Expenditure Minimisation Problem

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The expenditure minimisation problem (EMP) looks at the reverse side of the utility maximisation problem (UMP). The UMP considers an agent who wishes to attain the maximum utility from a limited income. The EMP considers an agent who wishes to find the cheapest way to attain a target utility. This approach complements the UMP and has several rewards:

- It enables us to analyse the effect of a price change, holding the utility of the agent constant.
- It enables us to decompose the effect of a price change on an agent's Marshallian demand into a substitution effect and an income effect. This decomposition is called the Slutsky equation.
- It enables us to calculate how much we need to compensate a consumer in response to a price change if we wish to keep her utility constant.

1 Model

We make several assumptions:

- 1. There are N goods. For much of the analysis we assume N=2 but nothing depends on this.
- 2. The agent takes prices as exogenous. We normally assume prices are linear and denote them by $\{p_1, \ldots, p_N\}$.

3. Preferences satisfy completeness, transitivity and continuity. As a result, a utility function exists. We normally assume preferences also satisfy monotonicity (so indifference curves are well behaved) and convexity (so the optima can be characterised by tangency conditions).

The expenditure minimisation problem is

$$\min_{x_1,\dots,x_N} \sum_{i=1}^N p_i x_i \quad \text{subject to} \quad u(x_1,\dots,x_N) \ge \overline{u}$$

$$x_i \ge 0 \quad \text{for all } i$$
(1.1)

The idea is that the agent is trying to find the cheapest way to attain her target utility, \overline{u} . The solution to this problem is called the **Hicksian demand** or compensated demand. It is denoted by

$$h_i(p_1,\ldots,p_N,\overline{u})$$

The money the agent must spend in order to attain her target utility is called her expenditure. The **expenditure function** is therefore given by

$$e(p_1, \dots, p_N, \overline{u}) = \min_{x_1, \dots, x_N} \sum_{i=1}^N p_i x_i$$
 subject to $u(x_1, \dots, x_N) \ge \overline{u}$
 $x_i \ge 0$ for all i

Equivalently, the expenditure function equals the amount the agent spends on her optimal bundle,

$$e(p_1,\ldots,p_N,\overline{u}) = \sum_{i=1}^N p_i h_i(p_1,\ldots,p_N,\overline{u})$$

1.1 Example

Suppose there are two goods, x_1 and x_2 . Table 1 shows how the agent's utility (the numbers in the boxes) varies with the number of x_1 and x_2 consumed.

To keep things simple, suppose the agent faces prices $p_1 = 1$ and $p_2 = 1$ and wishes to attain utility $\overline{u} = 12$. The agent can attain this utility by consuming $(x_1, x_2) = (6, 2)$, $(x_1, x_2) = (4, 3)$, $(x_1, x_2) = (3, 4)$ or $(x_1, x_2) = (2, 6)$. Of these, the cheapest is either $(x_1, x_2) = (4, 3)$ or $(x_1, x_2) = (3, 4)$. In either case, her expenditure is 4 + 3 = 7.

$x_1 \backslash x_2$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

Table 1: Utilities from different bundles.

Now suppose the agent faces prices $p_1 = 1$ and $p_2 = 3$ and still wishes to attain utility $\overline{u} = 12$. The combinations of (x_1, x_2) that attain this utility remain unchanged, however the price of these bundles is different. Now the cheapest is $(x_1, x_2) = (6, 2)$, and the agent's expenditure is $6 + 2 \times 3 = 12$.

While this "table approach" can be used to illustrate the basic idea, one can see that it quickly becomes hard to solve even simple problems. Fortunately, calculus comes to our rescue.

2 Solving the Expenditure Minimisation Problem

2.1 Graphical Solution

We can solve the problem graphically, as with the UMP. The components are also similar to that problem.

First, we need to understand the constraint set. The agent can choose any bundle where (a) the agent attains her target utility, $u(x_1, x_2) \ge \overline{u}$; and (b) the quantities are positive, $x_1 \ge 0$ and $x_2 \ge 0$. If preferences are monotone, then the bundles that meet these conditions are exactly the ones that lie above the indifference curve with utility \overline{u} . See figure 1.

Second, we need to understand the objective. The agent wishes to pick the bundle in the constraint set that minimises her expenditure. Just like with the UMP, we can draw the level curves of this objective function. Define an iso-expenditure curve by the bundles of x_1 and x_2 that deliver constant expenditure:

$$\{(x_1, x_2) : p_1x_1 + p_2x_2 = const\}$$

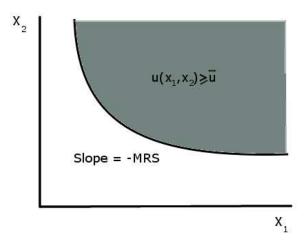


Figure 1: Constraint Set. The shaded area shows the bundles that yield utility \overline{u} or more.

These iso-expenditure curves are just like budget curves and so have slope $-p_1/p_2$. See figure 2

The aim of the agent is to choose the bundle (x_1, x_2) in the constraint set that is on the lowest iso-expenditure curve and hence minimises her expenditure. Ignoring boundary problems and kinks, the solution has the feature that the iso-expenditure curve is tangent to the target indifference curve. As a result, their slopes are identical. The tangency condition can thus be written as

$$MRS = \frac{p_1}{p_1} \tag{2.1}$$

This is illustrated in figure 3.

The intuition behind (2.1) is as follows. Using the fact that $MRS = MU_1/MU_2$, equation (2.1) implies that

$$\frac{MU_1}{MU_2} = \frac{p_1}{p_1} \tag{2.2}$$

¹Recall: $MU_i = \partial U/\partial x_i$ is the marginal utility from good i.

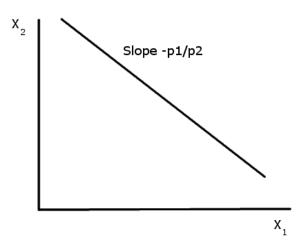


Figure 2: Iso-Expenditure Curve. This figure shows the bundles that induce constant expenditure.

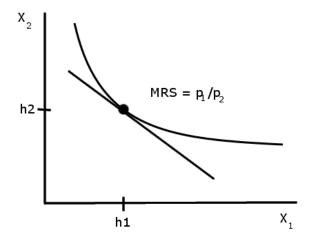


Figure 3: **Optimal Bundle.** This figure shows how the cheapest bundle that attains the target utility satisfies the tangency condition.

Rewriting (2.2) we find

$$\frac{p_1}{MU_1} = \frac{p_2}{MU_1}$$

The ratio p_i/MU_i measures the cost of increasing utility by one util, or the "cost–per–bang". At the optimum the agent equates the cost–per–bang of the two goods. Intuitively, if good 1 has a higher cost–per–bang than good 2, then the agent should spend less on good 1 and more on good 2. In doing so, she could attain the same utility at a lower cost.

If preferences are monotone, then the constraint will bind,

$$u(x_1, x_2) = \overline{u},\tag{2.3}$$

The tangency equation (2.2) and constraint equation (2.3) can then be used to solve for the two Hicksian demands.

If there are N goods, the agent will equalise the cost–per–bang from each good, giving us N-1 equations. Using the constraint equation (2.3), we can solve for the agent's Hicksian demands.

The tangency condition (2.2) is the same as that under the UMP. This is no coincidence. We discuss the formal equivalence in Section 4.2.

2.2 Example: Symmetric Cobb Douglas

Suppose $u(x_1, x_2) = x_1 x_2$. The tangency condition yields:

$$\frac{x_2}{x_1} = \frac{p_1}{p_2} \tag{2.4}$$

Rearranging, $p_1x_1 = p_2x_2$.

The constraint states that $\overline{u} = x_1 x_2$. Substituting (2.4) into this yields,

$$\overline{u} = \frac{p_1}{p_2} x_1^2$$

Solving for x_1 , the Hicksian demand is given by

$$h_1(p_1, p_2, \overline{u}) = \left(\frac{p_2}{p_1}\overline{u}\right)^{1/2}$$
 (2.5)

Similarly, we can solve for the Hicksian demand for good 2,

$$h_2(p_1, p_2, \overline{u}) = \left(\frac{p_1}{p_2}\overline{u}\right)^{1/2}$$

We can now calculate the agent's expenditure

$$e(p_1, p_2, \overline{u}) = p_1 h_1(p_1, p_2, \overline{u}) + p_2 h_2(p_1, p_2, \overline{u})$$

$$= 2(\overline{u}p_1 p_2)^{1/2}$$
(2.6)

2.3 Lagrangian Solution

Using a Lagrangian, we can encode the tangency conditions into one formula. As before, let us ignore boundary problems. The EMP can be expressed as *minimising* the Lagrangian

$$\mathcal{L} = p_1 x_1 + p_2 x_2 + \lambda [\overline{u} - u(x_1, x_2)]$$

As with the UMP, the term in brackets can be thought as the penalty for violating the constraint. That is, the agent is punished for falling short of the target utility.

The FOCs with respect to x_1 and x_2 are

$$\frac{\partial L}{\partial x_1} = p_1 - \lambda \frac{\partial u}{\partial x_1} = 0 \tag{2.7}$$

$$\frac{\partial L}{\partial x_2} = p_2 - \lambda \frac{\partial u}{\partial x_2} = 0 \tag{2.8}$$

If preferences are monotone then the constraint will bind,

$$u(x_1, x_2) = \overline{u} \tag{2.9}$$

These three equations can then be used to solve for the three unknowns: x_1 , x_2 and λ .

Several remarks are in order. First, this approach is identical to the graphical approach. Dividing (2.7) by (2.8) yields

$$\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = \frac{p_1}{p_2}$$

which is the same as (2.2). Moreover, the Lagrange multiplier is

$$\lambda = \frac{p_1}{MU_1} = \frac{p_2}{MU_2}$$

is exactly the cost-per-bang.

Second, if preferences are not monotone, the constraint (2.9) may not bind. If it does not bind, the Lagrange multiplier in the FOCs will be zero.

Third, the approach is easy to extend to N goods. In this case, one obtains N first order conditions and the constraint equation (2.9).

3 General Results

3.1 Properties of Expenditure Function

The expenditure function exhibits four important properties.

1. The expenditure function is homogenous of degree one in prices. That is,

$$e(p_1, p_2, \overline{u}) = e(\alpha p_1, \alpha p_2, \overline{u})$$

for $\alpha > 0$. Intuitively, if the prices of x_1 and x_2 double, then the cheapest way to attain the target utility does not change. However, the cost of attaining this utility doubles.

- 2. The expenditure function is increasing in (p_1, p_2, \overline{u}) . If we increase the target utility \overline{u} , then the constraint becomes harder to satisfy and the cost of attaining the target increases. If we increase p_1 then it costs more to buy any bundle of goods and it costs more to attain the target utility.
- 3. The expenditure function is concave in prices (p_1, p_2) . Fix the target utility \overline{u} and prices $(p_1, p_2) = (p'_1, p'_2)$. Solving the EMP we obtain Hicksian demands $h'_1 = h_1(p'_1, p'_2, \overline{u})$ and $h'_2 = h_2(p'_1, p'_2, \overline{u})$. Now suppose we fix demands and change p_1 , the price of good 1. This gives us a pseudo-expenditure function

$$\eta_{h_1',h_2'}(p_1) = p_1 h_1' + p_2' h_2'$$

This pseudo-expenditure function is linear in p_1 which means that, if we keep demands constant, then expenditure rises linearly with p_1 . Of course, as p_1 rises the agent can reduce her expenditure by rebalancing her demand towards the good that is cheaper. This means that

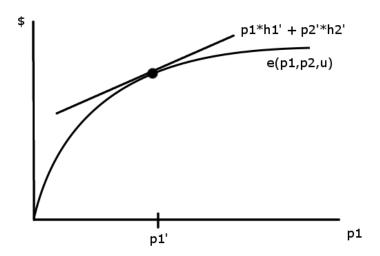


Figure 4: **Expenditure Function.** This figure shows how the expenditure function lies under the pseudo-expenditure function.

real expenditure function lies below the pseudo–expenditure function and is therefore concave. See figure 4.

More formally, the expenditure function is given by the lower envelope of the pseudo-expenditure functions. That is, for any bundle (x_1, x_2) , the cost of this bundle at prices (p_1, p_2) is given by

$$\eta_{x_1,x_2}(p_1,p_2) = p_1x_1 + p_2x_2$$

The expenditure function is then the minimum of these pseudo-expenditure functions given the bundle (x_1, x_2) attains the target utility. Mathematically,

$$e(p_1, p_2, \overline{u}) = \min\{p_1 x_1 + p_2 x_2 : u(x_1, x_2) = \overline{u}\}$$
(3.1)

Thus the expenditure function is the lower minimum of a collection of linear functions, and is therefore concave.² See figure 5.

4. Sheppard's Lemma: The derivative of the expenditure function equals the Hicksian demand. That is,

$$\frac{\partial}{\partial p_1} e(p_1, p_2, \overline{u}) = h_1(p_1, p_2, \overline{u}) \tag{3.2}$$

 $^{^2}$ Exercise: Show that the minimum of two concave functions is concave.

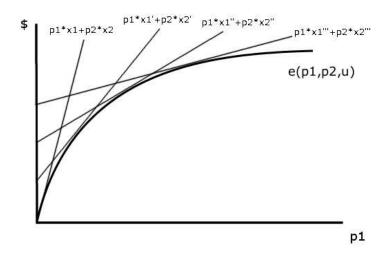


Figure 5: Envelope Property of Expenditure Function. This figure shows the expenditure function equals the lower envelope of the pseudo expenditure functions.

The idea behind this result can be seen from figure 4. At $p_1 = p'_1$ the expenditure function is tangential to the pseudo-expenditure function. The pseudo-expenditure is linear in p_1 with slope $h_1(p'_1, p'_2, \overline{u})$. Hence the expenditure function also has slope $h_1(p'_1, p'_2, \overline{u})$.

The intuition behind Sheppard's Lemma is as follows. Suppose an agent wishes to attain target utility $\overline{u}=25$ and faces prices $p_1=\$1$ and $p_2=\$1$. Furthermore, suppose that the cheapest way to attain the target utility is by consuming $h_1=5$ and $h_2=5$. Next, consider an increase in p_1 of 1¢. This change has a direct and indirect effect. The direct effect is that, holding demand constant, the agent's spending rises by $h_1 \times 1 = 5$; the indirect effect is that the agent will change her demands. However, the tangency condition illustrated in figure 3 shows that the agent is close to indifferent between choosing the optimal quantity and nearby quantities, so the rebalancing demand will will have a very small impact on her expenditure. We thus conclude that $\Delta e = h_1 \Delta p_1$, Rewriting,

$$\frac{\Delta e}{\Delta p_1} = h_1$$

This is the discrete version of equation (3.2).

Here is a formal proof of Sheppard's Lemma. By definition of the expenditure function,

$$e(p_1, p_2, \overline{u}) = p_1 h_1(p_1, p_2, \overline{u}) + p_2 h_2(p_1, p_2, \overline{u})$$

Differentiating with respect to p_1 yields

$$\frac{\partial}{\partial p_1} e(p_1, p_2, \overline{u}) = h_1(p_1, p_2, \overline{u}) + p_1 \frac{\partial h_1(p_1, p_2, \overline{u})}{\partial p_1} + p_2 \frac{\partial h_2(p_1, p_2, \overline{u})}{\partial p_1}$$
(3.3)

As discussed above, we have decomposed the effect of the price change into a direct effect (the first term) and an indirect effect (the second and third terms). We now wish to show the indirect effect is zero. From the agent's minimisation problem in Section 2.3, the FOCs are

$$p_i = \lambda \frac{\partial u(h_1, h_2)}{\partial x_i}$$

We also know that the agent's constraint binds:

$$u(h_1(p_1, p_2, \overline{u}), h_2(p_1, p_2, \overline{u})) = \overline{u}$$

$$(3.4)$$

Substituting the FOCs into (3.3)

$$\frac{\partial}{\partial p_1} e(p_1, p_2, \overline{u}) = h_1(p_1, p_2, \overline{u}) + \lambda \left[\frac{\partial u(h_1, h_2)}{\partial x_1} \frac{\partial h_1(p_1, p_2, \overline{u})}{\partial p_1} + \frac{\partial u(h_1, h_2)}{\partial x_2} \frac{\partial h_2(p_1, p_2, \overline{u})}{\partial p_1} \right] (3.5)$$

Differentiating (3.4) with respect to p_1 yields

$$\frac{\partial u(h_1, h_2)}{\partial x_1} \frac{\partial h_1(p_1, p_2, \overline{u})}{\partial p_1} + \frac{\partial u(h_1, h_2)}{\partial x_2} \frac{\partial h_2(p_1, p_2, \overline{u})}{\partial p_1} = 0$$
(3.6)

Substituting (3.6) into (3.5) yields Sheppard's Lemma.

3.2 Properties of Hicksian Demand

Hicksian demand has three important properties. These follow from the properties of the expenditure function derived above.

1. Hicksian demand is homogenous of degree zero in prices. That is,

$$h_1(p_1, p_2, \overline{u}) = h_1(\alpha p_1, \alpha p_2, \overline{u})$$

for $\alpha > 0$. Intuitively, doubling both prices does not alter the cheapest way to obtain the target utility \overline{u} .

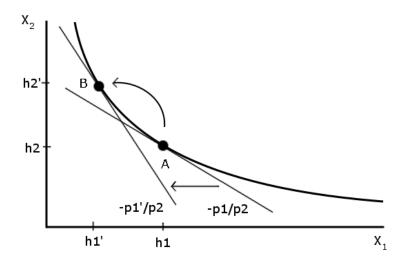


Figure 6: Hicksian Demand and Own Price Effects. This figure shows the effect of an increase in p_1 , from p_1 to p'_1 . The optimal bundle moves from A to B.

2. The Law of Hicksian Demand: The Hicksian demand for good i is decreasing in p_i . That is,

$$\frac{\partial}{\partial p_i} h_i(p_1, p_2, \overline{u}) \le 0$$

Intuitively, when p_1 rises the relative prices become tilted in favour of good 2. The cheapest way to attain the target utility then consists of less of good 1 and more of good 2. Graphically this can be seen from figure 6. As p_1 rises to p'_1 , the iso-expenditure function becomes steeper and the optimal bundle involves less of good 1 and more of good 2.³

A formal proof of this result uses the properties of the expenditure function:

$$\frac{\partial}{\partial p_1} h_1(p_1, p_2, \overline{u}) = \frac{\partial^2}{\partial p_1^2} e(p_1, p_2, \overline{u}) \le 0$$

where the equality comes from Sheppard's Lemma and the inequality follows from the concavity of the expenditure function.

This result highlights a big difference between Hicksian demand and Marshallian demand. An increase in p_1 always reduces the Hicksian demand for good 1 but may, in the case of a Giffen good, increase the Marshallian demand. This is because the effect of a price change on Marshallian demand has two effects: a substitution effect (a change in relative prices) and an

³The fact that the demand for good 2 always rises is an artifact of there only being 2 goods.

income effect (a change in the consumer's purchasing power). In comparison, the change in Hicksian demand isolates the substitution effect.

3. Hicksian demand has symmetric cross derivatives. That is,

$$\frac{\partial}{\partial p_2} h_1(p_1, p_2, \overline{u}) = \frac{\partial}{\partial p_1} h_2(p_1, p_2, \overline{u})$$

The proof of this result also uses the properties of the expenditure function.

$$\frac{\partial}{\partial p_2} h_1(p_1, p_2, \overline{u}) = \frac{\partial}{\partial p_2} \left[\frac{\partial}{\partial p_1} e(p_1, p_2, \overline{u}) \right] = \frac{\partial}{\partial p_1} \left[\frac{\partial}{\partial p_2} e(p_1, p_2, \overline{u}) \right] = \frac{\partial}{\partial p_1} h_2(p_1, p_2, \overline{u})$$

The first and third equalities come from Sheppard's Lemma and the second from Young's theorem.

We say goods x_1 and x_2 are **net substitutes** if

$$\frac{\partial}{\partial p_2} h_1(p_1, p_2, \overline{u}) > 0$$
 and $\frac{\partial}{\partial p_1} h_2(p_1, p_2, \overline{u}) > 0$

We say goods x_1 and x_2 are **net complements** if

$$\frac{\partial}{\partial p_2}h_1(p_1,p_2,\overline{u})<0 \qquad \text{and} \qquad \frac{\partial}{\partial p_1}h_2(p_1,p_2,\overline{u})<0$$

The symmetry of the cross derivatives means that we cannot have one cross–derivative positive negative and the opposite cross–derivative negative, as with gross substitutes and complements.⁴

4 Income and Substitution Effects

We are often interested in how price changes affect Marshallian demand. This matters to firms when choosing prices, to government when choosing tax rates and to economists when making forecasts. For example: how much will demand for ethanol increase if we lower the price by \$10?

We saw with the UMP that an increase in p_1 may lead to a large decrease in demand (if demand is elastic), may lead to a small decrease in demand (if demand is inelastic) or may lead to an increase in demand (in the case of a Giffen good). One major issue is that an increase in the

⁴Exercise: Suppose there are two goods. Show they must be net substitutes.

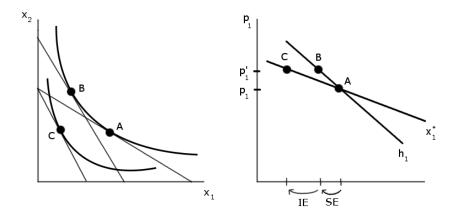


Figure 7: Substitution and Income Effects with Normal Good. With a normal good, both substitution effect (SE) and income effect (IE) are negative.

price of good 1 has two effects: it both makes good 1 relatively more expensive (the substitution effect) and reduces the agent's purchasing power (the income effect). This section will separate these effects. In Section 4.1 we do this graphically. In Section 4.3 we do this mathematically.

4.1 Pictures

Suppose we start at point A in figures 7 and 8. When p_1 increases, the budget line pivots around it's left end and demand falls from A to C. We can decompose this change into two effects.

- 1. A change in relative prices, keeping utility constant. This is the shift from A to B, and is called the **substitution effect**. This equals the change in Hicksian demand and, appealing to the Law of Hicksian Demand, is negative.
- 2. A change in income, keeping relative prices constant. This is the shift from B to C, and is called the **income effect**. This effect is positive if the good is normal, and negative if the good is inferior.

Exercise: draw the equivalent picture for a Giffen good.

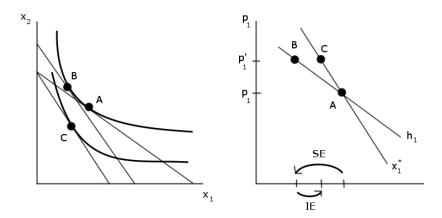


Figure 8: Substitution and Income Effects with Normal Good. With an inferior good, substitution effect (SE) is negative while the income effect (IE) is positive.

4.2 Relation between the UMP and EMP

The EMP and UMP are closely related. To illustrate, suppose the agent has \$10 to spend on two goods. Suppose her utility is maximised when $(x_1, x_2) = (5, 5)$ and she can attain 25 utils.⁵ What is the cheapest way for the agent to attain 25 utils? Given this information, the answer must be $(x_1, x_2) = (5, 5)$. Moreover, her expenditure is \$10. The reason is as follows. First, we know that the agent can obtain 25 utils from \$10, so the cheapest way to obtain 25 utils is at most \$10. That is, $e \le 10 . Now suppose, by contradiction, that the agent can obtain 25 utils for, say, \$8. Then, if preferences are monotone, she will be able to obtain strictly more than 25 utils with \$10, contradicting our initial assumptions.

We can state this result formally. Fix prices (p_1, p_2) and income m. Marshallian demand is given by $x_i^*(p_1, p_2, m)$ and indirect utility is $v(p_1, p_2, m)$. Consider the EMP:

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2$$
 subject to $u(x_1, x_2) \ge v(p_1, p_2, m)$

The induced Hicksian demand is given by $h_i(p_1, p_2, v(p_1, p_2, m))$ while the expenditure function is $e(p_1, p_2, v(p_1, p_2, m))$. Then using the reasoning above, one can show that

$$e(p_1, p_2, v(p_1, p_2, m)) = m (4.1)$$

$$h_i(p_1, p_2, v(p_1, p_2, m)) = x_i^*(p_1, p_2, m)$$
 (4.2)

⁵These numbers come from assuming $p_1 = 1$, $p_2 = 1$ and $u(x_1, x_2) = x_1 x_2$.

Suppose we start with income m. Equation (4.1) says that the minimum expenditure required to reach $v(p_1, p_2, m)$, the most utility from m, is just m. Equation (4.2) says that an agent who wishes to maximise her utility from m and one who wishes to find the cheapest way to attain $v(p_1, p_2, m)$ will buy the same goods. Intuitively, in both cases, they will spend m and will do so by equating the bang-per-buck from each good.

Equation (4.1) is practically useful. Fixing prices and omitting them from the arguments, it says that e(v(m)) = m. Since the expenditure function is increasing in \overline{u} , we can invert it and obtain:

$$v(m) = e^{-1}(m) (4.3)$$

Hence the indirect utility function equals the inverse of the expenditure function. To illustrate this result, suppose $u(x_1, x_2) = x_1 x_2$. From equation (2.6), we know that

$$e(\overline{u}) = 2\sqrt{\overline{u}p_1p_2}$$

We invert this equation by letting $m = e(\overline{u})$ and $v(m) = \overline{u}$, and solving for v(m). This yields

$$v(m) = \frac{m^2}{4p_1p_2}$$

One can verify that this indeed the indirect utility function.

We can also state a second, closely related, result. Fix prices (p_1, p_2) and target utility \overline{u} . Hicksian demand is given by $h_i(p_1, p_2, \overline{u})$ and the expenditure function is $e(p_1, p_2, \overline{u})$. Consider the UMP:

$$\max_{x_1, x_2} u(x_1, x_2)$$
 subject to $p_1 x_1 + p_2 x_2 \le e(p_1, p_2, \overline{u})$

The induced Marshallian demand is given by $x_i^*(p_1, p_2, e(p_1, p_2, \overline{u}))$ while the indirect utility is $v(p_1, p_2, e(p_1, p_2, \overline{u}))$. One can show that

$$v(p_1, p_2, e(p_1, p_2, \overline{u})) = \overline{u} \tag{4.4}$$

$$x_i^*(p_1, p_2, e(p_1, p_2, \overline{u})) = h_i(p_1, p_2, \overline{u})$$
(4.5)

Suppose we start with target utility \overline{u} . Equation (4.4) says that the most utility the agent can get from $e(p_1, p_2, \overline{u})$, the money required to reach \overline{u} , is just \overline{u} . Equation (4.5) says that an agent who wishes to find the cheapest way to attain \overline{u} and one who wishes to maximise her utility from $e(p_1, p_2, \overline{u})$ will buy the same goods. Intuitively, in both cases, they will attain utility \overline{u}

and will do so by equating the bang-per-buck from each good.

Fixing prices and omitting them from the arguments, equation (4.4) says that $v(e(\overline{u})) = \overline{u}$. Since the indirect function is increasing in m, we can invert it and obtain:

$$e(\overline{u}) = v^{-1}(\overline{u}) \tag{4.6}$$

Hence the expenditure function equals the inverse of the indirect utility function. Together, equations (4.3) and (4.6) mean we can move back and forwards between the expenditure function and indirect utility function.

4.3 Slutsky Equation: Own Price Effects

Suppose p_1 increases by Δp_1 . There are two effects:

- 1. Fixing the agent's utility, relative prices change causing demand to rise by $\frac{\partial h_1}{\partial p_1} \Delta p_1$. Since $\frac{\partial h_1}{\partial p_1} < 0$, this effect causes demand to fall. This is the substitution effect.
- 2. Fixing relative prices, the agent's income falls by $x_1^* \Delta p_1$. As a result, her demand falls by $x_1^* \frac{\partial x_1^*}{\partial m} \Delta p_1$. This is the income effect.

Putting these effects together, we have

$$\Delta x_1^* = \frac{\partial h_1}{\partial p_1} \Delta p_1 - x_1^* \frac{\partial x_1^*}{\partial m} \Delta p_1$$

Dividing by Δp_1 yields the Slutsky equation.

Theorem 1 (Own-Price Slutsky Equation). Fix prices (p_1, p_2) and income m, and let $\overline{u} = v(p_1, p_2, m)$ be the indirect utility. Then

$$\frac{\partial}{\partial p_1} x_1^*(p_1, p_2, m) = \frac{\partial}{\partial p_1} h_1(p_1, p_2, \overline{u}) - x_1^*(p_1, p_2, m) \frac{\partial}{\partial m} x_1^*(p_1, p_2, m)$$
(4.7)

A formal proof is reasonably straightforward. Using equation (4.5),

$$h_i(p_1, p_2, \overline{u}) = x_i^*(p_1, p_2, e(p_1, p_2, \overline{u}))$$

Differentiating with respect to p_1 yields

$$\frac{\partial}{\partial p_{1}} h_{1}(p_{1}, p_{2}, \overline{u}) = \frac{\partial x_{1}^{*}(p_{1}, p_{2}, e(p_{1}, p_{2}, \overline{u}))}{\partial p_{1}} + \frac{\partial x_{1}^{*}(p_{1}, p_{2}, e(p_{1}, p_{2}, \overline{u}))}{\partial m} \frac{\partial e(p_{1}, p_{2}, \overline{u})}{\partial p_{1}} \\
= \frac{\partial x_{1}^{*}(p_{1}, p_{2}, e(p_{1}, p_{2}, \overline{u}))}{\partial p_{1}} + \frac{\partial x_{1}^{*}(p_{1}, p_{2}, e(p_{1}, p_{2}, \overline{u}))}{\partial m} x_{1}^{*}(p_{1}, p_{2}, e(p_{1}, p_{2}, \overline{u})) \quad (4.8)$$

where the second line comes from Sheppard's Lemma. Using the definition of \overline{u} and equation (4.1),

$$e(p_1, p_2, \overline{u}) = e(p_1, p_2, v(p_1, p_2, m)) = m$$
 (4.9)

Substituting (4.9) into (4.8) and rearranging yields (4.7), as required.

4.4 Slutsky Equation: Cross Price Effects

Equation (4.7) analyses the effect of a change in p_1 on the demand for good 1. We can use the same approach to analyse the effect of a change in p_2 on the demand for good 1.

Suppose p_2 increases by Δp_2 . As before, there are two effects:

- 1. Fixing the agent's utility, relative prices change causing demand to rise by $\frac{\partial h_1}{\partial p_2} \Delta p_2$. Recall that $\frac{\partial h_1}{\partial p_2} > 0$ if the goods are net substitutes and $\frac{\partial h_1}{\partial p_2} < 0$ are net complements.
- 2. Fixing relative prices, the agent's income falls by $x_2^* \Delta p_2$. As a result, her demand falls by $x_2^* \frac{\partial x_1^*}{\partial m} \Delta p_2$.

Putting these effects together, we have

$$\Delta x_1^* = \frac{\partial h_1}{\partial p_2} \Delta p_2 - x_2^* \frac{\partial x_1^*}{\partial m} \Delta p_2$$

Dividing by Δp_2 yields the Slutsky equation for cross-price effects.

Theorem 2 (Cross-Price Slutsky Equation). Fix prices (p_1, p_2) and income m, and let $\overline{u} = v(p_1, p_2, m)$ be the indirect utility. Then

$$\frac{\partial}{\partial p_2} x_1^*(p_1, p_2, m) = \frac{\partial}{\partial p_2} h_1(p_1, p_2, \overline{u}) - x_2^*(p_1, p_2, m) \frac{\partial}{\partial m} x_1^*(p_1, p_2, m)$$
(4.10)

The proof is almost identical to that of (4.7). Using equation (4.5),

$$h_i(p_1, p_2, \overline{u}) = x_i^*(p_1, p_2, e(p_1, p_2, \overline{u}))$$

Differentiating with respect to p_2 yields

$$\frac{\partial}{\partial p_2} h_1(p_1, p_2, \overline{u}) = \frac{\partial x_1^*(p_1, p_2, e(p_1, p_2, \overline{u}))}{\partial p_2} + \frac{\partial x_1^*(p_1, p_2, e(p_1, p_2, \overline{u}))}{\partial m} \frac{\partial e(p_1, p_2, \overline{u})}{\partial p_2} \\
= \frac{\partial x_1^*(p_1, p_2, e(p_1, p_2, \overline{u}))}{\partial p_2} + \frac{\partial x_1^*(p_1, p_2, e(p_1, p_2, \overline{u}))}{\partial m} x_2^*(p_1, p_2, e(p_1, p_2, \overline{u})) \quad (4.11)$$

where the second line comes from Sheppard's Lemma. Using the definition of \overline{u} and equation (4.1),

$$e(p_1, p_2, \overline{u}) = e(p_1, p_2, v(p_1, p_2, m)) = m$$
 (4.12)

Substituting (4.12) into (4.11) and rearranging yields (4.10).

4.5 Slutsky Equation: Example

We illustrate the Slutsky equation with our running example. Let $u(x_1, x_2) = x_1x_2$. From the UMP we know that

$$x_1^*(p_1, p_2, m) = \frac{m}{2p_1}$$

 $v(p_1, p_2, m) = \frac{m^2}{4p_1p_2}$

From the EMP (see Section 2.2) we know that

$$h_1(p_1, p_2, \overline{u}) = \left(\overline{u} \frac{p_2}{p_1}\right)^{1/2}$$

 $e(p_1, p_2, \overline{u}) = 2(\overline{u}p_1p_2)^{1/2}$

The left hand side of the Slutsky equation states

$$\frac{\partial}{\partial p_1} x_1^*(p_1, p_2, m) = -\frac{1}{2} m p_1^{-2} \tag{4.13}$$

The right hand side is

$$\begin{split} \frac{\partial h_1}{\partial p_1} - x_1^* \frac{\partial x_1^*}{\partial m} &= -\frac{1}{2} \overline{u}^{1/2} p_1^{-1/2} p_2^{1/2} - \frac{1}{4} m p_1^{-2} \\ &= -\frac{1}{4} m p_1^{-2} - \frac{1}{4} m p_1^{-2} \end{split} \tag{4.14}$$

where the second line uses $\overline{u} = v(p_1, p_2, m)$.

Observe that (4.13) equals (4.14) as we would hope. Moreover, the two terms in equation (4.14) are identical. This means that the substitution and income effects are of equal size: both account for 50% of the fall in demand.

5 Consumer Surplus

It is often important to put a monetary value on the effect of a price change on an agent's utility. For example, the government may wish to evaluate the impact of a tax change; or a court may wish to evaluate the negative effect of collusion on consumers.

To gain some intuition, suppose the consumer has monetary valuations for each unit of the good. In particular suppose their valuations are given by table 2.

Unit	Valuation \$		
1	10		
2	8		
3	6		
4	4		
5	2		

Table 2: Agent's Valuations

Suppose the price of the good is initially $p_1 = 3$. Since the agent buys a unit if and only if her valuation exceeds the price, she will buy 4 units. Her consumer surplus, the difference between her willingness to pay and the price she pays, equals

$$CS = (10-3) + (8-3) + (6-3) + (4-3) = $16$$

Suppose the price rises to $p_1 = 7$. The agent then consumes 2 units and her consumer surplus is

$$CS = (10 - 7) + (8 - 7) = $4.$$



Figure 9: Consumer Surplus with Quasilinear Demand. The figure shows the agent's demand curve. The shaded area is the loss in CS due to the price increase.

Hence the agent would need to be compensated \$12 for this price increase. This is shown in figure 9.

This exercise is familiar from introductory economics courses: consumer surplus is the area under the agent's Marshallian demand curve. However this approach assumes the agent has quasilinear utility, allowing us to associate a monetary value to each unit demanded by the agent. In Section 5.1 we show that the welfare effect of a price change is determined by the area under the Hicksian demand curve rather than the Marshallian demand. In Section 5.2 we see that, when utility is quasi-linear then Hicksian demand and Marshallian demand coincide, justifying the approach taken above.

5.1 Compensating Variation

Suppose prices and income are initially (p_1, p_2, m) , and that p_1 increases to p'_1 . The **compensating variation** is defined by

$$CV = e(p'_1, p_2, \overline{u}) - e(p_1, p_2, \overline{u})$$

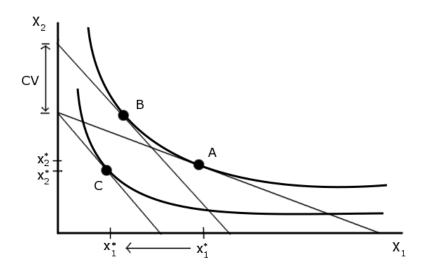


Figure 10: Compensating Variation and Indifference Curves. This figure shows the effect on an increase in p_1 . The Marshallian demand falls from A to C. The Hicksian demand moves from A to B. The compensating variation equals the difference between the consumer's original income and the income she would need to attain \overline{u} .

The CV is thus the extra spending needed to keep the agent at their original utility level. That is, an increase in income of CV completely compensates the agent for the price increase.⁶ This is shown in figure 10.

The compensating variation can be related to the Hicksian demand curve. Applying the fundamental theorem of calculus,⁷

$$CV = \int_{p_1}^{p_1'} \frac{\partial}{\partial p_1} e(\tilde{p}_1, p_2, \overline{u}) d\tilde{p}_1$$

$$= \int_{p_1}^{p_1'} h_1(\tilde{p}_1, p_2, \overline{u}) d\tilde{p}_1$$
(5.1)

where the second equation follows from Sheppard's Lemma. Equation (5.1) says that the lost welfare from the price change equals the area under the Hicksian demand curve. See figure 11.

⁶There is a closely related measure of welfare called the equivalent variation. We will not discuss this here.

⁷The fundamental theorem of calculus says that $f(b) - f(a) = \int_a^b f'(x) dx$.

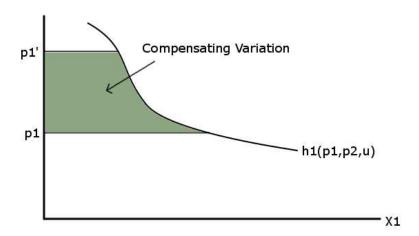


Figure 11: Compensating Variation and Hicksian Demand. This figure shows that CV equals the area under the demand curve.

5.2 Quasilinear Utilities

While we may wish to calculate the area under the Hicksian demand, it is often easier to calculate the area under the Marshallian demand curve. For example, in empirical applications, it is easy to estimate the Marshallian demand by looking at how much people buy at different prices.

Suppose utility is quasilinear in that it can be represented by a utility function of the form

$$u(x_1, x_2) = v(x_1) + x_2$$

where we assume $v(\cdot)$ is increasing and concave. Under this specification, the marginal utility of the second good is constant. For example, x_2 could be a general aggregate good or cash.

When utility is quasilinear we can think of an agent's utility in terms of dollar valuations, as at the start of this section. The argument is as follows. The agent's problem is to maximise her utility subject to her budget constraint, $p_1x_1 + p_2x_2 \leq m$. Since utility is monotone, the budget constraint will bind. Using the substitution method, the budget constraint becomes

$$x_2 = \frac{m}{p_2} - \frac{p_1}{p_2} x_1$$

Substituting this into the utility function, the agent maximises

$$v(x_1) - \frac{p_1}{p_2}x_1 + \frac{m}{p_2} \tag{5.2}$$

Notice the last term is a constant and can be ignored. If x_2 is interpreted as cash, we can normalise $p_2 = 1$. The agent then chooses x_1 to maximise

$$v(x_1) - p_1 x_1$$

The agent's choice is independent of m, so she acts as if she values x_1 units of good 1 at $v(x_1)$, independent of the units of x_2 being consumed. We can then think of $v'(x_1)$ as her valuation of the marginal unit of x_1 .

Under quasilinear utility, the Hicksian and Marshallian demands coincide. Ignoring boundary problems, the Marshallian demand is derived by maximising (5.2). The first-order condition implies that Marshallian demand is implicitly given by

$$v'(x_1^*(p_1, p_2, m)) = \frac{p_1}{p_2}$$
(5.3)

Turning to the EMP, the agent minimises

$$\mathcal{L} = p_1 x_1 + p_2 x_2 + \lambda [\overline{u} - v(x_1) - x_2]$$

The first first-order conditions are

$$p_1 = \lambda v'(x_1)$$
$$p_2 = \lambda$$

Looking at the ratio of these two equations, Hicksian demand is implicitly given by

$$v'(h_1(p_1, p_2, \overline{u})) = \frac{p_1}{p_2}$$
(5.4)

From equations (5.3) and (5.4) we see that Marshallian demand and Hicksian demand coincide. Hence the compensating variation is given by

$$CV = \int_{p_1}^{p_1'} h_1(\tilde{p}_1, p_2, \overline{u}) d\tilde{p}_1$$
$$= \int_{p_1}^{p_1'} x_1^*(\tilde{p}_1, p_2, m) d\tilde{p}_1$$

This result provides a foundation for the classical measure of consumer surplus.

6 Endowments of Goods

In the UMP we assume that agents are endowed with income m and use it to maximise their utility. While this is a useful model to address demand for retail products, it is sometimes more accurate to assume agents are endowed with goods which they can sell on the open market. There are two reasons for analysing this model:

- The model is important for understanding practical problems such as a worker's choice of labour supply (Section 6.1), and an agent's decision to smooth consumption over time (Section 6.2).
- When we analyse the entire economy, we will want to close the model. Hence we wish the agents who demand goods to also work for firms that make goods.

Suppose there are N goods and the agent starts with endowments $\{\omega_1, \ldots, \omega_N\}$, where $\omega_i \geq 0$ for all i. The consumer can sell these goods at market prices $\{p_1, \ldots, p_N\}$. For example, an agent may own a farm which produces vegetables and may sell the produce to buy meat. The agent has income

$$m = \sum_{i=1}^{N} p_i \omega_i \tag{6.1}$$

Given equation (6.1) the agent's problem is the same as that studied so far. We can derive her Marshallian demand and indirect utility (see figure 12). We can also derive her Hicksian demand and expenditure function (since these are independent of income)

The one major difference from the model with exogenous income is that a price change now affects the agent's income as well as the goods she buys. We study this in Section 6.3. We first consider two applications.

6.1 Labour Supply

Suppose an agent has utility $u(x_1, x_2) = x_1x_2$ over leisure x_1 and a general consumption good x_2 . The agent has exogenous income m and can also work at wage w. She has T hours which

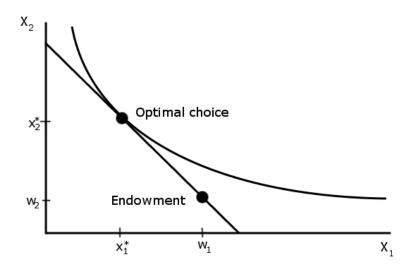


Figure 12: Marshallian Demand with Endowments This figure shows the optimal choice when the agent has endowments of the two goods.

she can allocate to either work or leisure. We normalise the price of x_2 to $p_2 = 1$.

The agent's budget constraint is

$$x_2 = w(T - x_1) + m$$

The left hand side equals the agent's spending on consumption; the right hand side equals her income. As a thought experiment, one can imagine the agent selling all T units of her labour and then buying x_1 units of it back at price w to be consumed as leisure. We can thus rewrite the budget constraint as

$$wx_1 + x_2 = wT + m$$

The left hand side is the goods consumed (including leisure, consumed at price w). The right hand side is the agent's endowment income, as in equation (6.1).

Ignoring boundary constraints,⁸ her problem is

$$\max_{x_1, x_2} \mathcal{L} = x_1 x_2 + \lambda [m + w(T - x_1) - x_2]$$

⁸With this problem the boundary constraints are slightly different to normal since the agent cannot consume more that T units of leisure. We thus have $T \ge x_1 \ge 0$ and $x_2 \ge 0$.

The FOCs are

$$x_2 = \lambda w$$

$$x_1 = \lambda$$

Taking the ratio of these FOCs, we see that

$$\frac{x_2}{x_1} = w$$

As before, the left hand side is the MRS, while the right hand side is the price ratio. Using the budget constraint the agent's demands are given by

$$x_1^* = \frac{1}{2w}[wT + m] \tag{6.2}$$

$$x_2^* = \frac{1}{2}[wT + m] \tag{6.3}$$

Equations (6.2) and (6.3) show that the consumer splits her endowment income of wT + m equally between leisure and consumption. This is just like the solution to the Cobb Douglas problem without endowments (see UMP notes), where we found that

$$x_1^* = \frac{1}{2p_1}m$$
 and $x_2^* = \frac{1}{2p_2}m$ (6.4)

We can now evaluate an effect of a change in wages. Differentiating (6.2) and (6.3),

$$\frac{\partial x_1^*}{\partial w} = \frac{1}{2w^2}wT - \frac{1}{2w^2}[wT + m] = -\frac{1}{2w^2}m\tag{6.5}$$

$$\frac{\partial x_2^*}{\partial w} = \frac{1}{2}T\tag{6.6}$$

From (6.5), we see an increase in the wage reduces the amount of leisure the agent consumes. There are two effects here: an increase in the wage raises the relative price of leisure and reduces demand (the substitution effect); it also makes the agent richer and increases the demand for leisure (the income effect). In this case the substitution effect dominates the income effect: we analyse this formally in Section 6.4.

From (6.6), we see an increase in the wage increase the amount of x_2 the agent consumes. This is because an increase in wages increase the value of the agent's endowment; in comparison, without endowments, equation (6.4) shows that x_2^* is independent of p_1 .

6.2 Intertemporal Optimisation

Suppose an agent allocates consumption (e.g. money) across two periods. Let the consumption in period 1 and 2 be x_1 and x_2 respectively. The agent's utility is

$$u(x_1, x_2) = \ln(x_1) + (1+\beta)^{-1} \ln(x_2)$$
(6.7)

where $\beta \geq 0$ is the agent's discount rate.

In periods 1 and 2 the agent is endowed with income m_1 and m_2 , respectively. The agent can save at interest rate $r \ge 0$, so that \$1 in period 1 is worth \$(1+r) in period 2. As a result, the agent's budget constraint is

$$m_1 + (1+r)^{-1}m_2 = x_1 + (1+r)^{-1}x_2$$
 (6.8)

The left hand side of (6.8) is the agent's lifetime income in terms of period 1 dollars. The right and side is the agent's lifetime spending. We say they are borrowing when $x_1 > m_1$ and saving when $x_1 < m_1$.

We can solve this problem just as we would solve a regular utility maximisation problem, where $p_1 = 1$ and $p_2 = (1 + r)^{-1}$. See figure 13. Using (6.7) and (6.8) the tangency condition, $MRS = p_1/p_2$, becomes

$$(1+\beta)\frac{1/x_1}{1/x_2} = (1+r)$$

Rearranging,

$$x_1^* = \frac{1+\beta}{1+r} x_2^* \tag{6.9}$$

Equation (6.2) immediately implies that if $r=\beta$ then the agent consumes the same in each period, $x_1^*=x_2^*$. Intuitively, since the agent's per–period utility is concave, she wishes to smooth her consumption across time. If $r=\beta$, then the agent is just as impatient as the market, so she will perfectly smooth her consumption across the two periods. If $\beta > r$ then the agent is more impatient than the market and she consumes more in the first period, $x_1^*>x_2^*$.

Using the budget constraint, demand is given by

$$x_1^* = \frac{1+\beta}{2+\beta}[m_1 + (1+r)^{-1}m_2]$$
 and $x_2^* = \frac{1+r}{2+\beta}[m_1 + (1+r)^{-1}m_2]$

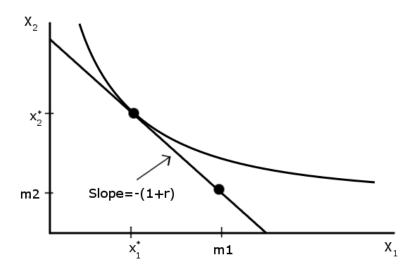


Figure 13: Intertemporal Optimisation This figure shows an agent who has a high income in period 1 and a low income in period 2. At the optimum, she saves in period 1.

6.3 Own Price Effects

Suppose there is an increase in p_1 . As with a fixed income m, the budget line becomes steeper. However, since the value of the endowment changes, it is no longer true that the budget set shrinks. Rather, the budget line pivots around the endowment: see figure 14.

As in Section 4, we can decompose the price change into a substitution and income effect. However, the income effect has to be adjusted for the change in the value of the endowment. Suppose p_1 increases by Δp_1 . Then there are two effects:

- 1. Fixing the agent's utility, relative prices change causing demand to rise by $\frac{\partial h_1}{\partial p_1} \Delta p_1$. Since $\frac{\partial h_1}{\partial p_1} < 0$, this effect causes demand to fall. This is the substitution effect.
- 2. Fixing relative prices, the agent's income rises by $(\omega_1 x_1^*)\Delta p_1$. This means that the agent's income rises if she is a net seller of the good (as in the labour example), and falls if she is a net buyer of the good. As a result, her demand rises by $(\omega_1 x_1^*)\frac{\partial x_1^*}{\partial m}\Delta p_1$. This is the income effect.

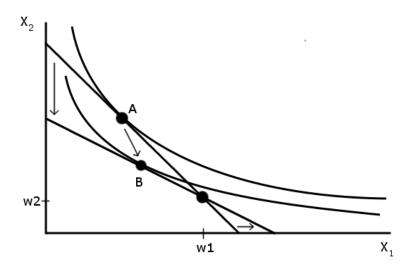


Figure 14: **Own-Price Effects with Endowments.** This figure shows the effect on a *decrease* in p_1 when the agent is endowed with $\{\omega_1, \omega_2\}$. Note the income looks like it goes down, even though prices fall. This is because, at point A, the agent owns more of good 1 than she buys, $\omega_1 > x_1^*$. Hence a decrease in p_1 reduces her purchasing power.

Putting these effects together, we have

$$\Delta x_1^* = \frac{\partial h_1}{\partial p_1} \Delta p_1 + (\omega_1 - x_1^*) \frac{\partial x_1^*}{\partial m} \Delta p_1$$

Dividing by Δp_1 yields the Slutsky equation.

Theorem 3 (Own–Price Slutsky Equation with Endowments). Fix prices (p_1, p_2) , income m and endowments (ω_1, ω_2) , and let $\overline{u} = v(p_1, p_2, m)$ be the indirect utility. Then

$$\frac{\partial}{\partial p_1} x_1^*(p_1, p_2, m) = \frac{\partial}{\partial p_1} h_1(p_1, p_2, \overline{u}) + (\omega_1 - x_1^*(p_1, p_2, m)) \frac{\partial}{\partial m} x_1^*(p_1, p_2, m)$$
(6.10)

This result follows from the regular Slutsky equation (4.7). All we need to do is define net demand for good 1 by $z_1^*(p_1, p_2, m) = x_1^*(p_1, p_2, m) - \omega_1$. We can then apply the regular Slutsky equation to the agent's net demand:

$$\frac{\partial}{\partial p_1} z_1^*(p_1, p_2, m) = \frac{\partial}{\partial p_1} h_1(p_1, p_2, \overline{u}) - z_1^*(p_1, p_2, m) \frac{\partial}{\partial m} z_1^*(p_1, p_2, m)$$
(6.11)

Since z_1^* and x_1^* differ by a constant term, we can put equation (6.11) back in terms of $x_i^*(p_1, p_2, m)$, yielding equation (6.10).

6.4 Labour Supply and the Slutsky Equation

We now apply the Slutsky equation (6.10) to the labour supply problem in Section 6.1. From equation (6.2), the Marshallian demand is

$$x_1^*(p_1, p_2, m) = \frac{1}{2w}[wT + m]$$
(6.12)

Using the (6.2) and (6.3) the indirect utility is

$$v(p_1, p_2, m) = x_1^* x_2^* = \frac{1}{4w} [wT + m]^2$$
(6.13)

From (2.5) and using $p_1 = w$ and $p_2 = 1$, the Hicksian demand is

$$h_1(p_1, p_2, \overline{u}) = \overline{u}^{1/2} w^{-1/2}$$
 (6.14)

We now have all the elements we need.

The left hand side of the Slutsky equation is

$$\frac{\partial}{\partial p_1} x_1^* = -\frac{1}{2w^2} m$$

The right hand side of the Slutsky equation is

$$\frac{\partial h_1}{\partial p_1} + (\omega_1 - x_1^*) \frac{\partial x_1^*}{\partial m} = -\frac{1}{2} w^{-3/2} \overline{u}^{1/2} + \left[T - \frac{1}{2w} [wT + m] \right] \frac{1}{2w}$$

$$= -\frac{1}{2} w^{-3/2} \frac{1}{2} w^{-1/2} [wT + m] + \frac{1}{4w^2} [wT - m]$$

$$= -\frac{1}{4w^2} [wT + m] + \frac{1}{4w^2} [wT - m]$$

$$= -\frac{1}{2w^2} m$$

where the first line uses (6.12) and (6.14), and the second uses $\overline{u} = v(p_1, p_2, m)$ and equation (6.13). We can therefore see that the substitution effect outweighs the income effect, and as m becomes smaller these two effects grow closer in magnitude. In the limit, as $m \to 0$, leisure demand (and hence labour supply) are independent of the wage.