Electronic Journal of Differential Equations, Vol. 2021 (2021), No. 06, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE AND BLOW UP OF SOLUTIONS FOR A STRONGLY DAMPED PETROVSKY EQUATION WITH VARIABLE-EXPONENT NONLINEARITIES

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Abstract. In this article, we consider a nonlinear plate (or beam) Petrovsky equation with strong damping and source terms with variable exponents. By using the Banach contraction mapping principle we obtain local weak solutions, under suitable assumptions on the variable exponents $p(\cdot)$ and $q(\cdot)$. Then we show that the solution is global if $p(\cdot) \geq q(\cdot)$. Also, we prove that a solution with negative initial energy and $p(\cdot) < q(\cdot)$ blows up in finite time.

1. Introduction

Let be Ω a bounded domain in R^n $(n \geq 1)$ with a smooth boundary $\partial\Omega$. We consider the initial boundary value problem

$$\begin{gathered}
u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{p(x)-2}u_t = |u|^{q(x)-2}u, \quad (x,t) \in \Omega \times (0,T) \\
u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega \\
u(x,t) = \partial_v u(x,t) = 0, \quad x \in \partial\Omega
\end{gathered} \tag{1.1}$$

where v is the unit outer normal to $\partial\Omega$, and the exponents $p(\cdot)$ and $q(\cdot)$ are measurable functions on Ω satisfying

$$\begin{gathered}
2 \leq p^- \leq p(x) \leq p^+ \leq p^* \\
2 \leq q^- \leq q(x) \leq q^+ \leq q^*,
\end{gathered} \tag{1.2}$$

where

$$\begin{gathered}
p^- = \operatorname{ess\,inf}_{x\in\Omega} p(x), \quad p^+ = \operatorname{ess\,sup}_{x\in\Omega} p(x) \\
q^- = \operatorname{ess\,inf}_{x\in\Omega} q(x), \quad q^+ = \operatorname{ess\,sup}_{x\in\Omega} q(x)
\end{gathered}$$

and

$$\begin{gathered}
2 < p^*, q^* < \infty \quad \text{if } n \leq 4, \\
2 < p^*, q^* < \frac{2n}{n-4} \quad \text{if } n > 4\,.
\end{gathered}$$

2010 *Mathematics Subject Classification.* 35A01, 35B44, 35L55.
Key words and phrases. Global solution; blow up; Petrovsky equation; variable-exponent nonlinearities.

Submitted May 20, 2020. Published January 29, 2021.

When $p(x)$ and $q(x)$ are constant and without strong damping $(-\Delta u_t)$, problem (1.1) becomes to the Petrovsky equation

$$\begin{gathered} u_{tt}+\Delta^2 u+|u_t|^{m-2}u_t=|u|^{p-2}u, \quad \text{in } Q_T=\Omega\times(0,T) \\ u=\partial u/\partial v=0, \quad \text{on } \Gamma_T=\partial\Omega\times[0,T) \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x) \quad \text{in } \Omega. \end{gathered} \tag{1.3}$$

Messaoudi [24] studied this problem and established an existence result and showed that the solution continues to exist globally if $m \geq p$, and that it blows up in finite time if $m < p$ and the initial energy is negative. This result was later improved by Chen and Zhou [9]. For more results related to the plate equations, we refer the reader to Lagnese [19], Horn and Lasiecka [16, 20].

Problem (1.1) with strong damping and p and q constants becomes

$$u_{tt}+\triangle^2 u-\Delta u_t+|u_t|^{p-2}u_t=|u|^{q-2}u. \tag{1.4}$$

Liu et al [21] showed the existence, decay and blow up of the solutions of (1.4) and proved global existence and blow up. In 2013 Pişkin and Polat [34] showed the global existence and the decay of the solutions for (1.4).

A considerable effort has been devoted to the study of (1.1) in case of constant and variable-exponent nonlinearities. In recent years, plate equations with lower order perturbation of p-Laplacian type in the form

$$u_{tt}+\Delta_x^2 u-\operatorname{div}(\phi(\nabla_x u))=F(u,u_t)$$

where $\phi(z) \approx |s|^{(p-2)}s$, $p \geq 2$, and $F(u,u_t)$ represents additional damping and forcing terms. This attracted attention of several authors. It is a prototype for some important models in real-world applications.

In the absence of the viscoelastic term $(g=0)$ and replacing the $\vec{p}(x,t)$-Laplacian by $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ (p is constant and $p \geq 2$), the equation

$$u_{tt}+\Delta^2 u-\operatorname{div}(|\nabla u|^{p-2}\nabla u)-\Delta u_t=h(x,u,u_t) \tag{1.5}$$

has been extensively studied and results concerning existence, nonexistence and long-time behaviour have been established; see [39, 40].

In one-dimension, (1.4) without damping or forcing terms is related to the model

$$\rho u_{tt}+\zeta u_{xxxx}+a(u_x^2)_x=0, \quad a>0 \ \zeta=\text{const}>0,$$

which describes elastoplastic-microstructure flows as discussed in [2, 3].

In two dimensions, with $p=4$ and weak damping, (1.4) corresponds to the so called model for nonlinear plates

$$u_{tt}+\Delta^2 u-div[|\nabla u|^2\nabla u]+ku_t=\sigma\Delta(u^2)-f(u).$$

This is indeed a limit of the Mindlin-Timoshenko plates as the shear modulus tends to infinity, as shows in [10]. Remarkable results were obtained in [10, 11], where the existence of finite- dimensional global attractors under a weak damping ku_t, instead of $-\Delta u_t$, was proved. Recently, the authors in [31] proved the blow up of solutions for a nonlinear viscoelastic wave equations with variable exponents,

$$u_{tt}-\Delta u+\int_0^t g(t-\tau)\Delta u(\tau)d\tau+|u_t|^{p(x)-2}u_t=|u|^{q(x)-2}u \quad (x,t)\in\Omega\times(0,T).$$

In the presence of the viscoelastic term $(g \neq 0)$, equation (1.1) with memory was first studied in [4].

The general decay of weak solutions $u = u(x,t)$ for plate equations with memory term and lower order perturbation of $\vec{p}(x,t)$-Laplacian type has been studied (see [14]). More precisely, we considered the problem

$$u_{tt} = \operatorname{div}(|\nabla u|^p \nabla u) \tag{1.6}$$

with constant exponent of nonlinearity $p \in (1,\infty)$. Equation (1.6) was intensively studied During the previous decades, and was casted for the role of a touchstone in the nonlinear PDEs. The existence of global a solution without an additional dissipation term is an still open problem.

We also mention the very important contribution in [5], where the author proved the existence and blow up for the weak solutions of a wave equation with $p(x,t)$-Laplacian and damping terms.

$$\begin{gathered}
u_{tt} = \operatorname{div}\Big(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u + \varepsilon\nabla u_t\Big) + b(x,t)|u|^{\sigma(x,t)-2}u + f(x,t),\\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,\\
u|_{\Gamma_T} = 0 \quad \Gamma_T = \partial\Omega \times (0,T),
\end{gathered}$$

where the coefficients a, b, f and the exponents p, σ are given measurable functions and $\varepsilon =$ const > 0. Such equations (with variable exponents of nonlinearities) are usually referred as equations with nonstandard growth conditions.

Equations with nonstandard growth conditions occur in the mathematical modeling of various physical phenomena, e.g., the flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, processes of filtration through a porous media and the image processing see [6] and references therein.

Note that in all papers (referring to the case $p \neq 0$) the viscous term $\varepsilon\Delta u_t$ plays a key role in the proof of the existence of local and global solutions (even if $p =$ const $\neq 2$). The principal difficulty remains in proving an existence theorem by considering the term $-\Delta_{\vec{p}(x,t)}u$. The viscous term $\varepsilon\Delta u_t$ (with $\varepsilon > 0$) facilitates the proof of existence theorems.

The authors in [7] improved the results from [4] by establishing local and global existence, as well as the uniqueness of the weak solution $u(x,t)$ to (1.1). Recently in [26], the author established the decay of solutions of a damped quasilinear wave equation with variable-exponent nonlinearities. Rivera et al. [28] considered the equation

$$u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds = 0 \quad \text{in } Q_T = \Omega \times (0,T),$$

with initial and dynamical boundary conditions and proved that the sum of the first and second energies decays exponentially (respectively polynomially) if the kernel g decays exponentially (respectively polynomially). Alabau-Boussouira et al. [1] worked on the problem

$$\begin{gathered}
u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)\,ds = f(u) \quad \text{in } Q_T = \Omega \times (0,T)\\
u = \partial u/\partial v = 0 \quad \text{on } \Gamma_T = \partial\Omega \times [0,T)\\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega
\end{gathered}$$

and established exponential and polynomial decay results for sufficiently small initial data. Lin and Li in [22] studied

$$u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)\,ds = \operatorname{div}(C(f(\nabla u)\nabla u))$$

in $Q_T = \Omega\times(0,T)$, with initial and dynamical boundary conditions similar to those imposed by Rivera et al. [28], and established similar decay results. Yang in [39], considered the problem

$$\begin{gathered}
u_{tt} + \Delta^2 u + \lambda u_t = \sum_{i=1}^n \frac{\partial}{\partial x_i}\sigma_i(\frac{\partial u}{\partial x_i}) \quad \text{in } Q_\tau = \Omega\times(0,T) \\
u = \partial u/\partial v = 0 \quad \text{on } \Gamma_T = \partial\Omega\times[0,T) \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega
\end{gathered}$$

for $\lambda \geq 0$ and σ_i nonlinear functions. He proved, under some conditions on nonlinear terms and initial data, that the problem admits a global weak solution and the solution decays exponentially to zero as $t\to\infty$.

Motivated by [8, 11, 16], we considered the existence of local and global solutions, and their blow up for nonlinear Petrovsky equation with variable exponents and strong damping. To the best of our knowledge, this is the first work dealing with equation (1.1) subject to the variable exponents and strong damping. Our aim in this work is to prove the existence of local and global solutions, and to find sufficient conditions on p, q for which the blow up takes place.

This article consists of five sections in addition to the introduction. In Section 2, we recall the definitions of the $L^{p(\cdot)}(\Omega)$, the Sobolev spaces $W^{1,p(\cdot)}(\Omega)$, as some of their properties. In Section 3, we prove the local existence of weak solutions for Problem (1). In Section 4, we establish a global existence. In Section 5,we state and prove our blow up result for solutions with negative initial energy are given.

2. Preliminaries

In this section, we state some results about the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ (see [12, 13, 18, 30]).

Let $p:\Omega\to[1,\infty]$ be a measurable function, where Ω is a domain of $\mathbb{R}^n$. We define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \{u:\Omega\to R:\ u \text{ is measurable in } \Omega \text{ and } \rho_{p(\cdot)}(\lambda u)<\infty \text{ for some } \lambda>0\},$$

where

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}dx.$$

The space $L^{p(\cdot)}(\Omega)$ equipped with the Luxemburg-type norm

$$\|u\|_{p(\cdot)} = \inf\big\{\lambda>0: \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \leq 1\big\}$$

becomes a Banach space [12]. The relation between the modular $\int_\Omega |f|^{p(x)}\,dx$ and the norm follows from

$$\min(\|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+}) \leq \int_\Omega |f|^{p(x)}\,dx \leq \max(\|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+}).$$

In the case $p(\cdot) = const > 1$, these inequalities transform into equalities. For all $f \in L^{p(\cdot)}(\Omega)$, $g \in L^{p'(\cdot)}(\Omega)$ with

$$p(x) \in (1, \infty), \quad p'(x) = \frac{p(x)}{p(x)-1}$$

the generalized Hölder inequality holds,

$$\int_\Omega |f\, g|\, dx \le \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)} \le 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}\,.$$

The variable exponent Sobolev space is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega)\}$$

with respect to the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ with respect to the norm $\|u\|_{1,p(\cdot)}$. For $u \in W_0^{1,p(\cdot)}(\Omega)$, we can define an equivalent norm

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)}.$$

Let the variable exponent $p(\cdot)$ satisfy the log-Hölder continuity condition

$$|p(x) - p(y)| \le \frac{A}{\log\frac{1}{|x-y|}}, \quad \text{for all } x, y \in \Omega \text{ with } |x-y| < \delta, \tag{2.1}$$

where $A > 0$ and $0 < \delta < 1$.

Lemma 2.1 (Poincare inequality [12]). *Let Ω be a bounded domain of $\mathbb{R}^n$ and $p(\cdot)$ satisfies log-Hölder condition, then*

$$\|u\|_{p(x)} \le c\|\nabla u\|_{p(x)}, \quad \text{for all } u \in W_0^{1,p(x)}(\Omega), \tag{2.2}$$

where $C = C(p^-, p^+, |\Omega|) > 0$.

Lemma 2.2 ([12]). *Let $p(\cdot) \in C(\overline{\Omega})$ and $q : \Omega \to [1, \infty)$ be a measurable function that satisfy*

$$\operatorname{ess\,inf}_{x\in\overline{\Omega}}(p^*(x) - q(x)) > 0.$$

Then the Sobolev embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact. Where

$$p^*(x) = \begin{cases} \frac{np^-}{n-p^-}, & \text{if } p^- < n \\ \text{any number in } [1,\infty), & \text{if } p^- \ge n. \end{cases}$$

If in addition $p(\cdot)$ satisfies log-Hölder condition, then

$$p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, & \text{if } p(x) < n \\ \text{any number in } [1,\infty), & \text{if } p(x) \ge n. \end{cases}$$

Remark 2.3. We denote by c various positive constants which may be different at different occurrences. Also, throughout this paper, we use the embedding

$$H_0^2(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^p(\Omega)$$

which implies

$$\|u\|_p \le C\|\nabla u\| \le C\|\Delta u\|,$$

where $2 \le p < \infty$ $(n = 1, 2)$, $2 \le p \le \frac{2n}{n-2}$ $(n \ge 3)$. Moreover,

$$\|u\|_p \le C\|\Delta u\|,$$

$$p = \begin{cases} \infty & \text{if } n < 4, \\ \text{any number in } [1, \infty), & \text{if } n = 4, \\ \frac{2n}{n-4} & \text{if } n > 4. \end{cases}$$

We will use also the Young inequality

$$ab \le \frac{1}{p}(\epsilon a)^p + \frac{p-1}{p}(\frac{b}{\epsilon}a)^{\frac{p}{p-1}}, \quad a, b \ge 0, \ \epsilon \in (0,1), \ 1 < p < \infty. \tag{2.3}$$

3. Existence of weak solutions

In this part, we prove a local existence result for (1.1). Firstly, we state the following lemma which can be obtained by exploiting the Feado-Galerkin method and using the similar arguments as in [27, 29].

Lemma 3.1. *Suppose that $p(\cdot)$ satisfies* (1.2) *and* (2.1)*, and that initial data satisfies $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$. Then there exists a unique local solution u of*

$$\begin{gathered} u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{p(x)-2} u_t = f(t,x), \quad (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = \partial_v u(x,t) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{3.1}$$

satisfying

$$u \in L^\infty((0,T), H_0^2(\Omega)), \quad u_t \in L^\infty((0,T), L^2(\Omega)) \cap L^{p(\cdot)}(\Omega \times (0,T)),$$

where $f \in L^2(\Omega \times (0,T))$.

Theorem 3.2. *Suppose that $p(\cdot)$ satisfies* (1.2) *and*

$$2 \le p^- \le p(x) \le p^+ \le 2 + \frac{4}{n-4} \quad (n > 4). \tag{3.2}$$

Furthermore assume that $q(\cdot)$ satisfies (1.2) *and*

$$\begin{gathered} 2 \le q^- \le q^+ < \infty \quad \text{if } n \le 4, \text{ and} \\ 2 \le q^- \le q^+ \le 2 + \frac{4}{n-4}, \quad \text{if } n > 4, \end{gathered} \tag{3.3}$$

$u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$. Then (1.1) *has a unique local solution*

$$u \in L^\infty((0,T), H_0^2(\Omega)), \quad u_t \in L^\infty((0,T), L^2(\Omega)) \cap L^{p(\cdot)}(\Omega \times (0,T)).$$

Proof. (Existence) Let $v \in L^\infty((0,T), H_0^1(\Omega))$ and $f(v) = |v|^{q(x)-2}v$. We have

$$\begin{aligned} \|f(v)\|^2 &= \int_{\Omega \cap (|v| \le 1)} |v|^{2(q(x)-1)} dx + \int_{\Omega \cap (1 < |v|)} |v|^{2(q(x)-1)} dx \\ &\le |\Omega| + \int_\Omega |v|^{2(q^+-1)} dx < \infty, \end{aligned}$$

since

$$2(q^- - 1) \le 2(q^+ - 1) \le \frac{2n}{n-2}.$$

Thus, for each $v \in L^\infty((0,T), H_0^1(\Omega))$, there exists a unique

$$u \in L^\infty((0,T), H_0^2(\Omega)), \quad u_t \in L^\infty((0,T), L^2(\Omega)) \cap L^{p(\cdot)}(\Omega \times (0,T)),$$

solving the problem

$$\begin{gathered}
u_{tt}+\Delta^2 u-\Delta u_t+|u_t|^{p(x)-2}u_t=f(v), \quad (x,t)\in\Omega\times(0,T),\\
u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\Omega,\\
u(x,t)=\partial_\upsilon u(x,t)=0, \quad x\in\partial\Omega.
\end{gathered} \tag{3.4}$$

We define the space

$$X_T:=\{w\in L^\infty((0,T);H_0^2(\Omega)): w\in L^\infty((0,T);L^2(\Omega))\}$$

which is a Banach space with respect to the norm

$$\|w\|_{X_T}=\|w\|_{L^\infty((0,T);H_0^2(\Omega))}+\|w\|_{L^\infty((0,T);L^2(\Omega))}.$$

We define the nonlinear map S as follows. For $v\in X_T$, $Sv=u$ is the unique solution (3.4).

We shall show that there exist $T>0$, such that

(i) $S:X_T\to X_T$

(ii) S is a contraction mapping in X_T.

To show (i), multiplying (3.4) by u_t and integrating over $\Omega\times(0,t)$, we obtain

$$\begin{aligned}
&\frac12\|u_t\|^2+\frac12\|\Delta u\|^2+\int_0^t\|\nabla u_\tau\|^2d\tau+\int_0^t\int_\Omega|u_\tau|^{p(x)}\,dx\,d\tau\\
&=\frac12\|u_1\|^2+\frac12\|\Delta u_0\|^2+\int_0^t\int_\Omega|v|^{q(x)-2}vu_\tau\,dx\,d\tau.
\end{aligned} \tag{3.5}$$

By the Young's, Sobolev-Poincare's inequalities and (3.3), we obtain

$$\begin{aligned}
\int_\Omega|v|^{q(x)-2}vu_\tau dx&\le\frac{\delta}{4}\int_\Omega u_t^2dx+\frac1\delta\int_\Omega|v|^{2q(x)-2}dx\\
&\le\frac{\delta}{4}\|u_t\|^2+\frac1\delta\Big[\int_\Omega|v|^{2(q^--1)}dx+\int_\Omega|v|^{2(q^+-1)}dx\Big]\\
&\le\frac{\delta}{4}\|u_t\|^2+\frac{C}{\delta}\big(\|\Delta v\|^{2(q^--1)}+\|\Delta v\|^{2(q^+-1)}\big).
\end{aligned} \tag{3.6}$$

Thus, by (3.5) and (3.6), we have

$$\begin{aligned}
&\frac12\|u_t\|^2+\frac12\|\Delta u\|^2+\int_0^t\|\nabla u_\tau\|^2d\tau+\int_0^t\int_\Omega|u_\tau|^{p(x)}\,dx\,d\tau\\
&\le\frac12\|u_1\|^2+\frac12\|\Delta u_0\|^2+\frac{\delta}{4}\int_0^t\|u_t\|^2d\tau\\
&\quad+\frac{C}{\delta}\int_0^t\big(\|\Delta v\|^{2(q^--1)}+\|\Delta v\|^{2(q^+-1)}\big)d\tau,
\end{aligned}$$

which implies

$$\begin{aligned}
&\sup_{t\in(0,T)}[\|u_t\|^2+\|\Delta u\|^2]\\
&\le\|u_1\|^2+\|\Delta u_0\|^2+\frac{\delta T}{2}\sup_{t\in(0,T)}\|u_t\|^2+\frac{CT}{\delta}[\|v\|_{X_T}^{2(q^--1)}+\|v\|_{X_T}^{2(q^+-1)}].
\end{aligned}$$

By taking $\delta T/2\le 1$, we have

$$\|u\|_{X_T}^2\le\lambda+\frac{CT}{\delta}\big[\|v\|_{X_T}^{2(q^--1)}+\|v\|_{X_T}^{2(q^+-1)}\ \big],$$

where $\lambda = \|u_1\|^2 + \|\Delta u_0\|^2$. At this point we choose M large enough, such that $\|v\|_{X_T} \le M$. Then

$$\|u\|^2_{X_T} \le \lambda + \frac{CT}{\delta} M^{2(q^+-1)} \le M^2$$

if $\lambda < M^2$ and $T \le T_0 < \frac{\delta(M^2-\lambda)}{CM^{2(q^+-1)}}$. Thus we have $S : X_T \to X_T$.

Next, we show S is a contraction mapping in X_T. For this purpose, we let $u_1 = Sv_1$ and $u_2 = Sv_2$, then $u = u_1 - u_2$ satisfies

$$\begin{gathered} u_{tt} + \Delta^2 u - \Delta u_t + [|u_{1t}|^{p(x)-2}u_{1t} - |u_{2t}|^{p(x)-2}u_{2t}] \\ = |v_1|^{q(x)-2}v_1 - |v_2|^{q(x)-2}v_2, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = \partial_v u(x,t) = 0, \quad x \in \partial\Omega. \end{gathered} \tag{3.7}$$

Multiplying by $u_t = u_{1t} - u_{2t}$ and integrating over $\Omega \times (0,t)$, we obtain

$$\begin{aligned} &\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \int_0^t \|\nabla u_\tau\|^2 d\tau \\ &+ \int_0^t \int_\Omega \big[|u_{1t}|^{p(x)-2}u_{1t} - |u_{2t}|^{p(x)-2}u_{2t}\big](u_{1t} - u_{2t})\,dx\,d\tau \\ &\le \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\Delta u_0\|^2 + \int_0^t \int_\Omega (f(v_1) - f(v_2))u_t\,dx\,d\tau. \end{aligned} \tag{3.8}$$

Since

$$[|u_{1t}|^{p(x)-2}u_{1t} - |u_{2t}|^{p(x)-2}u_{2t}](u_{1t} - u_{2t}) \ge 0,$$

inequality (3.8) yields

$$\begin{aligned} &\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \int_0^t \|\nabla u_\tau\|^2 d\tau \\ &\le \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\Delta u_0\|^2 + \int_0^t \int_\Omega (f(v_1) - f(v_2))u_t\,dx\,d\tau. \end{aligned} \tag{3.9}$$

We estimate the right-most term of as follows:

$$\int_\Omega |f(v_1) - f(v_2)|\,|u_t|dx = \int_\Omega |f'(\xi)|\,|v|\,|u_t|dx,$$

where $v = v_1 - v_2$ and $\xi = \alpha v_1 + (1-\alpha)v_2$, $0 \le \alpha \le 1$. Thanks to Young's inequality, (2.3), and since $f(v) = |v|^{q(x)-2}v$, we obtain

$$\begin{aligned} &\int_\Omega |f(v_1) - f(v_2)|\,|u_t|dx \\ &\le \frac{\delta}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2\delta}\int_\Omega |f'(\xi)|^2|v|^2 dx \\ &\le \frac{\delta}{2}\|u_t\|^2 + \frac{(q^+-1)^2}{2\delta}\int_\Omega |\alpha v_1 + (1-\alpha)v_2|^{2(q(x)-2)}|v|^2 dx \\ &\le \frac{\delta}{2}\|u_t\|^2 + c\Big(\int_\Omega |v|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{2}}\Big(\int_\Omega |\alpha v_1 + (1-\alpha)v_2|^{n(q(x)-2)}dx\Big)^{2/n} \\ &\le \frac{\delta}{2}\|u_t\|^2 + c\Big(\int_\Omega |v|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{2}}\Big[\Big(\int_\Omega |\alpha v_1 + (1-\alpha)v_2|^{n(q^+-2)}dx\Big)^{2/n} \end{aligned}$$

$$+\Big(\int_\Omega |\alpha v_1+(1-\alpha)v_2|^{n(q^- -2)}dx\Big)^{2/n}\Big]. \tag{3.10}$$

Thus by (3.2) and (3.3), we obtain

$$\begin{aligned}\int_\Omega |f(v_1)-f(v_2)|,|u_t|dx &\le \frac{\delta}{2}\|u_t\|^2 + C\|\Delta v\|^2\big[\|\Delta v_1\|^{2(q^+-2)}+\|\Delta v_1\|^{2(q^--2)}\\ &\quad + \|\Delta v_2\|^{2(q^+-2)}+\|\Delta v_2\|^{2(q^--2)}\big]\\ &\le \frac{\delta}{2}\|u_t\|^2 + 4CM^{2(q^+-2)}\|\Delta v\|^2\,.\end{aligned}$$

By the combining this inequality with (3.9), we obtain

$$\frac12\|u\|_{X_T}^2 \le \frac{\delta}{2}T_0\|u\|_{X_T}^2 + 4CM^{2(q^+-2)}T_0\|v\|_{X_T}^2.$$

By choosing δ small enough, we have

$$\|u\|_{X_T}^2 \le 8CM^{2(q^+-2)}T_0\|v\|_{X_T}^2.$$

Now, we choose T_0 sufficiently enough so that

$$0 < 8CM^{2(q^+-2)}T_0 < 1.$$

Thus, the map S is contraction. The Banach fixed point theorem implies the existence of a unique $u\in X_T$ satisfying $S(u)=u$. Obviously, it is a solution of (1.1).

(Uniqueness) Suppose that (1.1) have two solutions u and v. Then $w=u-v$ satisfies

$$\begin{gathered} w_{tt}+\Delta^2 w-\Delta w_t+|u_t|^{p(x)-2}u_t-|v_t|^{p(x)-2}v_t\\ =|u|^{q(x)-2}u-|v|^{q(x)-2}v,\quad (x,t)\in\Omega\times(0,T),\\ w(x,0)=0,\quad w_t(x,0)=0,\quad x\in\Omega,\\ w(x,t)=\partial_v w(x,t)=0,\quad x\in\partial\Omega. \end{gathered}$$

Multiplying by w_t and integrate over $\Omega\times(0,t)$, we obtain

$$\begin{aligned} &\frac12\|w_t\|^2+\frac12\|\Delta w\|^2+\int_0^t\|\nabla w_\tau\|^2d\tau\\ &+\int_0^t\int_\Omega\big(|u_t|^{p(x)-2}u_t-|v_t|^{p(x)-2}v_t\big)w_t\,dx\,d\tau\\ &=\int_0^t\int_\Omega\big(|u|^{q(x)-2}u-|v|^{q(x)-2}v\big)w_t\,dx\,d\tau. \end{aligned}$$

By using the inequality

$$(|a|^{p-2}-|b|^{p-2}b)(a-b)\ge 0,$$

for all $a,b\in\mathbb{R}^n$, $1<p<\infty$ and similarly (3.10), we have

$$\|w_t\|^2+\|\Delta w\|^2 \le C\int_0^t\int_\Omega(|w_t(\tau)|^2+|\Delta w(\tau)|^2)\,dx\,d\tau.$$

By Gronwall's inequality, we obtain

$$\|w_t\|^2+\|\Delta w\|^2=0.$$

Thus $w=0$. The proof is complete. □

4. Existence of global solutions

In this section, we obtain a global solution for (1.1) under suitable conditions on $p(\cdot)$ and $q(\cdot)$. In the presence of additional estimates, the proven local solution can be continued to an finite time interval.

Theorem 4.1. *Let the assumptions of Theorem 3.2 hold. If $u_0 \in H_0^2(\Omega)$, $u \in L^2(\Omega)$ and the exponents $p(\cdot)$ and $q(\cdot)$ satisfy one of the following two conditions*

$$1 < q(x) \leq 2, \quad 1 < p(x) < \infty \quad \textit{or}$$
$$2 \leq q(x) \leq p(x) < \infty.$$

Then problem (1.1) *has a global solution, with*

$$u \in L^\infty((0,T), H_0^2(\Omega))), \quad u_t \in L^\infty((0,T)L^2(\Omega)) \cap L^{p(\cdot)}(\Omega \times (0,T)).$$

Proof. To achieve the global existence of a solution, it suffices to show that

$$\sup_{t\in[0,T]} (\|u_t\|^2 + \|\Delta u\|^2) + \int_0^T \|\nabla u_\tau\|^2 d\tau + \int_0^T \int_\Omega |u_\tau|^{p(x)}\,dx\,d\tau \leq C$$

for any finite $T < \infty$. Multiplying (1.1) by u_t and integrating over $\Omega \times (0,t)$, we obtain

$$\begin{aligned}&\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \int_0^t \|\nabla u_\tau\|^2 d\tau + \int_0^t \int_\Omega |u_\tau|^{p(x)}\,dx\,d\tau \\ &= \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\Delta u_0\|^2 + \int_0^t \int_\Omega |u|^{q(x)-2} u u_\tau\,dx\,d\tau\,.\end{aligned} \tag{4.1}$$

First let us consider the case

$$q(x) \leq 2 \Leftrightarrow 2(q(x)-1) \leq 2, \quad 1 < p(x) < \infty. \tag{4.2}$$

We evaluate the term

$$\begin{aligned}|I| &= \Big|\int_0^t \int_\Omega |u|^{q(x)-2} u u_\tau\,dx\,d\tau\Big| \\ &\leq \frac{1}{2}\int_0^t \Big(\|u_t\|^2 + \int_\Omega |u|^{2(q(x)-1)}\,dx\,d\tau\Big) \\ &\leq \frac{1}{2}\int_0^t \big(\|u_t\|^2 + \|u\|^2\big)d\tau + \frac{1}{2}T|\Omega| \\ &\leq \frac{c}{2}\int_0^t (\|u_t\|^2 + \|\Delta u\|^2)d\tau + \frac{1}{2}T|\Omega|\end{aligned} \tag{4.3}$$

$2(q(x)-1) \leq 2 \Leftrightarrow q(x) \leq 2$. Introducing the function

$$Y(t) = \|u_t\|^2 + \|\Delta u\|^2$$

we arrive at integral inequality

$$Y(t) \leq C\int_0^t Y(\tau)d\tau + B, \ B = \|u_1\|^2 + \|\Delta u_0\|^2 + T|\Omega|.$$

Applying the Granwall inequality we derive the estimate

$$\sup_{t\in[0,T]} \big(\|u_t\|^2 + \|\Delta u\|^2\big) + \int_0^T \|\nabla u_\tau\|^2 d\tau + \int_0^T \int_\Omega |u_\tau|^{p(x)}\,dx\,d\tau \leq C \tag{4.4}$$

which holds for any finite $T < \infty$.

Now we consider the case

$$2 \le q(x) \le p(x) < \infty\,. \tag{4.5}$$

Applying the Young inequality (2.3) and (3.5) we derive

$$\begin{aligned}
&\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \int_0^t \|\nabla u_\tau\|^2 d\tau + \int_0^t \int_\Omega |u_\tau|^{p(x)}\,dx\,d\tau \\
&\le \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\Delta u_0\|^2 + \int_0^t \int_\Omega \frac{\varepsilon^{p^-}}{p^-}|u_\tau|^{p(x)}\,dx\,d\tau + I.
\end{aligned}$$

We evaluate the term I as follows,

$$\begin{aligned}
I &= \int_0^t \int_\Omega \frac{p-1}{p}\varepsilon^{\frac{p}{p-1}}|u|^{\frac{(q-1)p}{p-1}}\,dx\,d\tau \\
&\le C(\varepsilon, p^\pm)\int_0^t \int_\Omega |u|^{\frac{(q-1)p}{p-1}}\,dx\,d\tau \le \\
&\le C(\varepsilon, p^\pm, T, |\Omega|)\Big(\int_0^t \int_\Omega |u|^{p(x)}\,dx\,d\tau + 1\Big), \frac{(q-1)p}{p-1} \\
&\le p \\
&\Leftrightarrow q \le p.
\end{aligned}$$

Choosing $\varepsilon^p/p \le \varepsilon^{p^-}/p^- \le 1/2$ we obtain

$$\begin{aligned}
&\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \int_0^t \|\nabla u_\tau\|^2 d\tau + \frac{1}{2}\int_0^t \int_\Omega |u_\tau|^{p(x)}\,dx\,d\tau \\
&\le \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\Delta u_0\|^2 + \int_0^t \int_\Omega \frac{p-1}{p}\varepsilon^{\frac{p}{p-1}}|u|^{\frac{(q-1)p}{p-1}}\,dx\,d\tau \\
&\le \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\Delta u_0\|^2 + C(\varepsilon, p^\pm)\Big(\int_0^t \int_\Omega |u|^{p(x)}\,dx\,d\tau + 1\Big), \\
&\qquad \frac{(q-1)p}{p-1} \le p \Leftrightarrow q \le p.
\end{aligned}$$

Next we use the inequality

$$\begin{aligned}
\int_\Omega |u|^{p(x)}dx &= \int_\Omega \Big|\int_0^t u_t ds + u_0\Big|^{p(x)} dx \\
&\le C(p^\pm)\int_\Omega \Big(t^{p-1}\int_0^t |u_t|^{p(x)}ds + |u_0|^{p(x)p(x)}\Big) \\
&\le C(p^\pm)\Big(\int_0^t \int_\Omega t^{p(x)-1}|u_t|^{p(x)}\,dx\,ds + \int_\Omega |u_0|^{p(x)}dx\Big) \\
&\le C(p^\pm, T)\Big(\int_0^t \int_\Omega |u_t|^{p(x)}\,dx\,ds + \int_\Omega |u_0|^{p(x)}dx\Big) \\
&\le C(p^\pm, T)\Big(\int_0^t \int_\Omega |u_t|^{p(x)}\,dx\,ds + \int_\Omega |u_0|^{p(x)}dx\Big).
\end{aligned} \tag{4.6}$$

We introduce the function

$$Y(t) = \int_0^t \int_\Omega |u_t(x,s)|^{p(x)}\,dx\,ds.$$

Then (4.6), (4.6) lead us to the integral inequality

$$Y(t) \leq C\Big(\int_0^t Y(s)ds + \int_\Omega |u_0|^{p(x)}dx + \|u_1\|^2 + \|\Delta u_0\|^2 + 1\Big). \tag{4.7}$$

Applying the Granwall inequality we arrive at estimate (4.4). This estimate permits us to continue local solution for any finite interval of time. This completes the proof. $\square$

5. Blow up of solutions

In this part, we consider the blow up of the solution for problem (1.1). Firstly, we give following lemma.

Lemma 5.1 ([27])**.** *If $q : \Omega \to [1,\infty)$ is a measurable function and*

$$\begin{gathered} 2 \leq q^- \leq q(x) \leq q^+ < \infty \quad \text{for } n \leq 4, \\ 2 \leq q^- \leq q(x) \leq q^+ < \frac{2n}{n-2} \quad \text{for } n > 4 \end{gathered} \tag{5.1}$$

holds. Then, we have following inequalities:

$$\rho_{q(\cdot)}^{\frac{s}{q^-}}(u) \leq C\big(\|\Delta u\|^2 + \rho_{q(\cdot)}(u)\big), \tag{5.2}$$

$$\|u\|_{q^-}^s \leq C\big(\|\Delta u\|^2 + \|u\|_{q^-}^{q^-}\big), \tag{5.3}$$

$$\rho_{q(\cdot)}^{\frac{s}{q^-}}(u) \leq C\big(|H(t)| + \|u_t\|^2 + \rho_{q(\cdot)}(u)\big), \tag{5.4}$$

$$\|u\|_{q^-}^s \leq C(|H(t)| + \|u_t\|^2 + \|u\|_{q^-}^{q^-}), \tag{5.5}$$

$$C\|u\|_{q^-}^{q^-} \leq \rho_{q(\cdot)}(u) = \int_\Omega |u|^{q(\cdot)}dx \tag{5.6}$$

for any $u \in H_0^2(\Omega)$ and $2 \leq s \leq q^-$. Where $C > 1$ a positive constant and $H(t) = -E(t)$.

The functions $H(t), E(t)$ will be defined later. Now, we state and prove our blow up result.

Theorem 5.2. *Let the assumptions of Theorem 3.2, and Lemma 5.1 hold. Also let initial energy satisfy $E(0) < 0$, and the exponents $p(\cdot)$ and $q(\cdot)$ satisfy*

$$2 \leq p^- \leq p(x) \leq p^+ < q^- \leq q(x) \leq q^+ \leq 2 + \frac{4}{n-4}, \text{ if } n > 4\,.$$

Then the solution of (1.1) blows up in a finite time T^*, in the following sense

$$\Psi(t) \to \infty \quad \text{as } t \to T^* \leq \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}, \tag{5.7}$$

where $\xi \in (0,1)$, and $\Psi(t)$ and σ are given in (5.11) and (5.12) respectively.

Proof. Multiplying both sides of the equation in (1.1) by u_t, and integrating by parts, we have

$$\begin{gathered} \frac{d}{dt}\Big[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx\Big] = -\int_\Omega |u_t|^{p(x)}dx - \|\nabla u_t\|^2, \\ E'(t) = -\int_\Omega |u_t|^{p(x)}dx - \|\nabla u_t\|^2 \end{gathered} \tag{5.8}$$

where
$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\Delta u\|^2-\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx. \tag{5.9}$$
Set
$$H(t)=-E(t)$$
then $E(0)<0$ and (5.8) gives $H(t)\geq H(0)>0$. Also, by the definition $H(t)$, we have
$$\begin{aligned} H(t)&=-\frac{1}{2}\|u_t\|^2-\frac{1}{2}\|\Delta u\|^2+\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx\\ &\leq \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx\\ &\leq \frac{1}{q^-}\rho_{q(\cdot)}(u). \end{aligned} \tag{5.10}$$
We define
$$\Psi(t)=H^{1-\sigma}(t)+\varepsilon\int_\Omega uu_t\,dx+\frac{\varepsilon}{2}\|\nabla u\|^2, \tag{5.11}$$
where ε small to be chosen later and
$$0<\sigma\leq \min\big\{\frac{q^- - p^+}{(p^+-1)q^-},\frac{q^- -2}{2q^-}\big\}. \tag{5.12}$$
Differentiating $\Psi(t)$ with respect to t, and using (1.1), we have
$$\begin{aligned} \Psi'(t)&=(1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon\int_\Omega(u_t^2+uu_{tt})dx+\varepsilon\int_\Omega \nabla u\nabla u_t\,dx\\ &=(1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon\|u_t\|^2-\varepsilon\|\Delta u\|^2\\ &\quad+\varepsilon\int_\Omega |u|^{q(\cdot)}dx-\varepsilon\int_\Omega uu_t|u_t|^{p(\cdot)-2}dx. \end{aligned} \tag{5.13}$$
By using the definition of the $H(t)$, it follows that
$$\begin{aligned} -\varepsilon q^-(1-\xi)H(t)&=\frac{\varepsilon q^-(1-\xi)}{2}\|u_t\|^2+\frac{\varepsilon q^-(1-\xi)}{2}\|\Delta u\|^2\\ &\quad-\varepsilon q^-(1-\xi)\int_\Omega \frac{1}{q(x)}|u|^{q(\cdot)}dx, \end{aligned} \tag{5.14}$$
where $0<\xi<1$. Adding and subtracting (5.14) into (5.13), we obtain
$$\begin{aligned} \Psi'(t)&\geq(1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon q^-(1-\xi)H(t)\\ &\quad+\varepsilon\Big(\frac{q^-(1-\xi)}{2}+1\Big)\|u_t\|^2+\varepsilon\Big(\frac{q^-(1-\xi)}{2}-1\Big)\|\Delta u\|^2\\ &\quad+\varepsilon\xi\int_\Omega |u|^{q(\cdot)}dx-\varepsilon\int_\Omega uu_t|u_t|^{p(\cdot)-2}dx. \end{aligned} \tag{5.15}$$
Then, for ξ small enough, we obtain
$$\begin{aligned} \Psi'(t)&\geq\varepsilon\beta[H(t)+\|u_t\|^2+\|\Delta u\|^2+\rho_{q(\cdot)}(u)]\\ &\quad+(1-\sigma)H^{-\sigma}(t)H'(t)-\varepsilon\int_\Omega uu_t|u_t|^{p(\cdot)-2}dx \end{aligned} \tag{5.16}$$
where
$$\beta=\min\big\{q^-(1-\xi),\ \xi,\ \frac{q^-(1-\xi)}{2}-1,\ \frac{q^-(1-\xi)}{2}+1\big\}>0$$

and

$$\rho_{q(\cdot)}(u) = \int_\Omega |u|^{q(\cdot)} dx.$$

To estimate the last term in (5.16), we use the Young inequality (2.3). Consequently, applying the previous we have

$$\begin{aligned} \int_\Omega u|u_t|^{p(\cdot)-1} dx &\le \int_\Omega \frac{1}{p(x)} \delta^{p(x)} |u|^{p(x)} dx + \int_\Omega \frac{p(x)-1}{p(x)} \delta^{-\frac{p(x)}{p(x)-1}} |u_t|^{p(x)} dx \\ &\le \frac{1}{p^-} \int_\Omega \delta^{p(x)} |u|^{p(x)} dx + \frac{p^+-1}{p^+} \int_\Omega \delta^{-\frac{p(x)}{p(x)-1}} |u_t|^{p(x)} dx, \end{aligned} \tag{5.17}$$

where δ is constant depending on the time t and specified later. Inserting estimate (5.17) into (5.16), we obtain

$$\begin{aligned} \Psi'(t) &\ge \varepsilon\beta[H(t) + \|u_t\|^2 + \|\Delta u\|^2 + \rho_{q(\cdot)}(u)] \\ &\quad + (1-\sigma) H^{-\sigma}(t) H'(t) - \varepsilon \frac{1}{p^-} \int_\Omega \delta^{p(x)} |u|^{p(x)} dx \\ &\quad - \varepsilon \frac{p^+-1}{p^+} \int_\Omega \delta^{-\frac{p(x)}{p(x)-1}} |u_t|^{p(x)} dx. \end{aligned} \tag{5.18}$$

Let us choose δ so that

$$\delta^{-\frac{p(x)}{p(x)-1}} = k_1 H^{-\sigma}(t),$$

where $k_1, k_2 > 0$ are specified later, we obtain

$$\begin{aligned} &\Psi'(t) \\ &\ge \varepsilon\beta[H(t) + \|u_t\|^2 + \|\Delta u\|^2 \ + \rho_{q(\cdot)}(u)] \\ &\quad + (1-\sigma) H^{-\sigma}(t) H'(t) - \varepsilon k_2 H^{-\sigma}(t) H'(t) \\ &\quad - \varepsilon \frac{1}{p^-} \int_\Omega k_1^{1-p(x)} H^{\sigma(p(x)-1)}(t) |u|^{p(x)} dx - \varepsilon \frac{p^+-1}{p^+} \int_\Omega k_1 H^{-\sigma}(t) |u_t|^{p(x)} dx \\ &\ge \varepsilon\beta[H(t) + \|u_t\|^2 + \|\Delta u\|^2 \ + \rho_{q(\cdot)}(u)] \\ &\quad + (1-\sigma-\varepsilon k_2) H^{-\sigma}(t) H'(t) \\ &\quad - \varepsilon \frac{k_1^{1-p^-}}{p^-} H^{\sigma(p^+-1)}(t) \int_\Omega |u|^{p(x)} dx - \varepsilon \big(\frac{p^+-1}{p^+}\big) k_1 H^{-\sigma}(t) \int_\Omega |u_t|^{p(x)} dx \\ &\ge \varepsilon\beta[H(t) + \|u_t\|^2 + \|\Delta u\|^2 + \rho_{q(\cdot)}(u)] \\ &\quad + \big[(1-\sigma-\varepsilon k_2) - \varepsilon \big(\frac{p^+-1}{p^+}\big) k_1\big] H^{-\sigma}(t) H'(t) \\ &\quad - \varepsilon \frac{k_1^{1-p^-}}{p^-} H^{\sigma(p^+-1)}(t) \int_\Omega |u|^{p(x)} dx. \end{aligned} \tag{5.19}$$

By using (5.6) and (5.10), we obtain

$$
\begin{aligned}
&H^{\sigma(p^+-1)}(t)\int_\Omega |u|^{p(x)}dx\\
&\le H^{\sigma(p^+-1)}(t)\Big[\int_{\Omega_-}|u|^{p^-}dx+\int_{\Omega_+}|u|^{p^+}dx\Big]\\
&\le H^{\sigma(p^+-1)}(t)C\Big[\Big(\int_{\Omega_-}|u|^{q^-}dx\Big)^{\frac{p^-}{q^-}}+\Big(\int_{\Omega_+}|u|^{q^-}dx\Big)^{\frac{p^+}{q^-}}\Big]\\
&= H^{\sigma(p^+-1)}(t)C\big[\|u\|_{q^-}^{p^-}+\|u\|_{q^-}^{p^+}\big]\\
&\le C\Big(\frac{1}{q^-}\rho_{q(\cdot)}(u)\Big)^{\sigma(p^+-1)}\big[(\rho_{q(\cdot)}(u))^{\frac{p^-}{q^-}}+(\rho_{q(\cdot)}(u))^{\frac{p^+}{q^-}}\big]\\
&= C_1\big[(\rho_{q(\cdot)}(u))^{\frac{p^-}{q^-}+\sigma(p^+-1)}+(\rho_{q(\cdot)}(u))^{\frac{p^+}{q^-}+\sigma(p^+-1)}\big]
\end{aligned}
\tag{5.20}
$$

where $\Omega_-=\{x\in\Omega:|u|<1\}$ and $\Omega_+=\{x\in\Omega:|u|\ge 1\}$.

We then use Lemma 5.1 and (5.12), for

$$s=p^-+\sigma q^-(p^+-1)\le q^-$$

and for

$$s=p^++\sigma q^-(p^+-1)\le q^-,$$

to deduce, from (5.20), that

$$H^{\sigma(p^+-1)}(t)\int_\Omega |u|^{p(x)}dx\le C_1\big[\|\Delta u\|^2+\rho_{q(\cdot)}(u)\big]. \tag{5.21}$$

Thus, inserting estimate (5.21) into (5.19), we have

$$
\begin{aligned}
\Psi'(t)\ge{}&\varepsilon\Big(\beta-\frac{k_1^{1-p^-}}{p^-}C_1\Big)[H(t)+\|u_t\|^2+\|\Delta u\|^2+\rho_{q(\cdot)}(u)]\\
&+\big[(1-\sigma-\varepsilon k_2)-\varepsilon\big(\frac{p^+-1}{p^+}\big)k_1\big]H^{-\sigma}(t)H'(t).
\end{aligned}
\tag{5.22}
$$

Let us choose k_1 large enough so that

$$\gamma=\beta-\frac{k_1^{1-p^-}}{p^-}C_1>0,$$

and picking ε small enough such that

$$(1-\sigma-\varepsilon k_2)-\varepsilon\big(\frac{p^+-1}{p^+}\big)k_1>0$$

and

$$\Psi(t)\ge\Psi(0)=H^{1-\sigma}(0)+\varepsilon\int_\Omega u_0u_1dx+\frac{\varepsilon}{2}\|\nabla u_0\|^2>0,\quad \forall t\ge 0. \tag{5.23}$$

Consequently, (5.22) yields

$$
\begin{aligned}
\Psi'(t)&\ge\varepsilon\gamma\big[H(t)+\|u_t\|^2+\|\Delta u\|^2\ +\rho_{q(\cdot)}(u)\big]\\
&\ge\varepsilon\gamma\big[H(t)+\|u_t\|^2+\|\Delta u\|^2\ +\|u\|_{q^-}^{q^-}\big],
\end{aligned}
\tag{5.24}
$$

because of (5.6). Therefore

$$\Psi(t)\ge\Psi(0)>0,\ \text{for all } t\ge 0.$$

On the other hand, applying Hölder inequality, we obtain

$$\Big|\int_\Omega uu_t\,dx\Big|^{\frac{1}{1-\sigma}} \le \|u\|^{\frac{1}{1-\sigma}}\|u_t\|^{\frac{1}{1-\sigma}} \le C\Big(\|u\|_{q^-}^{\frac{1}{1-\sigma}}\|u_t\|^{\frac{1}{1-\sigma}}\Big).$$

Young inequality gives

$$\Big|\int_\Omega uu_t\,dx\Big|^{\frac{1}{1-\sigma}} \le C\Big(\|u\|_{q^-}^{\frac{\mu}{1-\sigma}} + \|u_t\|^{\frac{\theta}{1-\sigma}}\Big), \tag{5.25}$$

for $\frac{1}{\mu}+\frac{1}{\theta}=1$. We take $\theta=2(1-\sigma)$, to obtain $\frac{\mu}{1-\sigma}=\frac{2}{1-2\sigma}\le q^-$ by (5.12). Therefore, (5.25) becomes

$$\Big|\int_\Omega uu_t\,dx\Big|^{\frac{1}{1-\sigma}} \le C\Big(\|u_t\|^2 + \|u\|_{q^-}^{s}\Big),$$

where $\frac{2}{1-2\sigma}\le q^-$. By using (5.5), we obtain

$$\Big|\int_\Omega uu_t\,dx\Big|^{\frac{1}{1-\sigma}} \le C(\|u_t\|^2 + \|u\|_{q^-}^{q^-} + H(t)).$$

Thus, using the inequality

$$(a_1+a_2+\dots+a_m)^\lambda \le 2^{(m-1)/(\lambda-1)}(a_1^\lambda+a_2^\lambda+\dots+a_m^\lambda),$$

(for $a_1,a_2,\dots,a_m\ge 0$, $\lambda\ge 1$), we have

$$\begin{aligned}
\Psi^{\frac{1}{1-\sigma}}(t) &= \Big[H^{1-\sigma}(t)+\varepsilon\int_\Omega uu_t\,dx+\frac{\varepsilon}{2}\|\nabla u\|^2\Big]^{\frac{1}{1-\sigma}}\\
&\le 2^{\frac{\sigma}{1-\sigma}}\Big(H(t)+\varepsilon^{\frac{1}{1-\sigma}}\Big|\int_\Omega uu_t\,dx\Big|^{\frac{1}{1-\sigma}}\Big)\\
&\le C\Big(\|u_t\|^2+\|u\|_{q^-}^{q^-}+H(t)\Big)\\
&\le C\Big(H(t)+\|u_t\|^2+\|\Delta u\|^2\ +\|u\|_{q^-}^{q^-}\Big).
\end{aligned} \tag{5.26}$$

By combining of (5.24) and (5.26), we arrive at

$$\Psi'(t) \ge \xi\Psi^{\frac{1}{1-\sigma}}(t), \tag{5.27}$$

where ξ is a positive constant.

A simple integration of (5.27) over $(0,t)$ yields

$$\Psi^{\frac{\sigma}{1-\sigma}}(t) \ge \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0)-\frac{\xi\sigma t}{1-\sigma}},$$

which implies that the solution blows up in a finite time T^*, with

$$T^* \le \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

This completes the proof. □

Remark 5.3. Estimate (5.7) shows that the larger $\psi(0)$ is, the quicker the blow up takes place.

Conclusion. In this work, we obtained the local and global solutions and blow up in finite time for a nonlinear plate(or beam) Petrovsky equations with strong damping and source terms with variable exponents in a bounded domain. This improves and extends many results in the literature.

Acknowledgements. S. Antontsev was partially supported by the RSF grant no. 19-11-00069, and by the Foundation for Science and Technology of Portugal, under the project UID/MAT/04561/2019 (35% of the results in this article, statement of the problem, preliminaries, global solutions).

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