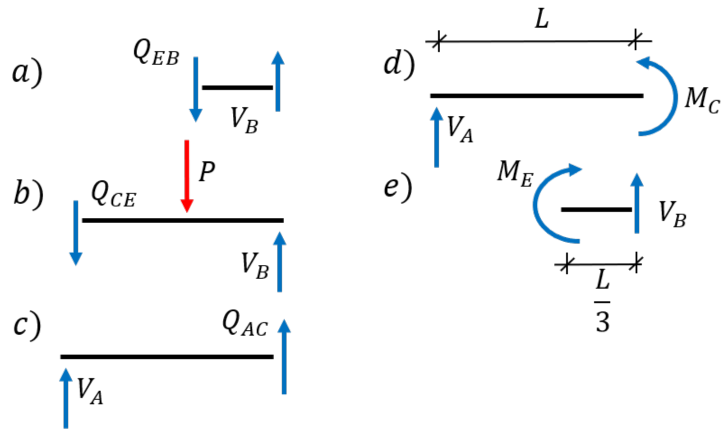


Fig. 3.4: Shear force and bending moment.

Additionally, no distributed loads are applied along the beam, thus the shear force is constant between the location where transverse forces are applied (e.g., supports and point loads). For this reason, the shear force distribution is determined by the values Q_{AC}, Q_{CE} and Q_{EB} and the external force P. Moving from right to the left along the beam axis, the shear forces Q_{EB}, Q_{CE}, Q_{AC} are determined by vertical force equilibrium equations (see Figs. 3.4a,b,c):

$$F_V: \quad -Q_{EB} + V_B = 0 \rightarrow Q_{EB} = \frac{1}{2}P.$$

$$F_V: \quad -Q_{CE} - P + V_B = 0 \rightarrow Q_{CE} = -\frac{1}{2}P.$$

$$F_V: \quad V_A + Q_{AC} = 0 \rightarrow Q_{AC} = \frac{1}{6}P.$$

The moment can be determined from its value at the four location A, B, C, D since its distribution is piecewise linear between the points where the transverse forces are applied. The moment is null at the end points A and B due to the support conditions. Thus, its distribution is fully determined from the two values M_C and M_E. The moment about C for the left part of the beam shown in Fig. 3.4d gives:

$$M_C: \quad M_C - V_A L = 0 \rightarrow M_C = -\frac{PL}{6}.$$

Similarly, considering a section at E, the moment about E for the right part of the beam (see Fig. 3.4e) yields:

$$M_E: \quad M_E - \frac{1}{3} V_B L = 0 \rightarrow M_E = \frac{PL}{6}.$$

The diagrams of internal forces, as per the calculation provided above, are provided in Fig. 3.5.

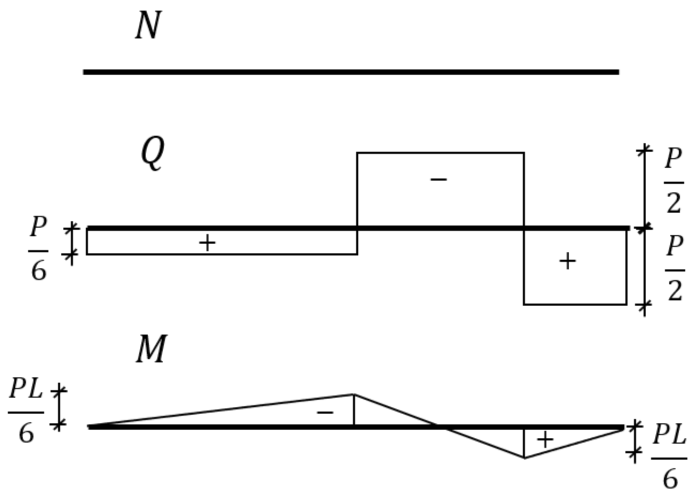


Fig. 3.5: Diagrams of the internal forces.

Note that the bending moment diagram is displayed on the side of the structure where the fibers of the beams are in tension. This is the convention generally used in the European countries.

3.2.3 Computation of displacements

Displacements via principle of virtual work

Similarly to what shown for a truss structure, the computation of the displacements for an elastic beam can be performed utilizing the principle of virtual works.

For slender beams and frames under bending, the work contributions due to axial and shear deformation of the beam elements can be neglected. As a result, the virtual internal work can be computed by considering the curvature of the beam element and the related bending moment only.

Let us compute the vertical displacement η_D of the cross section in D utilizing the principle of virtual work. To this purpose, we consider the beam of interest subjected to

the unitary vertical load $P^1 = 1$ in D (see Fig. 3.6a).
The reader can easily verify that the corresponding bending moment distribution M^1 is the one shown in Fig. 3.6b.

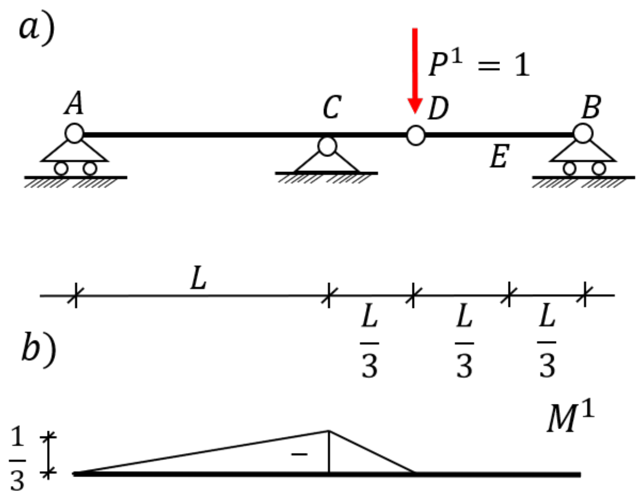


Fig. 3.6: a) Beam subjected to unitary vertical load in D. b) Bending moment diagram.

By exploiting the identity of the internal work and the external work given by the combination of the displacement field associated to the actual load configuration in Fig. 3.1, and the static field generated by the unitary force $P^1 = 1$, it is possible to calculate the vertical displacement of interest:

$$\eta_D P_1 = \int_S \frac{M(s)}{EI} M^1(s) ds = \int_{AC} \frac{M_{AC}(s)}{EI} M^1_{AC}(s) ds \quad + \int_{CD} \frac{M_{CD}(s)}{EI} M^1_{CD}(s) ds$$
$$+ \int_{DE} \frac{M_{DE}(s)}{EI} M^1_{DE}(s) ds \quad + \int_{EB} \frac{M_{EB}(s)}{EI} M^1_{EB}(s) ds$$

where s is the abscissa of a local reference system defined along each beam portion, e.g., from the generic cross-section X to the cross-section Y with $M_{XY}(s)$ the related bending moment. For the beam of interest the bending moments read:

$$M_{AC}(s) = -\frac{Ps}{6}, \quad M^1_{AC}(s) = -\frac{s}{3}, \quad M_{CD}(s) = \frac{P}{2}\left(s - \frac{L}{3}\right), \quad M^1_{CD}(s) = s - \frac{L}{3},$$
$$M_{DE}(s) = \frac{Ps}{2}, \quad M^1_{DE}(s) = 0, \quad M_{EB}(s) = \frac{Ps}{2}\left(\frac{L}{3} - s\right), \quad M^1_{EB}(s) = 0.$$

Substituting the above expressions of the bending moments, the vertical displacement[2] at node D reads:

$$\eta_D = \frac{2}{81}\frac{PL^3}{EI}.$$

The procedure can be replicated to calculate the rotation of the left and right cross sections at the internal hinge in D, $\phi_{D,DB}$ and , $\phi_{D,CD}$. The positive sign of the rotation ϕ is chosen according to the convention provided in Fig. 3.5. The unitary load configurations and the related moment distributions utilized for this purpose are shown in Fig. 3.3.

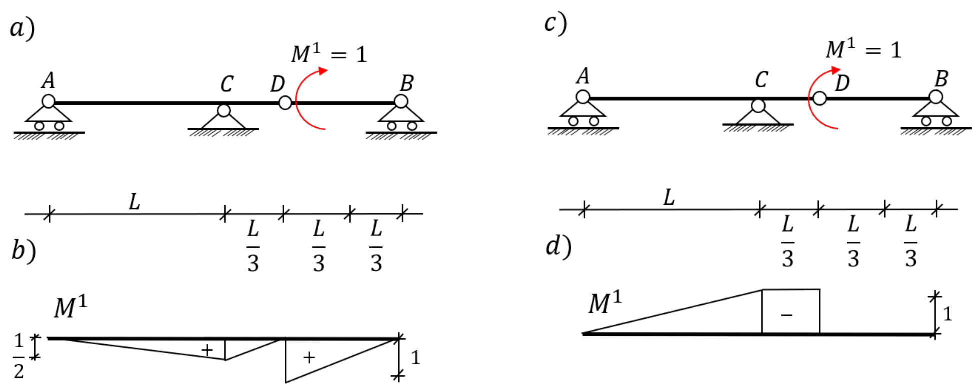


Fig. 3.7: a) Beam subjected to unitary moment at the right section of the internal hinge in D with the related b) bending moment diagram. c) Beam subjected to unitary moment at the left section of the internal hinge in D with the related d) bending moment diagram.

The reader can verify that the rotation $\phi_{D,DB}$ and $\phi_{D,CD}$ are, respectively:

$$\phi_{D,DB} = \int_{AB} \frac{M(s)}{EI} M^1(s)ds = \frac{1}{108}\frac{PL^2}{EI}$$

$$\phi_{D,CD} = \int_{AB} \frac{M(s)}{EI} M^1(s)ds = -\frac{PL^2}{12EI}$$

Displacements via elastic coefficients

The calculation of displacements and rotations can be often performed by exploiting the knowledge of some fundamentals beam deflection formulas. As an Appendix of

[2] Following the principle of virtual work, a positive displacement is directed towards the same direction of the applied unitary load, regardless the positive directions of the axes of the assumed reference system for the displacements.

these handouts, we have provided a short selection of these formulas which can be used during the ASM course for the calculation of displacements in beams and frames.

Let us compute the rotations at node A, B, C utilizing such formulas. To calculate the rotation in A and C, we consider the moment acting in C and utilize the beam deflection for a simply supported beam subjected to end moments (see Fig 3.8b). The cross section rotations at the supports can be calculated as function of the ends moment (see Appendix).

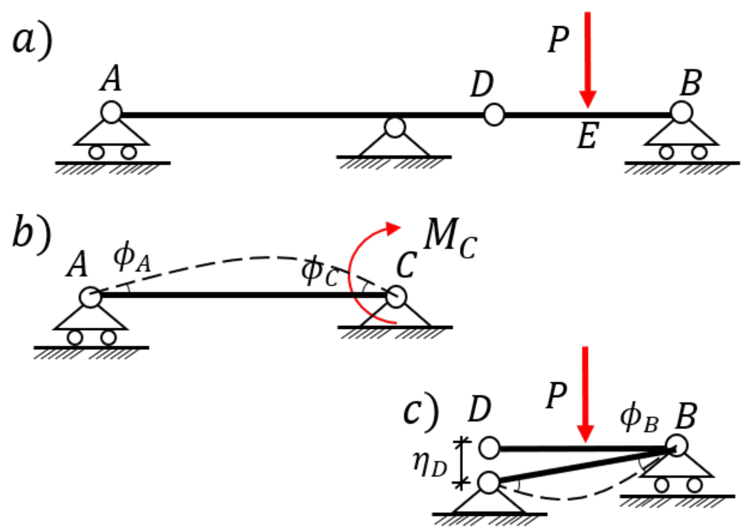


Fig. 3.8: a) Calculation of rotation via beam deflection formulas. b) Rotation a A and C. c) Rotation in B.

Thus, the rotation in A reads[3]:

$$\phi_A = \frac{M_C L}{6EI} = \frac{PL^2}{36EI}$$

Similarly the rotation in C reads:

$$\phi_C = \frac{M_C L}{3EI} = -\frac{PL^2}{18EI}$$

To calculate the rotation in B, we first account for the vertical displacement of the internal hinge in D (see Fig. 3.8c). This vertical displacement contributes as a rigid mechanism to the rotation of node B. In addition we evaluate the contribution of the vertical load P to the rotation in B utilizing the beam deflection formulas for a simply

[3]The rotation in A is positive according to the reference system for displacements defined in Fig. 3.3

supported beam subjected to a mid-span point load (see Appendix). As a result, the rotation in B reads:

$$\phi_B = \frac{3}{2L}\eta_D + \frac{P}{48EI}\frac{4L^2}{9} = \frac{7PL^2}{108EI}$$

The deformed shape of the beam is shown in Fig. 3.9.

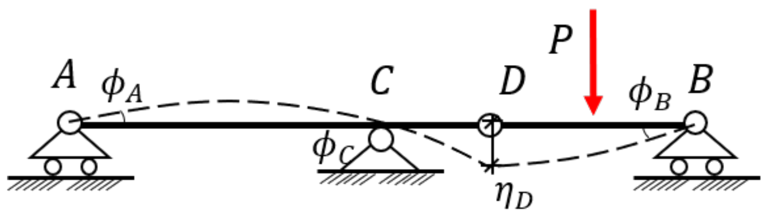


Fig. 3.9: Beam deformed shape.

3.3 Example E2: Frame with distributed loads

3.3.1 Analysis of the structure

The structure of interest is the simply supported angle frame[4] shown in Fig. 3.10a. As a single body in the $x-y$ plane, the frame has $DoF = 3$ and it is restrained by two hinges for a total of $DoR_{ext} = 4$. As the four external degrees of restraints are all effective, i.e., $DoR_{ext} = DoR = 4$, the structure is one time statically indeterminate ($DoF < DoR$). As such, its solution, namely the calculation of support reactions and internal forces, cannot rely on equilibrium equations only but it must consider a further equation based on the deformation of the structure.

The solution via method of forces requires the definition of an equivalent statically determinate structure. This structure can be obtained by releasing, for example, the rotation at node C, namely by inserting an internal hinge in C (see Fig. 3.10b). The true unknown internal moment at the cross-section C, i.e., M_C, is now represented by the unknown internal moment X at both sides of the hinge, the cross-sections C' and C''. The statically determined structure in Fig. 3.10b is equivalent to the one in Fig. 3.10a only if the two cross-sections C' and C'' rotate of the same amount:

$$\phi_{C'} = \phi_{C''}$$

[4] Adapted from the example 6.4 in Ref. [1].

This compatibility equation is exploited to find the equilibrium of the equivalent statically determined structure.

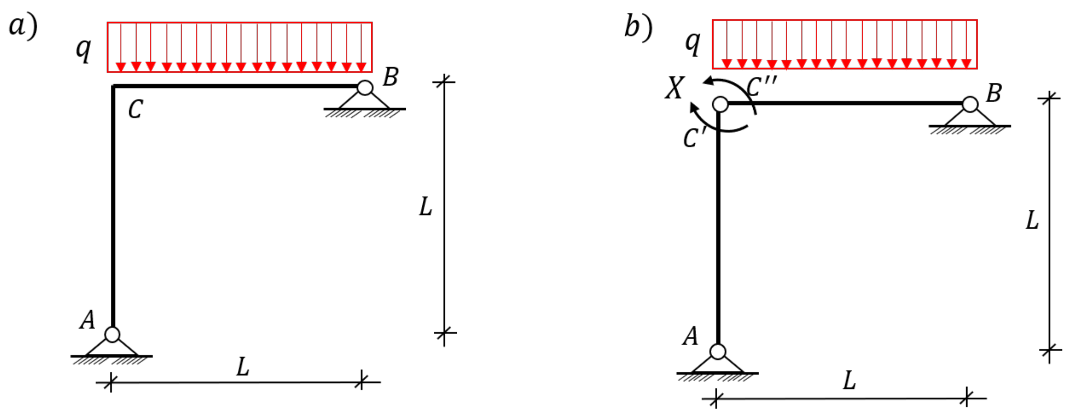


Fig. 3.10: a) Angle frame with distributed load. b) Associated statically determined structure.

3.3.2 Solution via Principle of Virtual Works

First, the moment distributions and the reaction forces are calculated for (i) the external load q without considering the unknown X, and (ii) for a unit moment $X = 1$ without considering the external load q. These two partial schemes are generally labeled as the $''0''$ configuration and the $''1''$ configuration, so the the related moment distributions are labeled as $M^0(s)$ and $M^1(s)$, respectively. Figures 3.11a and 3.11b shows these two load cases while the related moment distributions are displayed in Figures 3.11c and 3.11d.

For the configuration $''0''$, e.g., structure loaded with the distributed load q, one can easily verify that the horizontal reaction in A and in B are null and the vertical reaction in A and B are $V_A^0 = V_B^0 = \frac{1}{2}qL$. As a result, the moment M^0 along AC is null while the moment M^0 along CB reads:

$$M_{CB}^0(s) = V_A^0 s - \frac{qs^2}{2} = \frac{q(L-s)s}{2}$$

where s is the abscissa of a local reference system defined along the beam CB moving from C to B.

For the configuration $''1''$, e.g., structure loaded with the unitary load $X = 1$, one can easily obtain the reaction forces:

$$V_A^1 = \frac{X}{L} = \frac{1}{L}$$

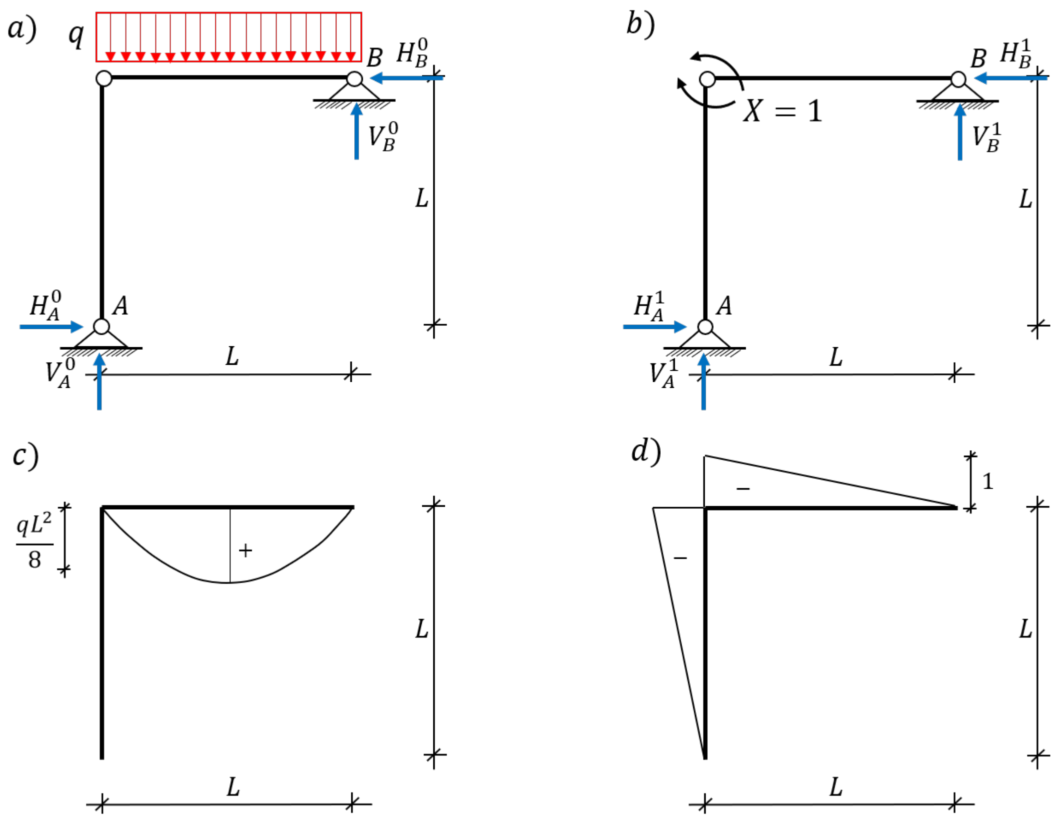


Fig. 3.11: a) Statically determined structure loaded with distributed load q. b) Statically determined structure loaded with unitary couple $X = 1$. c) Moment distribution $M^0(s)$ due to q. d) Moment distribution $M^1(s)$ due to $X = 1$.

$$H_A^1 = \frac{X}{L} = \frac{1}{L}$$

Thus, the moment M^1 along AC reads:

$$M_{AC}^1(s) = -H_A^1 s = -\frac{1}{L}s$$

while the moment M^1 along CB reads:

$$M_{CB}^1(s) = -\frac{L-s}{L}$$

By superposition, we can evaluate the actual moment in the equivalent isostatic structure as the sum of the moment given by the distributed load q and the one given by the couple X:

$$M(s) = M^0(s) + XM^1(s)$$

At this stage, we impose the identity of the external work and internal work $\delta W_{ext} = \delta W_{int}$, which in this case reads:

$$-1\phi_{C'} + 1\phi_{C''} = \int_{\mathcal{S}} M^1(s)\frac{M(s)}{EI}ds$$

where the rotations $\phi_{C'}$ and $\phi_{C''}$ are associated to the actual load configuration of the equivalent isostatic structure, namely distributed load q plus unknown couple X. Imposing the compatibility of rotations at node C, $\phi_{C'} = \phi_{C''}$, yields:

$$\int_{\mathcal{S}} M^1(s)\frac{M(s)}{EI}ds = 0$$

which can be rewritten as:

$$\int_{\mathcal{S}} M^1\frac{M^0}{EI}ds + \int_{\mathcal{S}} M^1\frac{XM^1}{EI}ds = 0$$

where:

$$\int_{\mathcal{S}} M^1\frac{M^0}{EI}dx = \int_{AB} \frac{-q(L-s)^2 s}{2EIL}ds = -\frac{qL^3}{24EI}$$

and:

$$\int_{\mathcal{S}} M_1\frac{XM^1}{EI}dx = \int_{AC} X\frac{s^2}{EIL^2}ds + \int_{CB} X\frac{(L-s)^2}{EIL^2}dx = \frac{2XL}{3EI}$$

Solving the equation for the unknown couple X, we obtain:

$$-\frac{qL^3}{24EI} + \frac{2XL}{3EI} = 0 \to X = \frac{qL^2}{16}$$

Knowledge of X allows calculating the reaction forces as the superposition of the individual load cases:

$$V_A = V_A^0 + XV_A^1 = \frac{9qL}{16}$$
$$V_B = V_B^0 + XV_B^1 = \frac{7qL}{16}$$
$$H_A = H_A^0 + XH_A^1 = \frac{qL}{16}$$

with:

$$H_B = H_A$$

From the reaction forces, the distribution of the internal forces are easily found:

$$N_{AC} = -V_A = -\frac{9qL}{16} \qquad Q_{AC} = H_A = \frac{qL}{16} \qquad M_{AC}(s) = -H_A s = -\frac{qLs}{16}$$

$$N_{CB} = -H_A = -\frac{qL}{16} \qquad Q_{CB}(s) = -V_A + qs = -\frac{9qL}{16} + qs \qquad M_{CB} = -\frac{qL^2}{16} + \frac{9qLs}{16} - \frac{qs^2}{2}$$

The internal force distributions are shown in Fig. 3.12.

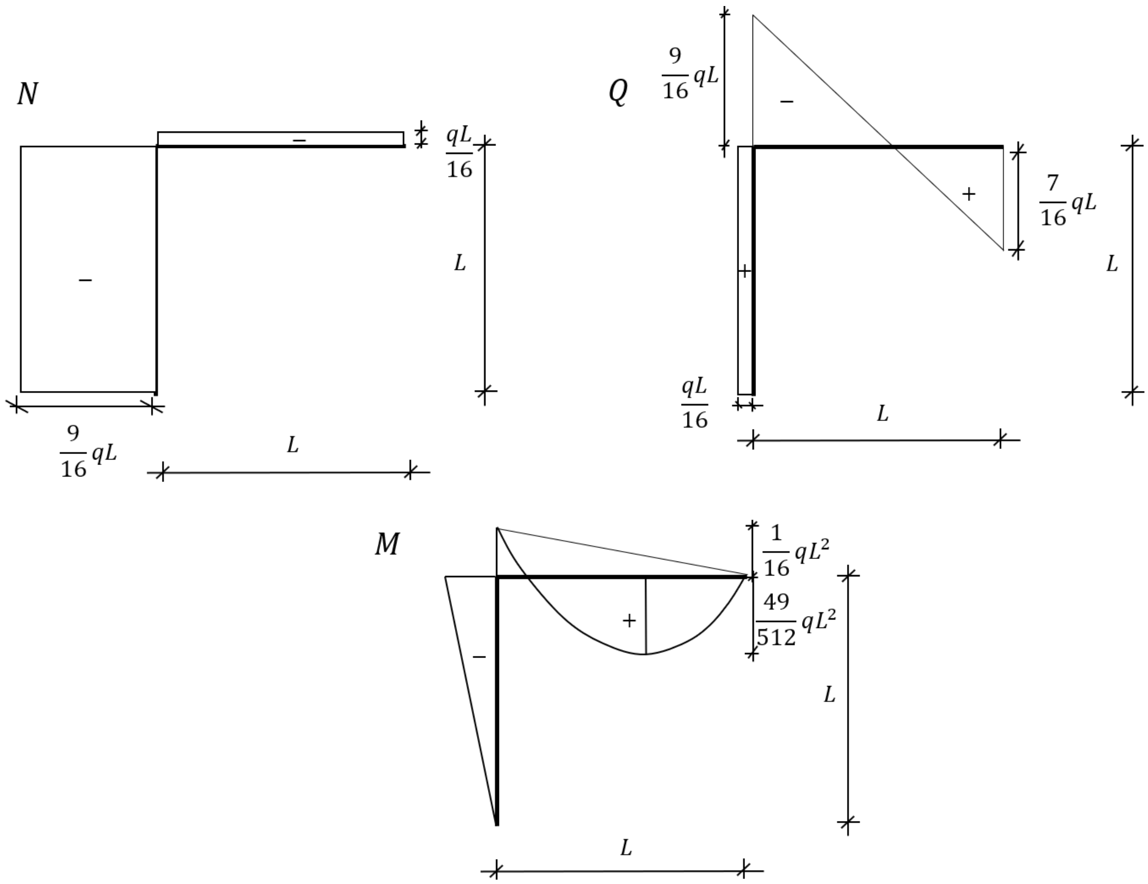


Fig. 3.12: a) Axial force b) Shear Force and c) Moment distribution.

Note that the moment distribution is linear in the vertical beam AC, while it is parabolic in the horizontal beam CB. The parabolic part shows a local maximum where the shear force is null:

$$Q_{CB}(s) = 0 \rightarrow -V_A + qs = 0 \rightarrow \bar{s} = \frac{9L}{16}$$

with:

$$M_{CB}\left(\frac{9L}{16}\right) = \frac{49}{512}qL^2.$$

3.3.3 Computation of generalized displacements via elastic coefficients

Let us calculate the rotations at some relevant cross-sections of the frame by exploiting the beam deflection formulas given in Appendix. Since the axial deformation of the frame can be neglected, we assume the node in C to be fixed. Hence, the rotation at node A, C and B can be calculated by resorting to known results for a simply supported beam. According to the schemes reported in Fig. 3.13, the rotation at node A reads:

$$\phi_A = \frac{M_C L}{6EI} = \frac{qL^3}{96EI}$$

Similarly the rotation in C' reads:

$$\phi_{C'} = -\frac{M_C L}{3EI} = -\frac{qL^3}{48EI}$$

which equates the rotation in C'', given by the contribution of the moment in C and the distributed load:

$$\phi_{C''} = \frac{M_C L}{3EI} - \frac{qL^3}{24EI} = \frac{qL^3}{48EI} - \frac{qL^3}{24EI} = -\frac{qL^3}{48EI}$$

Finally the rotation in B is given as:

$$\phi_B = -\frac{M_C L}{6EI} + \frac{qL^3}{24EI} = -\frac{qL^3}{96EI} + \frac{qL^3}{24EI} = \frac{3qL^3}{96EI}$$

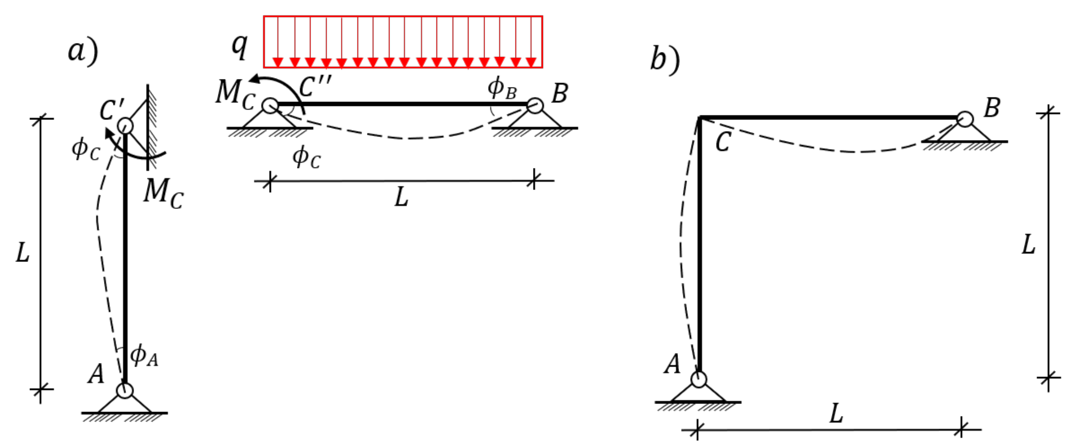


Fig. 3.13: a) Simply supported column AC subjected to moment M_C and simply supported beam CB subjected to moment M_C and distributed load q. b) Frame deformed shape

The deformed shape of the frame is shown in Fig. 3.13b.

4

Flexure, shear and torsion of beams

4.1 Fundamentals

- Axial stresses in beams under pure bending.
- Shear flow and shear stress due to bending: compact and thin-walled beams.
- Axial stresses for general bending.
- Shear stresses due to torsion: compact and thin-walled beams.

4.2 Examples

4.2.1 E1. Axial and shear stresses in a cantilever beam with thin-walled I-profile

Figure 4.1a shows a cantilever beam with a constant I-profile section and elastic modulus E[1]. The cross-section has height h and width w with $h = w = a$ and thickness t and $2t$, for the web and flanges, respectively. The cross-section is assumed to be thin-walled, i.e., $t << a$. The cross-section is double-symmetric with respect to the principal axes of inertia $y - z$ which intersect at cross-section centroid G. As such, the shear center C of the cross-section coincides with the centroid $C \equiv G$. The beam is loaded by a tip force P, acting in the y direction, and passing through the cross-section centroid. As a result, no torque is generated in the beam.

The distribution of strain and stress is determined at the cross-section placed at fixed support $x = 0$, where the axial force, shear force and bending moment values are:

$$N = 0, \quad Q_y = -P, \quad M_z = -PL$$

[1] Adapted from Example 11.4 in Ref. [1]

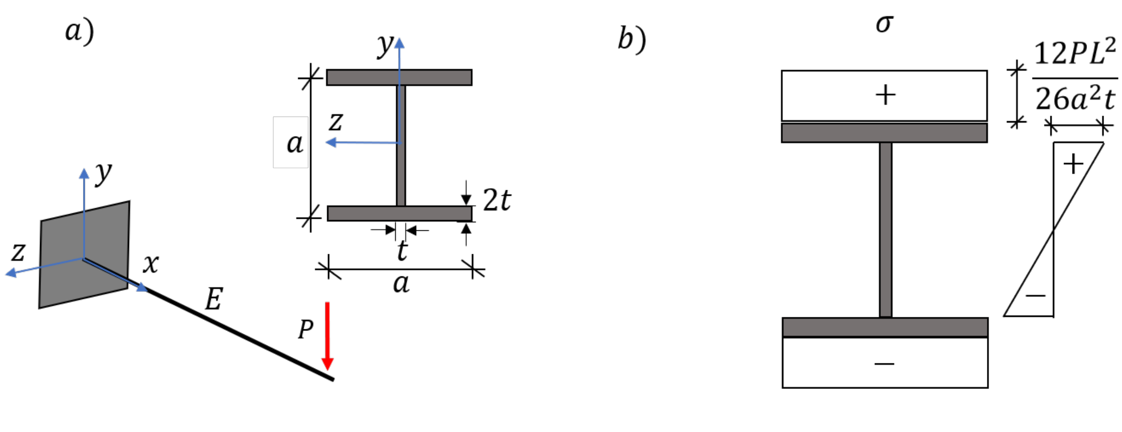


Fig. 4.1: a) Cantilever beam with a tip load P. Cross section details. b) Stress distribution due to bending at the fixed-support cross-section.

Normal stress

According to the Navier formula, the normal stress distribution over the beam cross-section induced by the bending moment is:

$$\sigma(y,z) = -\frac{M_z}{I_z} y$$

with I_z being the moment of inertia with respect to the z axis:

$$I_z = \frac{1}{12}a^3 t + 2\left(2ta\left(\frac{1}{2}a\right)^2\right) = \frac{13}{12}a^3 t$$

Thus, the normal stress distribution reads:

$$\sigma(y,z) = -\frac{M_z}{I_z} y = \frac{12}{13}\frac{PL}{a^3 t} y$$

Figure 4.1b shows the variation of the normal stress $\sigma(y,z)$ over the cross-section. The negative stress indicates that the bottom flange is in compression. The maximum positive (negative) value is reached at the uppermost (lowermost) fiber of the flange.

Shear stress

According to Grashof's formula (or Jourawsky's formula) for a cross-section subjected to a shear force Q_y the shear flow at a given cord of the cross-section is:

$$q = \frac{S_z^{A'}}{I_z} Q_y$$

where $S_z^{A'}$ is the static moment (first moment calculated with respect to the z axis) of the segment area A' of the cross-section delimited by the cord.

For the cross-section of interest, the shear flow is calculated by defining generic cords both in the flanges and in the web as shown in Fig. 4.2a. The cords are always orthogonal to the middle line of the cross-section, marked in black in Fig. 4.2a and Fig. 4.2b. First, a vertical cord is considered in the left part of the top flange at distance d_1 from the left end of the flange, as shown in Fig. 4.2a. The relative segment area A' is highlighted in blue.

The static moment of this segment with respect to the z-axis is:

$$S_z^{A'} = 2td_1\frac{a}{2} = atd_1$$

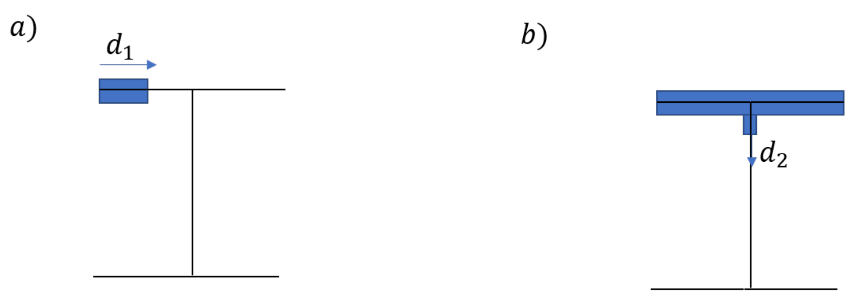


Fig. 4.2: a) Shear flow along the left flange. b) Shear flow along the web

Thus, the shear flow at the cord reads:

$$q_1(d_1) = \frac{S_z^{A'}}{I_z}Q_y = -\frac{12P}{13a^2}d_1$$

The minus sign indicates that the shear flow exits the segments area A'. The shear flow variation is linear with respect to the local coordinate d_1, and reaches its maximum at the flange center:

$$q_1\left(\frac{a}{2}\right) = \frac{S_z^{A'}}{I_z}Q_y = -\frac{6}{13a}P$$

The shear flow in the web is determined by taking a horizontal cord in the web, at distance d_2 from the centerline of the top flange, as shown in Fig. 4.2b. The static moment of this segment area contains contributions from the top flange and part of the web, namely:

$$S_z^{A'} = 2ta\frac{a}{2} + td_2\left(\frac{a}{2} - \frac{d_2}{2}\right) = ta^2\left(1 - \frac{d_2(d_2 - a)}{2a^2}\right)$$

It can be observed that the contribution of the web to the static moment presents a quadratic dependence with d_2, and that for $d_2 = 0$ and $d_2 = a$, namely at top and bottom locations of the web, this contribution vanishes.
As a result, the expression for the shear flow in the web is:

$$q_2(d_2) = \frac{S_y^{A'}}{I_y} Q_z = -\frac{12P}{13a}\left(1 - \frac{d_2(d_2-a)}{2a^2}\right)$$

The shear flow at the center of the web is:

$$q_2\left(\frac{a}{2}\right) = \frac{S_z^{A'}}{I_z} Q_y = -\frac{27}{26}\frac{P}{a}$$

The shear flow distribution along the whole section is provided in Fig. 4.3a.

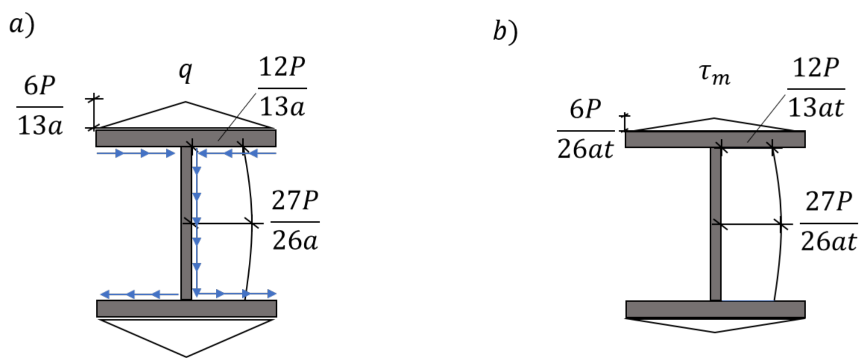


Fig. 4.3: a) Shear flow b) and shear stress distribution

The distribution of stress along the flanges and web section (Fig. 4.3b) is obtained by dividing the shear flow by the cord thicknesses.

4.2.2 E1. Torsion in a thin-walled C-profile.

A beam with a C-profile of height $h = 4a$ and width $w = 3a$ is subjected to torsion[2]. The thickness of the vertical web is $t_w = t$, while the thickness of the horizontal flanges is $t_f = 2t$. The cross-section is assumed to be thin-walled, so that $t_w, t_f << h, w$.

[2] Adapted from Example 11.12 in Ref. [1])

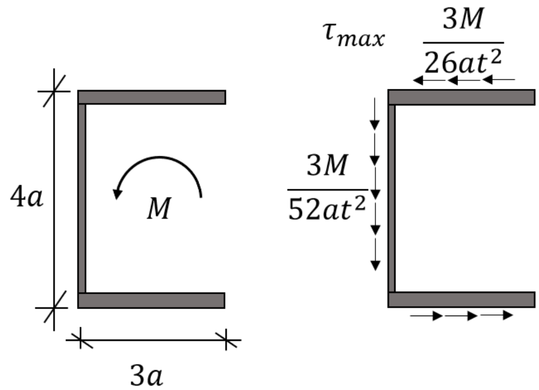


Fig. 4.4: a) C-Profile section. b) Maximum stress due to torsional moment M.

For the thin-walled section of interest, the torsional factor J_t is found by summation of the contribution from the two flanges and the web:

$$J_t = \frac{1}{3}(2(3a(2t)^3) + 4at^3) = \frac{52}{3}at^3$$

The torsional stiffness is given by the product of the torsional factor and the shear modulus G. The maximum stress in the flange is calculated as follows:

$$\tau_{max_f} = \frac{M_x}{GJ_t}t_f = \frac{3}{26}\frac{M_x}{at^2}$$

Similarly, the maximum stress in the web is calculated as

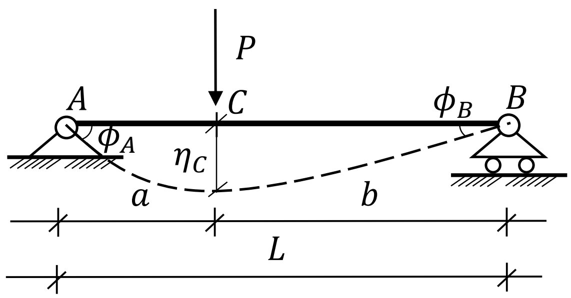
$$\tau_{max_w} = \frac{M_x}{GJ_t}t_w = \frac{3}{52}\frac{M_x}{at^2}$$

References

1. S. Krenk and J. Høgsberg. *Statics and Mechanics of Structures*. SpringerLink : Bücher. Springer Netherlands, 2013.

Appendix - Beam Deflection Formulas.

Simply-supported beam

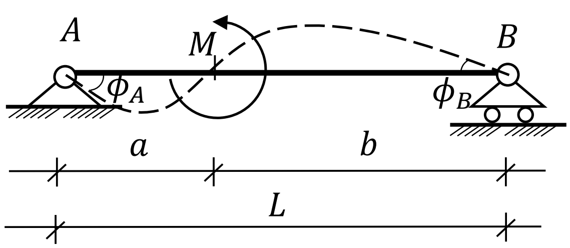


$$\phi_A = \frac{Pab(L+b)}{6LEI}; \phi_B = \frac{Pab(L+a)}{6LEI}$$

$$\eta_C = \frac{Pa^2b^2}{3EIL};$$

$$a = b = \frac{L}{2} \to \phi_A = \phi_B = \frac{PL^2}{16EI};$$

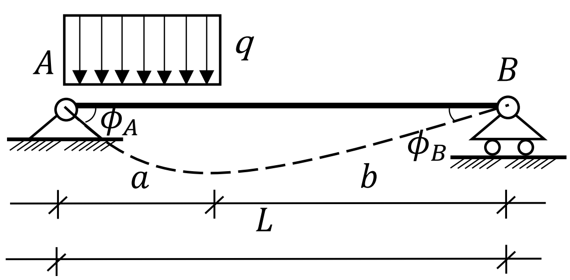
$$a = b = \frac{L}{2} \to \eta_C = \eta(L/2) = \frac{PL^3}{48EI}$$



$$\phi_A = \frac{M(3b^2 - L^2)}{6LEI}$$

$$\phi_B = \frac{M(3a - L^2)}{6LEI}$$

$$a = 0 \to \phi_A = \frac{ML}{3EI}, \phi_B = \frac{ML}{6EI}$$



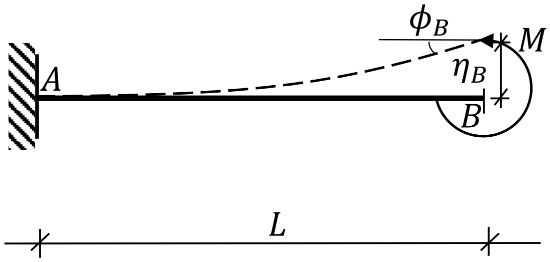
$$\phi_A = \frac{qa^2(a+2b)^2}{24LEI}$$

$$\phi_B = \frac{qa^2(2L^2 - a^2)}{24LEI}$$

$$a = L \to \phi_A = \frac{qL^3}{24EI}, \phi_B = \frac{qL^3}{24EI}$$

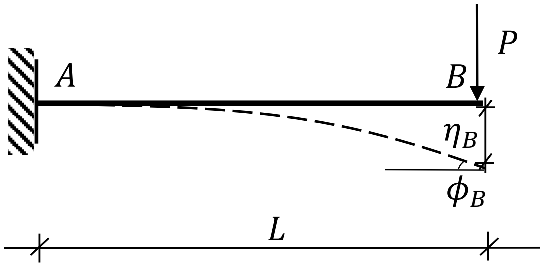
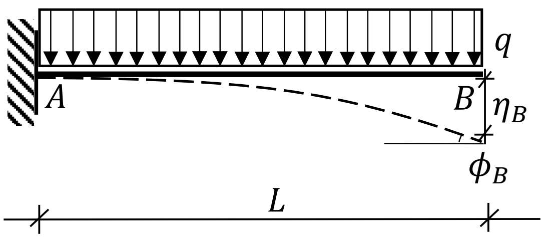
$$a = L \to \eta(L/2) = \frac{5qL^4}{384EI}$$

Cantilever beam

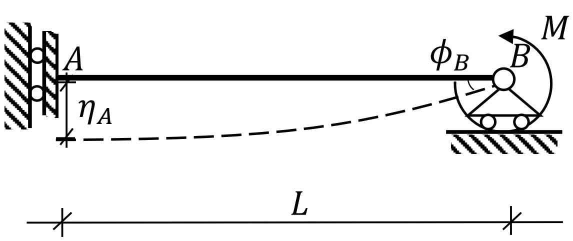
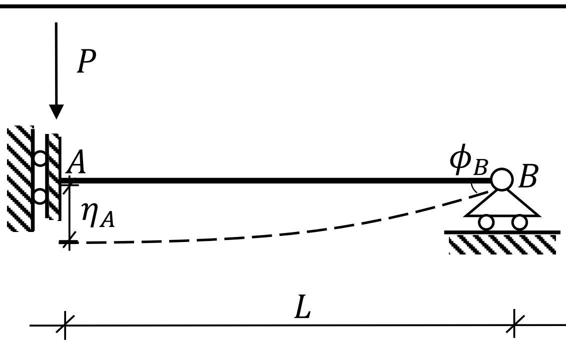


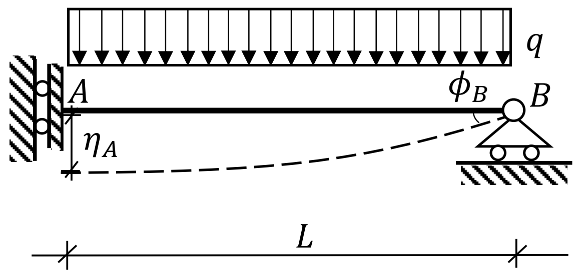
$$\phi_B = \frac{ML}{EI}$$

$$\eta_B = \frac{ML^2}{2EI}$$

	$\phi_B = \dfrac{PL^2}{2EI}$ $\eta_B = \dfrac{FL^3}{3EI}$
	$\phi_B = \dfrac{qL^3}{6EI}$ $\eta_B = \dfrac{qL^4}{8EI}$

Guided support-support beam

	$\phi_B = \dfrac{ML}{EI}$ $\eta_A = \dfrac{ML^2}{2EI}$
	$\phi_B = \dfrac{PL^2}{2EI}$ $\eta_A = \dfrac{PL^3}{3EI}$



$$\phi_B = \frac{qL^3}{3EI}$$

$$\eta_A = \frac{5qL^4}{24EI}$$

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