
Experiment VI: The LRC Circuit and Resonance

I. References

Halliday, Resnick and Krane, *Physics*, Vol. 2, 4th Ed., Chapters 38,39
Purcell, *Electricity and Magnetism*, Chapter 7,8

II. Equipment

Digital Oscilloscope	Digital Multimeter
Signal Generator	0.01 μ F Capacitor
Differential Amplifier	100 mH Inductor (2)
Circuit Breadboard	1 k Ω Resistor

III. Introduction

We are now ready to introduce one of the most important components of a radio, the AC LRC circuit. With both an inductor and capacitor in the same circuit, an LRC circuit essentially acts as a *bandpass* filter, a filter that only allows a narrow range of frequencies to pass. When tuned to the frequency of a particular radio station, it allows that station's signal to pass at the exclusion of all others. The characteristics of the “tuner” in a radio, including its Q value, are critical to its performance.

IV. Background and theory

Figure VI-1 shows a circuit with driving voltage $V(t) = V_0 \cos(\omega t)$ connected in series with a resistor, inductor, and capacitor. In this circuit, the resistance R includes every known resistance in the circuit, which can come from the internal resistance of the generator, the resistance of the inductor, and any real resistors connected.

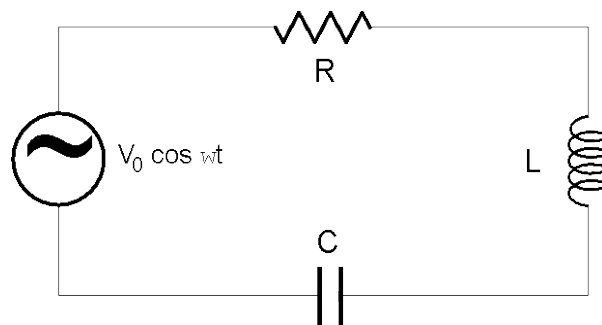


Figure VI-1: LRC circuit driven by a sine wave input.

As before, we analyze the current in the circuit by using Kirchoff's laws for complex resistances. The equivalent complex resistance is given by the series addition of R , $X_L = \omega L$, and $X_C = 1/\omega C$:

$$R_{eq} = R + iX_L - iX_C = R + i(X_L - X_C) = Ze^{i\theta}$$

where

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad (\text{VI-1})$$

and

$$\tan \theta = \frac{(X_L - X_C)}{R} \quad (\text{VI-2})$$

We can then find the current in the circuit using $V(t) = I(t)R_{eq}$:

$$I(\omega, t) = \frac{V_0 e^{i\omega t}}{Z(\omega) e^{i\theta(\omega)}} = \frac{V_0}{Z} e^{i(\omega t - \theta)}. \quad (\text{VI-3})$$

The current will therefore have an amplitude $|I(\omega)|$, which is a function of the frequency:

$$|I(\omega)| = \frac{V_0}{Z(\omega)} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}} \quad (\text{VI-4})$$

and which has its maximum at the resonance condition:

$$I_{MAX} = \frac{V_0}{R} \quad (\text{VI-5})$$

which is also when $X_C = X_L$. The resonance frequency $\omega = \omega_0$ is when $1/\omega_0 C = \omega_0 L$, or

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad (\text{VI-6})$$

Figure VI-2 shows the plot of the phase angle ϕ vs. ω/ω_0 . Note that when the frequency is very low, $X_L \rightarrow 0$ and $X_C \rightarrow \infty$, so $\tan \phi \rightarrow -\infty$ and $-\phi \rightarrow -\pi/2$. At very high frequencies, $X_C \rightarrow 0$ and $X_L \rightarrow \infty$, so $\tan \phi \rightarrow +\infty$, and $\phi \rightarrow +\pi/2$. At resonance, $X_L = X_C$, $\omega = \omega_0$, $\tan \phi \rightarrow 0$, and $\phi \rightarrow 0$ (no phase shift).

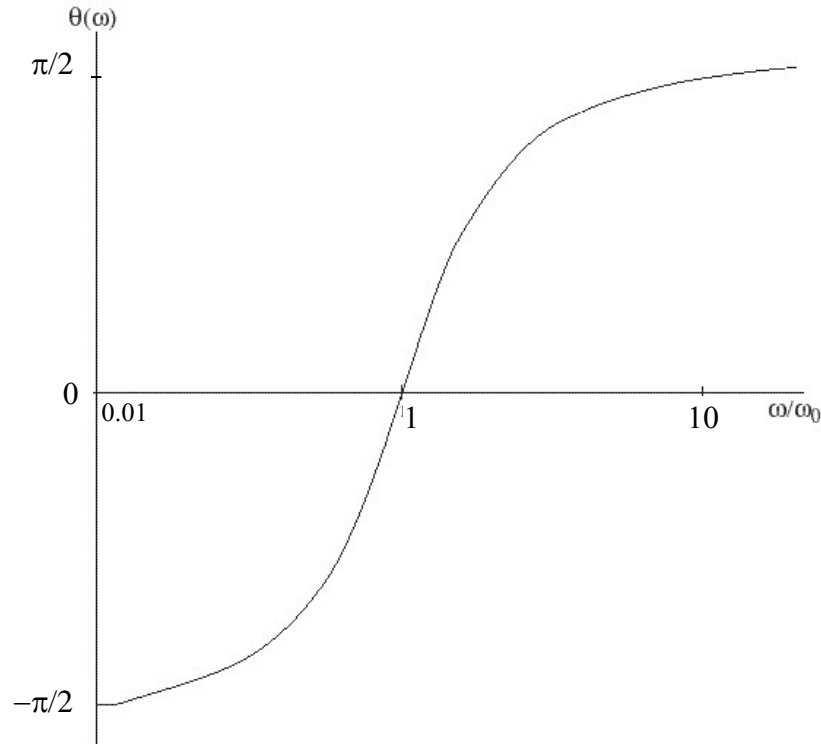


Figure VI-2: Behavior of the phase shift as a function of frequency.

Another feature is that at very low (high) frequencies, either X_C (X_L) gets very large, which increases the impedance Z , causing the amplitude of the current to become very small. Somewhere between low and high frequency is a resonance frequency ω_0 , at $X_C = X_L$. The plot of $|I(\omega)|$ vs. ω will therefore have a maximum at the “resonance frequency” $\omega = \omega_0$, as shown in figure VI-3.

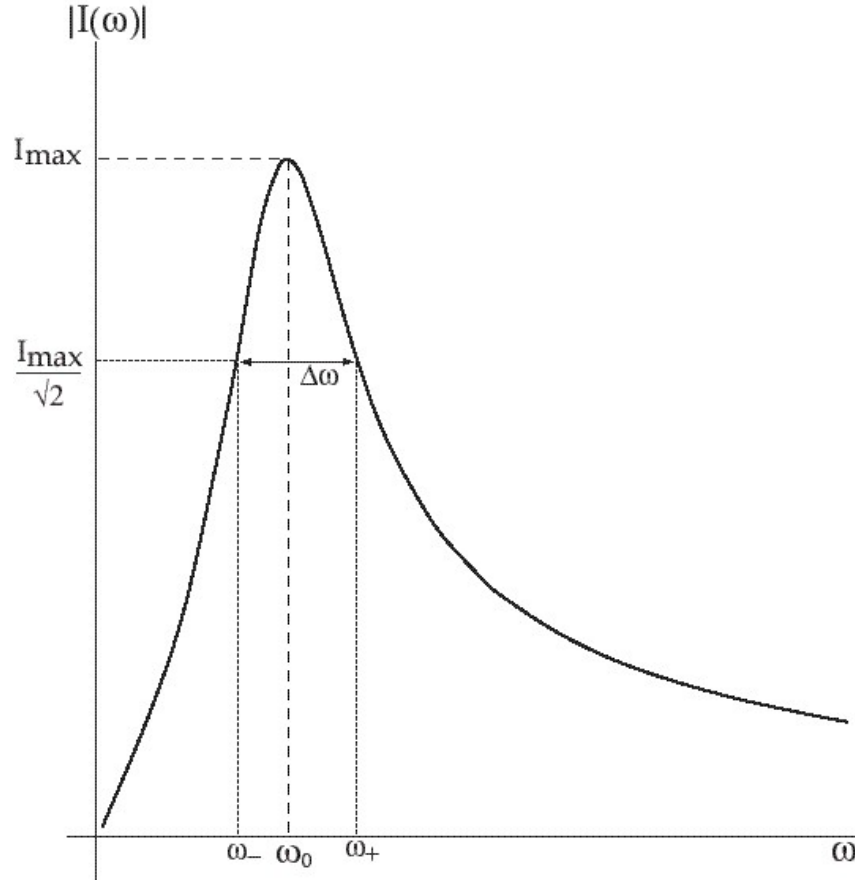


Figure VI-3: Behavior of the current as a function of frequency, showing a resonance peak.

The width of the resonance, $\Delta\omega$, is defined as the difference between the two angular frequencies ω_+ and ω_- , which are defined as those frequencies at which the amplitude of the current is $I_{MAX}/\sqrt{2}$. These two angular frequencies can be calculated using equations VI-4 and VI-5:

$$\frac{V_0}{\sqrt{2}R} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}$$

or equivalently:

$$\sqrt{1 + \frac{(X_L - X_C)^2}{R^2}} = \sqrt{2}$$

which translates to

$$\frac{(X_L - X_C)^2}{R^2} = 1.$$

One has to be careful taking the square root of both sides. If we define $\alpha = \pm 1$, then

$$X_L = X_C + \alpha R.$$

Using the equations $X_L \equiv \omega L$, $X_C \equiv 1/\omega C$, defining $\tau_L = L/R$, and $\omega_0 \equiv 1/\sqrt{LC}$ we get the quadratic equation

$$\omega^2 - \omega \frac{\alpha}{\tau_L} - \omega_0^2 = 0$$

which has the solutions

$$\omega = \frac{\alpha}{2\tau_L} \pm \sqrt{\omega_0^2 + \frac{1}{(2\tau_L)^2}}.$$

The negative sign here is unphysical since it produces negative frequencies, so we only take the positive solution. But the coefficient α can be either $+1$ or -1 , which gives us the two solutions ω_{\pm} :

$$\omega_{\pm} = \frac{\pm 1}{2\tau_L} + \sqrt{\omega_0^2 + \frac{1}{(2\tau_L)^2}}.$$

The width $\Delta\omega$ is the difference between the two:

$$\Delta\omega = \omega_+ - \omega_- = \frac{1}{\tau_L}. \quad (\text{VI-7})$$

So the width of the resonance depends only upon L and R . However, remember that the resonant frequency ω_0 also depends on L (and C). So the resonant frequency and the width of the resonance are linked by the value of the inductance used in the circuit. A more useful quantity is the “Quality factor”

$$Q \equiv \frac{\omega_0}{\Delta\omega}. \quad (\text{VI-8})$$

This quantity is essentially the ability of the circuit to limit the flow of high current to a small band of frequencies about the resonance frequency ω_0 . As $Q \rightarrow 0$, the current flowing through the circuit will never be large and a large range of frequencies will behave in the same way. As $Q \rightarrow \infty$, the circuit will resonate only for frequencies that are near the resonance frequency and the current through the circuit can become large. This type of resonant behavior is found to be extremely useful for things like radio reception, where a receiver LCR circuit can be tuned to a particular radio transmission frequency, eliminating interference from frequencies nearby.

Combining the above equations gives:

$$Q \equiv \frac{\omega_0}{\Delta\omega} = \omega_0 \tau_L = \frac{L}{R\sqrt{LC}} = \sqrt{\frac{L}{R^2 C}}. \quad (\text{VI-9})$$

Using this definition of Q we can rewrite equation VI-1 as:

$$Z = R \sqrt{1 + Q^2 \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)^2}.$$

This equation shows explicitly that for a large Q , the current will be very small (“off resonance”, $Z \rightarrow RQ$, therefore Z will be very large) except at the resonance frequency $\omega = \omega_0$ (where $Z = R$).

Knowing the current $I(t)$ we can calculate the voltage drops across the various components. The voltage drop across any resistor R_R (we use this notation to distinguish between a real resistive element R_R and a total resistance R):

$$V_R(t) = I(t)R_R = V_0 \frac{R_R}{Z} \cos(\omega t - \phi). \quad (\text{VI-10})$$

We see from this equation that $V_R(t)$ is out of phase with respect to the driving (input) voltage $V(t)$ by an amount ϕ which can be either positive or negative according to whether the angular frequency is less than or greater than the resonance angular frequency ω_0 as seen by equation VI-2. At resonance, $Z = R$ (total resistance in the circuit) and $\phi=0$, which gives

$$V_R(t) \Big|_{\omega=\omega_0} = V_0 \frac{R_R}{R} \cos \omega t ,$$

in phase with the input signal and reduced by an amount which depends upon the “other” resistive elements in the circuit.

Turning to the capacitor C and inductor L , the voltage drops are:

$$\begin{aligned} V_C(t) &= I(t)(-iX_C) = V_0 \frac{X_C}{Z} e^{-i\pi/2} e^{i(\omega t - \phi)} \\ &= V_0 \frac{X_C}{Z} \cos(\omega t - \phi - \pi/2) \\ &= +V_0 \frac{X_C}{Z} \sin(\omega t - \phi) \end{aligned} \quad (\text{VI-11})$$

and

$$\begin{aligned} V_L(t) &= I(t)(iX_L) = V_0 \frac{X_L}{Z} e^{i\pi/2} e^{i(\omega t - \phi)} \\ &= V_0 \frac{X_L}{Z} \cos(\omega t - \phi + \pi/2) \\ &= -V_0 \frac{X_L}{Z} \sin(\omega t - \phi) \end{aligned} \quad (\text{VI-12})$$

Using $X_L \equiv \omega L$, $X_C \equiv 1/\omega C$, $\tau_L \equiv L/R$, and $\omega_0 \equiv 1/\sqrt{LC}$ we can write

$$\frac{X_L}{Z} = \frac{\omega L}{Z} = \frac{\omega}{\omega_0} \omega_0 \frac{L}{R} \frac{R}{Z} = \frac{\omega}{\omega_0} \omega_0 \tau_L \frac{R}{Z} = Q \left(\frac{\omega}{\omega_0} \frac{R}{Z} \right)$$

and

$$\frac{X_C}{Z} = \frac{1}{Z\omega C} = \frac{\omega_0}{\omega} \frac{L}{\omega_0 LC} \frac{R}{RZ} = \frac{\omega_0}{\omega} \frac{1}{\omega_0 LC} \frac{L}{R} \frac{R}{Z} = \frac{\omega}{\omega_0} \omega_0 \tau_L \frac{R}{Z} = Q \left(\frac{\omega_0}{\omega} \frac{R}{Z} \right)$$

to get the following set of equations for the voltage drop across the resistor R_R , the inductor L , and the capacitor C :

$$V_R(t) = V_0 \frac{R_R}{Z} \cos(\omega t - \phi)$$

$$V_L(t) = -QV_0 \left(\frac{\omega}{\omega_0} \frac{R_R}{Z} \right) \cos(\omega t - \phi)$$

$$V_C(t) = +QV_0 \left(\frac{\omega}{\omega_0} \frac{R_R}{Z} \right) \sin(\omega t - \phi)$$

Figure VI-4 shows the voltages as a function of ωt . The input voltage V_{IN} (dot dashed line) starts at V_0 when $t=0$. The voltage drop across the resistor V_R is offset by the phase angle $\phi = 0.5$. Since in this case ϕ is positive, V_R lags V_{IN} . The voltage across the capacitor V_C and inductor V_L are shown as dashed and solid lines respectively. Note that these are completely out of phase with each other, and both are out of phase with respect to V_R (by an amount $\pm\pi/2$).

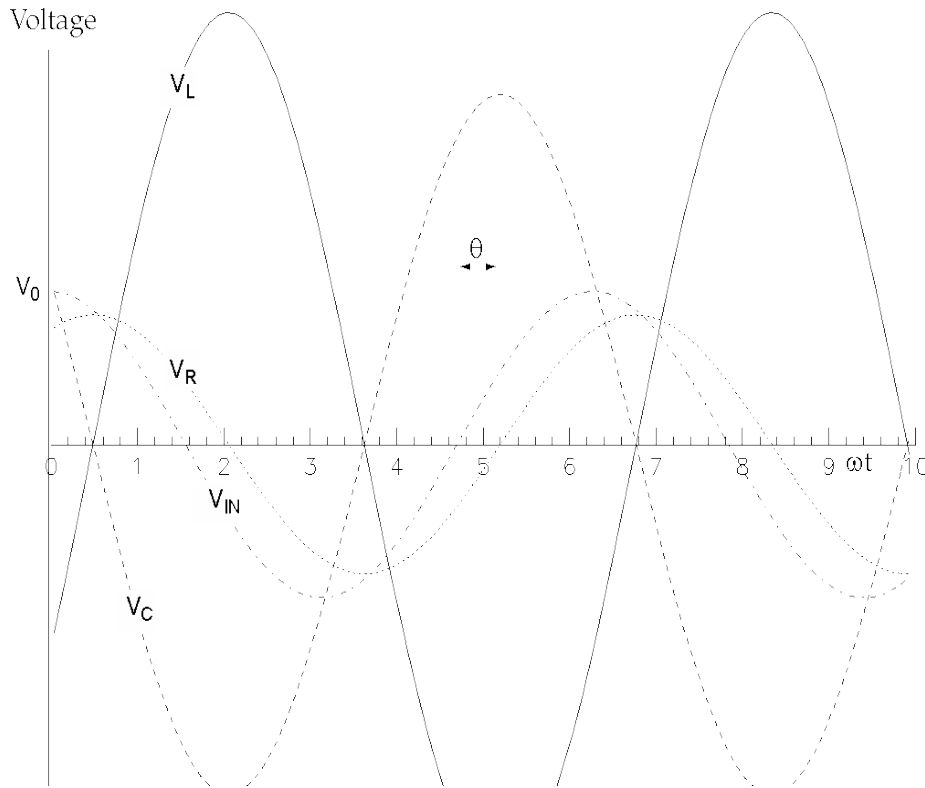


Figure VI-4: Behavior of the voltages across the various circuit elements. Note that all waveforms have the same frequency.

Notice that the amplitudes for V_C and V_L are both greater than that for V_{IN} . To understand what is going on, consider what happens at resonance where $X_L = X_C$, $\omega = \omega_0$, and $\phi = 0$. The equations for the voltage drops are

$$V_R(t)\big|_{\omega=\omega_0} = V_0 \frac{R_R}{R} \cos \omega_0 t$$

$$V_L(t)\big|_{\omega=\omega_0} = -QV_0 \frac{R_R}{R} \sin \omega_0 t$$

$$V_C(t)\big|_{\omega=\omega_0} = +QV_0 \frac{R_R}{R} \sin \omega_0 t.$$

Written this way, we can clearly see that at resonance, the amplitude for V_L and V_C are proportional to the driving amplitude V_0 and Q , which is “tuned” to be as large as possible. Figure VI-5 shows the corresponding curves. Notice that V_R is in phase with V_{IN} , but with a slightly lower amplitude. This is because V_R is proportional to the voltage drop across the resistor R_R , whereas the impedance Z takes into account the total real resistance (R) in the circuit. For these cases, R_R is taken to be 80% of the total resistance R .

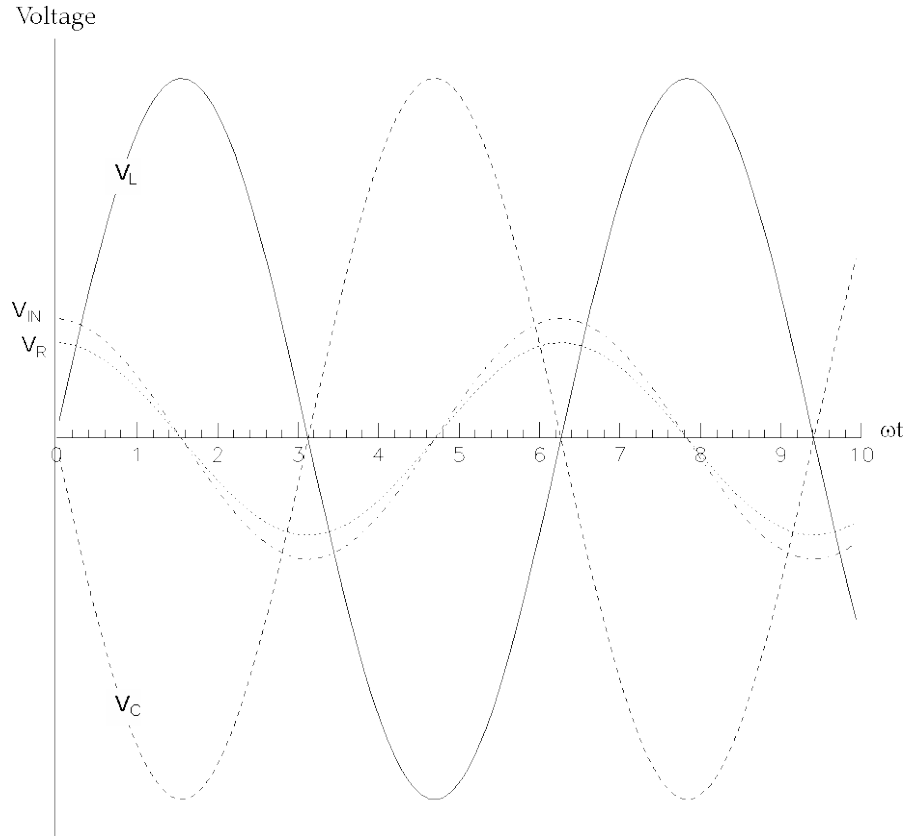


Figure VI-5: Voltages when the circuit is resonant

The large amplitudes for V_L and V_C can be understood by turning to classical mechanics. Imaging a mass m on a spring with spring constant k , driven by an external driving force $F(t)$ at some angular frequency ω . The equation of motion is:

$$F(t) = m\ddot{x} + kx$$

where x measures the displacement from equilibrium. The system will oscillate with greatest amplitude when $\omega = \omega_0$ with ω_0 , the natural frequency given by $\omega_0 = \sqrt{k/m}$. Now, add a piece of cardboard attached to the mass, with a large area such that wind resistance comes into play. The wind resistance force will be proportional to the velocity, which modifies the equation of motion to be:

$$F(t) = m\ddot{x} + kx + b\dot{x}$$

The wind resistance is a “damping” term $b\dot{x}$: all energy will be lost only through this term, not from the “inertial” term $m\ddot{x}$ or the “elastic” term kx . In words, the equation of motion is then:

Driving Force ($F(t)$) = inertial term ($m\ddot{x}$) + elastic force (kx) + damping force ($b\dot{x}$)

Now turn back to the RLC circuit, and use Kirchoff’s law with regards to voltages around the loop and write the differential equation:

$$\begin{aligned} V_{IN}(t) &= V_L + V_C + V_R \\ &= L\dot{I} + \frac{q}{C} + IR \\ &= L\ddot{q} + \frac{1}{C}q + R\dot{q} \end{aligned}$$

where q is the charge on the capacitor plate and $I = \dot{q}$. We can then equate V_L with the inertial term (inertia involved in building up the magnetic field inside the inductor), V_C with the “elastic” term (the elasticity involved in moving an equal and opposite charge on and off the plates of a capacitor) and V_R with the “damping” term (the resistor dissipates energy whether the current goes in one end and out the other or vice versa). All energy is dissipated through the resistor. Note that in any mechanically oscillating system, it is quite natural for the resonance oscillations to be larger in amplitude than the driving force: energy is being added to the system little by little, dissipation is minimal, and the energy causes the total oscillation amplitude to build up. The classic example of this mechanical system is a person on a swing, being pushed. The pushing ($F(t)$) amplitude need not be large compared to the oscillation, the system builds up a lot of energy at resonance.

IV. The AC LRC circuit

Part A: Initial Setup

A-1: Set up a circuit with an inductor, resistor and capacitor in the loop as in Fig. VI-1. For components, start with a 1 k Ω resistor, a 0.01 μ F capacitor, and the 200 mH inductor. As input, use a 2 kHz sine wave with a maximum amplitude of about 1.5 V. If necessary, eliminate any DC offset of the input voltage by using the DC offset adjust.

For all measurements, be sure that all inputs to the oscilloscope are DC coupled.

A-2: Measure the values of L and C using the LCR meter. Measure the value of the resistor R_R and the resistance of the inductor (R_L) using the digital multimeter. (The inductor consists of a wound wire which has some small resistance that will affect the total resistance of the circuit.) Determine the total resistance of the circuit, $R = R_R + R_L + R_0$, where R_0 is the internal resistance of the waveform generator (which is $50 \pm 2 \Omega$). From these measured values, determine the resonance (angular) frequency ω_0 and the gain Q .

Part B: Resonant frequency and Q value from voltages

B-1: Look at the input voltage V_0 and the voltage across the resistor, V_R , on the oscilloscope. Be sure to pay attention to where “ground” is located in your circuit and use the instrumentation amplifier if necessary. Determine the resonant frequency $f_0 = \omega_0/2\pi$ by looking for the frequency at which the phase shift ϕ goes to 0. Estimate the uncertainty in f_0 from your estimate of how well you can determine the zero crossing.

B-2: Measure the amplitude of V_R at the resonant frequency, and then find the two frequencies f_{\pm} at which the amplitude drops to $1/\sqrt{2}$ of its maximum value. Record these two frequencies, and compute the width $\Delta\omega = \omega_+ - \omega_-$ of the resonance. From the width and the determination of ω_0 , determine the Q value and its uncertainty. Compare this value to what you got in part A, and comment on any discrepancy.

Part C: Amplitude and phase vs. frequency

C-1: Vary the input frequency using the following values:

$$f = f_0 \times (0.1, 0.3, \mathbf{0.5}, 0.7, \mathbf{0.9}, 0.95, \mathbf{1.0}, 1.05, 1.075, \mathbf{1.1}, 1.3, \mathbf{1.5}, 1.7, \mathbf{1.9}, 2.0, 2.3)$$

and use your measured f_0 from B-1 in the above formula. For each value, record the amplitudes of V_0 , V_R , as well as the frequency f and the phase shift ϕ (from the time shift of the zero crossings) between V_0 and V_R (keep track of the sign!). For the frequencies highlighted in bold, also record V_C and V_L .

C-2: Looking at the data taken in Part C-1, how does V_L - V_C vary with frequency? Also, describe the relationship between V_0 and V_R as a function of frequency.

C-3: Plot the phase shift ϕ vs. ω . From the zero crossing of this graph, determine the resonant frequency ω_0 . Qualitatively compare the value from your graph to what you computed in part A above.

C-4: Plot the amplitude of $V_R(t)$ vs. ω . From the graph, determine the resonance frequency ω_0 , and the width of the resonance $\Delta\omega$, and from these determine the Q value. Compare to the values determined in parts A and B and comment on any discrepancies. Why are these values so critical for the performance of a radio?