Explaining *Logarithms*

$$\log(x * y) = \log x + \log y$$

$$\log(x / y) = \log x - \log y$$

A Progression of Ideas Illuminating an Important Mathematical Concept
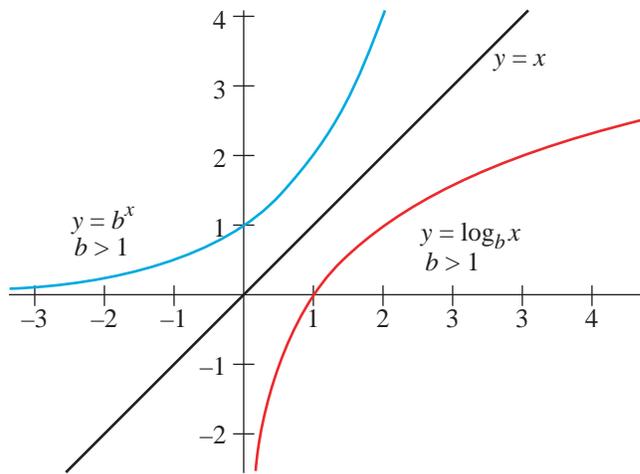
$$\log_b b^x = x$$

$$b^y = x \text{ is equivalent to } y = \log_b x$$

$$b^{\log_b x} = x$$

$$\log b^m = m \log b$$

$$\log_p x = \frac{\log_q x}{\log_q p}$$

By Dan Umbarger

www.mathlogarithms.com

Dedication

This text is dedicated to every high school mathematics teacher whose high standards and sense of professional ethics have resulted in personal attacks upon their character and/or professional integrity. Find comfort in the exchange between Richard Rich and Sir Thomas More in the play A Man For All Seasons by Robert Bolt.

Rich: "And if I was (a good teacher) , who would know it?"

More: "You, your pupils, your friends, God. Not a bad public, that …"

In Appreciation

I would like to acknowledge grateful appreciation to Mr. (Dr.?) Greg VanMullem, who authored the awesome freeware graphing package at *mathgv.com* that allowed me to communicate my ideas through many graphical images. A picture is truly worth 1,000 words.

Also a big "Thank you" to Dr. Art Miller of Mount Allison University of N.B. Canada for explaining the "non-integer factoring technique" used by Henry Briggs to approximate common logarithms to any desired place of accuracy. I always wondered about how he did that! Four colleagues, Deborah Dillon, Hae Sun Lee, and Fred Hurst, and Tom Hall all graciously consulted with me on key points that I was unsure of. "Thank you" Paul A. Zoch, author of Doomed to Fail, for finally helping me to understand the parallel universe that we public high school teachers are forced to work in. "Thank you" Shelley Cates of thetruthnetwork.com for helping me access the www. And the biggest "Thank you" goes to John Morris of Editide (info@editide.us) for helping me to clean up my manuscript and change all my 200 dpi figures to 600 dpi. All errors, however, are my own.

Explaining *Logarithms*

A Progression of
Ideas Illuminating an
Important Mathematical Concept

By Dan Umbarger

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John Napier, <u>Canon of Logarithms</u>, 1614

   "Seeing there is nothing that is so troublesome to mathematical practice, nor doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hindrances….Cast away from the work itself even the very numbers themselves that are to be multiplied, divided, and resolved into roots, and putteth other numbers in their place which perform much as they can do, only by addition and subtraction, division by two or division by three."

   As quoted in "When Slide Rules Ruled" by Cliff Stoll, <u>Scientific American Magazine</u>, May 2006, pgs. 81

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Foreword

Many, if not most or all, high school math and science teachers have had the experience of hearing a student exclaim something comparable to the following: "$234 \times 4{,}192 = 8{,}219$ because the calculator said so." Clearly the magnitude of such a product should have at least 5 places past the leading digit, $200 \times 4{,}000 = 800{,}000$ … 2 zeros + 3 zeros = 5 zeros, etc. That's not "rocket science." While only a savant can perform the exact calculation above in their heads most educated people can estimate simple expressions and "sense" when either bad data was entered into the calculator (GIGO—garbage in, garbage out) or that the order of operation for an expression was incorrectly entered. Similarly I have read of an experiment whereby calculators were wired to give answers to multiplication problems that were an order of magnitude off and then given to elementary students to see if they noticed the errors. They didn't.

What is happening here? Many people would say that the culprit is the lack of *number sense* in our young people. They say that four-function calculators are given to students too early in the grade school before number sense is developed. There is a school of thought that abstraction, a component of number sense, must be developed in stages from concrete, to pictorial, to purely abstract. Learning that $5 + 2 = 7$ needs to start with combining 5 coins (popsicle sticks, poker chips, etc.) with 2 coins resulting in 7 coins. From that experience, the student can proceed to learn that the photographic/pictorial images of 5 coins (popsicle sticks, poker chips, etc.) combined with the photographic/pictorial images of 2 coins results in 7 coin images. Similarly, 5 tally marks combined with 2 tally marks results in 7 marks. Finally, one internalizes the abstraction $5 + 2 = 7$ … concrete, pictorial, abstraction … concrete, pictorial, abstraction. Giving calculators too early in an attempt to shortcut the learning progression robs the student of the chance to learn or internalize number sense. The result of not being required to develop number sense and not memorizing the basic number facts at the elementary school level manifests itself daily in upper school math and science classrooms. There are people responsible who should know better. An "expert" for math curriculum for a local school district attaches the following words of wisdom to every email message she sends: "Life is too short for long division!!" … but I won't even go there.

Calculators make good students better but they do not compensate for a lack of number sense and knowing the basic number facts from memory. They do not make a poor math student into a good one!

The introduction of the handheld "trig" calculator (four operations combined with all the trig and log and exp functions) into the math curriculum has had similar impact on the student's ability to learn concepts associated with logarithms. Thank the engineers at HP and TI for that! Life **is** too short to spend on log tables, using them to find logs and antilogs (inverse logs), and interpolating to extend your log table decimal value from four positions out to five! Yuck! However, by completely eliminating the traditional study of logarithms, we have deprived our students of the evolution of ideas and concepts that leads to deeper understanding of many concepts associated with logarithms. As a result, teachers now could hear

"$(5.2)^y = 30.47$, $y = 6.32$ because the calculator says so," ($5^2 = 25$ for goodness sakes!!)
or "$y = \log_{4.8}(714.6)$, $y = 22.9$ because the calculator says so." ($5^4 = 625$, $5^5 = 3125$!!)

Typically, today's students experience teachers incanting: "The log of a product is the sum of the logs." "The log of a quotient is the difference of the logs." The students see the rules

$$1.) \quad \log(a \times b) = \log a + \log b \quad \text{or}$$

$$2.) \quad \log\left(\frac{a}{b}\right) = \log a - \log b \quad \text{or}$$

$$3.) \quad \log b^m = m \log b$$

with little development of ideas behind them or history of how they were used in conjunction with log tables (or slide rules which are mechanized log tables) to do almost all of the world's scientific and engineering calculations from the early 1600s until the wide-scale availability of scientific calculators in the 1970s. All three of these rules were actually taught in Algebra I, but in another format. Little effort is made in textbooks to make a connection between the Algebra I format (rules for exponents) and their logarithmic format. It is just assumed that the student sees and understands the connection. With the use of log tables and slide rules there was a daily, although subtle, reminder of the connection between these three rules and their parallel Algebra I "Rules of exponents."

Algebra 1 Rule	Associated Log Rule
$b^m * b^n = b^{m+n}$	$\log_b(m * n) = \log_b m + \log_b n$
$b^m / b^n = b^{m-n}$	$\log_b(m / n) = \log_b m - \log_b n$
$(b^m)^n = b^{mn}$	$\log b^m = m \log b$

"Black-box" calculator programming has obscured much of this connection. As a result, the progression of ideas associated with logarithms that existed for hundreds of years has been abbreviated. For really bright students, the curricular changes have not been a problem. For some students, however, the result has been confusion.

Let me give you a specific example. The following quote is taken verbatim from http://mathforum.org/library/drmath/view/55522.html (website viable June., 2010)

The Math Forum, "Ask Dr. Math." "*I have a bunch of rules for logs, properties and suchlike, but I find it hard to remember them without a proof. My precalculus book has no proof of why logs work or even what they are, nor does my calculus book. I understand what logs are … but I don't understand why they are what they are. Please help me.*"

 This plea for help is from a calculus student who (presumably) has credit on their transcript for mastery of precalculus!! Yet, clearly he or she does not even know enough about logarithms to articulate a question regarding what they would like to know.

My all time favorite magic log formulas are :

$$1.) \quad \log_b b^x = x$$

$$\text{and } 2.) \quad b^{\log_b x} = x$$

Where did those two formulas come from? There is some pretty simple logic behind these mysterious identities but teachers are always in a hurry to get to the "good stuff" … applying the rules to solve exponential equations with variable exponents. They don't have time or take the time to develop and explain these "rules". And most books are not helpful with their terse presentation of these ideas. These formulas are still vital even today. The calculator has not made them obsolete in the way that the four function calculator has rescued us from the tyranny of the log tables and all the drudgery associated with them. Without these formulas we cannot **knowledgeably** use our scientific calculators to solve equations of the form $(5.2)^y = 30.47$ or $y = \log_{4.8} (714.6)$. If the student does not understand the log rules, then he or she can still apply them and "get answers" just like the teacher. But unlike the teacher, some students really do not understand what is happening. If they make a severe error in their work they do not have the number sense that will enable them to catch unreasonable answers and they will be baffled in a later math class when the topic comes up again.   Chapter 2 is totally dedicated to understanding these two later rules.

All the formulas shown above just seem to appear in the math books like "Athena jumping out of the head of Zeus" … *deus ex machina*!!! There is none of the development of ideas and evolution of thought that used to exist in the high school curriculum. The high school pre-calculus teacher may understand fully what is going on with these formulas and ideas and the class genius may also but Joe Shmick and Betty Shmoe do not! Many students are just sitting there working with abstractions that have not been developed and fully understood. It's all magic … magic formulas and magic transformations. They are building "cognitive structures" without proper foundations.

When students do not fully understand mathematical ideas they tend to quickly forget all the tricks that got them past their unit test and that "knowledge" is not there when a later math teacher asks them to recall and apply it. Also they do not have the number sense to know when their answers are not reasonable.

Mathemagic is the learning of tricks that help a student to pass their immediate unit test. Mathemagic is confusing and quickly forgotten. Mathemagic is rigid. All problems that a student can solve using mathemagic must be in the exact same format as the problems the teacher used when teaching the unit. *Mathematics* is the learning **and understanding** of ideas, theories, and rules that stay with you for years or even decades and allow you to attack and solve problems that are not in the exact same format as the problems the teacher solved when teaching the material. Mathematics is a disciplined, organized way of thinking.

If a student fully understands the ideas behind working with logarithms, then correct answers, comfort with logarithmic situations, and multiyear retention will result. This is not an if-and-only-if relation. If a student can get correct answers on her/his immediate unit test that does not mean that s/he understood the concepts or that retention will occur so that the necessary recognition and skills will be there for the student should a future occasion (math, science, and business classes) require them.

The omnipresence of scientific calculators today means that even most teachers have not experienced the joys ☺ of working with log tables or working with a slide rule ☺. For the most part that is good. I would not wish my worst enemy to have to learn about logarithms the way I did, using log tables to find logs and anti-logs and interpolating to tweak out one more decimal value for both. There was also the special case situation of using a log table to determine the log of x where $0 < x < 1$. All the preceding was a real a "pain in the patootie" which we are spared today. The calculator allows us to concentrate on the application and not be distracted by the mechanics and minutia of the arithmetic! I do feel, however, that in the education world there is a need to develop the ideas and history associated with logarithms prior to expecting the students to work with them. Doing so will replace the mystery of the study of logarithms with a deep appreciation and understanding of log ideas and concepts that will stay with the student for an extended period of time. That is the motivation behind this material.

Note to Teachers

This text is not written for you. With the exception of parts of chapters 5, 6, and 7 and Appendix A, I assume that you already understand all the ideas presented. This is a book written for students who do not understand logarithms **even if they can apply the rules and get correct answers.** However, it would greatly gratify me if a teacher were to tell me that he or she enjoyed my organization and presentation.

I am a high school math teacher, not a mathematician. As such, I live and work in a world where sequence and progression of concepts leading to key ideas, along with pacing, "anticipatory sets," evolution and organization of ideas, reinforcement, examples and counterexamples, patterns, visuals, repeated threading and spiraling of concepts, and, especially, repetition, repetition, and repetition are all more important than *rigor*. It has always seemed ironic that authors and teachers, so knowledgeable about mathematical sequences, could be so insensitive and clumsy about the sequencing of curriculum … how they could be so knowledgeable about continuity of functions but so discontinuous in their writing.

There are plenty of materials available on teaching logarithms that are mathematically rigorous. I believe that "rigor before readiness" is counter-productive for all but the most gifted students. As such, I present many, many examples to help the student to see patterns and only then do I present the abstraction which will allow for generalization to all cases. Induction is a powerful teaching tool. Because of economy imposed by the publisher or perhaps because the material is so "obvious" to the authors most textbooks present the abstraction (generalization) first with little attempt to develop the rationale behind it or to connect the material to previous material such as the Algebra I Laws of Exponents or the history of logarithms. Those texts then proceed hurriedly to applying the abstraction to specific situations.

I believe that the best way to introduce a new idea is to somehow relate it to previous ideas the student has been using for some time. Using this approach, new concepts are an extension of previous ideas … a logical progression. Logarithms are a way to apply many of the laws of exponents taught in Algebra I. It is important that the students understand that!! **I also believe in introducing an idea in one chapter and revisiting that idea repeatedly in different ways throughout the book.**

The materials presented here are usually spread over two years of math instruction: precalculus and calculus. Doing so, however, separates ideas and examples that are helpful in the synthesis that leads to a deeper understanding of logarithms. For example, most high school text books seem to shy away from a meaningful discussion of why scientists and other professionals prefer to work with base *e*, the natural log, rather than the more intuitive common base, base 10. They do so because the pre-calculus student has not yet been exposed to the ideas that are necessary to justify the use of base *e*. If the goal is "rigor" then indeed many ideas associated with *e* must be postponed until calculus. But if your goal is to create familiarity with logarithms and appreciation of the number *e*, I do not believe that all that rigor is required. I have tried to bring all those ideas down to the pre-calculus level. I hope that I have done so. My approach, however, has been done at the expense of rigor. If I get consigned to one of the levels of Dante's <u>Inferno</u> because of my transgression it will be worth it if I am able to help young students past what, for me, was an unnecessarily difficult multiyear journey. When I did make an attempt at "rigor," I chose the formal two column proof over the abbreviated paragraph proof.

I see three different audiences for this text: 1.) students who have never worked with logarithms before, 2.) those students in calculus or science who did not manage to master logarithms during their algebra/pre-calculus instruction, and 3.) summer reading for students preparing for calculus. The former students will need to receive instruction, but the second and third group of students, if sufficiently motivated, should be able to read these materials on their own with little or no help. There are questions at the end of each chapter to use to evaluate student understanding. **Heavy emphasis is placed upon practicing estimation skills!!!**

Chapter 1: Logarithms Used to Calculate Products

For hundreds of years scientists and mathematicians did their calculations using the standard approach currently taught in elementary school.

```
    361                    5  3 . 1  1   7    etc.
  ×  25              17 | 9 0 3 . 0  0   0
  1 8 0 5                8 5
  7 2 2                  5 3
  9 0 2 5                5 1
                           2  0
                           1  7
                              3  0
                              1  7
                              1  3  0
                              1  1  9
                                 1  1    etc.
```

Not only were all these calculations tedious and prone to error, but the time spent in doing those calculations took away from the tasks requiring those calculations … astronomy, navigation, etc. People were always looking for a way to aid in the calculation process.

For now, define logarithms as a technique developed to aid in the drudgery of doing long and tedious calculations. In 1614, a Scottish mathematician, John Napier (1550–1617), published his table of logarithms and revolutionized the calculation process. (Joost Burgi, a Swiss watchmaker who interacted and worked with the famous astronomer Johann Kepler, also seems to have independently discovered logarithms, but Napier was the first to publish and he is usually given credit for their discovery and development.) For reasons that are distracting to the flow of ideas in this book, we will instead focus on the approach to logarithms by English mathematician Henry Briggs (1561–1630) who consulted with and was inspired by Mr. Napier's insight and original ideas.

The term logarithm is a portmanteau word … a word made of two smaller words. In this case, logarithm is made of two Greek words (*logos,* ratio and *arithmos*, number). In brief, a logarithm is nothing more than an exponent. In the equation $5^y = 10$ the "y" is a logarithm.

Logarithms Used to Multiply

For years, mathematicians had noticed a certain pattern held for sequences of exponentials with fixed bases.

For example:

Exponential	2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
Exponent	0	1	2	3	4	5	6	7	8	9	10
Value	1	2	4	8	16	32	64	128	256	512	1,024

Notice that $\quad 8 \times 32 = 256$

$\qquad\qquad 2^3 \times 2^5 = 256$

or $\qquad 2^3 \times 2^5 = 2^8$

or

Exponential	3^0	3^1	3^2	3^3	3^4	3^5	3^6	3^7	3^8
Exponent	0	1	2	3	4	5	6	7	8
Value	1	3	9	27	81	243	729	2,187	6,561

Notice that $\quad 9 \times 243 \quad = \quad 2,187$

$\qquad\qquad 3^2 \times \quad 3^5 \quad = \quad 2,187$

or $\qquad 3^2 \times \quad 3^5 \quad = \qquad 3^7$

From before $2^3 \times \quad 2^5 \quad = \qquad 2^8$

and now $\qquad 3^2 \times \quad 3^5 \quad = \qquad 3^7$

By induction, we move from the specific to the general:

Product of Common Base Factors Rule (rule applies for all m & n)
$$b^m \times b^n = b^{(m+n)}$$

Another way to think of this rule is to apply the definition of exponentiation ….

$$\overbrace{\qquad\qquad}^{\text{m times}}$$

Definition of exponentiation: $b^m = b * b * b * \ldots \qquad\qquad * b \quad$ (*b* times itself *m* times)

For example: $\qquad\qquad b^4 \qquad\quad \times \qquad\quad b^3 \qquad =$

$\qquad\qquad (b \times b \times b \times b) \quad \times \quad (b \times b \times b) = $ (definition of exponentiation)

$\qquad\qquad\quad b \times b \times b \times b \times b \times b \times b \qquad = $ (associative property of multiplication)

$\qquad\qquad\qquad\quad b^7 \qquad\qquad\qquad\quad$ (definition of exponentiation)

By transitive $\quad b^4 \times b^3 = b^7$ or $b^m \times b^n = b^{(m+n)}$ $\qquad$ **Product of Common Base Factor Rule**

When monomials with the same base are multiplied, one can obtain the result by adding the respective exponents. Napier (and later Briggs) saw from this pattern the possibility of converting a complicated, difficult multiplication problem into an easier, far less error-prone, addition problem. For example:

$\qquad\qquad 4,971.26 \qquad \times \qquad 0.2459 \quad =$

$\qquad\qquad 10^m \qquad\quad \times \qquad\quad 10^n \qquad = 10^{(m+n)}$

Where $\quad 3 \ < \ m \ < \ 4 \qquad\qquad$ and $\qquad -1 \ < \ n \ < \ 0$

Because $\qquad\qquad 10^4 \ = \ 10,000 \qquad$ and $\qquad 10^0 \ = \ 1$

$\qquad\qquad\qquad 10^m \ = \ 4,971.26 \quad$ and $\qquad 10^n \ = \ 0.2459$

$\qquad\qquad\qquad 10^3 \ = \ 1,000 \qquad$ and $\qquad 10^{-1} \ = \ \frac{1}{10} = 0.1$

This approach follows immediately from the pattern noted before

$b^m \times b^n = b^{(m+n)}$ $\qquad\qquad$ **Product of Common Base Factors Rule**

Mr. Briggs devoted a great deal of the last 20 years of his life to identifying those values of y whereby $10^y = x$. In the equation $10^y = x$, the exponent y came to be know as the logarithm of the number x using a base of 10, $y = \log_{10}(x)$. Hence $10^y = x$ is equivalent to $y = \log_{10} x$. For example, $\sqrt{10} = 10^{(1/2)} = 10^{0.5} = 3.162277$. In English … "0.5 is the base 10 logarithm of 3.162277."

$10^y = x$ is equivalent to $y = \log_{10} x$ **Equivalent Symbolism Rule**

Appendix A goes into detail about some of the ingenious techniques Mr. Briggs used to develop his logarithmic information. The curious reader is referred there because a discussion of those ideas here would distract from the more important goal of explaining how logarithms were used to convert tedious multiplication problems into simpler addition problems.

Mr. Briggs organized his work into tables. Discussing that organization and adding the new vocabulary words (characteristic, mantissa, antilogarithm) necessary to use the table would also distract from the discussion at hand and is mostly omitted from this book. See Appendix A, pg. 1 for a hint. Suffice to say that in the table of logarithms that Mr. Briggs developed was information comparable to the following:

Logarithm	Exponent Form	Number	
0	10^0	1	$(\log_{10} 0 \quad = 1)$
0.08720	$10^{0.08720}$	1.222	$(\log_{10} 1.222 = 0.087)$
0.39076	$10^{0.39076}$	2.459	$(\log_{10} 2.459 = 0.39076)$
0.69644	$10^{0.69644}$	4.971	$(\log_{10} 4.971 = 0.69644)$
1	10^1	10	$(\log_{10} 10 \quad = 1)$

Thus, the problem originally posed can be evaluated as follows:

4,971.26	$\times$		0.2459		$=$	
4.97126	$\times$	10^3	$\times$	2.459	$\times$ $10^{(-1)}$ $=$	(scientific notation)
$10^{0.69644}$	$\times$	10^3	$\times$	$10^{0.39076}$	$\times$ $10^{(-1)} =$	(exponent values taken from table)
				$10^{3.08720} =$		$(b^m b^n b^o b^p = b^{(m+n+o+p)})$
				10^3	$\times$ $10^{0.08720} =$	(see $10^{0.08720}$ in box above)
				1,000	$\times$ 1.222 $=$	1,222

By calculator $4{,}971.26 \times 0.2459 = 1{,}222.432834$ which compares very favorably with the answer obtained using Mr. Briggs' logarithm technique. Three additional thoughts here: 1.) Mr. Briggs' log table had as many as 13 decimal places (more than a TI-83 calculator), which would have made our work greatly more accurate had we used his raw data. 2.) Scientists and engineers are usually happy with "close" answers as long as the answers are close enough for the work they are doing to succeed. The number 1.414213562 would make most engineers very happy, but for the mathematician only the $\sqrt{2}$ would be acceptable. 3.) There are complications involved in using a log table when finding the log of x when $0 < x < 1$. Fortunately, the scientific calculator saves us from having to deal with those complications. See Appendix D if you are curious about this matter.

Notice the relationship between Briggs' logarithmic approach to multiplying numbers and the form of math called *scientific notation*.

Multiply Avogadro's number by the mass of an electron. (It's probably not good science, but it is good math.)

Avogadro's number   ×   mass of an electron

$600{,}000{,}000{,}000{,}000{,}000{,}000{,}000 \times$ $0.0000000000000000000000000000009 =$

$6 \quad \times \quad 10^{23} \quad \times \quad 9 \quad \times \quad 10^{(-31)} =$

$54 \times 10^{(-8)} =$

$5.4 \times 10^{(-7)} = 0.00000054 \ \text{kg}$

Your turn. Use your scientific calculator to evaluate the following product using the logarithmic technique shown on the previous page. Use the × button on your calculator to check your work.

$274{,}246 \quad \times \quad 0.0005461 \quad =$

$10^m \quad \times \quad 10^n \quad =$

$10^{\log 274246} \quad \times \quad 10^{\log 0.0005461} \quad =$ (using calculator twice for log m and log n)

$10^{(\log 274246 \ + \ \log 0.0005461)} \quad =$ **Product of Common Base Factors Rule**

$$b^m \times b^n = b^{(m+n)}$$

etc.,   use your calculator to finish and check

(Using a log table to obtain the log of a number less than one (1) involves some ideas that used to be very important but which are all dealt with now by the black-box code inside those marvelous scientific calculators. For a more complete discussion, please see Appendix D.)

Evaluate using the rule $b^m \times b^n = b^{(m + n)}$. Use a calculator to determine necessary logs. Check your work.

1.) $3{,}451{,}234 \quad \times \quad 9{,}871{,}298{,}345 \quad =$

2.) $56{,}819{,}234{,}008 \quad \times \quad 0.004881234 \quad =$

3.) $0.00003810842 \quad \times \quad 0.000000089234913 \quad =$

It is important to make a connection between the **Product of Common Base Factors Rule** and a new rule that will be called the **Log of a Product Rule**:

$b^m \times b^n = b^{(m+n)}$	**Product of Common Base Factors Rule**
and $\quad \log(x \times y) = \log x + \log y$	**Log of a Product Rule**

These rules are two different forms of the same idea. The latter simply states that if two numbers x and y are being multiplied, they can both be expressed as exponentials with a common base. Once the exponents of the respective factors are added, the resulting exponent can be used to determine the result of the original problem by using that exponent sum as a power (antilog or inverse) of the common base. On the calculator, the antilog or inverse button is marked 10^x. **We use symbols to avoid convoluted statements like these!**

$$a \times b = 10^{(\log a + \log b)}$$

By **Product of Common Base Factors Rule**	By **Log of a Product Rule**

By **Product of Common Base Factors Rule**

$$5 \quad \times \quad 7 \quad = \quad x$$

$$10^{0.69897} \quad \times \quad 10^{0.84509} \quad = \quad x$$

$$10^{1.54406} \quad = \quad x$$

$$34.99935 \quad = \quad x$$

By **Log of a Product Rule**

$$5 \quad \times \quad 7 \quad = \quad x$$

$$\log (5 \quad \times \quad 7) \quad = \quad \log (x) \textbf{ iff Log Rule}$$
$$(\textbf{\textit{m}} = \textbf{\textit{n}}) \textbf{ iff } (\log m = \log n)$$

$$\log 5 \quad + \quad \log 7 \quad = \quad \log (x) \textbf{ Log of a Product}$$

$$0.69897 \quad + \quad 0.84509 \quad = \quad \log (x)$$

$$1.54406 = \quad \log (x)$$

Applying the intuitive rule $\textbf{\textit{m}} = \textbf{\textit{n}}$ **iff** $\textbf{10}^{\textbf{\textit{m}}} = \textbf{10}^{\textbf{\textit{n}}}$

$$10^{(1.54406)} = \quad 10^{\log (x)}$$

Finally applying the decidedly nonintuitive **Antilog (Inverse) Log Rule** … $10^{\log x} = x$ (<u>discussed later in chapter 2</u>) on the right side and a calculator on the left side $34.99935 = x$ (by calculator)

(Note to the reader. For all my work to fit on the page I restricted my precision to 5 decimals. Be assured that the use of 10 decimals does result in a product of 35 as would the use of Mr. Briggs' 13 place log tables.)

With practice, the steps shown at the right to calculate 5×7 can be shortcut as follows:

To multiply two numbers add their respective logs and take the antilog of the sum.

$$4971.26 \times 0.2459 = \text{antilog} (\log 4{,}971.26 + \log 0.2459)$$

Shortly after the appearance of log tables, two English mathematicians, Edmund Gunter and William Oughtred, had the insight to mechanize the process of obtaining log and antilog values. This picture shows a modern slide rule. The magic behind how the slide rule multiplies values is the rule $a \times b = \text{antilog} (\log a + \log b)$.

Source: The Museum of HP Calculators http://www.hpmuseum.org
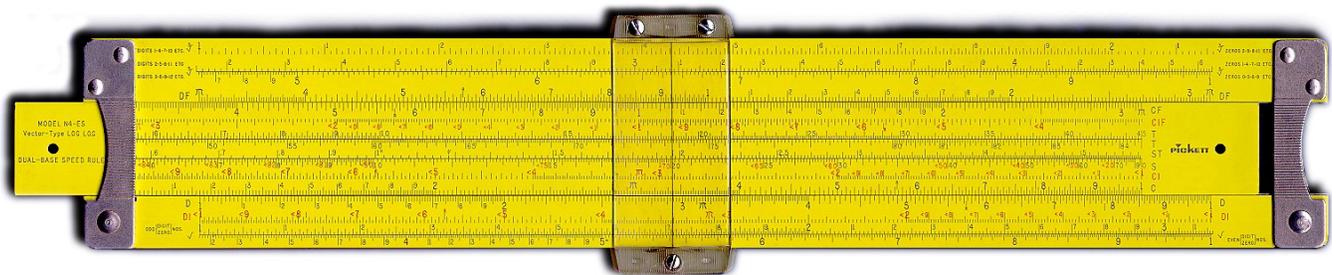
Chapter 1 Summary—**From the early 1600s to the late 1990s, one of the main applications of logarithms was to obtain the result of difficult or tedious multiplication problems through the easier, less error-prone operation of addition. Using log tables, one could multiply two numbers by adding their respective logs and taking the antilog of the sum.** (**_Do you see how awkward the wording of the procedure to use logarithms to multiply two numbers is? That is why we use rules. The use of symbolic rules allows us to focus on the process and ideas without getting confused with words_**)*. In the words of John Napier, "Cast away from the work itself even the very numbers themselves that are to be multiplied,... and putteth other numbers in their place which perform much as they can do, only by addition..." Source: John Napier, Cannon of Loagarthms in "When Slide Rules Ruled", by Cliff Stoll, Scientific American, May, 2006, pg. 83*

Symbolically

$a \times b =$ **antilog (log a + log b)**   **or** $a \times b =$ **inverse log of (log a + log b)**
(the antilog button is marked 10^x on some calculators and "inv log" on others)

$123 \times 4{,}567 = 10^{(\log 123 \,+\, \log 4567)}$ **or** $123 \times 4{,}567 =$ **inverse log of (log 123 + log 4,567)**

> The Algebra I rule $b^m \times b^n = b^{(m+n)}$ and the log rule
> $\log(a \times b) = \log a + \log b$ are two different forms of the
> same idea. <u>Although it is not proved they work for
> both integer and real values</u>

Just in case it slipped by you, the function $y = \log_{10} x$ is the inverse of the function $y = 10^x$ and the function $y = 10^x$ is the inverse of the function $y = \log_{10} x$.

1.) The function $y = \log_{10} x$ is the inverse of exponential function $y = 10^x$.
2.) The function $y = 10^x$ is the inverse of the log function $y = \log_{10} x$.

There is much, much more on this in chapter 2! The entire chapter 2 is written to clarify and emphasize these last two ideas!!

Log Rules through chapter 1	
$b^m \times b^n = b^{(m+n)}$	**Product of Common Base Factors Rule**
$\log(x \times y) = \log x + \log y$	**Log of a Product Rule**
$m = n$ iff $b^m = b^n$	**iff Antilog (10^x) Rule**
$m = n$ iff $\log m = \log n$	**iff Log Rule**
$b^y = x$ is equivalent to $y = \log_b x$	**Equivalent Symbolism Rule**

Chapter 1 Exercises

1.) Approximate $\log_{10} 285{,}962$ by bracketing it between two known powers of 10 as shown in Chapter 1.

$10^? \quad = \quad \underline{\hspace{2cm}}$

$10^? \quad = \quad 285{,}962$

$10^? \quad = \quad \underline{\hspace{2cm}}$

2.) Approximate $\log_{10} 0.000368$ by bracketing it between two known powers of 10 as shown in Chapter 1.

$10^? \quad = \quad \underline{\hspace{2cm}}$

$10^? \quad = \quad 0.000368$

$10^? \quad = \quad \underline{\hspace{2cm}}$

3.) Mentally approximate using the Equivalent Symbolism Rule, $10^y = x$ is equivalent to $y = \log_{10} x$. *Check yourself using a calculator.*

e.g., $\log_{10} 200 \approx 2\text{--}3 \quad$ because $\ 10^2 = 100 \ < \ 200 \ < \ 1{,}000 = 10^3$

a.) $\log_{10} 56$ b.) $\log_{10} 687$ c.) $\log_{10} 43{,}921$
d.) $\log_{10} 0.0219$ e.) $\log_{10} 0.0000038$ f.) $\log_{10} 0.00007871$

(There are special case ideas associated with using a log table to find the log of a number x, where 0 < x < 1. These ideas used to be important, but they are all dealt with by the black-box code inside those wonderful scientific calculators. See Appendix D if you are curious.)

4.) Using your calculator to obtain log values, multiply the following numbers using the technique $a \times b = 10^{(\log a + \log b)}$. Show each step as you would have had to do before calculators. Use your calculator, however, to obtain the necessary log and antilog values. Check yourself using the $\times$ button on your calculator.

a.) $4{,}526 \times 104{,}264 =$

b.) $0.061538 \times 40{,}126.7 =$

c.) $0.015872 \times 0.000000183218 =$

5.)   The Equivalent Symbolism Rule was presented as follows:

$10^y = x$ is equivalent to $y = \log_{10} x$ **Equivalent Symbolism Rule**

In this case, the base of the exponentiation is 10. In practice, it could be any number. More generally, the rule would look like the following:

$$b^y = x \text{ is equivalent to } y = \log_b x$$

Generalized Form Equivalent Symbolism Rule

Use the **Generalized Equivalent Symbolism Rule** to change each of the following equations into its "equivalent form."

a.) $y = 3^x$ f.) $y = \log_8 x$
b.) $5 = 2^x$ g.) $y = \log_3 x$
c.) $y = 7^x$ h.) $y = \log_7 x$
d.) $y = p^q$ i.) $8 = \log_2 x$
e.) $g = w^{3.2}$ j.) $9 = \log_x 11$

Chapter 2: The Inverse Log Rules

There is no escaping it … one must learn and feel comfortable applying several math rules when working with logarithms. These rules symbolize in abstract form very sophisticated ideas that cannot easily be put into a few words. We have already seen, discussed, and applied several. They are reviewed here along with a new one, the **iff Log Rule (if and only if Log Rule)**

Log Rules through chapter 1	
$b^m \times b^n = b^{(m+n)}$	**Product of Common Base Factors Rule**
$\log(x \times y) = \log x + \log y$	**Log of a Product Rule**
$m = n$ iff $b^m = b^n$	**iff Antilog Rule**
$m = n$ iff $\log m = \log n$	**iff Log Rule**
$b^y = x$ is equivalent to $y = \log_b x$	**Equivalent Symbolism Rule**

Two more rules, I call the Inverse Log Rules, are presented in most textbooks with only very terse explanation or clarification:

1.)	$\log_b b^x = x$ and	**Inverse Log Rule #1** (Log of an Exponential Rule)
2.)	$b^{\log_b x} = x$	**Inverse Log Rule #2** (Power of a Base Rule)

There are several ideas that build to an understanding of these *Inverse Log Rules*. For those readers who already know all this material, please skip ahead. I am not writing this material for you.

Idea #1: A function refers to two sets, called domain and range, together with a rule that matches each member of the domain to exactly one member of the range. ("Domain" refers to allowable x values while "range" refers to allowable y values.)

Rule

$$y = 3x + 1$$

x (domain)	y (range)
−2	−5
−1	−2
0	1
1	4
2	7
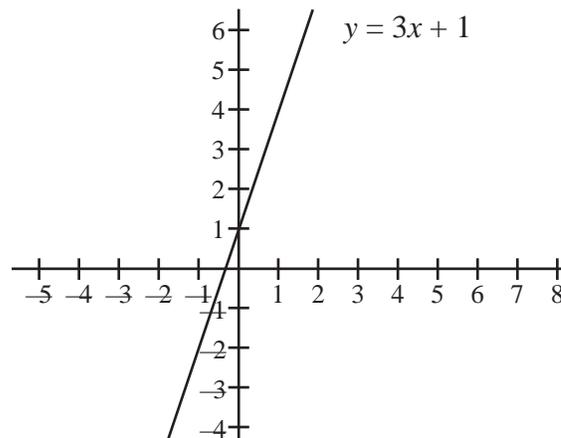
Idea #2: An inverse function, if it exists, of a given function can be found by exchanging the x and y variables in the given function. For $y = 3x + 1$, we get $x = 3y + 1$. We then traditionally solve this new equation for y ... $y = (x - 1)/3$. There is an interesting geometric relationship between the graph of the original function and the graph of it's inverse function. Both graphs are symmetric around the line $y = x$. If you fold the graph along the line $y = x$ the graph of both functions fall upon each other.

e.g., Original function $y = 3x + 1$
     Inverse function $x = 3y + 1$
        or $x - 1 = 3y$
        or $3y = x - 1$
        or $y = (x - 1)/3$
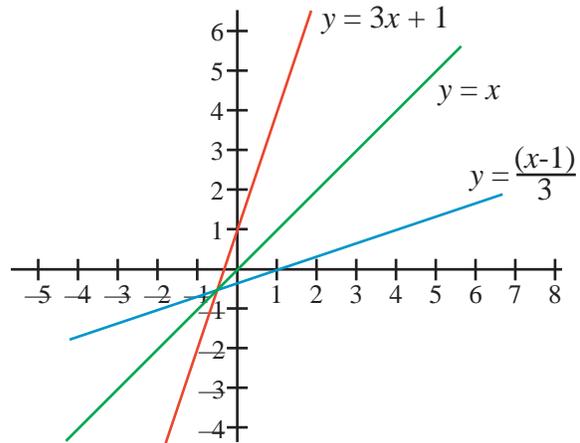
x (domain of inverse function)	y (range of inverse function)
−5	−2
−2	−1
1	0
4	1
7	2

Placing the table of (x, y) values for the original function ... $y = 3x + 1$ ... side by side with the table of

(x, y) values of the inverse function ... $y = \dfrac{x-1}{3}$ we notice that the x and y values of each pair have been exchanged.

Original function

$$y = 3x + 1$$

Inverse function

$$y = \frac{x-1}{3}$$

x (domain)	y (range)		x (domain)	y (range)
−2	−5	⇐compare⇒	−5	−2
−1	−2	⇐compare⇒	−2	−1
0	1	⇐compare⇒	1	0
1	4	⇐compare⇒	4	1
2	7	⇐compare⇒	7	2

This should not be surprising. The inverse function was formed by exchanging the x and the y in the original function. This is what causes the two graphs to be symmetric around the line $y = x$.

Idea #3 The exponential equation $y = b^x$ is a function.

e.g., $y = 2^x$ (base $b > 1$)

x	Y
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8
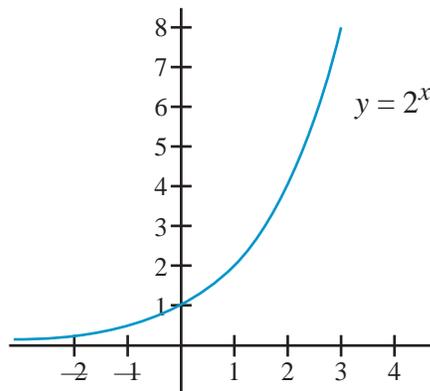
Idea # 4 Exchanging the x and y values in the exponential equation $y = b^x$ results in its inverse $x = b^y$. For graphing purposes, we traditionally solve equations for y. You enter graphing mode by pressing the "$y =$" button, right? You specify the graph you want graphed by filling in the "$y =$" field that results, right? We solve for y using techniques taught in Algebra I: 1.) the Addition/Subtraction Property of Equality, 2.) the Multiplication/Division Properties of Equality, 3.) a combination of the Addition/Subtraction Properties of Equality with the Multiplication/Division Properties of Equality, 4.) raising both sides of an equation to a power, and 5.) taking the root of both sides of an equation.

6.) How do we solve for y in the equation $x = b^y$? The techniques that we learned to solve for y in Algebra I all fail to solve an equation for "y" when it is an exponent.

(Sub. Prop. Of Eq.)	*(Div Prop. of Eq.)*	*(Sub. & Div. Prop. Of Eq.)*
1.) $\quad x = y + 2$ $x - 2 = y + 2 - 2$ $x - 2 = y$ $y = x - 2$	2.) $\quad x + 2 = 3y$ $\dfrac{x+2}{3} = \dfrac{3y}{3}$ $y = \dfrac{x+2}{3}$	3.) $\quad x = 3y + 2$ $x - 2 = 3y$ $\dfrac{x-2}{3} = \dfrac{3y}{3}$ $y = \dfrac{x-2}{3}$
(Square both sides)	*(Take the sq. root both sides)*	*(How to solve for y?)*
4.) $\quad x - 1 = \sqrt{y}$ $(x-1)^2 = (\sqrt{y})^2$ $y = (x-1)^2$	5.) $\quad y^2 = x - 5$ $\sqrt{(y^2)} = \sqrt{(x-5)}$ $y = \pm\sqrt{(x-5)}$	6.) $\quad x = b^y$ $???$ $y = ????$

This problem of solving for y in equation #6 above is overcome by what is essentially a definition. "y" is defined to be the exponent of a base (b) which results in a desired value (x). Hence, $x = b^y$ is equivalent to $y = \log_b x$. In this book, this is known as **The Equivalent Symbolism Rule.**

$b^y = x$ is equivalent to $y = \log_b x$ **Equivalent Symbolism Rule**

When you graph $b^y = x$ (a.k.a. $y = \log_b x$) you are basically graphing $b^x = y$ but with all the ordered pairs exchanged.

The graph at the right below shows the graphs of two functions—$y = 2^x$ and its inverse, $x = 2^y$—both plotted on the same x–y axis. Again notice that folding the graph along the line $y = x$ causes the two inverse functions to match up with each other. The two functions are symmetric around the line $y = x$. Notice the domain and range of $y = 2^x$ and notice that the domain and range restrictions have been exchanged for $x = 2^y$ (a.k.a. $y = \log_2 x$)

$y = 2^x$ (base $b > 1$) $x = 2^y$ or $y = \log_2 x$ (base > 1)

x	y
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

x	Y
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

($y = 2^x$
domain: all real
range: $y > 0$)

$x = 2^y$
($y = \log_2 x$
domain: $x > 0$
range: all real)
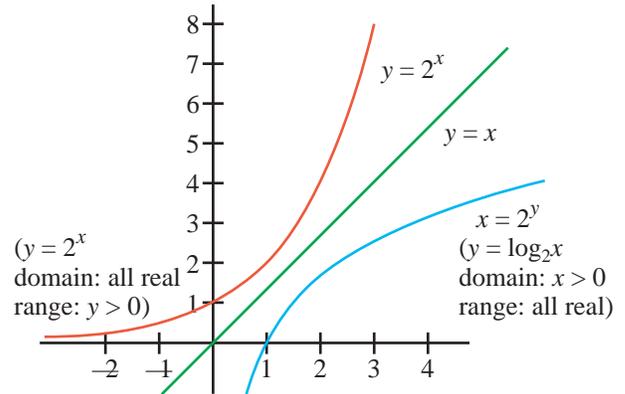
Think of the graph $b^y = x$ (a.k.a. $y = \log_b x$) as graphing $b^x = y$ but with all the ordered pairs exchanged.

Ideas #3 and #4 for base < 1

$y = \frac{1}{2}^x$
domain: all real
range: $y > 0$

$y = \log_{\frac{1}{2}} x$
$(x = \frac{1}{2}^y)$
domain: $x > 0$
range: all real

$y = \left(\frac{1}{2}\right)^x$

$x = \left(\frac{1}{2}\right)^y$

$\left(y = \log_{\frac{1}{2}} x\right)$

x	y
-2	4
-1	2
0	1
1	$\frac{1}{4}$
2	$\frac{1}{2}$
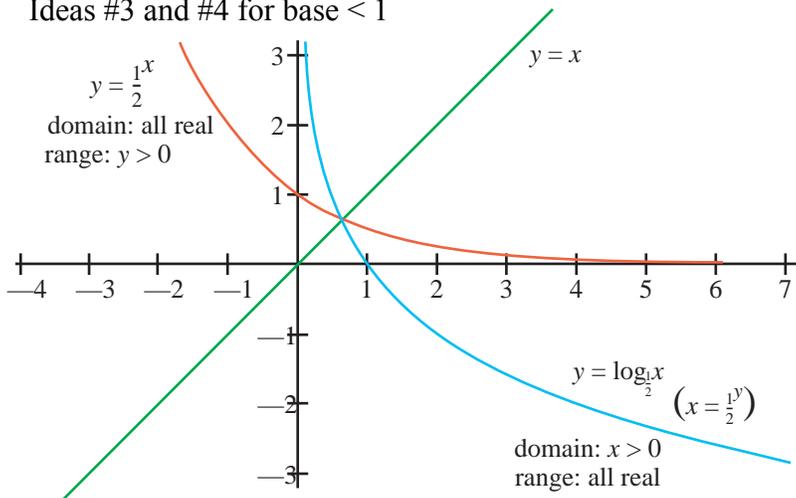
x	Y
4	-2
2	-1
1	0
$\frac{1}{4}$	1
$\frac{1}{2}$	2

**Think of the graph $b^y = x$ (a.k.a. $y = \log_b x$) as graphing $b^x = y$ but with all the ordered pairs exchanged.**

Repeating for emphasis:

1.) For the graph y = b^x, the domain is all real numbers and the range is positive.

2.) For the graph x = b^y (a.k.a. y = log$_b$ x), the domain is positive

**and the range is all real numbers.**

Idea #5 As we are discussing restrictions on the domain and range for the exponential and log functions, this would be a good time to discuss the restrictions on b … namely $b > 0$. What would it mean to have a function $y = b^x$ with $b < 0$? Let's experiment for $y = (-2)^x$. Recall that raising a negative number to an even power results in a positive value whereas a negative number raised to an odd power results in a negative result.

$$y = (-2)^x$$

x	Y
-2	$\frac{1}{4}$
-1	$-\frac{1}{2}$
0	1
1	-2
2	4
3	-8

$y = (-2)^x$

x is an integer
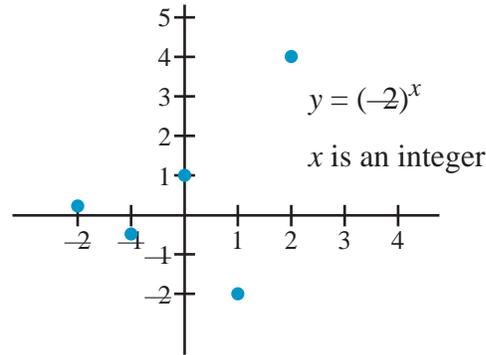
Is this function continuous? How do you connect these points? The chart above only shows x for selected integer values. The domain for $y = b^x$ is all real. What if we had fractions and decimals and irrational numbers for x in the chart of x–y values? Let's try an experiment.

Enter $(-2)^{(3/2)}$ or $(-2)^{(1.5)}$ or $(-2)^{\pi}$ into your calculator. Be sure to place parenthesis about the (-2). The TI-83 Plus gives ERR: Non-Real Answer. Now since the log function $y = \log_{(-2)} x$ is the inverse function of $y = (-2)^x$, what does all this discussion mean for our log function? Maybe we should just avoid the whole situation by requiring our base, b, to be nonnegative. What if $b = 0$? e.g., $y = 0^x$? Well, you can actually raise 0 to positive powers but 0^0 is not defined and for negative powers, $0^{-1} = 1/(0^1) = 1 / 0$, you get division by zero!! So clearly b must be positive in the two functions $y = b^x$ and $y = \log_b x$.

What about $b = 1$? b must be positive and we have seen graphs for both $y = b^x$ with $0 < b < 1$ and $y = b^x$ for $b > 1$. What would the graphs of $y = 1^x$ and its inverse $y = \log_1 (x)$ look like? $y = 1^x$ would actually be OK although it would be written more simply as $y = 1$, the special case horizontal line. For $y = 1^x$ exchange x and y resulting in $x = 1^y$ (a.k.a. $y = \log_1 x$) or more simply, $x = 1$. Notice that $x = 1$ is a vertical line and therefore not a function. A function cannot have more than one y value for any given x value. Obviously $y = \log_1 (x)$ fails the vertical line test and cannot be a function.
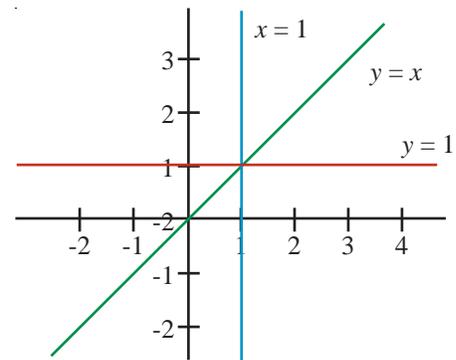
Conclusion: For $y = b^x$, $b > 0$. For $y = \log_b x$, $b > 0$ and $b \neq 1$.

One last thing!! The equation $y = b^x$ for $b < 0$ is not allowed, but that is not the same thing as $y = -(b^x)$ for $b > 0$. $y = -(b^x)$ is a reflection of $y = b^x$ about the x axis and is allowed. There will be more on this in chapter 9. Stay tuned.
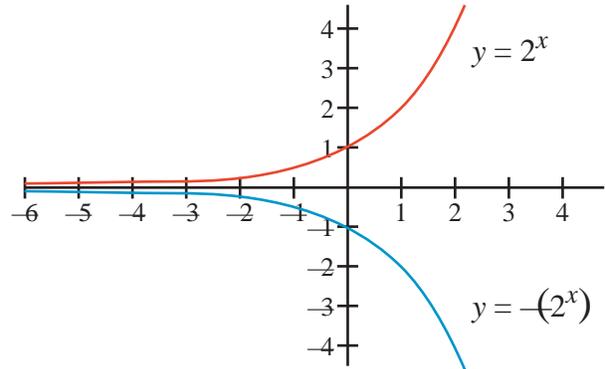
Let's review: Idea #4 … the domain for the exponential function is all real, the range for the exponential function is $y > 0$ … the domain for the log function is $x > 0$, the range for the log function is all real …. Idea #5 … the base requirement for the exponential function is $b > 0$, the base requirement for the log function is $b > 0$, $b \neq 1$. These ideas are all important, but they can be confusing. Let's use a chart to summarize and review them.

Function	Domain	Range	Base (b)
$y = b^x$	$-\infty < x < \infty$	$y > 0$	$b > 0$
$y = \log_b (x)$ $(x = b^y)$	$x > 0$	$-\infty < y < \infty$	$b > 0, b \neq 1$
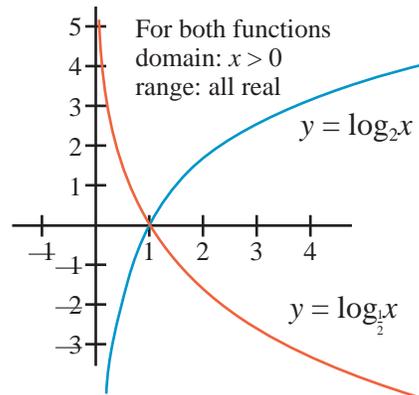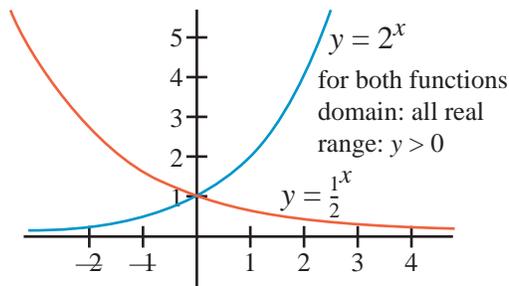
The fact that $b > 0$ for both the exponential and the log functions gives us another way to understand the domain and range restrictions on both those functions. For $y = b^x$, we see a positive number b ($b > 0$, remember?) raised to a power. Since exponentiation is repeated multiplication and the set of positive numbers is closed under multiplication, b^x must be positive. Therefore, the range (y values) of the exponential function is positive. For $y = \log_b (x)$ the base is also a positive number, $b > 0$, $b \neq 1$. It follows that $b^y = x$ means that a positive number is repeatedly multiplied so $b^y > 0$. Therefore, the domain (x values) of the function $y = \log_b x$ must be positive.

Idea #6: Composition of functions occurs when the result of one function is used as input to another.

e.g., $f(x) = 2x + 1$ $g(x) = 3x - 1$

x	$f(x)$	$g(f(x))$	$g(x)$	$f(g(x))$
−1	−1	**−4**	−4	**−7**
0	1	**2**	−1	**−1**
1	3	**8**	2	**5**
		*		*

Compare
$$g(f(x)) \neq f(g(x))$$

Idea #7: The order of composition of functions is important.
$g(f(x))$ might not equal $f(g(x))$.
In the chart above compare $g(f(x))$ with $f(g(x))$.

Also notice in the graph at right that the graphs of
$g(f(x))$ and $f(g(x))$ do not match up when folded across
the line $y = x$.
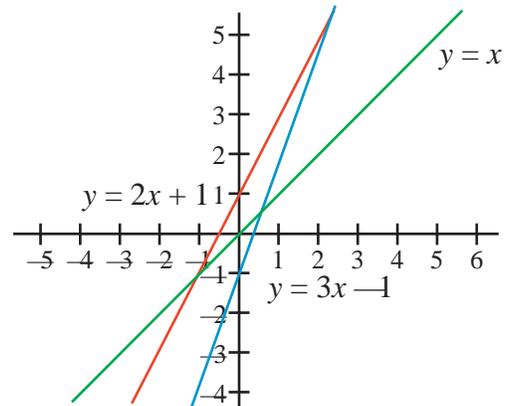
Idea #8: Sometimes the graphs of $f(x)$ and $g(x)$ do match
up when folded across the line $y = x$.

$f(x) = 3x + 1$ $g(x) = \dfrac{x-1}{3}$

x	$f(x)$	$g(f(x))$	$g(x)$	$f(g(x))$
−1	−2	**−1**	$-\frac{2}{3}$	**−1**
0	1	**0**	$-\frac{1}{3}$	**0**
1	4	**1**	0	**1**

Compare
$$x = g(f(x)) = f(g(x))$$
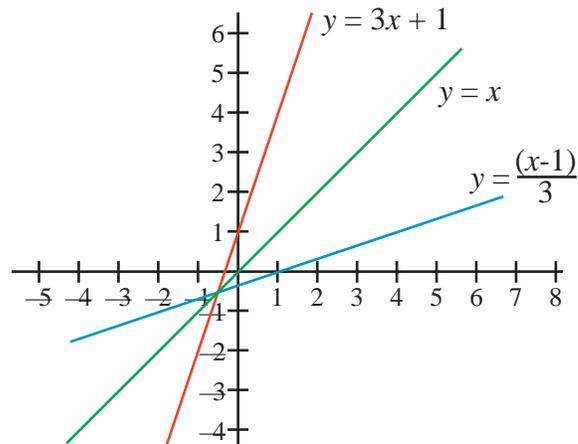
    Inverse functions are symmetric with the line $y = x$ and composition of inverse functions will result in
x regardless of the order of composition. That is, $x = f(g(x)) = g(f(x))$

Idea #9: The exponential function and the log function are inverse functions of each other. $y = b^x$ is an exponential function … $x = b^y$ the inverse …a.k.a. $y = \log_b x$, logarithmic form of the inverse
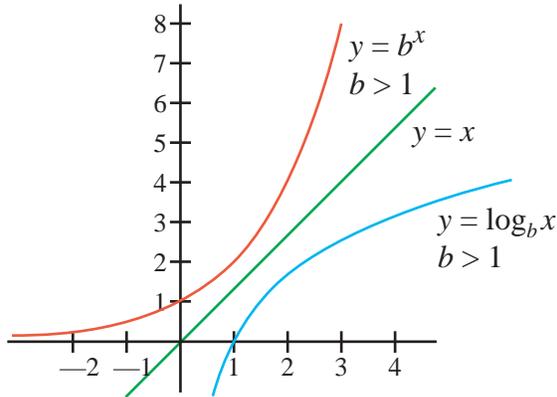
Let $f(x) = b^x$ exponential function
and $g(x) = \log_b x$ inverse of b^x in log form

Then $f(g(x)) = x$ (because they are inverse functions)   and $g(f(x)) = x$ (because they are inverse functions)

$\quad\quad f(\log_b x) = x$ $\quad\quad g(b^x) = x$

$\quad\quad b^{\log_b x} = x$ **Inverse Log Rule #2** $\log_b b^x = x$ **Inverse Log Rule #1**
$\quad\quad\quad\quad\quad\quad\quad$ **(Power of a Base Rule)** $\quad\quad\quad\quad\quad$ **(Log of an Exponential Rule)**

Restating the Inverse Log Rules together, we get

$$\log_b b^x = x \quad\quad \textbf{Inverse Log Rule \#1 (Log of an Exponential Rule)}$$
$$\text{and } \quad b^{\log_b x} = x \quad\quad \textbf{Inverse Log Rule \#2 (Power of a Base Rule)}$$

These rules pop up in the most unexpected situations. For example, refer back to the last few lines of chapter 1.

$$5 \quad\times\quad 7 \quad= x$$
$$\log(5 \quad\times\quad 7) \quad= \log(x) \quad\quad \textbf{iff Log Rule, } \boldsymbol{m = n} \textbf{ iff } \boldsymbol{\log m = \log n}$$
$$\log 5 \quad+\quad \log 7 = \log(x) \quad\quad \textbf{Log of a Product}$$
$$0.69897 \quad+\quad 0.84509 = \log(x) \quad\quad \text{by calculator}$$
$$1.54406 \quad\quad\quad = \log(x)$$

Applying the intuitive rule $\boldsymbol{m = n}$ **iff** $\boldsymbol{10^m = 10^n}$ (equivalent to saying if $3 = 3$ then $10^3 = 10^3$)

$$10^{(1.54406804)} \quad\quad = 10^{\log(x)}$$

*****   And the decidedly nonintuitive **Inverse Log Rule #2** … $10^{\log x} = x$ *********

$$34.99935 \quad\quad = x \quad\quad\quad \text{(using a calculator for } 10^{1.54406804}\text{)}$$

Following is an example of applying **Inverse Log Rule #1, $\log_b b^x = x$**

$$10^x \quad = \quad 35$$

$$\log_{10} 10^x \quad = \quad \log_{10} 35 \qquad \text{Taking the log of both sides, } \textbf{iff Log Rule} \ldots \textbf{\textit{m} = \textit{n} iff } \log_{10} \textbf{\textit{m}} = \log_{10} \textbf{\textit{n}}$$

$$x \quad = \quad \log_{10} 35 \qquad \textbf{Inverse Log Rule \#1 (Log of an Exponential Rule)}$$

$$x \quad = \quad 1.54406 \qquad \text{by calculator (ck: } 10^{1.54406} = 34.99935)$$

You should be aware that many textbooks and teachers will shortcut the previous work because they expect that you have fully internalized the log rules and are prepared for shortcuts.

Compare the two following approaches to solve $10^x = 35$:

As presented here | As frequently presented

1.) $\quad 10^x \quad = \quad 35$ 1.) $\quad 10^x = \quad 35$

2.) $\log_{10} 10^x \quad = \quad \log_{10} 35$ 2.) $\quad x \quad = \quad \log_{10} 35$

3.) $\quad x \quad = \quad \log_{10} 35$ 3.) $\quad x \quad = \quad 1.54406$

4.) $\quad x \quad = \quad 1.54406$ (ck: $10^{1.54406} = 34.99935171$)

The problem $10^x = 35$ is actually a bit contrived. The solutions shown immediately above would not be applicable if the problem had been $23^x = 35$…or $17^x = 100$…or $30^x = 2456$, etc.

$$23^x \quad = \quad 35$$

$$\log_{10} 23^x \quad = \quad \log_{10} 35$$

$$????$$

Here we can go no further as the Log of a Power Rule, $\log_b b^x = x$, cannot be applied to the situation $\log_{10} 23^x$. The base of the log must be the same as the base of the log's argument for the rule "$\log_b b^x = x$" to work. In a later chapter, we will learn how to solve for an exponent in an equation where this requirement is no longer necessary in order to solve for an unknown exponent (eg. . $23^x = 35$). That is called solving for a "general case logarithm."

Often when learning new rules, concepts, and ideas it is helpful to look at them in different ways. For example, on previous pages the two inverse log rules were shown to hold by function composition: $f(g(x)) = x$ and $g(f(x)) = x$. Here is another way to look at those same two rules.

I $\qquad y \quad = \quad y$

$\qquad\qquad b^y \quad = \quad b^y \qquad\qquad$ **iff Antilog Rule:** $\quad m = n \text{ iff } b^m = b^n$

$\qquad\qquad b^y \quad = \quad x \qquad\qquad$ arbitrary substitution, let $x = b^y$, you will see why in two more steps

$\qquad\qquad y \quad = \quad \log_b x \qquad\qquad$ **Equivalent Symbolism Rule** $b^y = x$ is equivalent to $y = \log_b x$

$\qquad\qquad y \quad = \quad \log_b b^y \qquad\qquad$ back substituting $y = b^x$ results in **Inverse Log Rule #1**
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ **The Log of an Exponential Rule**

$\qquad$ or $\quad x \quad = \quad \log_b b^x \qquad\qquad$ another arbitrary substitution, let $y = x$

II x = x

$\log_b x$ = $\log_b x$ **iff Log Rule** Take the log of both sides. This is like saying
$$100 = 100 \text{ iff } \log 100 = \log 100 \ (2 = 2)$$

$\log_b x$ = y arbitrary substitution, let $y = \log_b x$, you will see why in two more steps

b^y = x **Equivalent Symbolism Rule** $b^y = x$ is equivalent to $y = \log_b x$

Finally $b^{\log_b x}$ = x back substituting $y = \log_b x$ results in **Inverse Log Rule #2**
                                         The Power of a Base Rule

---

All the rules learned to this point are gathered together and listed below for reference

$b^m \times b^n = b^{(m+n)}$ **Product of Common Base Factors Rule**

$\log(x \times y) = \log x + \log y$ **Log of a Product Rule**

$m = n$ iff $b^m = b^n$ **iff Antilog Rule**

$m = n$ iff $\log m = \log n$ **iff Log Rule**

$b^y = x$ is equivalent to $y = \log_b x$ **Equivalent Symbolism Rule**

$\log_b b^x = x$ **Inverse Log Rule #1** (Log of an Exponential Rule)

$b^{\log_b x} = x$ **Inverse Log Rule #2** (Power of a Base Rule)

Chapter 2 Summary—People who write mathematics books have worked extensively over the years with logarithms and they tend to forget that there are people who do not have their background and familiarity with logarithms. The result is that they will omit steps in their explanations because the step was "obvious," expecting the reader to understand what was done. This is particularly the case with the two iff Log rules and the two Inverse Log rules.

$m = n$ iff $b^m = b^n$ **iff Antilog Rule**
                              $4 = 4$ iff $10^4 = 10^4$

$m = n$ iff $\log m = \log n$ **iff Log Rule**
                              $3 = 3$ iff $\log 3 = \log 3$

$\log_b b^x = x$ and   **Inverse Log Rule #1 (Log of an Exponential Rule)**

$b^{\log_b x} = x$ **Inverse Log Rule #2 (Power of a Base Rule)**

These latter two rules hold true because they are inverses of each other and hence, by the definition of inverse functions, $f(g(x)) = g(f(x)) = x$.

When reading passages talking about logarithms, one must constantly be on guard for applications of one of these "stealth" Inverse Log and iff Log rules.

Chapter 2 Exercises

1.) Given $y = 2x + 5$. Fill in the following chart and graph.

$y = 2x + 5$

x	y
-2	
-1	
0	
1	
2	

2.) Exchange the x and y variables in the equation $y = 2x + 5$ and solve for y. Use the values of y in the previous chart as your x values in the chart below, complete the chart.

$x = 2y + 5$

x	y

3.) Graph the relations for #1 and #2 above on the same x–y axis. What do you notice?

4.) Given $r(x)$ and $s(x)$ as inverse functions, complete the following statement.
$$r(s(x)) =$$
and
$$s(r(x)) =$$

5.) If two functions $f(x)$ and $g(x)$ are inverse functions then $f(g(x)) = g(f(x))$. Is this an "iff" (if and only if) relation? That is, "If $f(g(x)) = g(f(x))$, are $f(x)$ and $g(x)$ inverse functions? Do their graphs reflect about the line $y = x$?"

Hint: a.) Try with $f(x) = 3x$ and $g(x) = 3x$.
b.) Try with $f(x) = 2x$ and $g(x) = 3x$
c.) Try with $f(x) = x^2$ and $g(x) = x^3$.

6.) State the two **Inverse Log Rules** from memory.

7.) Given $p = q$ state the **iff Antilog Rule.**

8.) Given $p = q$ state the **iff Log Rule.**

9.) Convert each of the following using the **Equivalent Symbolism Rule**.
a.) $x = (-5)^y$ b.) $y = \log_{(-2)} 7$

10.) Use a scientific calculator to find the log of a number x, x > 1. Use the result as a power of 10. Repeat this activity a few times. What are you demonstrating?

Chapter 3: Logarithms Used to Calculate Quotients

For hundreds of years scientists and mathematicians did their calculations using the standard approach currently taught in elementary school.

```
    361                        5 3 . 1  1   7    etc.
  ×  25              17 │ 9 0 3 . 0  0   0
  1 8 0 5                  8 5
  7 2 2                    5 3
  9 0 2 5                  5 1
                             2  0
                             1  7
                                3  0
                                1  7
                                1  3   0
                                1  1   9
                                   1   1    etc.
```

The log tables and log rules that were so helpful in finding products can also be applied to quotients.

Logarithms Used to Find Quotients

For years, mathematicians had noticed a certain pattern held for sequences of exponentials with fixed bases.

For example:

Exponential	2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
Exponent	0	1	2	3	4	5	6	7	8	9	10
Value	1	2	4	8	16	32	64	128	256	512	1024

Notice that $32 \ / \ 8 \ = \ 4$

$2^5 \ / \ 2^3 \ = \ 4$

or $2^5 \ / \ 2^3 \ = \ 2^2$

or

Exponential	3^0	3^1	3^2	3^3	3^4	3^5	3^6	3^7	3^8
Exponent	0	1	2	3	4	5	6	7	8
Value	1	3	9	27	81	243	729	2187	6561

Notice that $2,187 \ / \ 27 \ = \ 81$

$3^7 \ / \ 3^3 \ = \ 81$

or $3^7 \ / \ 3^3 \ = \ 3^4$

From before 2^5 / 2^3 = 2^2

and now 3^7 / 3^3 = 3^4

By induction, we move from the specific to the general case: $\dfrac{b^m}{b^n} = b^{(m-n)}$

Another way to think of this rule is to apply the definition of exponentiation ….

$$\overbrace{}^{\text{m times}}$$

Definition of exponentiation: $b^m = b * b * b * \ldots$ $* b$ (*b* times itself *m* times)

For example: b^8 / b^3 =

$\dfrac{\cancel{b} \times \cancel{b} \times \cancel{b} \times b \times b \times b \times b \times b}{\cancel{b} \times \cancel{b} \times \cancel{b}}$ =   (definition of exponentiation)

$b \times b \times b \times b \times b$ = b^5

By transitive $b^8 / b^3 = b^5$ or

$\dfrac{b^m}{b^n} = b^{(m-n)}$ **Quotient of Common Bases Rule**

Although it is not proved this rule holds true for all m & n .

When monomials with the same
base are divided, one can obtain the result by subtracting the respective exponents. Napier (and later Briggs) saw from this pattern the possibility of converting a complicated, difficult division problem into an easier, far less error-prone, subtraction problem. For example:

4,971.26   /   0.2459   =

10^m / 10^n $=10^{(m-n)}$

Where   3 < *m* < 4   and   −1 < *n* < 0

Because 10^4 = 10,000   and 10^0 = 1

10^m =   4,971.26   and 10^n = 0.2459

10^3 =   1,000   and 10^{-1} = $\frac{1}{10}$ = 0.1

This approach follows immediately from
the pattern noted before

$\dfrac{b^m}{b^n} = b^{(m-n)}$ **Quotient of Common Bases Rule**

From previous discussion and from Appendix A, we know that from a table of logarithms (or today from a calculator) we can find the following information.

Logarithm	Exponent Form	Number	
0	10^0	1	($\log_{10} 1$ = 0)
0.30568	$10^{0.30568}$	2.022	($\log_{10} 2.022 = 0.30568$)
0.39076	$10^{0.39076}$	2.459	($\log_{10} 2.459 = 0.39076$)
0.69644	$10^{0.69644}$	4.971	($\log_{10} 4.971 = 0.69644$)
1	10^1	10	($\log_{10} 10$ = 1)

Thus the problem originally posed

$$\frac{4971.26}{0.2459} =$$

$$\frac{4.97126 \times 10^3}{2.459 \times 10^{(-1)}} =$$

$$\frac{10^{0.69644} \times 10^3}{10^{0.39076} \times 10^{(-1)}} = \qquad \text{(from the table on the previous page)}$$

$$\frac{10^{3.69644}}{10^{(-0.60924)}} = \qquad \textbf{Product of Common Bases Rule, } \boldsymbol{b^m \times b^n = b^{(m+n)}}$$

$$10^{(3.69644 - (-0.60924))} = \qquad \textbf{Quotient of Common Bases Rule, } \boldsymbol{\frac{b^m}{b^n} = b^{(m-n)}}$$

$$10^{4.30568} =$$

$$10^4 \times 10^{0.30568} = \quad 20{,}220 \quad (10^{0.30568} = 2.022 \text{ from the table on the previous page})$$

By calculator 4,971.26 / 0.2459 = 20,216.59211 which approximates the answer obtained using Mr. Briggs' logarithm technique. As stated before in chapter 1: 1.) Mr. Briggs' log table had as many as 13 decimal places, which would have made our work greatly more accurate had we used his raw data. 2.) Scientists and engineers are usually happy with "close" answers as long as the answers are close enough for the work they are doing to succeed. The number 1.414213562 would make most engineers very happy, but for the mathematician only $\sqrt{2}$ would be acceptable. 3.) There are special-case complications when using a log table to obtain the log of a number between 0 and 1. These issues are dealt with by the black-box code inside scientific calculators. (See Appendix D.)

Once again, notice the relationship between Briggs' logarithmic approach to dividing numbers and scientific notation.

Divide Avogadro's number by the mass of an electron. (It's probably not good science, but it is good math.)

Avogadro's number   /   mass of an electron

600,000,000,000,000,000,000,000 /   0.00000000000000000000000000000009

$$(6 \quad \times \quad 10^{23}) \quad / \quad (9 \quad \times 10^{(-31)}) \quad =$$

$$\tfrac{2}{3} \quad \times \quad 10^{54} \quad =$$

$$0.667 \times \quad 10^{54} \quad = 6.667 \times 10^{53} \text{ kg}^{-1}$$

Your turn. Use your scientific calculator to evaluate the following product using the logarithmic technique shown on the previous page. Use the $\times$ button on your calculator to check your work.

$$274{,}246 \quad / \quad 0.0005461 \quad =$$

$$10^m \quad / \quad 10^n \quad =$$

$$10^{\log 274246} \quad / \quad 10^{\log 0.0005461} \quad = \qquad \text{(using calculator twice for log } m \text{ and log } n\text{)}$$

$$10^{(\log 274246 \quad - \quad \log 0.0005461)} \quad = \qquad \textbf{Quotient of Common Bases Rule, } \boldsymbol{\frac{b^m}{b^n} = b^{(m-n)}}$$

etc.,   use your calculator to finish and check

Do the same for the following problems using the rule $\dfrac{b^m}{b^n} = b^{(m-n)}$. Use your calculator to obtain values m and n and $10^{(m-n)}$. Check yourself using the "/" operation on your calculator.

1.)   3,451,234   /   9,871,298,345   =

2.)   56,819,234,008 /   0.004881234   =

3.)   0.00003810842 / 0.000000089234913 =

It is important to notice that the two formulas,

$$\frac{b^m}{b^n} = b^{(m-n)}$$ 　　　　　　**Quotient of Common Bases Rule**

and　　　$$\log\left(\frac{x}{y}\right) = \log x - \log y$$ 　　　**Log of a Quotient Rule,**

are two different forms of the same idea. The latter simply states that if two numbers x and y are being divided they can both be expressed as exponentials with a common base. Once the exponents of the respective numbers are subtracted the resulting exponent can be used to determine the quotient of the original problem by using that exponent difference as a power (antilog) of the common base. ***This convoluted wording is a classic example of why we use symbols in math to communicate ideas.***

$$\frac{a}{b} = \text{antilog}(\log a - \log b)$$

$$\frac{a}{b} = 10^{(\log a - \log b)}$$

$$\frac{a}{b} = \text{inverse}\log(\log a - \log b)$$ 　　　**These rules apply to both integers and reals.**

The example from chapter 1 is recycled here to demonstrate this: 5 / 7

By **Quotient of Common Base Factors Rule**	By **Log of a Quotient Rule**
$\dfrac{5}{7} = x$	$\dfrac{5}{7} = x$
$\dfrac{10^{0.69897}}{10^{0.84509}} = x$	$\log\dfrac{5}{7} = \log(x)$ **iff Log Rule**
	$(m = n)$ iff $(\log m = \log n)$
$10^{(-0.14613)} = x$	$\log 5 - \log 7 = \log(x)$ **Log of a Product**
$0.7142824907 = x$	$0.69897 - 0.84509 = \log(x)$
	$-0.14613 = \log(x)$
	Applying the intuitive rule $m = n$ iff $10^m = 10^n$
	$10^{(-0.14613)} = 10^{\log(x)}$
	And the decidedly nonintuitive **Antilog Log Rule**
	$0.7142824907 = x$ (by calculator)

With practice the steps shown above to calculate 5 / 7 can be shortcut as follows:

$$\frac{5}{7} = \text{antilog (log } 5 - \text{log } 7) \text{ or } \frac{5}{7} = 10^{(\log 5 - \log 7)} \ldots \text{inverse log (log } 5 - \text{log } 7)$$

$$\frac{a}{b} = \text{antilog (log } a - \text{log } b) \text{ or } \quad \frac{a}{b} = 10^{(\log a - \log b)} \ldots \text{inverse log (log } a - \text{log } b)$$

To divide two numbers subtract their respective logs and take the antilog of the difference.

As stated in chapter 1, the development of the slide rule mechanized the process of obtaining logs and antilogs. The magic behind how the slide rule divides values is the rule $a / b = \text{antilog (log } a - \text{log } b)$.

Source: The Museum of HP Calculators http://www.hpmuseum.org
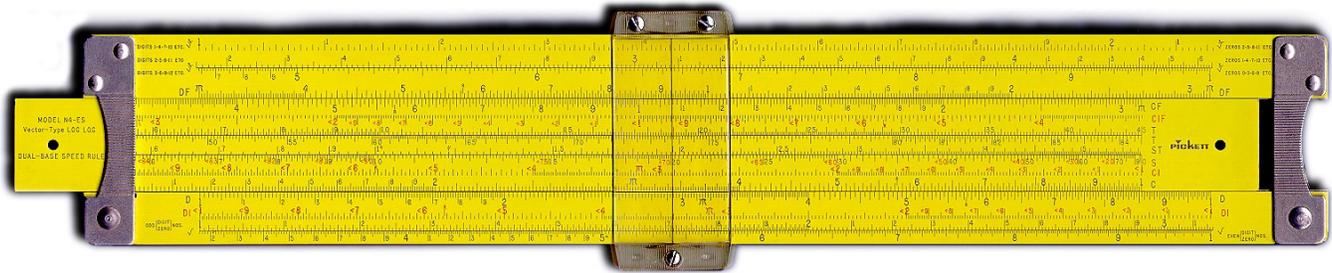
Chapter 3 Summary—From the early 1600s to the late 1990s, one of the main applications of logarithms was to obtain the result of difficult division problems through the easier, less error-prone operation of subtraction. To divide two numbers, subtract their respective logs and take the antilog (10^x) of the difference. *In the words of John Napier, "Cast away from the work itself even the very numbers themselves that are to be divided,… and putteth other numbers in their place which perform much as they can do, only by… subtraction…"* **Source: "When Slide Rules Ruled", by Cliff Stoll, Scientific American, May, 2006, pg. 83**

The Algebra I Rule, $\dfrac{b^m}{b^n} = b^{(m-n)}$, and the log rule,

$$\log\left(\frac{x}{y}\right) = \log x - \log y, \text{ are two different forms of the same idea.}$$

These rules apply to both integer and real numbers.

$$\frac{a}{b} = \text{antilog}(\log a - \log b) \qquad \frac{a}{b} = 10^{(\log a - \log b)} \qquad \frac{a}{b} = \text{inverse log}(\log a - \log b)$$

Chapter 3 Exercises

1.) Using your calculator to obtain log values, divide the following numbers using the **Rule to Divide Using Logarithms, $\dfrac{a}{b} = 10^{(\log a - \log b)}$.** Check yourself using the / operator on your calculator.

a.) $\dfrac{676}{94283}$ b.) $\dfrac{0.000000676}{94.283}$ c.) $\dfrac{6.76}{0.94283}$

For each quotient above what do you notice about the pattern of significant digits. Explain.

Chapter 4: Solving for an Exponent—The General Case

All the rules learned to this point are gathered together and listed below for reference

$b^m \times b^n = b^{(m+n)}$	**Product of Common Base Factors Rule**
$\dfrac{b^m}{b^n} = b^{(m-n)}$	**Quotient of Common Bases Rule**
$\log(x \times y) = \log x + \log y$	**Log of a Product Rule**
$\log\left(\dfrac{x}{y}\right) = \log x - \log y$	**Log of a Quotient Rule,**
$m = n$ iff $b^m = b^n$	**iff Antilog Rule**
$m = n$ iff $\log m = \log n$	**iff Log Rule**
$b^y = x$ is equivalent to $y = \log_b x$	**Equivalent Symbolism Rule**
$\log_b b^x = x$	**Inverse Log Rule #1** (Log of an Exponential Rule)
$b^{\log_b x} = x$	**Inverse Log Rule #2** (Power of a Base Rule)
$x \times y = 10^{(\log x + \log y)}$	**Rule to Multiply Using Logarithms**
$\dfrac{x}{y} = 10^{(\log x - \log y)}$	**Rule to Divide Using Logarithms**

In chapter 2, we showed how to solve for an exponent if the base was 10.

$$10^x \quad = \quad 35$$
$$\log_{10} 10^x \quad = \quad \log_{10} 35 \quad \text{Taking the log of both sides, } \textbf{iff Log Rule} \dots \textbf{\textit{m} = \textit{n} iff log}_{\textbf{10}} \textbf{\textit{m}} = \textbf{log}_{\textbf{10}} \textbf{\textit{n}}$$
$$x \quad = \quad \log_{10} 35 \quad \textbf{Inverse Log Rule \#1 (Log of an Exponential Rule)}$$
$$x \quad = \quad 1.54406 \quad \text{by calculator}$$

However, we were stymied, at that time, about how to solve for a general-case exponent where the base being exponentiated was not 10 .

$$23^x \quad = 35 \qquad (23^1 = 23 \text{ so clearly } 1 < x < 2)$$

$$\log_{10} 23^x \quad = \quad \log_{10} 35 \qquad \text{What next?}$$

Or even better, find $23^{1/7}$ (i.e., $\sqrt[7]{23}$). How do you proceed?

As has been stated before, for hundreds of years one of the main uses of logarithms was to obtain the answer of a difficult problem by somehow transforming the necessary calculation to an easier operation. We have seen how to obtain the answer to difficult multiplication problems by the easier operation of addition (of logarithms). We have seen how to obtain the answer to difficult division problems by the easier subtraction (of logarithms) operation. We are now going to find how to evaluate exponential situations by converting them to an easier multiplication operation. We start by reviewing the first log rule we learned:

$$\log(x \times y) = \log x + \log y \quad \textbf{Log of a Product Rule}$$

Everything about this rule screams out that it can be generalized as follows:

$$\overbrace{\qquad}^{m \ \text{times}} \qquad\qquad \overbrace{\qquad}^{m \ \text{times}}$$

$$\log\ b^m \ = \ b \times b \times b \times \ldots\ldots \times b \ = \ \log b + \log b + \log b + \ldots + \log b \ = \ m \log b$$

By the transitive rule, $\textbf{log } b^m = m \textbf{ log } b$ \qquad **Log of a Base Raised to a Power Rule**

> **The rules** \qquad **1.) $(b^m)^n = b^{mn}$ and 2.) log b^m = m log b**
>
> **are two different forms of the same idea!**
> **They apply to both integer and real m and n.**

Now this rule can be used to solve or evaluate the two problems posed on the previous page:

23^x	$=$	35	$23^{1/7}$	$=$	x	(i.e., $\sqrt[7]{23}$ = ?)

$$23^x \ = \ 35 \qquad\qquad 23^{1/7} \ = \ x \qquad (\text{i.e., } \sqrt[7]{23} = ?)$$
$$\log 23^x \ = \ \log 35 \qquad \log(23^{1/7}) \ = \ \log(x) \qquad \text{take the log of both sides}$$
$$x \log 23 \ = \ \log 35 \qquad \tfrac{1}{7}\log 23 \ = \ \log x \qquad \log(x) \textbf{ Log of a Base…}$$
$$x \ = \ \frac{\log 35}{\log 23} \qquad \log(x) \ = \ \tfrac{1}{7}\log 23 \qquad \text{Algebra Symmetry Rule}$$
$$x \ = \ \frac{1.544068044}{1.361727836} \qquad \log(x) \ = \ \tfrac{1}{7}(1.361727836) \ \text{by calculator}$$
$$x \ = \ 1.133903562 \qquad \log(x) \ = \ 0.194532548 \quad \text{by calculator}$$

Check
$$23^{1.133903562} \ = \qquad\qquad 10^{\log x} \ = \ 10^{0.194532548} \quad \text{taking the antilog of both sides}$$
$$34.999999950$$
$$x \ = \ 1.565065608 \quad \text{left: } \textbf{Power of a Base}, \text{ right: calculator}$$
$$\text{Check } 23^{(1/7)} \ = \ 1.565065608 \quad \text{by calculator}$$

Your turn:

Solve or evaluate the following using logarithm skills. Check yourself using a calculator.

1.) \qquad $5.97^x = 250$. Solve for x.

2.) \qquad Find $\sqrt[5]{824^3}$. Recall that $\sqrt[5]{b^3} = b^{3/5}$. Therefore x = $824^{(3/5)}$

Chapter 4 Summary—In this chapter we learned a new log rule

$\log b^m = m \log b$ <u>**Log of a Base Raised to a Power Rule**</u>

(The formulas $\log b^m = m \log b$ and $(b^m)^n = b^{mn}$ are two different forms of the same idea.

They apply to both integer and real numbers.)

---

and used it in two ways:

1) **we learned how to solve an equation with a variable exponent and arbitrary base. (This skill is still very relevant today!!!) and**

2) **we learned how *n*th roots and fractional roots were extracted for hundreds of years until the calculator gave us an alternative.   In the words of John Napier, "*Cast away from the work itself even the very numbers themselves that are to be…resolved into roots, and putteth other numbers in their place which perform much as they can do, only by… division by two or division by three*." Source: <u>Cannon of Logarithms</u> by John Napier, 1614, as quoted by Cliff Stoll, "When SlideRules Ruled", <u>Scientific American</u>, May, 2006, pg. 83.**

After chapter 2, we could only solve for variable exponents when the base was 10:

$$10^x = 20.$$

We now have a way to solve for the exponent of all exponential equations, not just the ones with a base of 10:

$$7^x = 10.$$

Also we have learned how to use logarithms to extract any desired integer root, $x^{1/5}$, or rational root, $x^{3/5}$. I won't even attempt to put this process (algorithm) into words. That is why we use symbols in math … to avoid having to put complicated ideas into words.

If $b^x = y$ **then** $x = \dfrac{\log y}{\log b}$ Recall the restrictions on b and log b

$23^x = 35$ **then** $x = \dfrac{\log 35}{\log 23}$

$b^{p/q} = x$ **then** $x = 10^{p/q \times \log b}$

$23^{1/7} = x$ **then** $x = 10^{1/7 \times \log 23}$

Chapter 4 Exercises

1.) Approximate $\log_4 200$ by bracketing it between powers of 4 as shown in chapter 1 for powers of 10.

$$4^? = \underline{\hspace{2cm}}$$
$$4^? = \quad 200$$
$$4^? = \underline{\hspace{2cm}}$$

2.) Approximate x in the following equation and then solve for a more precise answer applying the Log of a Base Raised to a Power Rule and a calculator.

$$10^x = 14{,}290$$

3.) Approximate $\log_{17} 14{,}290$ by bracketing it between powers of 17 as shown in chapter 1 for powers of 10.

$$17^? = \underline{\hspace{2cm}}$$
$$17^? = \quad 14{,}290$$
$$17^? = \underline{\hspace{2cm}}$$

 4.) Attempt to solve $17^x = 14{,}290$ using the iff Log Rule, $m = n$ iff $\log m = \log n$. What is the problem with your approach?

 5.) Solve the equation $17^x = 14{,}290$ using the **iff Log Rule** and the **Log of a Base Raised to a Power Rule.** Is your answer consistent with the work you did in #3?

6.) Check your work in #5 using a calculator … $17^{\text{your answer in \#5}} = $ ….

7.) Evaluate $\sqrt[5]{621}$ using the logarithmic approach. Check yourself using a calculator … $621^{1/5}$.

8.) Evaluate $\sqrt[9]{621}^{\,7}$ using the logarithmic approach. Check yourself using a calculator … $621^{7/9}$.

Chapter 5: Change of Base, *e*, the Natural Logarithm

Change of Base

So far the impression has been given that logarithmic representation of values are always given with a base of 10. Working with a base of 10 is intuitive to most people. For example, estimate log 450.

$$10^2 = 100$$
$$10^x = 450$$
$$10^3 = 1{,}000$$

Since $100 < 450 < 1{,}000$

$$10^2 < 10^x < 10^3$$

then $2 < x < 3 \ldots x = 2.\text{something}$

$$10^{2.\text{something}} = 450$$

Working with powers of 10 is easy because of the relationship between the power of ten and the last several digits of the result.

$10^2 = 100$,   power of $10 = 2$,   2 zeros after the 1
$10^5 = 100{,}000$,   power of $10 = 5$,   5 zeros after the 1

$10^{-1} = 0.1$,   power of $10 = -1$,   no zeros before the one   (abs $(-1) - 1 = 0$)
$10^{-2} = 0.01$ power of $10 = -2$,   1 zero before the one   (abs $(-2) - 1 = 1$)
$10^{-3} = 0.001$ power of $10 = -3$,   2 zeros before the one   (abs $(-3) - 1 = 2$)

However, the logarithmic base could be something other than 10. For example, you could use logarithms with a base 5. To do so one would have to indicate the fact that you are using a different base because the default base for working with logarithms is 10. For example, log 450 is understood to be $\log_{10} 450$. If you wanted to let other people know that you were assuming a base of 5, you would have to explicitly indicate the desired base as 5 somehow. The standard format to do so is **$\log_5 450$**.

Estimate $\log_5 450$

$$5^3 = 125$$
$$5^x = 450$$
$$5^4 = 625$$

Since $125 < 450 < 625$, and $5^3 < 5^x < 5^4$

then $\log_5 450 = 3.\text{something}$

Therefore **$5^{3.\text{something}} = 450$**

Exactly what is $\log_5 450$? Applying skills that have been developed in this book

$$5^x = 450$$
$$\log(5^x) = \log(450) \qquad \text{taking the log of both sides, \textbf{iff Rule of Logs}}$$
$$x \log(5) = \log(450) \qquad \textbf{Log of a Power Rule}$$
$$x = \frac{\log 450}{\log 5}$$
$$x = \frac{2.653212514}{0.6989700043} \qquad \text{by calculator}$$
$$x = 3.795888948$$

Check: $5^{3.795888948} = 450.000000691$

Summarizing, $\log_5 450 = \dfrac{\log 450}{\log 5}$ or $\dfrac{\log_{10} 450}{\log_{10} 5} = 3.795888948$

Summarizing, $\log_5 450 = \dfrac{\log 450}{\log 5}$ or $\dfrac{\log_{10} 450}{\log_{10} 5}$

or generalizing, $\log_p 450$ (for any base *p*) $= \dfrac{\log 450}{\log p}$ or $\dfrac{\log_{10} 450}{\log_{10} p}$

$$\boxed{\;\log_p x = \dfrac{\log_q x}{\log_q p}\quad \textbf{Change of Base Log Rule … change from base } \textbf{\textit{p}} \textbf{ into base } \textbf{\textit{q}}.\quad \textbf{(Often, base } \textbf{\textit{q}} \textbf{ is either 10 or } \textbf{\textit{e}}.)}$$

| **e and the Natural Log** | Now that we have seen that the choice of 10 as the base of a log function is arbitrary and based upon our predilection of working with powers of |

10, it is not so big a step to consider another base: *e* named after the famous Swiss mathematician, Euler.

e is an irrational number that begins as follows: 2.718281828.

Hence, while log (450) = 2.653212514 because $10^{2.653212514} = 450$,

$\qquad$ $\log_e (450)$ = 6.109247583 because $2.718281828^{6.109247583} = 449.999999642$

Because *e* is used so often as a base for work with logarithms, there is special symbolism to indicate its use … "ln." We write "ln *x*" instead of "$\log_e x$." In this case the omitted base of "ln *x*" is, by convention *e*, just as the omitted base of "log *x*" is 10.

$\qquad\qquad$ $\ln x = \log_e x$ and $\log x = \log_{10} x$

It is important to note that ln *e* = 1 because ln *e* = $\log_e e$ and by the Equivalent Symbolism Law $e^1 = e$.

This equivalence can be used in solving for the "6.109247583" that appeared magically above.

when obtaining $\log_e$ 450

$e^x = 450$ Equivalent Symbolism Rule

$\ln e^x = \ln 450$ iff ln Rule

x ln e = ln 450   ln of a Base Raised to a Power Rule

x (1) = ln 450   ln e = 1…here it is!

$\quad$ x = ln 450

$\quad$ x = 6.109247583 (by calculator)

$\quad$ **Why use base *e*? It seems like a very curious choice. The short answer is that there are numerous situations that arise in the physical world that involve *e* or that that involve a function with a base of *e*. Following are a few examples.**

1.) Continuous compound interest $S = P\,e^{rt}$

We could eliminate the e value and express this same idea in our comfortable base 10 as follows:

Set $e = 10^x$

$\log e = \log 10^x$ taking the log of both sides,   **iff Log Rule**

$\log e = x \log (10)$ **Log of a Power Rule**

$x = \dfrac{\log e}{\log 10}$ log (e) by calculator … can you evaluate log 10 mentally?

$x = 0.4342944819$

Check: $10^{(0.4342944819)} = 2.718281828$

So now the original formula $S = P\,e^{rt}$ could be rewritten with base 10 as

$$S = P\,(10^{0.4342944819})^{rt}$$

$$\text{or } S = P \times 10^{0.4342944819rt}$$

If we let $k = 0.4342944819$ then the equation $S = P\,e^{rt}$ with $e = 2.718281828$

$$\text{becomes } S = P\,(10)^{krt} \text{ with } k = 0.4342944819$$

We have in effect exchanged one mysterious number ($e = 2.718281828\ldots$) with another ($k = 0.4342944819\ldots$) which is itself derived from e. **In other words the expression of many real world phenomenon is dependent upon a sort of "universal constant" … e.**

2.) Exponential Growth and Decay

There are many quantities out there in the world that are governed (at least for a short time period) by the equation,

$$f = i \times a^{kt},$$

where f represents the final quantity, i represents the initial quantity, k represents a constant of proportionality, and t represents a unit of time. If k is positive, then the function will grow without bound and is called the **exponential growth** equation. Likewise, if k is negative the function will die down to zero and is called the **exponential decay** equation. **Often the base (a) in such equations is the number e. (see #3 and # 7 below!)**

3.) Bell/Normal Curve

The formula for the *bell curve* is clearly dependent on the value of the mysterious *e*.

$$y = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$$

The shape of the bell curve is like the silhouette of a bell, hence the name.
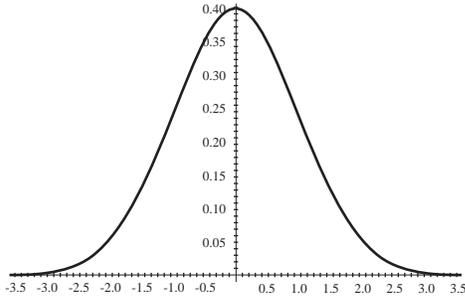
4.) Fast Fourier transform

A **fast Fourier transform** (**FFT**) is an efficient algorithm to compute the discrete Fourier transform (DFT) and its inverse. FFTs are of great importance to a wide variety of applications, from digital signal processing to solving partial differential equations to algorithms for quickly multiplying large integers. Let x_0, …, x_{n-1} be complex numbers. The DFT is defined by the formula

$$f_j = \sum_{k=0}^{n-1} x_k e^{-\frac{2\pi i}{n} jk} \quad j = 0,\ldots,n-1.$$

Note the *e* in this series definition.

Source: Wikipedia, the free encyclopedia

(http://en.wikipedia.org/wiki/Fast_Fourier_transform)

5.) Logarithmic spiral

The equation for the logarithmic spiral is $r = e^{at}$ (polar form with "*r*" = radius, "*a*" being a constant, and "*t*" = theta.)   Again notice the base *e*.

Alternatively, since $e = 10^{0.4342944819}$, the equation could be written with a base of 10 as $r = 10^{0.4342944819at}$. Notice the replacement of base *e* in the original formula results in a mysterious 0.4342944819 in the exponent. And, since that number is derived from solving $e = 10^x$, then we have just eliminated the mysterious number *e* by substituting another mysterious number derived from *e*.

$r = e^{0.10\,\theta}$
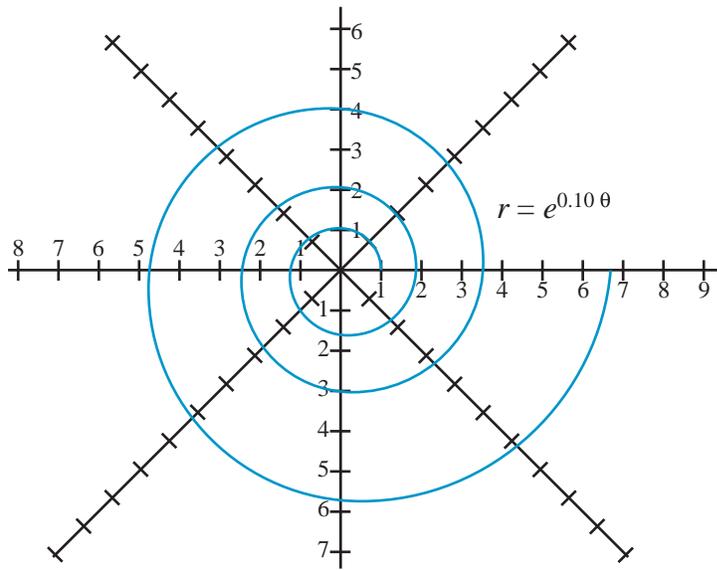
$$r = e^{at}$$

or

$$r = 10^{(0.4342944819at)}$$

6.) Catenary Curve, $y = \dfrac{e^x + e^{-x}}{2}$

When a flexible wire or chain is supported at each of its ends gravity will shape the wire or chain in what looks like a concave up parabola, $y - k = (x - h)^2$ but is actually a different curve called a *catenary* … derived from the Latin word for *chain*. The equation for a catenary curve is $y = (e^x + e^{-x}) / 2$. Notice the number *e* in the formula. *e* seems to be some sort of universal number built into the design of the universe. The St. Louis Arch closely approximates an inverted catenary.
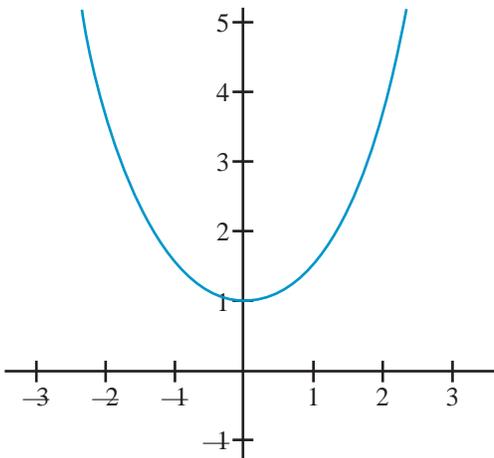
Newton's Law of Cooling is used to model the temperature change of an object of some temperature

| 7.) Newton's Law of Cooling |

placed in an environment of a different temperature. A formula that grows out of Newton's Law is

$$T(t) = T_m + (T_i - T_m) \times e^{(-kt)},$$

where $T(t)$ is the temperature of the object at time t, T_m is the temperature of the surrounding medium, T_i is the initial temperature of the object and k is a constant of proportionality. What this law says is that the rate of change of temperature is proportional to the difference between the temperature of the object and that of the surrounding environment. Notice the *e* in the formula.

Chapter 5 Summary—The reason that scientists and mathematicians prefer working with base *e* is that it appears frequently in nature and its use makes for a simpler, more aesthetic equation than would the same equation if it were written with a base of 10. Nicholas Mercator was the first person to describe $\log_e x$ (or ln x) as log naturalis. The number *e* seems to be a sort of universal constant in the way that π (pi) and φ (phi) are. Simply put, *e* is somehow built into the fabric or design of *nature* … of the universe. The function $y = \log_e x$ (a.k.a. $y = \ln x$) is said to be the natural logarithm while the function $y = \log_{10} x$ is called the common logarithm. Since we can solve for *x* in the equation $e = 10^x$, we could replace the *e* values in each of the above situations involving *e* with an equivalent form involving our more comfortable base 10. However, in doing so we will be replacing one unfamiliar number, 2.71828, with another, $10^{0.4342944819}$, which is dependent upon *e* to begin with. A religious person might describe *e* as "God's number." As $\log_e e = 1$ is equivalent to ln $e = 1$ you can substitute 1 anywhere you see ln *e*.

Chapter 5 Exercises

1.) Estimate each of the following for y. Justify your estimate.
   a.) $y = 5^{2.7}$ b.) $y = 8.64^{2.13}$

2.) Obtain exact answers for the problems in #1. Compare with your estimates in #1.

3.) Estimate x for each of the following. Justify your estimate.
   a.) $32.7 = 4^x$ b.) $117 = 5^x$

4.) Solve for exact answers for the problems in #3. Compare with your estimates in #3.

5.) Estimate each of the following. Justify your estimate.
   a.) $10 = x^{2.6}$ b.) $62.73 = x^{4.31}$

6.) Solve for exact answers for the problems in #5. Compare with your estimates in #5.

7.) Estimate each of the following values for x.
   a.) $5^{4.6} = 7^x$ b.) $8^{2.7} = 6^x$

8.) Solve for exact answers for the problems in #7. Compare with your estimates in #8.

9.) Estimate y for each of the following. Justify your estimate.
   a.) $y = \log_2 50$ b.) $y = \log_3 28$.

10.) Solve for exact answers for the problems in #9. (Hint. Either change to equivalent exponential form and solve for y or use the Change of Base rule to change the problem to one involving only $\log_{10}$.) Compare with your estimates in #9.

11.) Estimate the following values of x. Justify your estimate.
   a.) $6.1 = \log_2 x$ b.) $\log_9 x = 5.1$

12.) Solve for exact answers for the problems in #11. Compare with your estimates in #11.

13.) Estimate for x. Justify your estimate.
   a.) $4.9 = \log_x 37.1$ b.) $\log_x 126.21 = 3.207$

14.) Solve for exact answers in #13. Compare with your estimates in #13.

15.) Given

$$y = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$$

    Without using a graphing calculator capability (only the function/operation keys) find the two values of x that will result in a y value of 0.2. Check using your graphing calculator (intersection of $y = f_1$ and $y = f_2$).
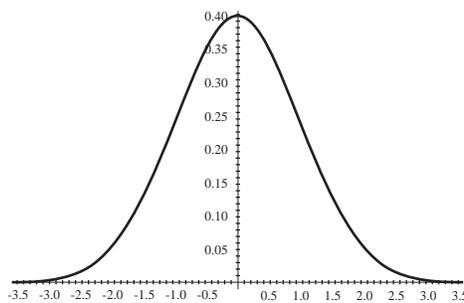
16.) Recall Newton's Law of Cooling: $T(t) = T_m + (T_i - T_m) \times e^{(-kt)}$

    A hard-boiled egg at temperature 90 °C is placed in water at 20 °C to cool. Three minutes later the temperature of the egg is 60 °C.

    Step 1: Use the given information to solve for k.

    Step 2: Use the k that you solved for in Step 1 to determine when the egg will be 30 °C.

17.) The equation for the logarithmic spiral is $r = e^{at}$ (polar form with r = radius, a being a constant, and t = theta.) For the spiral at right. Estimate theta when $r = 4$. Solve for theta when radius = 4.

$r = e^{0.10\,\theta}$
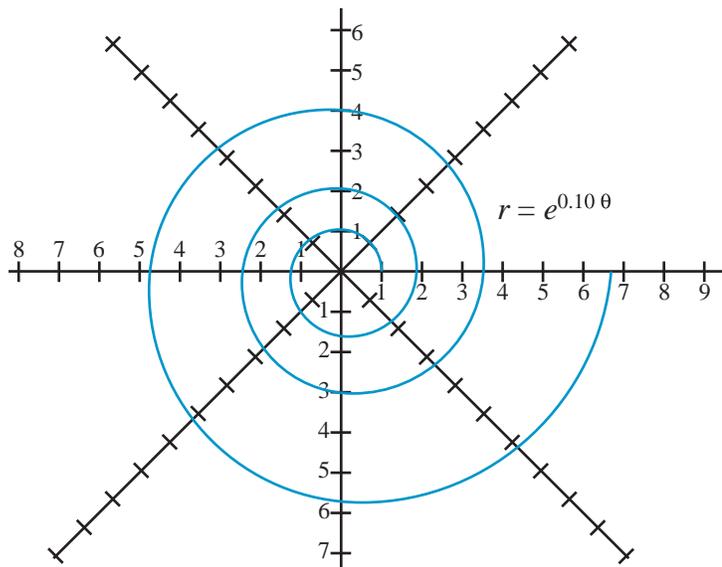
18.) The formula for calculating pH is:

$$pH = -\log_{10}\left[H^+\right]$$

    where pH is the acidity of the solution and H^+ is the hydrogen ion concentration.

a.) If Grandma's lye soap has a hydrogen ion concentration of $9.2 \times 10^{(-12)}$ what is its pH?
b.) If the pH of a tomato is 4.2 what is its hydrogen ion concentration?

Chapter 6: "When will we ever use this stuff?"

All the rules learned to this point are gathered together and listed below for reference

$b^m \times b^n = b^{(m+n)}$	**Product of Common Base Factors Rule**
$\dfrac{b^m}{b^n} = b^{(m-n)}$	**Quotient of Common Bases Rule**
$\log(x \times y) = \log x + \log y$	**Log of a Product Rule**
$\log\left(\dfrac{x}{y}\right) = \log x - \log y$	**Log of a Quotient Rule,**
$m = n$ iff $b^m = b^n$	**iff Antilog Rule**
$m = n$ iff $\log m = \log n$	**iff Log Rule**
$b^y = x$ is equivalent to $y = \log_b x$	**Equivalent Symbolism Rule**
$\log_b b^x = x$	**Inverse Log Rule #1** (Log of an Exponential Rule)
$b^{\log_b x} = x$	**Inverse Log Rule #2** (Power of a Base Rule)
$\log b^m = m \log b$	**Log of a Base Raised to a Power Rule**
$x \times y = 10^{(\log x + \log y)}$	**Rule to Multiply Using Logarithms**
$\dfrac{x}{y} = 10^{(\log x - \log y)}$	**Rule to Divide Using Logarithms**
$\log_p x = \dfrac{\log_q x}{\log_q p}$	**Change of Base Log Rule … change from base p to q.**
If $b^x = y$ then $x = \dfrac{\log y}{\log b}$	**Rule to Solve for an Exponent**

There many areas in science, sociology, economics, etc., that require knowledge of logarithms.

Compound interest, exponential growth and decay, pH, depreciation, measurement of the magnitude of volume, of earthquakes, of sound, of the efficiency of algorithms and of fractional dimensions for fractals are all examples of the need to be able to understand and work with logarithms. Use the natural log function when working with an expression involving a base of e. Use the common logarithm when the expression involves a base of "10."

One application of logarithms is to work problems involving compound interest. Interest is a fee paid or received for the lending of money. Interest is usually calculated in terms of percent. Say, for example, that you wished to determine how long it would take $1,000 to double if invested at 20% interest compounded annually.

Year	Amount	Interest = PRT at end of year ($I = PR$ if $T = 1$)
0	$1,000	$200
1	1,200	240
2	1,440	288
3	1,728	345.6
4	** 2,073.6 **	

Apparently at 20% interest compounded yearly it will take almost four (4) years for the $1,000 to double in value. What if you wished to compound the interest 1.) for more than 4 years or 2.) several times a year for several years? What if, for example, you wished to compound (calculate interest and add it to the principal) interest for 120 years? How much would the original amount of money be worth? We would not want to develop the chart above to find out our answer as that would be 120 rows!! Let's look at the problem above again and see if we can see a pattern.

Let P_n = principal (amount of money) during year n. Therefore, for P_0 (the first year of the loan) the money involved was $1,000 ... P_1 (the second year of the loan) the principal was worth $1,200, etc.

P_0 = $1,000

P_1 = $1,200 = $P_0 (1 + 0.20)$

P_2 = $1,440 = $P_1 (1 + 0.20)$
= $\{P_0 (1 + 0.20)\} (1 + 0.20)$ ... substituting for P_1
= $P_0 (1 + 0.20)^2$

P_3 = $1,728 = $P_2 (1 + 0.20)$
= $\{P_0 (1 + 0.20)^2\} (1 + 0.20)$ ... substituting for P_2
= $P_0 (1 + 0.20)^3$

P_4 = $2,073.6 = $P_3 (1 + 0.20)$
= $\{P_0 (1 + 0.20)^3\} (1 + 0.20)$ ... substituting for P_3
= $P_0 (1 + 0.20)^4$

This pattern suggests the formula $P_f = P_0 (1 + r)^y$ where P_f = final principal after y years

P_0 = initial principal
and r = interest rate of loan
and y = number of years of the loan.

Normally interest is not compounded yearly but for a smaller time interval ... say quarterly. In this case the formula that would be used would be

$$P_f = P_0 \left[\left(1 + \frac{r}{k} \right)^k \right]^y$$

where
P_f = final principal after y years
P_0 = original principal
r = annual interest rate of loan
k = number of times per year that the interest is calculated and compounded
y = number of years of the loan

$$P_f = P_0\left[\left(1+\frac{r}{k}\right)^k\right]^y$$

where P_f = final principal after y years
P_0 = original principal
r = annual interest rate of loan
k = number of times per year that the interest is calculated and compounded
y = number of years of the loan

Here we have one equation with five different unknowns. If k and y are both known values as well as any two of the remaining variables then, combining Algebra I skills with the power of a calculator, it is not usually difficult to solve for whatever the remaining unknown is. However, if either k or y is unknown then you will need to have knowledge of the log rules in order to solve for the unknown power.

How many years will be required for $1,000 to double if 5% interest is paid and interest is compounded quarterly. Here $P_f = 2,000$, $P_0 = 1,000$, $r = 0.05$, $k = 4$.

$$2,000 = 1,000\left(1+\frac{0.05}{4}\right)^{4y}$$

2	=	1 $(1.0125)^{4y}$	divide both sides by 1,000
log 2	=	log $(1.0125)^{4y}$	take the log of both sides, **iff Log Rule**
log 2	=	$4y$ log (1.0125)	**Log of a Base Raised to a Power Rule**
0.3010299957	=	$4y$ (0.0053950319)	calculator values for log 2 and log (1.0125)
$4y$	=	$\dfrac{0.3010299957}{0.0053950319}$	division property of equality
y	=	13.949 years	by calculator

There is a part of the formula for compound interest that involves the compounding of interest for k time intervals over one year. It is shown in bold at the right.

$$P_f = P_0\left[\left(1+\frac{r}{k}\right)^k\right]^y$$

Lets do an experiment. Let $r = 1$ in the subformula and let k get bigger and bigger. Watch what happens.

The fact that k gets larger and larger in the formula above means that our formula is compounding interest for more and more time intervals over one (1) year period of time. In fact, as k approaches infinity we say that we are computing continuous compound interest.

$\left(1+\dfrac{1}{10}\right)^{10}$	=	2.593742460
$\left(1+\dfrac{1}{100}\right)^{100}$	=	2.704813829
$\left(1+\dfrac{1}{1,000}\right)^{1,000}$	=	2.716923932
$\left(1+\dfrac{1}{10,000}\right)^{10,000}$	=	2.718145927
$\left(1+\dfrac{1}{100,000}\right)^{100,000}$	=	2.718268237
$\left(1+\dfrac{1}{1,000,000}\right)^{1,000,000}$	=	2.718280469

Since, for very large k and $r = 1$, $\left(1 + \dfrac{r}{k}\right)^k = 2.71828$ we can substitute $e = 2.71828 \ldots$ in the formula

$$P_f = P_0\left[\left(1 + \dfrac{r}{k}\right)^k\right]^y \quad \text{(assuming } r = 1 \text{ and } k \text{ is very large)}$$

giving the formula

$$P_f = P_0[e]^y \quad \text{(assuming } r = 1\text{), which would be the formula for computing continuous}$$

compound interest at a 100% annual rate of interest. That would be a really nice rate of return for the lender, but it's hardly practical for the one borrowing

.

Lets do another experiment. Let $r = 0.05$ in the highlighted portion of the formula and let k get bigger and bigger. Watch what happens.

$$\left(1 + \dfrac{r}{k}\right)^k$$

$$\left(1 + \dfrac{0.05}{10}\right)^{10} \qquad = 1.05114013$$

$$\left(1 + \dfrac{0.05}{100}\right)^{100} \qquad = 1.05125796$$

$$\left(1 + \dfrac{0.05}{1,000}\right)^{1,000} \qquad = 1.05126978$$

$$\left(1 + \dfrac{0.05}{10,000}\right)^{10,000} \qquad = 1.05127097$$

$$\left(1 + \dfrac{0.05}{100,000}\right)^{100,000} \qquad = 1.05127108$$

$$\left(1 + \dfrac{0.05}{1,000,000}\right)^{1,000,000} \qquad = 1.05127109$$

But since $e = 2.71828 \ldots$ and $e^{0.05} = 1.051271096$ it appears that

$$\left(1 + \dfrac{0.05}{1,000,000}\right)^{1,000,000} = 1.05127109 \text{ (from the table above)} \approx e^{0.05}$$

By transitive

$$\left(1 + \dfrac{0.05}{1,000,000}\right)^{1,000,000} = e^{0.05}$$

In other words $\left(1 + \dfrac{r}{k}\right)^k = e^r$

So in the original formula we developed above

$$P_f = P_0\left[\left(1 + \dfrac{r}{k}\right)^k\right]^y =$$

$$P_f = P_0[e^r]^y =$$

$$P_f = P_0 e^{ry} \qquad \text{where} \quad \begin{aligned} P_f &= \text{ final principal after } y \text{ years} \\ P_0 &= \text{ original principal} \\ r &= \text{ annual interest rate of loan} \\ k &= \text{ number of times per year that the interest is calculated} \\ &\quad\ \text{ and compounded} \\ y &= \text{ number of years of the loan} \end{aligned}$$

$$P_f = P_0 e^{ry}$$

Here we have one equation with five unknowns. If any four of the unknowns can be determined from the problem conditions then the fifth can also be determined. Again, as before, **if either of the exponents is unknown, then knowledge of logarithms must be applied to solve the equation**. Repeating the same conditions as before (finding years necessary to double principal) for continuous compounding, we get

2,000	=	$1,000\ (e^{0.05y})$	substituting known values into appropriate formula
2	=	$e^{0.05y}$	divide both sides by 1,000
$\ln 2$	=	$\ln e^{0.05y}$	take the natural log of both sides,**iff Log Rule**
$\ln 2$	=	$0.05y\ (\ln e)$	**Log of a Base Raised to a Power Rule**
$0.05y$	=	$\ln 2$	reflexive property, and $\ln e = 1$
$0.05y$	=	0.6931471806	by calculator
y	=	13.86294361 yrs	divide both sides by 0.05

Continuous compounding doubles our money slightly faster than quarterly compounding.

> **Exponential Growth**

When calculating compound interest one sees the result of a quantity growing larger and larger at predictable rates. This happens a lot in mathematics and science. Let's say that a colony of 100 bacteria grows 10 times larger every 5 days. In chart form, we get the following.

Day	Number of bacteria					
0	100	d_0	=	100		
5	1,000	d_5	=	1,000	$= 10\ d_0$	
10	10,000	d_{10}	=	10,000	$= d_5\ (10)^1$	
			=	$(10\ d_0)\ (10)$	… substituting for d_5	
			=	$d_0\ (10)^2$		
15	100,000	d_{15}	=	100,000	$= d_{10}\ (10)$	
			=	$[d_0 (10)^2)\ (10)]$	… substituting for d_{10}	
			=	$d_0 (10)^3$		

This is the classic pattern of what is known as a geometric progression: Each term in a sequence is a fixed multiple of the previous term:

$$t_n = 10 \times t_{(n-1)}$$

Also, if you do not have access to the previous term's value but do know the initial term and the geometric ratio, you could have $t_n = t_0 \times r^{(n-1)}$, where n is the term number and r is the ratio of change from term to term

There are many quantities out there in the world that are governed (at least for a short time period) by the equation,

$$f = i \times r^{kt} \text{ (or } f = i \times b^{kt} \text{ with } b = \text{constant ratio of change)},$$

where f represents the final quantity, i represents the initial quantity, **k represents a constant of proportionality** (which varies from problem to problem) and t represents a unit of time. If k is positive, then the equation will grow without bound and is called the **exponential growth** equation. Likewise, if k is negative the equation will die down to zero and is called the **exponential decay** equation.
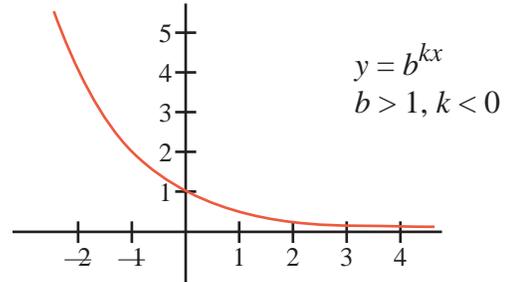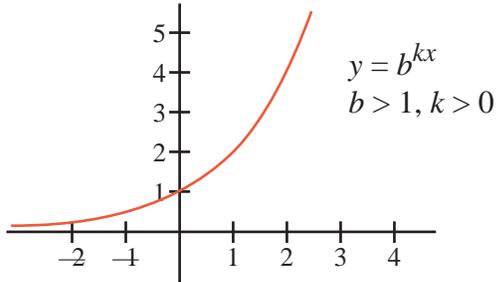
Short-term population growth is often modeled by the exponential growth equation and the decay of a radioactive element is governed by the exponential decay equation.

Say the number of bacteria per square millimeter in a culture in a biology lab is increasing six-fold daily. On Monday, there are 2,000 bacteria per square millimeter. On Wednesday, the number has increased to 4,500 per square millimeter.

a.) Use the given information to obtain the constant of proportionality, k.

$$f \quad = \quad i \times b^{kt}$$

$$f \quad = \quad i \times 6^{kt}$$

$$4,500 \quad = \quad 2,000 \times 6^{k(2)} \text{ (where } t \text{ is in days) substituting given information into equation}$$

$$\frac{4,500}{2,000} \quad = \quad 6^{2k} \quad \text{division property of equality}$$

$$\log (^9\!/\!_4) \quad = \quad \log 6^{2k} \quad \text{taking the log of both sides, \textbf{iff Log Rule}}$$

$$\log 2.25 \quad = \quad 2k \,(\log 6) \quad \textbf{log } b^m = m \textbf{ log } b \textbf{ rule}$$

$$0.4525887711 \quad = \quad 2k \quad \text{using a calculator}$$

$$k \quad = \quad 0.2262943855$$

   Notice that if k = 1 in the equation $i \times 6^{kt}$ then $i \times 6^t$ and since t = 2 (Monday to Wednesday = 2) then the number of bacteria would be f = 2000 * 6^2 = 72,000. (Monday = 2000, Tuesday = 12,000, Wednesday = 72,000) However the number of bacteria is given to be 4,500 on Wednesday. The "k" or constant of proportionality affects the rate of growth. Basically it allows for a fine tune modification on the t in the equation.   With k < 1 one should expect for fewer than $i \times 6^t$ bacteria to be present for any given t.

b.) Use the work in part a.) to predict how many bacteria there will be on Saturday (5 days from the start of the experiment).

$$f \quad = \quad i \times 6^{kt} \qquad \text{same formula}$$

$$= \quad 2{,}000 \times 6^{(0.2262943855 \times 5)} \quad \text{substitution of given information into the formula}$$

$$= \quad 2{,}000 \times 6^{1.131471928}$$

$$= \quad 2{,}000 \times 7.59375$$

$$= \quad 15{,}187 \qquad \text{bacteria per square millimeter}$$

Exponential Decay Carbon-14 Testing

Photosynthesis is the chemical process that takes place in the green leaves of a plant. During photosynthesis the plant takes in carbon dioxide (CO_2) and gives off oxygen (O_2). Some of the CO_2 that the plant takes in contains the carbon isotope called carbon 14. Carbon 14 results when the sun strikes the CO_2 in the atmosphere in just the right way. When the plant dies, the carbon 14 in the plant slowly decays, changing into nitrogen 14. Assume that it takes 5,750 years for the amount of carbon 14 originally in the plant to steadily decay to one-half ($\frac{1}{2}$) of its original amount. What would be the expression giving the amount of carbon 14 t years after the tree died? Assume the common ratio to be e.

$$f \quad = \quad i \times e^{kt}$$

$$f \quad = \quad i \times e^{k \times 5750}$$

$$50\% \, i \quad = \quad i \times e^{5750 \times k}$$

$$0.50 \quad = \quad e^{5750k} \qquad \textbf{divide both sides by "i"}$$

$$\ln(0.50) \quad = \quad \ln e^{5750k} \qquad \textbf{iff log rule for } b = e,\ m = n \textbf{ so } \ln m = \ln n$$

$$-0.6931471806 \quad = \quad 5{,}750k\,(\ln e) \qquad \textbf{log of a power rule}$$

$$-0.6931471806 \quad = \quad 5{,}750k \qquad \textbf{(} \ln e = 1 \textbf{)}$$

$$k \quad = \quad \frac{-0.6931471806}{5750}$$

$$k \quad = -0.0001205473358 \quad \textbf{\textit{k} < 0 indicating exponential decay, not growth}$$

$$f \quad = i \times e^{-0.0001205473358t} \qquad \textbf{final amount of carbon 14 \textit{t} years after plant dies}$$

An artifact in a museum was displayed as being the masthead of a Viking ship dating back to the 1200s. An analysis of a sample of that artifact revealed that it had 90% of its carbon 14 remaining. Could that artifact have dated back to the 1200s?

f	$=$	$i \times e^{kt}$	i = initial amount of carbon 14, f = final amount
f	$=$	$i \times e^{k \times 5750}$	
$90\% \, i$	$=$	$i \times e^{-0.0001205473358 \, t}$	
0.90	$=$	$e^{-0.0001205473358 \, t}$	**divide both sides by i**
$\ln(0.90)$	$=$	$\ln e^{-0.0001205473358 \, t}$	**iff log rule for $b = e$, $m = n$ so $\ln m = \ln n$**
-0.1053605157	$=$	$-0.0001205473358t \, (\ln e)$	**log of a power rule**
-0.1053605157	$=$	$-0.0001205473358t$	**($\ln e = 1$)**
t	$=$	$\dfrac{-0.1053605157}{-0.0001205473358}$	
t	$\approx$	**874 years**	Yes, the results of the carbon 14 test support the idea that the artifact dates back to the 1200s.

| **Earthquake Intensity** |

The **Richter scale** is commonly used to measure the intensity of an earthquake. It was developed by C. Richter in the 1930s. There are many different ways of computing this scale based on a variety of different quantities. Here is one based upon a, the amplitude (in micrometers) of the vertical ground motion at the receiving station, T, the period of the seismic wave (in seconds), and B, a factor that accounts for the weakening of the seismic wave with increasing distance from the epicenter of the earthquake.

The energy released, measured in joules, during an earthquake is proportional to the antilog of the magnitude and magnitude of the earthquake is given by,

$$R = \log\left(\frac{a}{T}\right) + B$$

Here we have one equation with four unknowns. There are no unknown exponents involved, so given any three of the variables in the equation we can solve for the fourth variable using only algebra and a scientific calculator. However, feeling comfortable with the concept of logarithm gives you the number sense to understand the formula and to better know if your answers are reasonable.

Richter Scale numbers are a really good example for you to test yourself on your understanding of logarithmic numbers. The San Francisco earthquake of 1906 measured 8.25 on the Richter scale. The Seattle earthquake of 1965 measured 7 on the Richter scale. Richter scale numbers are logarithmic numbers. How much stronger was the San Francisco quake than the Seattle quake? A simple ratio of the two quakes' Richter scale values (8.25 / 7) would make it seem as if the San Francisco quake was only 1.1785 times as severe as the Seattle quake. That would be misleading as the Richter scale measurements are both base 10 logarithmic numbers. **The answer 1.1785 is not reasonable!** If we were to compare the numbers 1,000 (10^3) and 100 (10^2) by comparing the two exponents 3 against 2 we would conclude that 1,000 is 1.5 times as much as 100 when obviously the ratio of 1,000 to 100 is 10 to 1. To accurately compare the numbers 1,000 and 100 we do not compare the 3 against the 2 but the base 10 antilog of 3 (1000) against the base 10 antilog of 2 (100).

How much stronger was the San Francisco quake than the Seattle quake?

$$\frac{10^{8.25}}{10^7} = \frac{177827941}{10000000} = 17.7827941 \text{ times as severe!!!}$$

Measurement of pH
Measurement of pH **Example of a Log Function**

pH is the measure of activity of hydrogen ions in a solution

The formula for calculating pH is $\mathbf{pH = -\log_{10}[H+]}$

[H⁺] denotes the activity of H⁺ ions (or more accurately written, [H₃O⁺], the equivalent hydronions), measured in moles per liter (also known as molarity). The pH factor determines whether a substance is classified as acidic, neutral, or alkaline depending on if the pH < 7, pH = 7, or pH > 7. Tomato juice has H+ = $6.3 \times 10^{(-5)}$. What is the pH value?

$$\mathbf{pH = -\log(6.3 \times 10^{(-5)})}$$
$$\mathbf{\approx 4.2} \quad \text{Therefore tomato juice is acidic.}$$

Again, feeling comfortable with the concept of logarithms gives you the number sense to understand the formula and to better know if your numbers do not make sense.

Decibel scales … example of a log function

In 1825, a German physiologist, Ernst Weber, formulated a mathematical law that was introduced to measure the human response to physical stimuli such as weight, pressure, or sound. The decibel scale is a result of Mr. Weber's law. The decibel is not a unit in the sense that a meter or a second is. Meters and seconds are objectively defined quantities of distance and time. You can go to the National Bureau of Standards and see their definition. They never change and are not subjective in the way that they are quantified. Decibels, however, are in accordance with Mr. Weber's laws, defined to be the logarithm of the ratio of two different sounds.

$$Db = 10 \times \log\left(\frac{power_A}{power_B}\right)$$

Logarithmic scales are often used for comparing quantities of greatly disparate values. For example, if the softest audible sound has a power of 0.000000000001 W/sq ft and the "threshold of pain" is about 1 W/sq ft, then we would evaluate the decibels of the later to be

$$Db = 10 \times \log\left(\frac{1 \, W/_{ft^2}}{10^{-12} \, W/_{ft^2}}\right)$$

$$= 10 \times \log 10^{12}$$

$$= 10 \times 12 \qquad \textbf{(Inverse Log Rule \#1, Log of an Exponential Rule)}$$

$$= 120 \text{ decibels}$$

Notice that, for logarithmic scales every 10-fold increase of a quantity results in a scale increase of only one. Think powers of 10 (a logarithm is a power, right?): $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1,000$, etc.

| Measurement of efficiency of algorithms |

Understanding logarithmic ideas and concepts comes in handy in an area of computer science called "measuring the efficiency of algorithms … Big O."   Big O is a type of category comparison system … not the linear type of rating system you are probably used to with scalars. In a category comparison system, items are grouped into categories. You can meaningfully compare categories against one another and by inference you can compare items of one category against the items in another. But you cannot compare items within a category against other items within the same category.

For example, if you were to do a sequential search for the value sv (search value) on the following set of randomly ordered numbers … $x_1, x_2, x_3, x_4, …, x_{100}$ you would compare sv against the first item of the list, x_1, then against the second item of the list, x_2, then against the third, x_3, etc. until either you have a match or until you run out of numbers to compare against. search you would find a match for sv in the first item of data, x_1. This would be considered an O(1) efficiency. In the In the best case worst case you would not find a match of sv until x_{100}, the nth number. This would be considered an O(n) efficiency … order of efficiency of n. In an O(n) category efficiency algorithm, the effort or work of the algorithm, in this case the number of comparisons, increases proportionally with the number (n) of data values.

If the numbers were in order, that is, $x_1 < x_2 < x_3 < x_4 < … < x_{100}$ it is possible to take advantage of the ordering of the numbers to do a more efficient search called the binary search. For example let's assume that the number we are looking for is in the 78th number of the set of x's, but we don't know that ahead of time because we are dealing with variable values. We set variable *left* to indicate the leftmost position of numbers being searched, variable *right* to indicate the rightmost position of numbers being searched, and variable *mid* to (left + right)/2.

$$\left(\begin{array}{l} x_1 < x_2 < x_3 < x_4 < \cdots < x_{50} < \cdots < x_{100} \\ \text{left} \qquad\qquad\qquad\qquad \text{mid} \qquad \text{right} \end{array} \right)$$

Compare the search value, sv, against the value in the position marked mid. If sv is equal to x_{50}, you may cease your search. If sv < the value in position marked mid, then because of the ordering of the array, you need no longer search in positions mid and up. Accordingly, you would pull in the rightmost bound to the location to the left of mid:

$$\left(\begin{array}{l} x_1 < x_2 < x_3 < x_4 < \cdots < x_{49} \\ \text{left} \qquad\qquad\qquad\quad \text{right} \end{array} \right) < x_{50} < \cdots < x_{100} \\ \text{mid}$$

On the other hand if sv > the value in the position marked mid, then you would no longer need to search in positions mid and down. Accordingly, you would pull in the leftmost bound to the location to the right of mid:

$$x_1 < x_2 < x_3 < x_4 < \cdots < x_{50} < \left(\begin{array}{l} x_{51} < \cdots < x_{100} \\ \text{left} \qquad \text{right} \end{array} \right) \\ \text{mid}$$

This process is continued until you either locate sv in the list or determine that it is not in the list. Notice that each time you apply this search algorithm you effectively eliminate ½ of the remaining numbers. Of the original set of 100 ordered numbers you originally considered, each step leaves

Step 0:   all 100

Step 1:   then only 50

Step 2:   then only 25

Step 3:   then only 13

                 etc.

Number of data							
100,	50,	25	13,	7,	4,	2,	1

Number of comparisons						
1	2	3	4	5	6	7

A binary search of 100 numbers will take, in the worst-case scenario, 7 search attempts until the search value, sv, is found (or determined not to be in the set of data.)

There really must be a more efficient way of determining how many search attempts will be necessary to search 100 ordered numbers other than enumerating and counting the phases.

Look at the following:

$$2^6 = 64$$
$$2^? = 100$$
$$2^7 = 128$$

Using the binary search, the number of comparison searches necessary to locate data in an ordered list is, are you ready for this? … ta da!! The answer is $\log_2 100$ because $2^x = 100$. Logarithms! You gotta love 'em! (Actually, since searching steps are discrete, the value is 7 … not $\log_2 100$ which would be a decimal … 6.64385619.) The logarithmic efficiency of the binary search algorithm is said to be $O(\log_2 n)$, where n represents the number of items in the ordered list that are being searched. It can be said (for large n) all algorithms of algorithmic efficiency rating $O(\log_2 n)$ are more efficient than all algorithms of with an $O(n)$ efficiency rating.

Determining the Fractional Dimension of a Fractal

In the spirit of George Cantor, the 19th-century mathematician who took the historical concept of infinity and extended it to include different kinds of infinity … $aleph_0$, $aleph_1$, etc. Benoit Mandelbrot, a 20th-century mathematician, took the historical concept of dimension and extended it to include *rational dimensions*. To do so, he needed to use the knowledge of logarithms that we have developed in this text.

In the book *The Golden Ratio* by Mario Livio, Random House, 2002, the author prepares the reader for the concept of rational dimensions of fractals by first reviewing some ideas from the traditional dimensions: 1-space, 2-space, and 3-space.

One-Space

Take a line segment (1 dimension) and divide it into halves. There are two equal subparts.

Take a line segment (1 dimension) and divide it into thirds. There are three equal subparts.
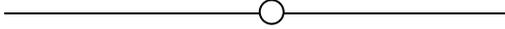
The quantities

d (dimension, 1),
n (number of subparts, 2),
and f (fraction of each part, $\frac{1}{2}$)

can be related as follows:

$$2 = \left(\frac{1}{\frac{1}{2}}\right)^1$$

or in general

$$n = \left(\frac{1}{f}\right)^d$$

The quantities

d (dimension, 1),
n (number of subparts, 3)
and f (fraction of each part, $\frac{1}{3}$)

Can be related as follows:

$$3 = \left(\frac{1}{\frac{1}{3}}\right)^1$$

where d = dimension(s) of object
f = fraction each side is divided into
n = number of subparts after division

Two Space

Take a square (2 dimensions) and divide each side into halves. There are now four equal subparts.

Take a square (2 dimensions) and divide each side into thirds. There are now nine equal subparts.
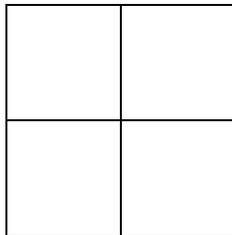
The quantities

d (dimension, 2),
n (number of subparts, 4),
and f (fraction of each part, $\frac{1}{2}$)

can be related as follows:

$$4 = \left(\frac{1}{\frac{1}{2}}\right)^2$$

or in general

$$n = \left(\frac{1}{f}\right)^d$$
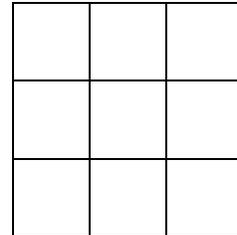
The quantities

d (dimension, 2),
n (number of subparts, 9)
and f (fraction of each part, $\frac{1}{3}$)

Can be related as follows:

$$9 = \left(\frac{1}{\frac{1}{3}}\right)^2$$

where d = dimension(s) of object
f = fraction each side is divided into
n = number of subparts after division

Three Space

Take a cube (3 dimensions) and divide each side into halves. There are now eight equal subparts.

Take a cube (3 dimensions) and divide each side into thirds. There are now 27 equal subparts.
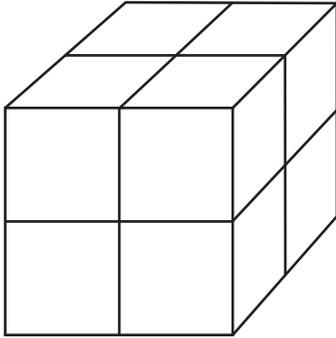
The quantities

$$d \text{ (dimension, 3)},$$
$$n \text{ (number of subparts, 8)},$$
$$\text{and } f \text{ (fraction of each part, } \tfrac{1}{2})$$

can be related as follows:

$$8 = \left(\frac{1}{\frac{1}{2}}\right)^3$$

or in general

$$n = \left(\frac{1}{f}\right)^d$$
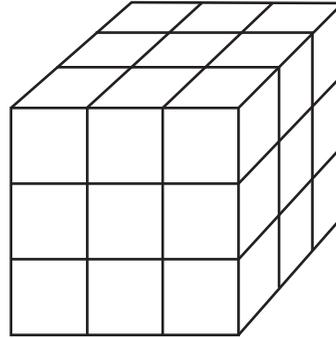
The quantities

$$d \text{ (dimension, 3)},$$
$$n \text{ (number of subparts, 27)}$$
$$\text{and } f \text{ (fraction of each part, } \tfrac{1}{3})$$

Can be related as follows:

$$27 = \left(\frac{1}{\frac{1}{3}}\right)^3$$

where d = dimension(s) of object
f = fraction each side is divided into
n = number of subparts after division

Mandelbrot is particularly known for his work with fractals … self-similar shapes. Fractals are shapes that repeat an identical pattern over and over. Following is an example of a snowflake fractal. Start with an equilateral triangle. Divide each side into three (3) equal parts.
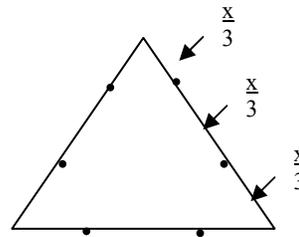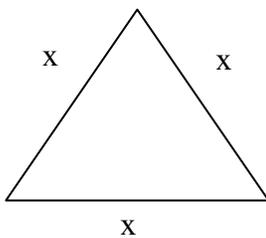
On each side of the triangle form a new equilateral triangles whose base is $\frac{1}{3}$ the length of a side of the original triangle.
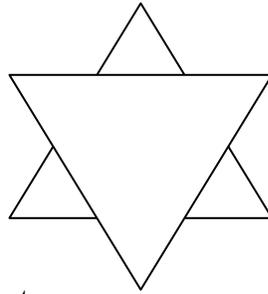
Repeating this process, we now get
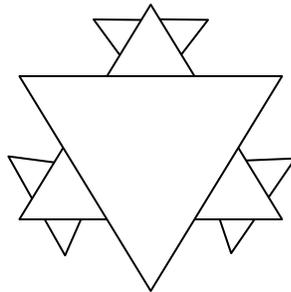
It occurred to Mandelbrot that the area enclosed by this snowflake area was greater than the area under a 1 dimensional line but less that the area of a bounding square that could enclose it.

$1 < \text{dimension} < 2$

Notice above that each time a side is divided into three (3) equal parts, four subparts are formed.
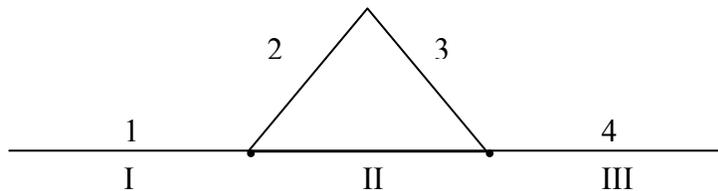
Applying the formula we developed previously, we get

$$n = \left(\frac{1}{f}\right)^d$$

d = dimension(s) of object
f = fraction each side is divided into
n = number of subparts after division

$$4 = \left(\frac{1}{\frac{1}{3}}\right)^d$$

$$4 = 3^d$$

Using logarithms, we can solve for the dimension of this "snowflake" fractal.

$$\log 4 = \log 3^d$$

$$\log 4 = d \log 3$$

$$\frac{\log 4}{\log 3} = d$$

$$d = 1.261859507$$

The fractional dimension of a fractal is "a measure of the wrinkliness of the fractal, or of how fast length, surface, or volume increases if we measure it with respect to ever-decreasing scales." (Livio, *The Golden Ratio*) Logarithms!!

Geometric Series	A "geometric series" is defined to be an addition of numbers (terms) such that each term is a fixed multiple of the preceding one.

$3 +$ 9 $+ 27 +$ 81 $+ 243 + 729 + 2{,}187$ etc. is a series because each term is 3 times the previous
$3 + 3(3) + 3(3^2) + 3(3^3) + 3(3^4) + 3(3^5) + 3(3^6) + \ \dots$
$t_1 + \ t_2 \ + \ t_3 \ + \ t_4 \ + \ t_5 \ + \ t_6 \ + \ t_7 \ + \ \dots$

In general a finite geometric sum for n terms looks like the following:

Eq. 1 $S_n \ = \ a \ + a(r) + a(r^2) + a(r^3) + \ \dots \ + a(r^{n-1})$

Multiplying both sides of Eq. 1 by r

$S_n \ = \ a \ + a(r) + a(r^2) + a(r^3) + \ \dots \ + a(r^{n-1})$

we get

$r\,S_n = r \times [\ a \ + a(r) + a(r^2) + a(r^3) + \ \dots \ + \ a(r^{n-1})\ ]$

Eq. 2 $r\,S_n = \ ar \ + a(r^2) + a(r^3) + \ \dots \ + a(r^{n-1}) + \ a(r^n)$

Placing eq.1 and eq. 2 together we get

Eq. 1 $S_n \ = \ a \ + \ a(r) + a(r^2) + a(r^3) + \ \dots \ + A(r^{n-1})$

Eq. 2 $r\,S_n = \qquad\quad a(r) + a(r^2) + a(r^3) + \ \dots \ + A(r^{n-1}) + a(r^n)$

Subtracting Eq. 1 – Eq. 2, we get

$S_n \ - \ r\,S_n = \ a \ - \ a(r^n)$

$S_n(1 - r) \ = \ a(1 - r^n)$

$S_n = a\dfrac{1 - r^n}{1 - r}$ Sum of n terms of the geometric series with common ratio r and first term a.

For the series $2 + 2 \times (\tfrac{3}{2})^1 + 2 \times (\tfrac{3}{2})^2 + 2 \times (\tfrac{3}{2})^3 + 2 \times (\tfrac{3}{2})^4 + \dots,$

$$2 + \quad 3 \quad + \quad (\tfrac{9}{2}) \quad + \quad (\tfrac{27}{4}) \quad + \quad (\tfrac{81}{8}) \quad + \dots$$

find how many terms will be necessary to result in a sum > 100.

$$S_n < a\frac{1-r^n}{1-r}$$

$$100 < \frac{2\left[1-(\tfrac{3}{2})^n\right]}{1-\tfrac{3}{2}}$$

$$50 < \frac{1-(\tfrac{3}{2})^n}{1-\tfrac{3}{2}}$$

$$50 < \frac{1-(\tfrac{3}{2})^n}{-\tfrac{1}{2}}$$

$$-25 > 1-(\tfrac{3}{2})^n$$

$$-26 > -(\tfrac{3}{2})^n$$

$$26 < (\tfrac{3}{2})^n$$

$$\log(26) < \log(\tfrac{3}{2})^n$$

$$\log(26) < n\log(\tfrac{3}{2})$$

$$1.414973348 < n\,(0.1760912591)$$

$$n > 8.03545454773 \text{ terms}$$

check: $2+3+ (\tfrac{9}{2}) + (\tfrac{27}{4}) + (\tfrac{81}{8}) + (\tfrac{243}{16}) + (\tfrac{729}{32}) + (\tfrac{2187}{64}) =$
$\quad\quad 5 + 4.5 + 6.75 + 10.125 + 15.1875 + 22.78125 + 34.171875 =$
$\quad\quad\quad\quad 16.25 + \quad 25.3125 \quad + \quad 56.953125 \quad =$
$$98.515625$$

Since the "number of terms" is discrete data, it will take 9 terms before the sum > 100.

Once again, we need logarithms to solve this problem.

Examples Ad Nauseum, Ad Infinitum

A **power law** relationship between two scalar quantities x and y is any such relationship that can be written as:

$$y = ax^k$$

Power laws are observed in many fields, including physics, biology, geography, sociology, economics, linguistics, war, and terrorism. Power laws are among the most frequent scaling laws that describe the scale invariance found in many natural phenomena.

Examples of power law relationships:

The Stefan–Boltzmann law
The Gompertz Law of Mortality
The Ramberg–Osgood stress–strain relationship
Gamma correction relating light intensity with voltage
Kleiber's law relating animal metabolism to size
Behavior near second-order phase transitions involving critical exponents
Frequency of events or effects of varying size in self-organized critical systems: e.g., Gutenberg–Richter
Law of earthquake magnitudes and Horton's laws describing river systems
Proposed form of experience curve effects
Scale-free networks where the distribution of links is given by a power law (in particular, the World Wide Web)
The differential energy spectrum of cosmic-ray nuclei

Examples of power law probability distributions:

The Pareto distribution
Zipf's law
Weibull distribution

Source: http://en.wikipedia.org/wiki/Power_law

In each case if the exponent is unknown, it must be solved for using logarithmic skills.

Chapter 6 Summary—There many areas in science which require knowledge of logarithms. Compound interest, exponential growth and decay, ph, depreciation, measurement of the magnitude of volume, of earthquakes, of sound, of the efficiency of algorithms, and of fractional dimensions for fractals are all examples of the need to be able to understand and work with logarithms. Use the natural log function when working with an expression involving a base of *e*. Use the common logarithm when the expression involves a base of "10."

Chapter 6 Exercises

1.)   How long will it take an amount of money to triple in value if the money is compounded monthly at 7%?

2.)   What rate of interest would be necessary for $900 to be compounded to $1,500 in 10 years if compounded continuously?

3.)   In chapter 6, there was an example where $1,000 was compounded annually at 20% interest. The money doubled in value in just under 4 years. Find out exactly the day of the year that the money will have doubled. Assume there are 365 days in a year. Ignore the inconsistency of identifying a specific date using a formula that compounds annually.

4.)   When Mary was 6 years old, her father invested a sum of money at 5% interest compounded semiannually so that Mary received $10,000 from that investment when she graduated from med school at age 26. How much money was invested?

5.)   A certain radioactive material decays at a rate given by the formula $Q_f = Q_i \times 10^{(-kt)}$ where Q_f represents the final amount of material in grams and $Q_i = 500$ grams is the initial amount. Find k if $Q_f = 400$ grams when t = 1,000 years.

6.)   Use the value of k you found in problem #5 to find Q_f when $t = 2,000$ years.

7.)   If t is the thickness of a material, k is an absorption coefficient that results from the physical characteristics of the material, and I the intensity of a beam of gamma radiation, the intensity of the gamma beam after passing the material is given by $I_f = I_i \times 10^{(-kt)}$. Find the absorption coefficient k of a material for which 9.4 cm thickness reduces a beam of 1 million electron volts to 100,000 electron volts intensity.

8.)   For question #7 find the thickness of the material that will reduce the initial I value, I_o, by half.

9.)   Say an air conditioner puts out 65 decibels of sound. Use the formula

$$Db = 10 \times \log\left(\frac{power_A}{power_B}\right)$$

to evaluate the noise of that air conditioner in watts per square foot of power. Use data for "softest audible sound" given in the text for $power_B$.

10.) "Moore's law" states that the number of transistors on an integrated circuit will double every two years. Its mathematical form is

$$n_2 = 2^{\frac{y_2-y_1}{2}} \times n_1$$

where $n_2 = $ # transistors in base year y_1 and
   $n_1 = $ # transistors in later year y_2
If a microprocessor had 376 million transistors in 2007 approximate what year it will have 12,024,000,000 transistors on it?

11.) You wish to locate a telephone number in the Dallas phone book for John Q. Public. You open the phone book to the middle and find names beginning with M. You reason that the name you are searching for is in the set of pages to the right of M. You move your left hand to the Ms and proceed to open the pages M to Z to the middle again searching for John Q. Public. This time you find that you are on the names beginning with S. Since Public starts with P you decide to ignore all the pages to the right of the S names and do so by moving your right hand to the Ss. You continue this process until you locate the desired telephone number. Assuming there are 1,050,000 people in the Dallas phone book and that there are 200 names per page, how many times will you have to search until you locate the number?

12.) The Koch snowflake had a fractal dimension between 1 and 2. By extension, what dimension do you estimate a self-similar polyhedron will have?

13.) A wooden spear is found at an ancient burial site. It is found to have only 80% the amount of carbon-14 that a live tree has. How many years ago did the tree live that was used to make that spear? Assume the half-life for carbon-14 is 5,750 years.

14.) There are 64 teams competing in a basketball tournament. Each tournament "round" eliminates $\frac{1}{2}$ of the remaining teams. Write the exponential equation that describes how many rounds will be necessary to decide the winner of the tournament. Solve that equation.

15.) Following is a chart showing how many biological relatives Bobby has going back several generations.

	Parents	Grandparents	Great-grandparents	GG-grandparents	GGG-grandparents, etc.
Bobby	2	4	8	16	32
	2^1	2^2	2^3	2^4	2^5

How many Gs (GGGG … G) are there in the generation where Bobby had 8,192 ancestors?
Do not solve by enumeration. Solve using logarithmic skills.

16.) A capacitor is a device that can store an electrical charge for later use. An example of such use would be the charge needed for a camera flash or for computer back-up if power fails. The voltage of a capacitor decreases steadily over time (exponential decay). Use the following formula

$$v_f = v_i \times e^{-t/RC}$$

where $v_f = $ final voltage
   $t = $ time
   $R = $ resistance in ohms
   $C = $ capacitance in millifarads
and $v_i = $ initial voltage

to determine how much time will elapse until the voltage falls to 10% of its initial voltage if capacitance $= 35$ µF and resistance $= 120$ Ω.

# Chapter 7: More about *e* and the Natural Logarithm

The number *e*, when viewed as a base for an exponential expression has some very interesting properties. 1.) the rate of change of the exponential equation $y = e^x$ is itself e^x. 2.) The area under the curve $y = e^x$ from negative infinity to $x = 1$ is *e* square units. 3.) The rate of change for the inverse of the function $y = e^x$ (i.e., the function $y = \ln x$) is $1/x$. 4.) The area under the curve $y = (1/x)$ from $x = 1$ to $x = e$ is equal to 1. 5.) The functions $f(x) = e^x$ and $f(x) = \ln (x)$ can both be evaluated using esthetically pleasing infinite-series polynomials. 6.) The sequence $(1 + 1/n)^n$ approaches *e* as *n* approaches ∞. 7.) As *x* gets close to 0 by powers of 10, the function log $(1 + x)$ gets closer and closer to a "diminished magnitude" of log *e*. 8.) Finally, as *x* gets close to 0 by powers of 10, the function 10^x gets closer and closer to the sum of 1 plus a diminished magnitude of ln 10. **Say what?!?**

These ideas will require some prerequisite background.

Prerequisite idea: How do you find the slope of a tangent line to a curve? In Algebra I, we learned how to find the slope of a line connecting two points. Since, by definition, a secant line crosses a curve at two points we merely identify those two points (x_1, y_1) and (x_2, y_2) and use the slope formula, to find the slope of a secant line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

However, a tangent line has only one point in common with a curve. Since the slope formula requires two points and the tangent line has only one point we must "get at" the desired information (slope of a tangent line) through successively approximating it with secant line slopes which are formed by moving point p_2 (and its coordinate pair (x_2, y_2)) closer and closer to point p_1 (and its coordinate pair (x_1, y_1)). The sequence of secant-line slopes sec_1, sec_2, sec_3, gets closer and closer to the desired tangent-line slope.
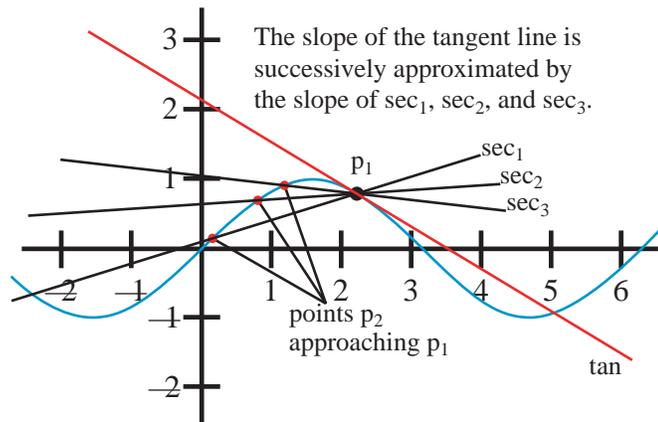
Now why do we care about the slope of a tangent line? Ostensibly, there is no reason to find the slope of a line tangent to a curve! Remarkably, though, that skill is mathematically equivalent to an abstract idea called the *instantaneous rate of speed,* which has great utility in physics and engineering. The connection between the two ideas, slope of a line tangent to a curve at a specified point and the instantaneous rate of speed comes about because the formula for the slope of a tangent line found by approaching secant lines using the formula $m = (y_2 - y_1)/(x_2 - x_1)$ as point p_2 approaches point p_1 is equivalent to the formula for average rate of speed, $r = d/t$ as time intervals decrease to zero.

**This is shown in great detail on the following page**.

RELATIONSHIP BETWEEN
1.) The slope of a tangent to a curve at a given point
and
2.) Instantaneous rate of speed

PROBLEM: A missile is fired at a target 64 miles away. The distance in miles that the missile has traveled from its starting point is given by the function $f(t) = t^2/4$. Determine the speed of the missile at the precise instant when it strikes the target—its instantaneous rate of speed.

$$\text{distance} = \text{rate} \times \text{time} \qquad \text{therefore}$$

$$d = r \times t \qquad\qquad rate = \frac{distance}{time} \qquad \text{this is a } \underline{\text{ratio}} \text{ just like}$$

$$\text{e.g., 100 miles} = 50 \text{ mph} \times 2 \text{ hours} \qquad r = \frac{d}{t} \qquad \text{ratios} \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

$f(t) = \frac{1}{4}t^2$

x	$f(x)$
0	0
2	1
4	4
6	9
8	16
10	25
12	36
14	49
15	$\frac{225}{4}$
$15\frac{1}{2}$	$\frac{\left(\frac{31}{2}\right)^2}{4} = \frac{\left(\frac{961}{4}\right)}{4} = \frac{961}{16}$
$15\frac{3}{4}$	$\frac{\left(\frac{63}{4}\right)^2}{4} = \frac{\left(\frac{3969}{16}\right)}{4} = \frac{3969}{64}$
16	64

Distance in Miles

(16,64)
(15½,961/16)
(15,225/4)
(14,49)
(12,36)
(8,16)
(4,4)
(2,1)
(0, 0)

Time in Minutes
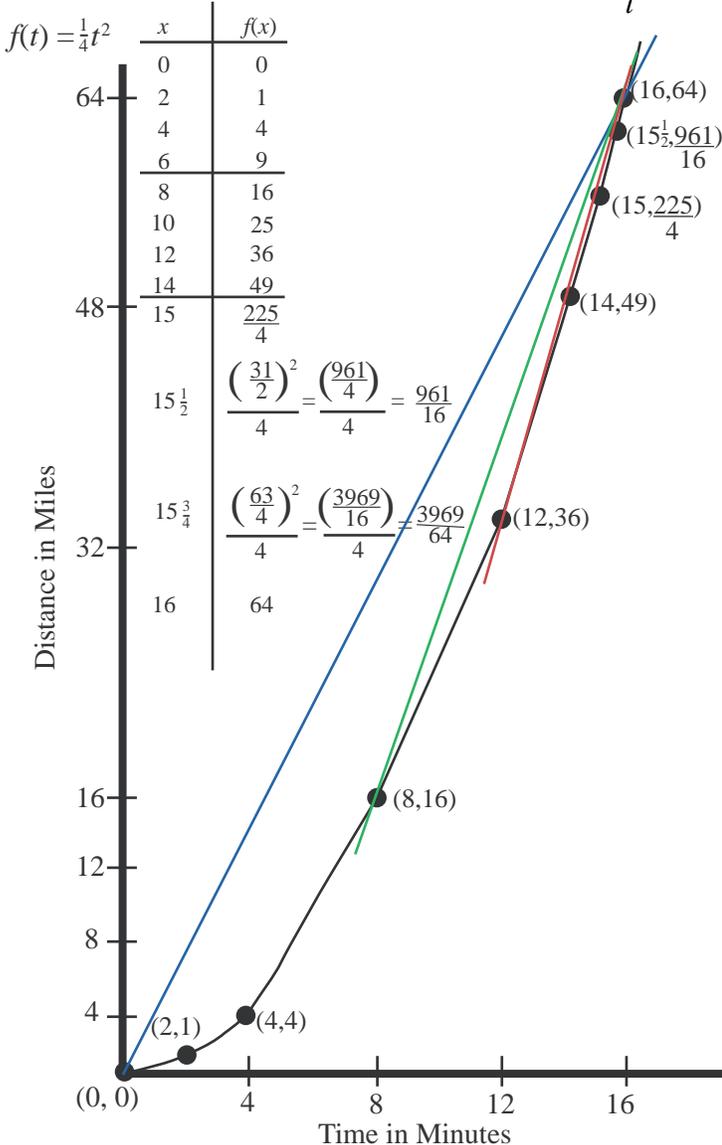
1.) Average rate of speed from $t = 0$ to $t = 16$:
$$r = \frac{d}{t} = \frac{64 - 0}{16 - 0} = \frac{64}{16} = 4 \text{ mpm average speed.}$$
Slope of secant line from (0, 0) to (16, 64):
$$m = \frac{y_2 - y_2}{x_2 - x_2} = \frac{64 - 0}{16 - 0} = \frac{64}{16} = 4$$

2.) Average rate of speed from $t = 8$ to $t = 16$:
$$r = \frac{d}{t} = \frac{64 - 16}{16 - 8} = \frac{48}{8} = 6 \text{ mpm average speed.}$$
Slope of secant line from (8, 16) to (16, 64):
$$m = \frac{y_2 - y_2}{x_2 - x_2} = \frac{64 - 16}{16 - 8} = \frac{48}{8} = 6$$

3.) Average rate of speed from $t = 12$ to $t = 16$:
$$r = \frac{d}{t} = \frac{64 - 36}{16 - 12} = \frac{28}{4} = 7 \text{ mpm average speed.}$$
Slope of secant line from (12, 36) to (16, 64):
$$m = \frac{y_2 - y_2}{x_2 - x_2} = \frac{64 - 36}{16 - 12} = \frac{28}{4} = 7$$

4.) Average rate of speed from $t = 14$ to $t = 16$:
$$r = \frac{d}{t} = \frac{64 - 49}{16 - 14} = \frac{15}{2} = 7\frac{1}{2} \text{ mpm average speed.}$$

5.) Average rate of speed from $t = 15$ to $t = 16$:
$$r = \frac{d}{t} = \frac{64 - \frac{225}{4}}{16 - 15} = \frac{\frac{256}{4} - \frac{225}{4}}{16 - 15} = \frac{\frac{31}{4}}{1} = 7\frac{3}{4}.$$

6.) Average rate of speed from $t = 15\frac{1}{2}$ to $t = 16$:
$$r = \frac{d}{t} = \frac{64 - \frac{961}{16}}{16 - 15} = \frac{\frac{1024}{16} - \frac{961}{16}}{16 - 15} = \frac{\frac{63}{16}}{\frac{1}{2}} = \frac{63}{8} = 7\frac{7}{8}.$$

Notice the progression of "average rates of" speeds (or equivalently the progression of secant slope lines) m = 4, m = 6, m = 7, m = 7½, m = 7¾, m = 7⅞. As these average speeds keep getting calculated over distances that are decreasing to zero you are approaching the "instantaneous rate of speed." Or, stated another way, the slopes of the secant lines are approaching the slope of the tangent line at that point

Now that you have an understanding of the idea of "slope of a line tangent to a curve at a specific point" we can get back to the first idea on the first page of Chapter 7 regarding the number e.

1.) The function $y = e^x$ at point x evaluates to e^x. The slope of the tangent line to the curve $y = e^x$ at any point x is also e^x.

At left below, we see the graph of the curve $y = x^2$, and in the rightmost graph we see the graph of the curve $y = x^3$. Notice for the two curves below that $f(x)$ does not equal the slope of the line tangent at that point.

At point (1,1), m = 2 and f(1) = 1. The slope does not equal the function value.

$f(x) = x^2$

At point (1,1), m = 3 and f(1) = 1. The slope does not equal the function value.

$f(x) = x^3$
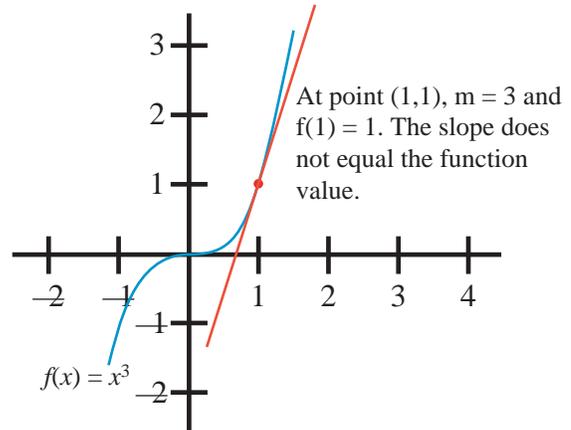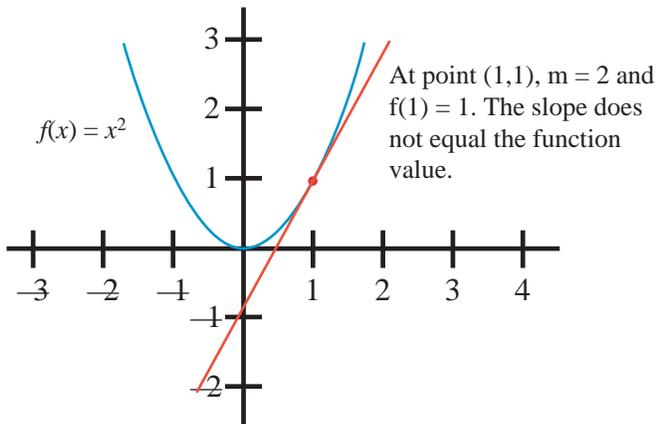
However, for the curve $y = e^x$, for any x, the slope of the tangent line at that point (x, y) is e^x. Also the value of the function at x is e^x. That's pretty neat!!

$f(x) = e^x$

At point (x, e^x), m = e^x. At point x, $f(x) = e^x$. Therefore, $f(x)$ equals the slope of the tangent at point (x, e^x).
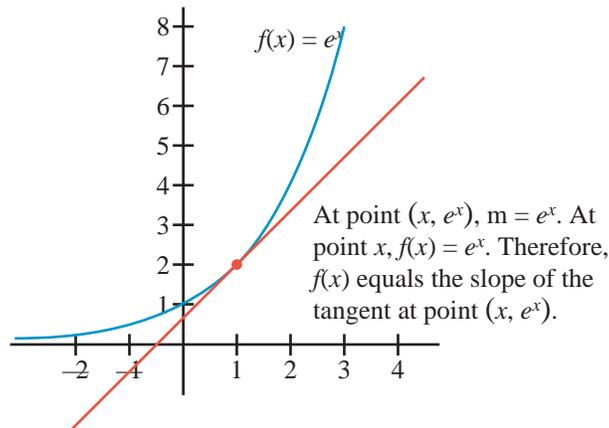
2.) The area under the curve y = *e^x* from –∞ to x = 1 is *e* units. Not 2, not 5, not 10 but *e* units. That's neat too!

$y = e^x$

The area under the curve is *e* units.
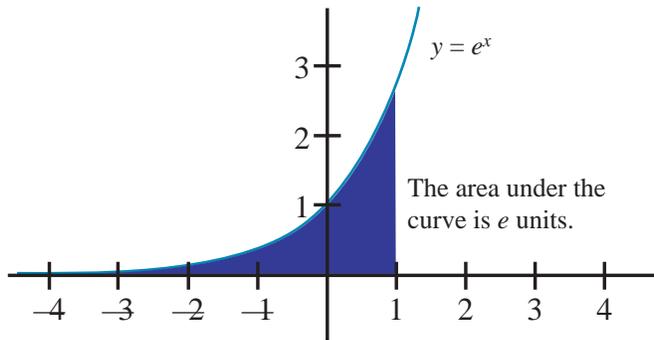
This remarkable pattern continues. The area under the curve $y = e^x$ *from -∞ to x* for any *x* is e^x square units. See the figure at right.

$y = e^x$

area = e^2
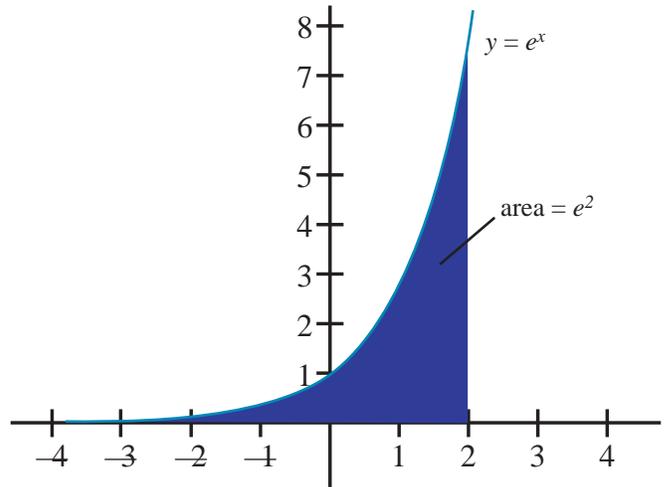
3.) The rate of change (the slope of a tangent at any given point) for the inverse of the function of y = *e^x*—i.e., the log function, y = ln(x)—is 1/x. That's interesting!

In the graph of the log function shown below, you can see that a line drawn tangent to the log function between 0 and 1 is very, very steep. As *x* grows larger and larger, the slope grows more and more flat. That is, *m* is large when the denominator of 1/*x* is between 0 and 1 but becomes smaller and smaller as *x* grows larger and larger. For the curve $y = \ln(x)$ the slope of the tangent line at any point is 1/*x*. !!!

at $x = 3$
$m = \frac{1}{3}$

at $x = \frac{1}{2}$
$m = 2$

The slope at any point on $y = \ln x$ is $\frac{1}{x}$!!
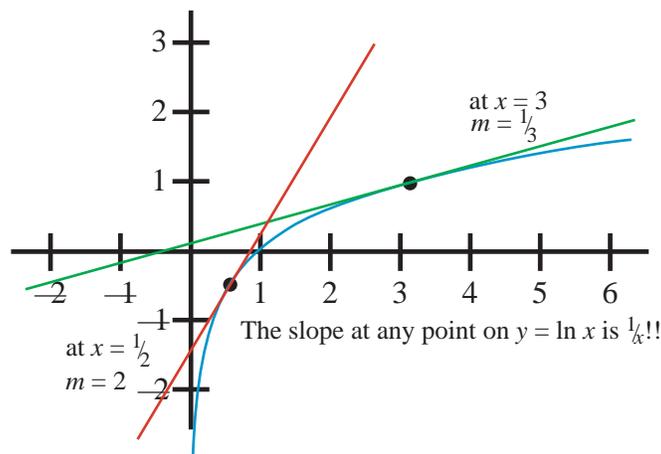
4.) **The area under the curve $y = 1/x$ from $x = 1$ to $x = e$ is equal to 1 square unit. It only makes sense that there must be a value of x that will be cause the area under the curve $y = 1/x$ from 1 to x to be 1 square unit but who would have thought that the value would be e? Why e?! Is that a coincidence or what!**
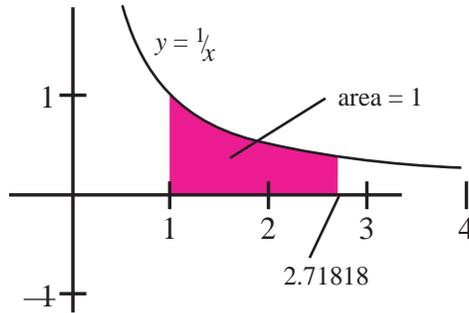
5.) **The functions $f(x) = e^x$ and $f(x) = \ln x$ can both be evaluated using esthetically pleasing infinite series polynomials.**

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$

Check it out:

$e_1 \approx 1$

$e_2 \approx 1 + 1 = 2$

$e_3 \approx 1 + 1 + \dfrac{1}{2} = 2.5$

$e_4 \approx 1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} = 2.66667$

$e_5 \approx 1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} = 2.70833$

$e_6 \approx 1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \dfrac{1}{120} = 2.7166667$

$e_7 \approx 1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \dfrac{1}{120} + \dfrac{1}{720} = 2.71805556$, **etc.**

The terms in the sequence **1, 2, 2.5, 2.66666, 2.70833, 2.7166633, 2.7180555 …** would seem to be converging to e, 2.718281828 …

5.) **continued**

$$\ln x = \frac{1}{1}\frac{(x-1)^1}{x^1} + \frac{1}{2}\frac{(x-1)^2}{x^2} + \frac{1}{3}\frac{(x-1)^3}{x^3} + \frac{1}{4}\frac{(x-1)^4}{x^4} + \frac{1}{5}\frac{(x-1)^5}{x^5} + \dots$$

(x > 1)

Check it out (let x = 5 then (x-1)/x = ⁴⁄₅ = 0.8):

$$\ln 5 \approx (0.8)^1 + \frac{1}{2}(0.8)^2 + \frac{1}{3}(0.8)^3 + \frac{1}{4}(0.8)^4 + \frac{1}{5}(0.8)^5 + \frac{1}{6}(0.8)^6$$

ln (5) ≈ 0.8
ln (5) ≈ 0.8 + 0.32 = 1.12
ln (5) ≈ 0.8 + 0.32 + 0.17066 = 1.29067
ln (5) ≈ 0.8 + 0.32 + 0.17066 + 0.1024 = 1.39307
ln (5) ≈ 0.8 + 0.32 + 0.17066 + 0.1024 + 0.65536 = 1.458602667
ln (5) ≈ 0.8 + 0.32 + 0.17066 + 0.1024 + 0.65536 + 0.043690667 = 1.502293333
ln (5) ≈ 1.50229333 + 0.0299593 = 1.532252647
ln (5) ≈ 1.532252647 + 0.02097152 = 1.5532241676
ln (5) ≈ 1.5532241676 + 0.0149130808 = 1.5681372485
ln (5) ≈ 1.5681372485 + 0.01073741824 = 1.57887466675
ln (5) ≈ 1.57887466674 + 0.00780903145 = 1.58668369820
ln (5) ≈ 1.58668369819 + 0.00572662306 = 1.59241032126
ln (5) ≈ 1.59241032125 + 0.00422889088 = 1.59663921213
ln (5) ≈ 1.59663921213 + 0.00314146179 = 1.59978067393
ln (5) ≈ 1.59978067392 + 0.00234562480 = 1.6021262987

   The sequence above 0.8, 1.12, 1.29066, 1.39306, 1.458602666, etc. is slowly converging to ln 5 = 1.60943791

As discussed before

> **6.) The sequence defined by** $\left(1+\dfrac{1}{k}\right)^k$ **converges to *e*.**

$$\left(1+\frac{1}{10}\right)^{10} = 2.59374246$$

$$\left(1+\frac{1}{100}\right)^{100} = 2.704813829$$

$$\left(1+\frac{1}{1000}\right)^{1000} = 2.716923932$$

$$\left(1+\frac{1}{10000}\right)^{10000} = 2.718145927$$

$$\left(1+\frac{1}{100000}\right)^{100000} = 2.718268237$$

$$\left(1+\frac{1}{1000000}\right)^{1000000} = 2.718280469$$

7.) As *x* gets close to 0 by powers of 10, the function log(1 + *x*) gets closer and closer to a diminished magnitude of log *e*!

---

while the sequence defined by $\left(1+\dfrac{x}{k}\right)^k$ converges to e^x as *k* gets larger and larger.

We know that log 1 = 0 because $10^0 = 1$. So it is not surprising that log (1 + *x*) is not 0. However, there is a fascinating, highly predictable sequence of numbers that results when taking log values that get closer and closer to 1 by powers of 10 … that is the function *y* = log (1 + *x*) as *x* decreases to 0 by powers of 10. That is surprising! Let's do an experiment taking log values of numbers that get closer and closer to 1 by powers of 10 and see if a pattern results that we recognize.

Number	Log of number	Distance "x" of Number from 1
1.1	0.0**41**3926852	0.1
1.01	0.00**43**213738	0.01
1.001	0.000**434**0774793	0.001
1.0001	0.0000**4342**727686	0.0001
1.00001	0.000004**342923**104	0.00001
1.000001	0.0000004**342942**647	0.000001
1.0000001	0.00000004**342944**604	0.0000001

Numbers decreasing by powers of 10	Numbers here decreasing to zero as expected

Deja vu! Those bold face numbers in the middle column look familiar. Back in chapter 5, we discussed the mysterious number *e* (2.718281828) that frequently occurs in nature and business and how it could be replaced with a power of 10. The formula for continuous interest, $S = P\,e^{rt}$, was rewritten as $S = P\,(10)^{0.4342944819rt}$. We were able to rewrite that formula by the following process:

$$e = 10^x$$
$$\log e = \log 10^x$$
$$\log e = x \log 10$$
$$\log e = x \times 1 \qquad\qquad \text{(do you understand why log 10 = 1?)}$$
$$\log e = x$$
$$x = \textbf{\textit{log e = 0.4342944819}} \qquad \text{by calculator}$$

Now do you remember? Revisit the table above and see what is happening. As *x* decreases to 0 by powers of 10 the function log (1 + *x*) decreases to 0 as expected. (Recall that log 1 = 0.) But also at the same time the significant digits of the log (1 + *x*) get closer and closer to **log *e*, 0.4342944819**.

0.0**41**3926852,   0.00**43**213738,   0.000**434**0774793,   0.0000**4342**727686,
0.000004**342923**104, 0.0000004**342942**647, 0.00000004**342944**604 …

At each step as *x* is diminished by powers of 10, the resulting term decreases to zero, while the number of significant digits of log *e* is increased. Symbolically, log (*1* + *x*) = *x* (log *e*) = log e^x as *x* gets smaller and smaller. The number *e* (2.718281828) or in this case log *e* (0.4342944819) seems to pop up over and over and over in the most unexpected situations! The existence of a numerical sequence decreasing to zero for log (1 + *x*) should have been anticipated but did you anticipate the "*e* based pattern of significant digits" for each term in the sequence? Cute!

and finally

8.) **As *x* gets close to 0 by powers of 10, the function 10^x gets closer and closer to 1 + *a* diminished magnitude of ln 10.**

We know that $10^0 = 1$, so it not surprising that $10^x \neq 1$ when $x \neq 0$. However, there is a highly predictable sequence of numbers that results by evaluating 10^x for values that get closer and closer to 0 by powers of 10. That is surprising! Let's do an experiment by obtaining values 10^x for *x* values getting closer and closer to zero and see if we can recognize a pattern.

x	10^x
0.1	1.**2589254117941673**
0.01	1.0**23292992280754**
0.001	1.00**23052380778996**
0.0001	1.000**2302850208247**
0.00001	1.0000**230261160268**
0.000001	1.000002**302587744**
0.0000001	1.0000002**302585358**
0.00000001	1.0000000**230258512**

Since $10^0 = 1$ it is not surprising that the sequence of numbers in column 2 is decreasing to 1. What is surprising is the predictable pattern of significant digits in the numbers in column 2. Compare that sequence of numbers in the second column with ln 10 = 2.302585092994046. Recall that ln 10 is equivalent to $\log_e 10$. There's that number *e* again!!

Chapter 7 Summary—The number *e*, when viewed as a base for an exponential expression has some very interesting properties.

1.) **The rate of change of the function $y = e^x$ is itself e^x.**

2.) **The area under the curve $y = e^x$ from negative infinity to $x = 1$ is *e* units.**

3.) **The rate of change of the natural log function with base *e* at any point is 1/x.**

4.) **The area under the curve $y = \dfrac{1}{x}$ from $x = 1$ to $x = e$ is exactly 1.**

5.) **The functions $f(x) = e^x$ and $f(x) = \ln(x)$ can both be evaluated using esthetically pleasing infinite series polynomials.**

6.) **The sequence defined by $\left(1 + \dfrac{1}{k}\right)^k$ converges to e^1 while the sequence defined by $\left(1 + \dfrac{x}{k}\right)^k$ converges to e^x.**

7.) **As *x* gets closer and closer to 0 by powers of 10, the function $\log(1 + x)$ gets closer and closer to a diminished magnitude of $\log e$ ... $\log(1 + x) = x \log e$.**

8.) **As *x* gets closer and closer to 0 by powers of 10, the function 10^x gets closer and closer to 1 + a diminished magnitude of ln 10 (i.e., $1 + x \times \log_e 10$)**

Chapter 7 Exercises

1.)   On the second page of this chapter, there was a discussion of instantaneous rate of speed. From that discussion of the rate of speed over decreasing intervals of time, how fast do you think the missile will be traveling at the "instant" that it strikes the target? Explain your answer.

2.)   In the graph of $y = e^x$ (at right) you should see several secant lines all drawn through the same end point, $(1, e)$. By repeatedly applying the slope-between-two-points formula, $m = (y_2 - y_1)/(x_2 - x_1)$, find the slope of sec1, sec2, sec3, and sec4.

Describe what is happening to the successive secant slopes as the initial points approach the fixed end point at $(1, e)$.
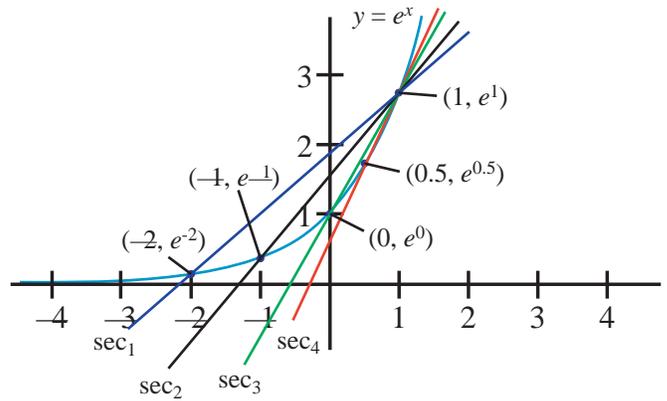
3.)   For the graph at right, $y = 1/x$, approximate the area under the curve from 1 to 2.71828 by adding up the areas of the four rectangles, $r_1 + r_2 + r_3 + r_4$.
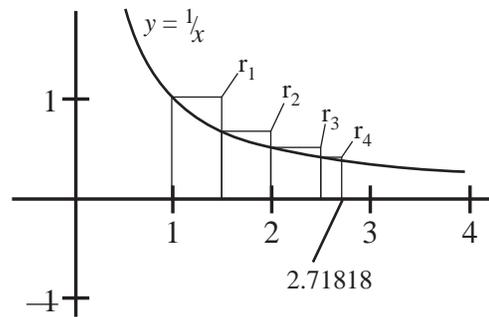
4.)   For the graph at right, $y = 1/x$, approximate the area under the curve from 1 to 2.71828 by adding up the areas of the four rectangles, $r_1 + r_2 + r_3 + r_4$.
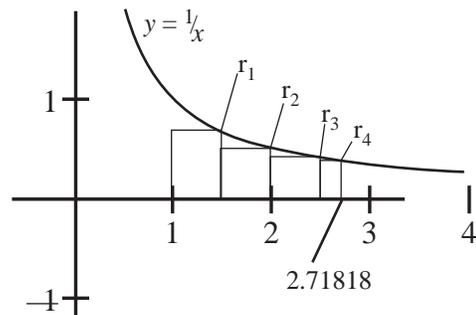
5.) Average the two areas you got in questions #4 and #5. What did you get? What do you think might happen if the number of rectangles is increased to say 8, 16, etc.?

6.) You know that $\log_{10} 1 = 0$. Use your calculator to find each of the following values:

$\log_{10} 0.9 =$
$\log_{10} 0.99 =$
$\log_{10} 0.999 =$
$\log_{10} 0.9999 =$
$\log_{10} 0.99999 =$
$\log_{10} 0.999999 =$
$\log_{10} 0.9999999 =$
$\log_{10} 0.99999999 =$
$\log_{10} 0.999999999 =$

Evaluate these *y* values.
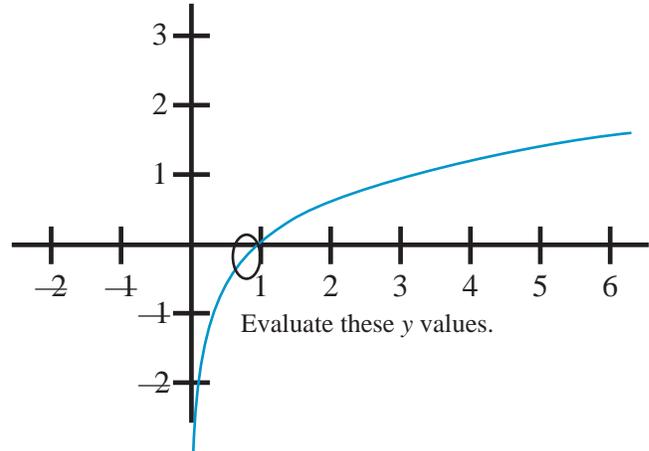
What do you notice about the pattern of your answers?

7.) You are on a game show … "The Weakest Link." You are asked to enumerate eight fun facts about the number *e*, the exponential curve $y = e^x$, or the log curve $y = \ln x$. How many can you name?

Chapter 8: More Log Rules

All the rules learned to this point are gathered together and listed below for reference

$b^m \times b^n = b^{(m+n)}$	**Product of Common Base Factors Rule**
$\dfrac{b^m}{b^n} = b^{(m-n)}$	**Quotient of Common Bases Rule**
$\log(x \times y) = \log x + \log y$	**Log of a Product Rule**
$\log\left(\dfrac{x}{y}\right) = \log x - \log y$	**Log of a Quotient Rule,**
$m = n$ iff $b^m = b^n$	**iff Antilog Rule**
$m = n$ iff $\log m = \log n$	**iff Log Rule**
$b^y = x$ is equivalent to $y = \log_b x$	**Equivalent Symbolism Rule**
$\log_b b^x = x$	**Inverse Log Rule #1** (Log of an Exponential Rule)
$b^{\log_b x} = x$	**Inverse Log Rule #2** (Power of a Base Rule)
$\log b^m = m \log b$	**Log of a Base Raised to a Power Rule**
$x \times y = 10^{(\log x + \log y)}$	**Rule to Multiply Using Logarithms**
$\dfrac{x}{y} = 10^{(\log x - \log y)}$	**Rule to Divide Using Logarithms**
$\log_p x = \dfrac{\log_q x}{\log_q p}$	**Change of Base Log Rule … change from base p to q.**
If $b^x = y$ then $x = \dfrac{\log y}{\log b}$	**Rule to Solve for an Exponent**
If $b^{x/y} = q$ then $q = 10^{\frac{x}{y}\log b}$	**Rule to Raise a Constant to a Power**

Depending on the curriculum you are in, it is possible that you might see other log rules. For example,

a.) $\log_a b = \dfrac{1}{\log_b a}$

b.) $(\log_a b)(\log_b c) = \log_a c$

In general you should approach each of these formulas as follows:

1.) Get a feel for the log rule by choosing arbitrary values for a, b, and c and check to how the formula holds for your values.

2.) See if you can apply a combination of the above-listed Log Rules to prove the new rule. Be creative and flexible.

For example: a.) $\log_a b = \dfrac{1}{\log_b a}$

1.) Get a feel for the log rule by choosing arbitrary values for a and b and check how the rule holds for your values. Let $a = 10$ and $b = 450$

so $\qquad \log_a b = \dfrac{1}{\log_b a}$

becomes $\qquad \log_{10} 450 \overset{?}{=} \dfrac{1}{\log_{450} 10}$

$\qquad\qquad 2.653212514 \overset{?}{=} \dfrac{1}{\log_{450} 10}$

Solving for $\log_{450} 10 = \dfrac{\log_{10} 10}{\log_{10} 450}$ $\qquad$ **Change of Base Rule**

$\qquad\qquad\qquad = \dfrac{1}{2.653212514}$ $\qquad$ **Inverse Log Rule #1**

$\qquad\qquad\qquad = 0.3769015843.$

Substituting 0.3769015843 for $\log_{450} 10$ into the previous equation,

$\qquad\qquad 2.653212514 \overset{?}{=} \dfrac{1}{0.3769015843}$

$\qquad\qquad 2.653212514 = 2.653212514 \qquad$ check

2.) See if you can apply a combination of the above-listed Log Rules to prove the new rule. Be creative and flexible.

$\qquad\qquad\qquad \log_a b \overset{?}{=} \dfrac{1}{\log_b a}$

$\qquad \log_a b \times \log_b a = 1 \qquad\qquad$ cross-multiply

$\qquad\qquad\quad \text{let } k = \log_a b \qquad\qquad$ arbitrary substitution, you will see why in two more steps

$\qquad\qquad k \times \log_b a = 1 \qquad\qquad$ substitution

$\qquad\qquad\quad \log_b a^k = 1 \qquad\qquad$ **Log of a Base Raised to a Power Rule**

$\qquad\quad \log_b a^{\log_a b} = 1 \qquad\qquad$ back substitution

$\qquad\qquad\quad \log_b b = 1 \qquad\qquad$ **Power of a Base Rule (Inverse Log Rule 2)**

$\qquad\qquad\qquad b^1 = b \qquad\qquad$ **Equivalent Symbolism Rule   QED**

Your turn.

Process the log rule $(\log_a b)(\log_b c) = \log_a c$

    1.) Get a feel for the log rule by choosing arbitrary values for a, b, and c and checking how the rule holds for your values. Say $a = 3$, $b = 5$, and $c = 7$. Hint, convert each term into Base 10 using

$$\log_p x = \frac{\log_{10} x}{\log_{10} p}$$ **Change of Base Log Rule … change from base p into base 10**

    2.) prove $(\log_a b)(\log_b c) = \log_a c$
    Since two of the three terms in the log rule **$\underline{(\log_a b)}$** $(\log_b c) = \underline{\log_a c}$ have a base of "a" Maybe changing the remaining term, $(\log_b c)$ to base "a" using the **Change of Base Log Rule** and substituting appropriately might be helpful.

$$\log_p x = \frac{\log_q x}{\log_q p}$$ **Change of Base Log Rule … change from base p into base q**

Chapter 8 Summary—Chapter 8 discussed the fact that there are more log rules than have been discussed up to now. Should you ever encounter them, apply the following two steps to learning about them and feeling comfortable about them.

    1.) Get a feel for the log rule by choosing arbitrary values for whatever variables are involved and checking how the formula holds for your values.
    2.) See if you can apply a combination of the above-listed Log Rules to prove the new rule. Be creative and flexible.

Chapter 8 Exercises

Given the log rule $\log_{b^n} x^n = \log_b x$

Show that you can apply the process discussed in this chapter to the new log rule above. Hint: Apply the Equivalent Symbolism Rule for logs and the Power of a Power Rule from algebra to prove this rule.

Chapter 9: Asymptotes, Curve Sketching, Domains & Ranges

The astute reader may have noticed something about all the exponential and log curves we have seen so far. The exponential curves have always had nonnegative y values. This comes about because the base of an exponential curve is positive by definition. In that case, $y = b^x$ will always be positive due to the closure property of multiplication. Notice that when $x < 1$ the corresponding y is small, and as x becomes larger than 1 y becomes large very fast. At least, this is true when the base (2 in this case) is greater than one.

$y = 2^x$

x	y
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8
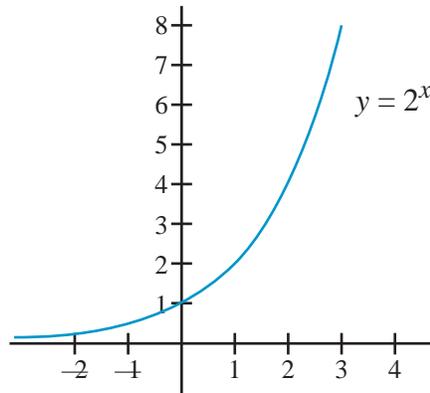
However when $0 < \text{base} < 1$ the curve's appearance is as follows.

$y = \frac{1}{2}^x$

x	y
-3	8
-3	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
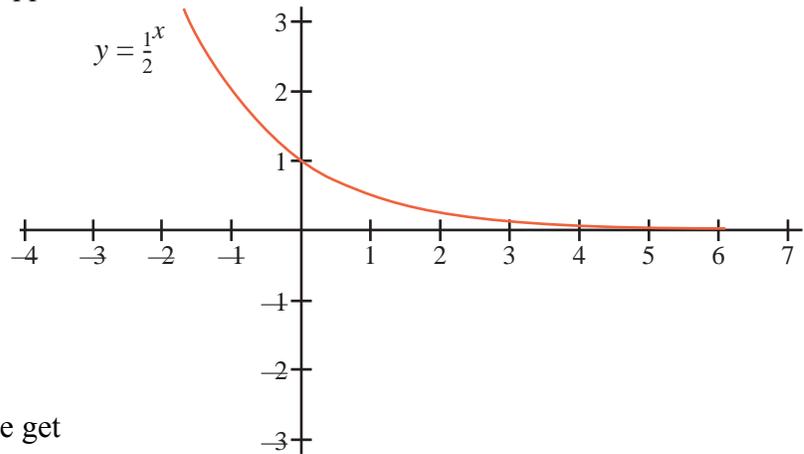
Plotting both curves on the same axis, we get the figure at right.
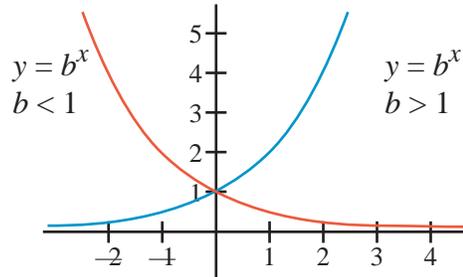
Regardless of the value of b (as long as b is positive) the resulting graph of the exponential function $y = b^x$ never drops below the x axis. Its domain is all real numbers and its range is positive. We say that the function $f(x) = b^x$ is **asymptotic with the x axis**.

Forming the inverse of $y = 2^x$, we get its associated log curve. **As the two curves are inverses, it should not surprise us when the asymptotes for the two curves are exchanged along with the exchange of the x and y values.**

$$y = 2^x \qquad\qquad x = 2^y$$
$$(y = \log_2 x)$$

x	y		x	y
-2	$\tfrac{1}{4}$		$\tfrac{1}{4}$	-2
-1	$\tfrac{1}{2}$		$\tfrac{1}{2}$	-1
0	1		1	0
1	2		2	1
2	4		4	2
3	8		8	3

$(y = 2^x$
domain: all real
range: $y > 0)$

$y = 2^x$

$y = x$

$x = 2^y$
$(y = \log_2 x$
domain: $x > 0$
range: all real)
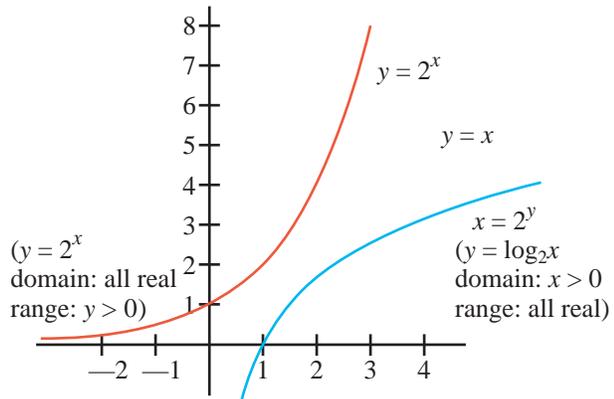
Choosing b, where $0 < b < 1$, and again placing the exponential curve on the same x–y axis as its inverse we get the figure at right. Again the asymptotes are exchanged.

$$y = \tfrac{1}{2}^x \qquad\qquad x = \tfrac{1}{2}^y$$
$$(y = \log_{\frac{1}{2}} x)$$

$y = \tfrac{1}{2}^x$
domain: all real
range: $y > 0$

x	y		x	y
-3	8		8	-3
-3	4		4	-3
-1	2		2	-1
0	1		1	0
1	$\tfrac{1}{2}$		$\tfrac{1}{2}$	1
2	$\tfrac{1}{4}$		$\tfrac{1}{4}$	2

$y = x$

$y = \log_{\frac{1}{2}} x$
$(x = \tfrac{1}{2}^y)$

domain: $x > 0$
range: all real
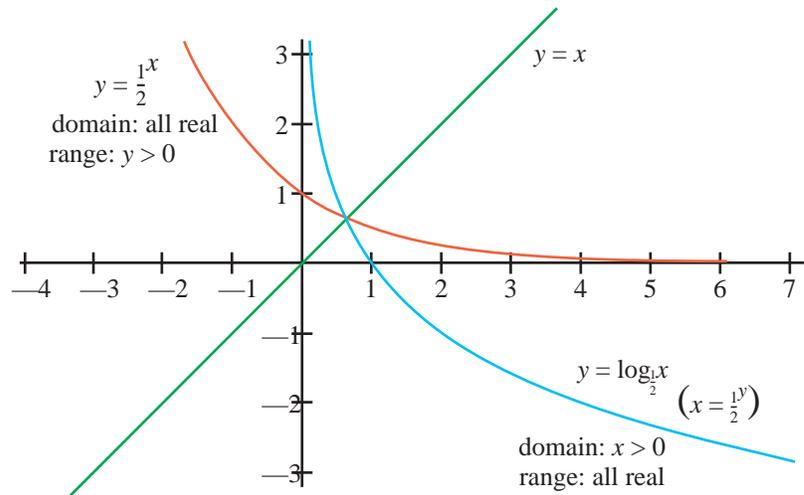
All the curve-sketching tricks you learned with polynomial and trig graphs work with exponential and log curves as well. For $y = a \times b^x$ the curve stretches vertically for $a > 1$ and flattens for $a < 1$. The curve $y = a \times \log(x)$ stretches vertically for $a > 1$ and flattens for $a < 1$. <u>Notice that the asymptote of neither curve is affected by the multiplier!!</u>

$y = ab^x$
$a > 1$

$y = ab^x$
$a = 1$

$y = ab^x$
$a < 1$

$y = a\log_b x$
$a > 1$

$y = a\log_b x$
$a = 1$

$y = a\log_b x$
$a < 1$
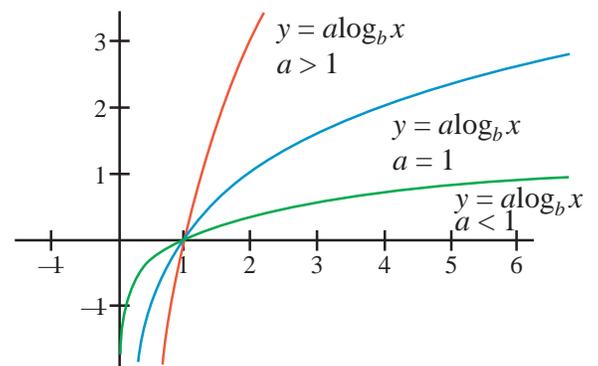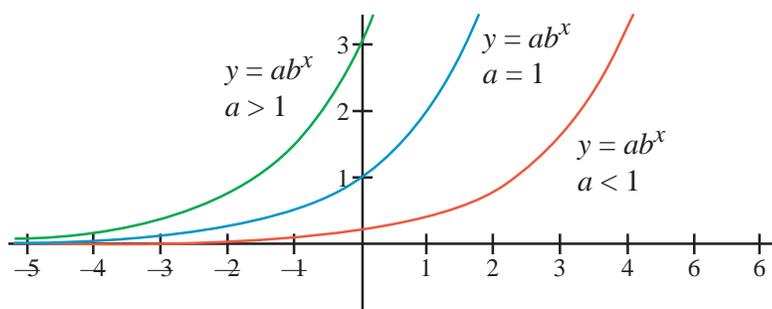
Continuing our experiment with curve-sketching techniques, we illustrate the vertical shift for $y = b^x$ and for $y = \log_b x$. Here the asymptote for the exponential functions is affected but not those of the log functions. Why is the asymptote for the exponential functions affected by a vertical shift while the asymptote for the log function is not? Well, to begin with, a vertical shift is either up or down and that would not change a vertical asymptote. Another way of thinking about this is to change from $y = \log_2 (x)$ into $x = 2^y$. Now a vertical shift only changes the y value. As long as the base is positive (2 in this case), the x value will have to remain positive.
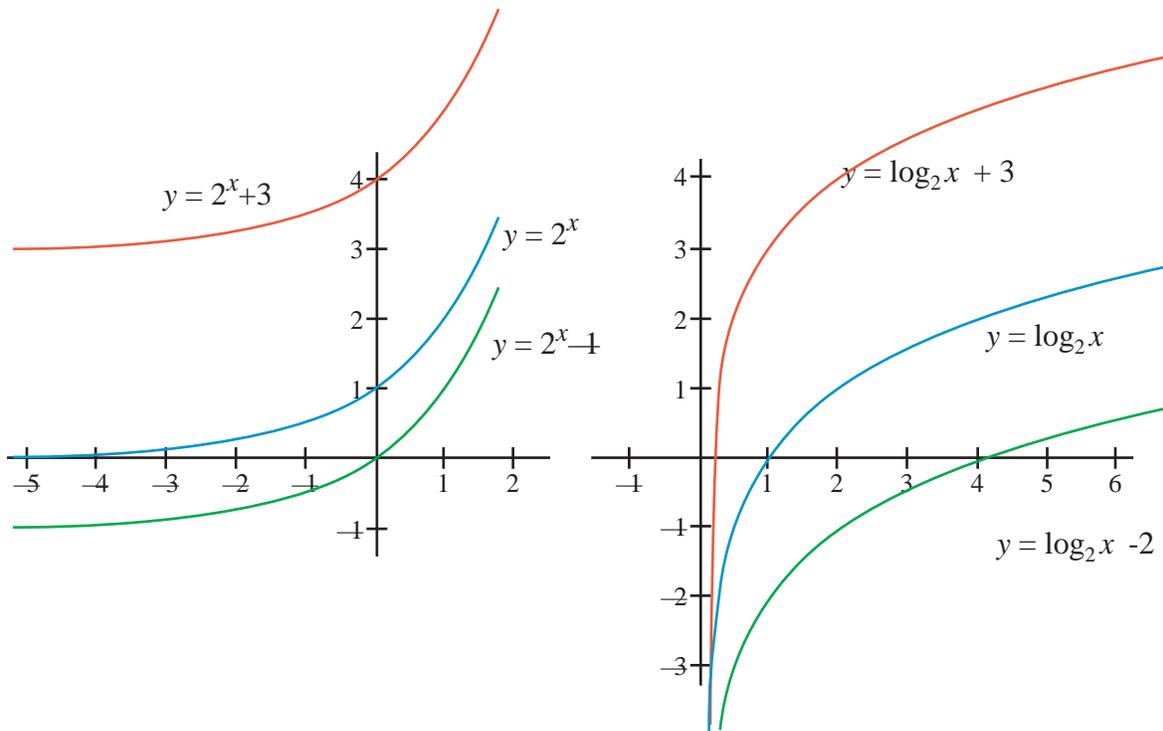
And of course we need to experiment with the horizontal shift for both the exponential and the logarithmic functions. Notice that the exponential functions' asymptotes are unchanged after the horizontal shift, but the horizontal shift **does** affect the asymptote for the log function. Why would this be?
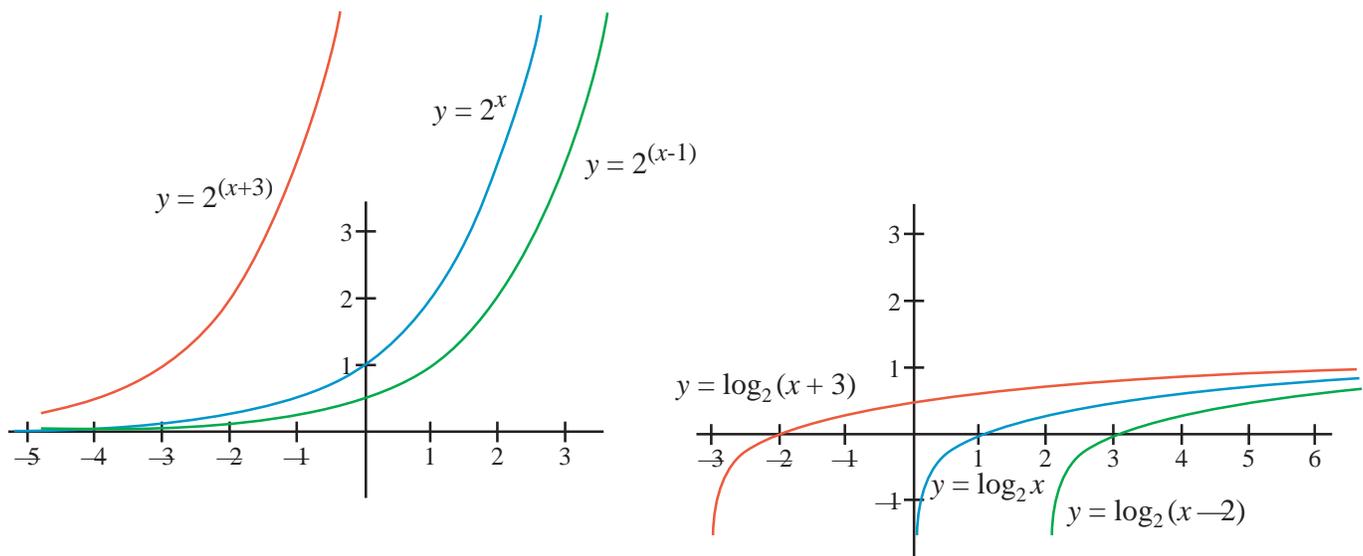
We have been examining all the curve-sketching skills as they apply to graphing the exponential and log curves. In general form, we have now analyzed $y = a \log_b (x - h) + k$. Is there anything else? Well, in the example given above, the multiplier was positive. What would happen if the multiplier were negative? Do you see that the rising log curve gets reflected about the x axis? We should have anticipated that a negative multiplier would change any previously positive values to negative values and vice versa.
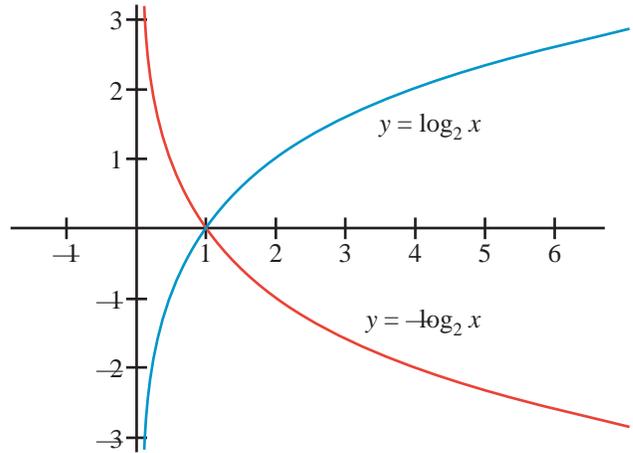
Let's put all this together. Graph

$$y = -\frac{1}{3}\log_{\frac{1}{2}}(x+4)+2 \text{ without a graphing calculator.}$$

A quick look at this functions shows that there are 6 things to take into consideration:

1.) This is a log function for $b > 1$

$y = \log_b x$ with b > 1

Vertical asymptote here is $x = 0$.
Domain: $x > 0$
Range: $-\infty < y < \infty$

2.) This is a log function with a base less than one (say $\frac{1}{2}$). This requires using the change of base formula for work with a calculator.

$y = \log_b x$ with b < 1
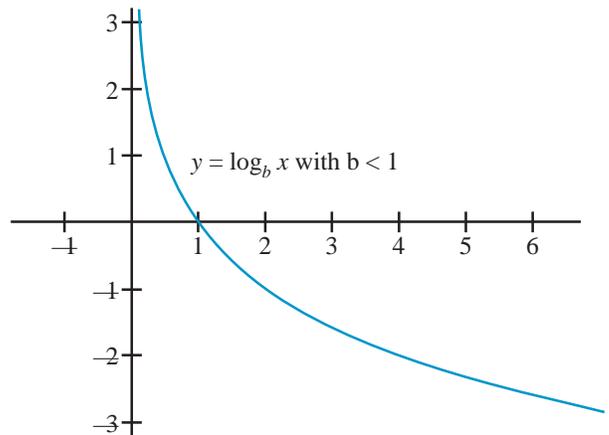
Vertical asymptote here is still $x = 0$.
Domain: $x > 0$
Range: $-\infty < y < \infty$

3.) The negative multiplier will reflect the curve about the x axis.

4.) There is a "shrink" or flattening effect because the absolute value of the multiplier is less than 1.

$y = -\log_{\frac{1}{2}} x$

$y = -\tfrac{1}{3}\log_{\frac{1}{2}} x$
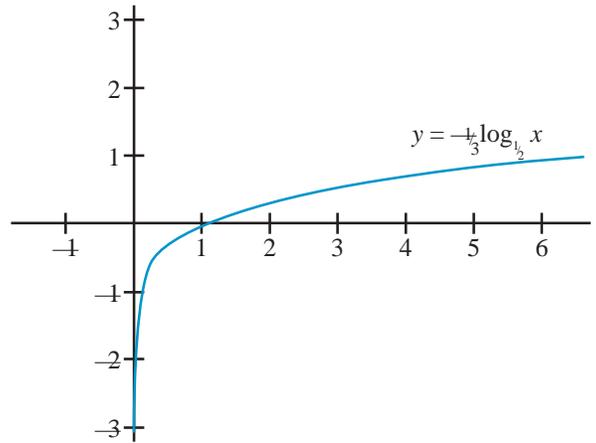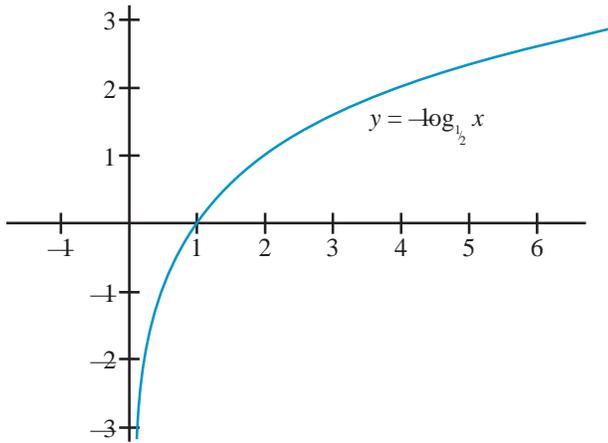
Vertical asymptote here is $x = 0$.
Domain: $x > 0$
Range: $-\infty < y < \infty$

Vertical asymptote here is still $x = 0$.
Domain: $x > 0$
Range: $-\infty < y < \infty$

5.) There is a horizontal shift of -4.

6.) And finally, there is a vertical shift of 2.
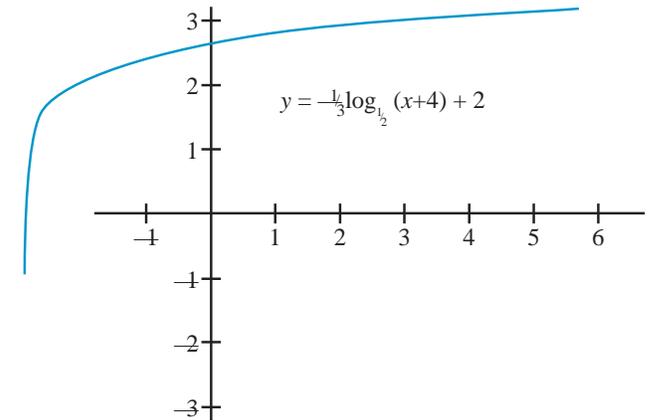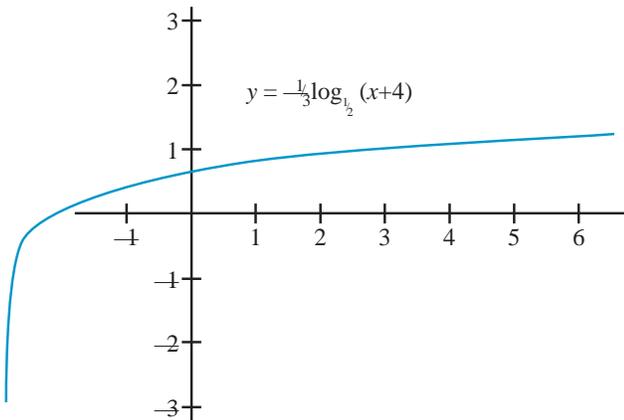
$y = -\tfrac{1}{3}\log_{\frac{1}{2}} (x+4)$

$y = -\tfrac{1}{3}\log_{\frac{1}{2}} (x+4) + 2$

Vertical asymptote here is still $x = -4$.
Domain: $x > -4$
Range: $-\infty < y < \infty$

Vertical asymptote here is now $x = -4$.
Domain: $x > -4$
Range: $-\infty < y < \infty$

From inspection of the figure in #6, what are the domain and range of the final function?

$$y = -\frac{1}{3}\log_{\frac{1}{2}}(x+4)+2$$

From looking at the generalized formula $y = a \times \log_b (x + h) + k$ what are the domain, range, and asymptote of that generalized function in terms of h and k?

Your turn: In stages, as shown above, use what you know about curve sketching to find the domain and range of the function $f(x) = 5 \times \log_7 (x + 7) - 4$. Sketch the curve labeling all important information.

Problem: Give the domain and range of the relation $\log_x (y - 2) = \log_x (4 - x)$

This sort of problem requires some thought.

Step 1:   The term *domain* pertains to allowable x values.

Step 2:   In chapter 2, we found, for $y = \log_b x$, that $b > 0$, $b \neq 1$, and $x > 0$ (in a log function the domain is positive and the base is positive and not equal to 1).

In both sides of the equation above the base $= x$, therefore **$x > 0$, $x \neq 1$**

Also, in $\log_b k$, $k > 0$ (see chapter 2). therefore, for $y = \log_x (4 - x)$, $(4 - x) > 0$

So $4 > x + 0$
$x < 4$

1.) $x > 0$
2.) $x \neq 1$
3.) $x < 4$

Or   **$0 < x < 4, x \neq 1$**

Step 3:   The term *range* pertains to allowable y values.

Step 4:   Since, for $\log_b k$, $k > 0$ (See chapter 2).
Therefore, for $y = \log_x (y - 2)$, $y - 2 > 0$, so $y > 2$

$\log_x (x-4)$
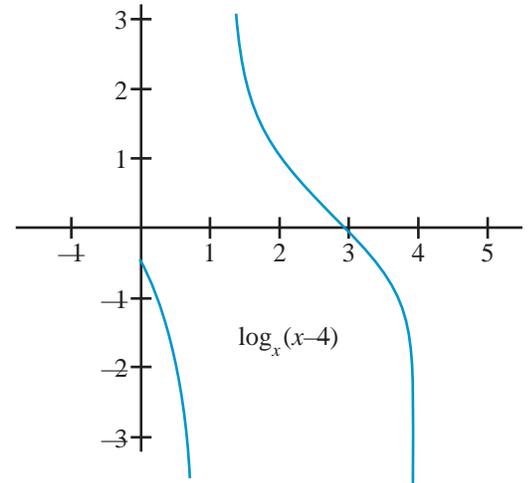
Applying the **iff Log Rule ($a = b$ iff log $a =$ log b)** to the original equation in order to put into graphing form "y ="

$\log_x (y-2) = \log_x (4-x)$
      $y - 2 = 4 - x$

$y = -x + 6$ with $0 < x < 4$, $x \neq 1$

X	y
0	Err
0.5	5.5
1	Err
1.5	4.5
2	4
2.5	3.5
2.5	3.5
3	3
3.5	2.5
4	Err

$0 < x < 4, x > 1$

$y > 2$

$\log_x (y-2) = \log_x (x-4)$
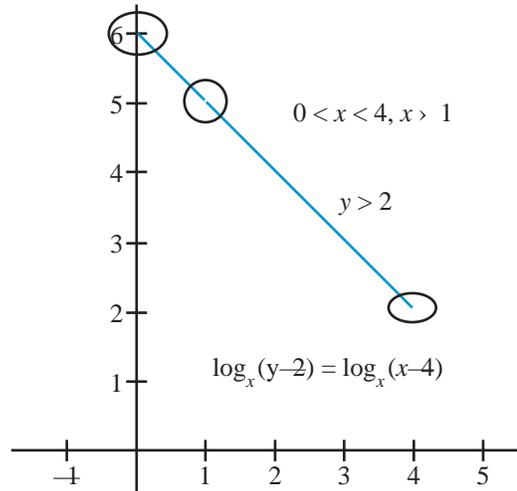
Chapter 9 Summary—We have found that the curve-sketching techniques that work with both polynomials and trig curves also work with the exponential and logarithmic curves. **We have found that exponential curves are always asymptotic with a horizontal line and log curves are always asymptotic with a vertical line. We reviewed that the base of an exponential function and its inverse function (the log function) is positive (but $\neq 1$ for the log function). We also reviewed that the domain for the exponential function was all real numbers while the range for the log function was all real numbers. Finally we reviewed that the range for the exponential function is positive while the domain for the log function is positive.**

Chapter 9 Exercises

1.) In chapter 9, we learned that after a vertical shift the asymptotes for the exponential function are affected, but not those of the log functions. The text tries to explain why that would be. Then we saw figures that showed a horizontal shift for the exponential and log functions. Notice that the exponential functions' asymptote are unaffected after the horizontal shift, but the horizontal shift does affect the asymptote for the log function. Explain.
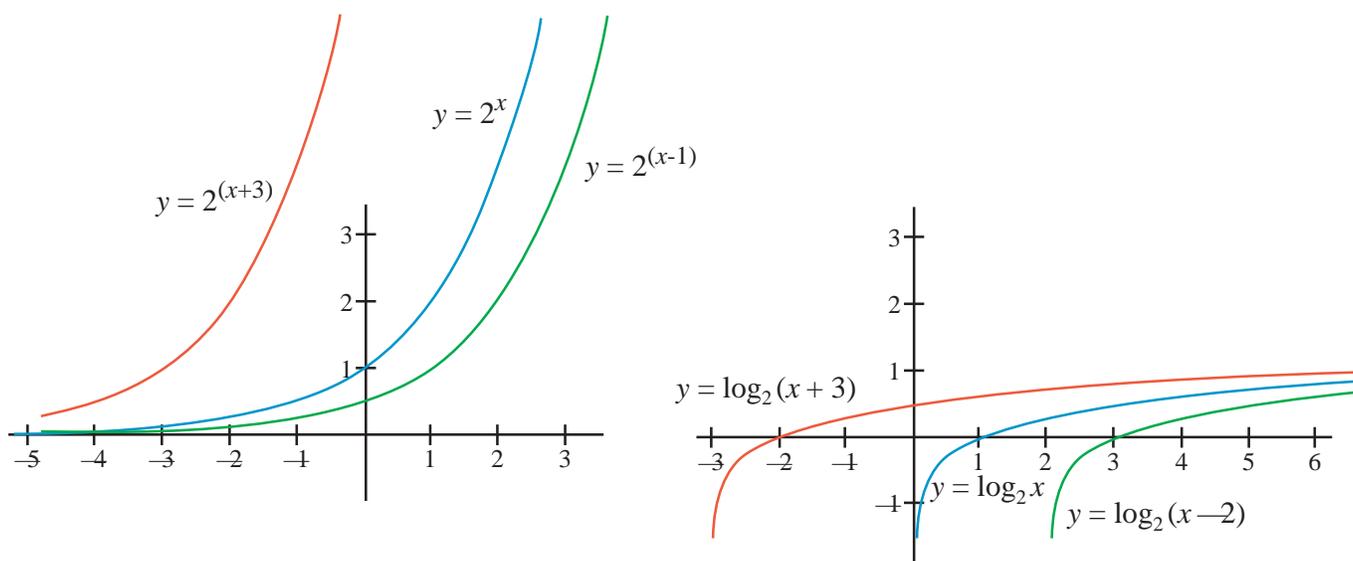
2.) In chapter 9, you saw the step by step graphing of the log function $y = -\frac{1}{3}\log_{\frac{1}{2}}(x+4)+2$. Show

that you understand progressive graphing by graphing the exponential curve $y = -\frac{1}{5}2^{(x-3)} - 4$. At

each graphing stage, indicate the asymptote, the domain, and the range for the function at that step.

3.) In chapter 5, we were shown the formula for the measurement of pH.

> **Measurement of PH**

pH is the measure of activity of hydrogen ions in a solution.

The formula for calculating pH is:

$$pH = -\log_{10}\left[H^+\right]$$

Graph this relation with pH as the dependent variable.

Chapter 10 … Practice, Practice, Practice

As with any skill proficiency requires practice, practice, and more practice.   This entire chapter is dedicated to practicing all the skills learned to this point in different ways and situations.

All the rules learned to this point are gathered together and listed below for reference

$b^m \times b^n = b^{(m+n)}$	**Product of Common Base Factors Rule**
$\dfrac{b^m}{b^n} = b^{(m-n)}$	**Quotient of Common Bases Rule**
$\log(x \times y) = \log x + \log y$	**Log of a Product Rule**
$\log\left(\dfrac{x}{y}\right) = \log x - \log y$	**Log of a Quotient Rule,**
$m = n$ iff $b^m = b^n$	**iff Antilog Rule**
$m = n$ iff $\log m = \log n$	**iff Log Rule**
$b^y = x$ is equivalent to $y = \log_b x$	**Equivalent Symbolism Rule**
$\log_b b^x = x$	**Inverse Log Rule #1** (Log of an Exponential Rule)
$b^{\log_b x} = x$	**Inverse Log Rule #2** (Power of a Base Rule)
$\log b^m = m \log b$	**Log of a Base Raised to a Power Rule**
$x \times y = 10^{(\log x + \log y)}$	**Rule to Multiply Using Logarithms**
$\dfrac{x}{y} = 10^{(\log x - \log y)}$	**Rule to Divide Using Logarithms**
$\log_p x = \dfrac{\log_q x}{\log_q p}$	**Change of Base Log Rule … change from base *p* to *q*.**
If $b^x = y$ then $x = \dfrac{\log y}{\log b}$	**Rule to Solve for an Exponent**
If $b^{x/y} = q$ then $q = 10^{\frac{x}{y}\log b}$	**Rule to Raise a Constant to a Power**

Problem #1:

Graph $\log y = -\log x$

$\log y = \log x^{-1}$ Log of a Base Raised to a Power Rule

$y = x^{-1}$ iff Log Rule

$y = \frac{1}{x}$
for $y > 0$
$x > 0$
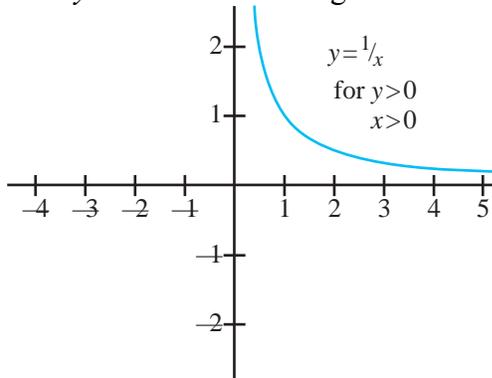
Problem #2:

Graph $\log y = 2 \log x$

$\log y = \log x^2$ Log of a Base Raised to a Power Rule

$y = x^2$ iff Log Rule

$y = x^2$
for $y > 0$
$x > 0$
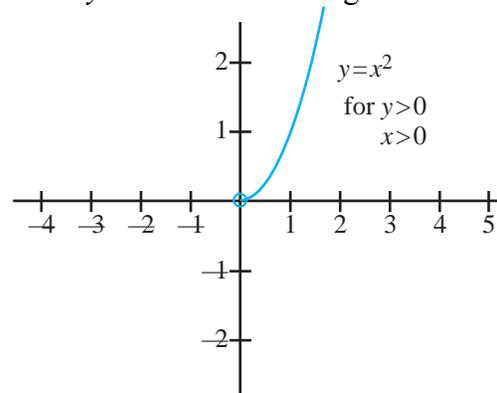
3.) Evaluate $\log_6 6^{-20}$ without a calculator

Applying the rule $\log_b b^x = x$ this is easy … –20

4.) Evaluate $3 \log_2 4$ without a calculator

Let $3 \log_2 4 = x$

Then $\log_2 4^3 = x$ Log of a Base Raised to a
                              Power Rule

$2^x = 4^3$ Equivalent Symbolism
                Rule

$2^x = (2^2)^3$

$2^x = 2^6$

$x = 6$ iff Antilog Rule

5.)   Estimate $\log_3 15$. Then evaluate using the Change of Base Rule and your calculator.
          Estimate as follows: $\boxed{9 = 3^2 < 3^x < 3^3 = 27}$

Now applying the **Change of Base Log Rule … change from base p into base q**

$$\log_p x = \frac{\log_q x}{\log_q p}$$

then $\log_3 15 = \dfrac{\log 15}{\log 3}$

$= \dfrac{1.17609}{0.47712}$

$= 2.464974$

Check: $\log_3 15 = x$ can be rewritten as $3^x = 15$ by the Equivalent Symbolism Rule Then $3^{2.464974}$
         $= 15.00007898$ by calculator verifies our work

6.)   Simplify $(\log_x y)(\log_y x)$
       When faced with a logarithm problem with different bases it is a good rule of thumb that you
       convert one of them so that both terms have the same base.
       Applying the **Change of Base Log Rule … change from base p into base q,**

$$\log_p x = \frac{\log_q x}{\log_q p} \quad \text{(general rule)}$$

$$\log_x y = \frac{\log_y y}{\log_y x} \quad \text{(applying the general rule to } (\log_x y) \text{ and substituting}$$

Then $\dfrac{\log_y y}{\log_y x} \cancel{\log_y x} = 1$ (can you see that $\log_y y = 1$?)

By the transitive rule, $(\log_x y)(\log_y x) = 1$ … the original two terms were multiplicative inverses!!

7.) Simplify $4 \log_5 p - \log_5 q$

$$\log_5 p^4 - \log_5 q \qquad \text{Log of a Base Raised to a Power Rule}$$

$$\log_5 \frac{p^4}{q} \qquad \text{Log of a Quotient Rule}$$

8.) Solve $\log_2 (x - 3) + \log_2 (x + 1) = 5$

$$\log_2 (x - 3)(x + 1) = 5 \quad \text{Log of a Product Rule}$$

$$\log_2 (x^2 - 2x - 3) = 5 \qquad\qquad \text{FOIL}$$

$$2^5 = x^2 - 2x - 3 \qquad \text{Equivalent Symbolism Rule}$$

$$x^2 - 2x - 3 = 32$$

$$x^2 - 2x - 35 = 0$$

$$(x - 7)(x + 5) = 0 \qquad\qquad \text{Reverse FOIL}$$

$$x = 7, -5$$

check: $\log_2 (7 - 3) + \log_2 (7 + 1) = 5$? $\qquad \log_2 (-5 - 3) + \log_2 (-5 + 1) = 5$?

$$\log_2 4 + \log_2 8 = 5 ? \qquad\qquad \log_2 (-8) + \log_2 (-4) = 5 ?$$

$$\log_2 (4 \times 8) = 5 ? \qquad\qquad \log_2 (-8 \times -4) = 5 ?$$

$$\log_2 32 = 5 ? \qquad\qquad\qquad \log_2 32 = 5 ?$$

$$2^5 = 32 \text{ ck!} \qquad\qquad\qquad 2^5 = 32 \text{ck! ??????}$$

What is wrong with the second answer, $x = -5$? Let's look at the graph of the conditions. Hopefully you can see for the sum graph that only values $x > 3$ are allowed in the solution set. So we will have to reject the solution $x = -5$ as extraneous.

We should have checked on the domain in the equation at first. Because $\log_b x$ requires $x > 0$ then $\log_2 (x - 3)$ requires $(x - 3) > 0$ and $\log_2 (x + 1)$ requires $(x + 1) > 0$. Therefore $(x > 3) \cap (x > -1)$ results in a domain of $x > 3$.
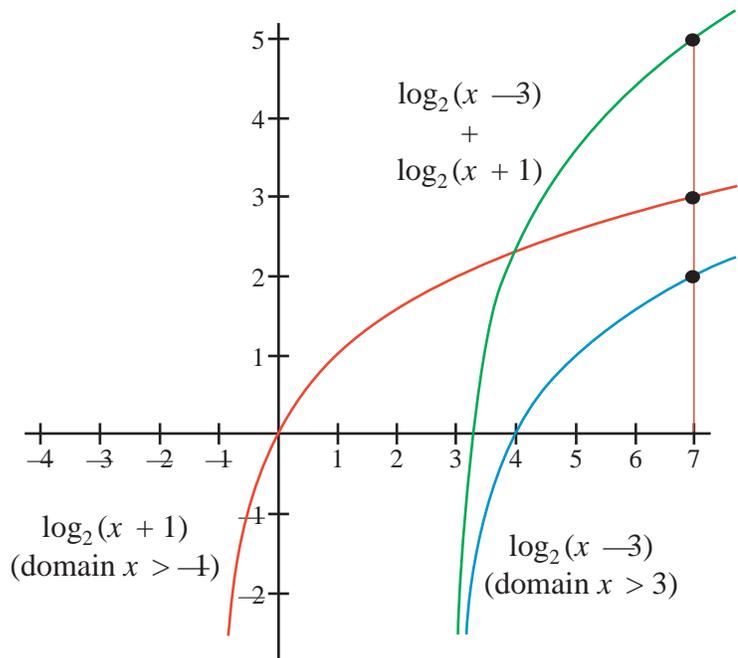
9.)   Expand $\log\left(\dfrac{p\sqrt{q}}{c}\right)$ **using the laws of logarithms.**

$\log p + \frac{1}{2} \log (q) - \log c$ by Log of Product, Quotient, and a Base Raised to a Power Rules

10.)   Given $\log_2 3 = x,\ \log_2 5 = y,\ \log_2 7 = z$

Express $\log_2 \dfrac{15}{7}$ in terms of x, y, and z

Let	$k = \log_2 \dfrac{15}{7}$	
	$k = \log_2 \dfrac{3 \times 5}{7}$	Inverse Log Rule #1 … Log of a Power Rule
	$k = \log_2 (3) + \log_2 (5) - \log_2 (7)$	Log of a Product and Quotient Rules
	$k = x + y - z$	Substitution

11.)   Simplify $\log_3 54 - \log_3 8 + \log_3 4$

$\log_3 54 - \log_3 8 + \log_3 4 =$

$\log_3 (54 / 8) + \log_3 4 =$	Log of a Quotient Rule
$\log_3 ((27 / 4) \times 4) =$	Log of a Product Rule
$\log_3 27 = x$	
$3^x = 27$	Equivalent Symbolism Rule
$x = 3$	

12.)   State the values of x for which the following identity is true.

$$\log_5 (x{+}1) + \log_5 (x{-}4) = \log_5 (x^2 - 3x - 4)$$

$\log_5 ((x{+}1) \times (x{-}4)) = \log_5 (x^2 - 3x - 4)$	Log of Product Rule
$(x + 1) \times (x - 4) = x^2 - 3x - 4$	iff Log Rule
$x^2 - 3x - 4 = x^2 - 3x - 4$	FOIL

all real values of x

Well, maybe not. $\log_5 (x + 1)$ has a domain of $x > -1$ and $\log_5 (x - 4)$ has a domain of $x > 4$

$$\{\, x > -1 \} \cap \{x > 4\} = \{x > 4\}$$

Let's see what the function $\log_5 (x^2 - 3x - 4)$ looks like:

$y = \log_5 (x^2 - 3x - 4)$
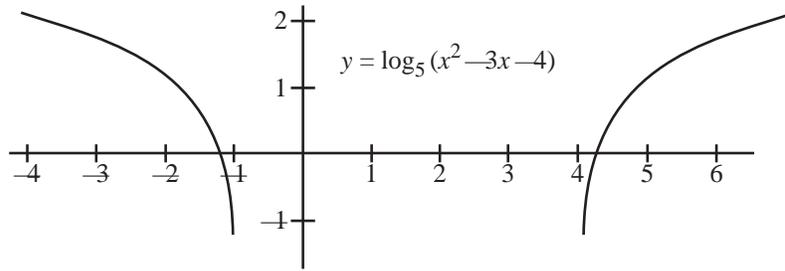
Wow! Who would have thought?! Where did that log graph come from anyway? We have never taken the log of a quadratic expression before, only linear ones. Let's graph the parabola $x^2 - 3x - 4$ and compare the log curve above with this parabola.

$y = \log_5 (x^2 - 3x - 4)$
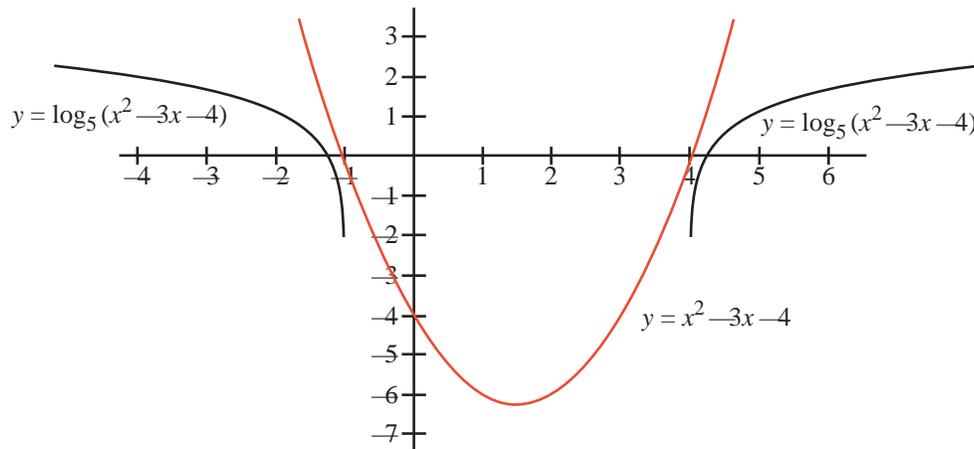
$y = \log_5 (x^2 - 3x - 4)$

$y = x^2 - 3x - 4$

By visualizing vertical asymptotes at $x = -1$ and $x = 4$, we notice that whenever $y = x^2 - 3x - 4 < 0$ ($-1 < x < 4$) the log curve $y = \log (x^2 - 3x - 4)$ is undefined. That makes sense as the log function is only defined for when the log argument > 0 (Chap. 2 remember?). So we conclude that $\log_5 (x^2 - 3x - 4)$ implies that $x^2 - 3x - 4 > 0$. Therefore $(x + 1)(x - 4) > 0$ which is only true when $x > -1$ and $x > 4$ (or when x < -1 and x < 4)

$\{x > -1\} \cap \{x > 4\} = \{x > 4\}$ so reject $x > -1$ for the domain of the identity.

Also $\{x < -1\} \cap \{x < 4\} = \{x < -1\}$ so reject x < 4 for the domain of the identity

Domain for function $\log_5 (x^2 - 3x - 4)$ is: { x < -1 } union { x > 4 }

13.) Solve $4^{2x} (8^{x+3}) = 32^{(4-x)}$

$$(2^2)^{2x} (2^3)^{(x+3)} = (2^5)^{(4-x)}$$

$$2^{4x} \times 2^{(3x+9)} = 2^{(20-5x)}$$

$$2^{(7x+9)} = 2^{(20-5x)}$$

$$7x + 9 = 20 - 5x$$

$$12x = 11$$

$$x = {}^{11}\!/_{12}$$

check: $4^{(22/12)} \times 8^{(11/12 + 3)} = 32^{(4 - 11/12)}$???

$$(2^2)^{(22/12)} \times (2^3)^{(47/12)} = (2^5)^{(37/12)} \text{ ???}$$

$$2^{(44/12)} \times 2^{(141/12)} = 2^{(185/12)} \text{ ???}$$

$$2^{(185/12)} = 2^{(185/12)} \text{ check}$$

14.) Solve $4 \times 3^{(3x)} = 9^{(x+1)}$

Somehow combine the x terms

$$\frac{4 \times 3^{3x}}{3^{3x}} = \frac{9^{x+1}}{3^{3x}}$$

$$4 \times 1 = \frac{\left(3^2\right)^{x+1}}{3^{3x}}$$

$$4 = \frac{3^{2x+2}}{3^{3x}}$$

$$4 = 3^{(-x+2)} \qquad\qquad \frac{b^m}{b^n} = b^{m-n}$$

$$\log (4) = \log 3^{(-x+2)} \qquad \textbf{iff Log Rule}$$

$$\log (4) = (-x+2) \log 3 \qquad \textbf{Log of a Base Raised to a Power Rule}$$

$$\frac{\log 4}{\log 3} = -x + 2$$

$$x = 2 - \frac{\log 4}{\log 3}$$

$$x = 2 - \frac{0.60206}{0.47712}$$

$$x = 2 - 1.2618$$

$$x = 0.7382$$

ck: $4 \times 3^{2.2146} = 9^{1.7382}$???

$45.5714 = 45.5685$ \qquad\qquad ck (to within the rounding error of the calculations)

15.) Solve for x: $(\log x)^2 = a^2$

$$\log x = \pm (a)$$

$$10^{(\pm a)} = x$$

$$x = 10^a,\ 10^{-a}$$

16.) Prove $\log_a x = -\log_{1/a} x$

Get rid of negative coefficient and $(1/a)$ base.

$$\log_a x = \log_{1/a} x^{-1} \qquad\qquad \textbf{Log of a Base Raised to a Power Rule}$$

$$\log_{1/a} x^{-1} = \log_a x \qquad\qquad \textbf{Symmetric Property}$$

$$\frac{1}{a}^{\log_a x} = x^{(-1)} \qquad\qquad \textbf{Equivalent Symbolism Rule}$$

$$\left(a^{-1}\right)^{\log_a x} = x^{(-1)} \qquad\qquad (\tfrac{1}{a}) = a^{(-1)}$$

$$a^{-\log_a x} = x^{(-1)} \qquad\qquad (b^m)^n = b^{mn}$$

$$a^{\log_a x^{-1}} = x^{(-1)} \qquad\qquad \textbf{Log of a Base Raised to a Power Rule}$$

$$x^{(-1)} = x^{(-1)} \qquad\qquad \textbf{Inverse Rule \#2 Power of a Log Rule} \qquad \textbf{Q.E.D}.$$

17.) Solve $\dfrac{\ln x}{x} = \dfrac{\ln 2}{2}$

      Approach #1:   Approach #2 Cross multiply

$$\frac{1}{x}\ln x = \frac{1}{2}\ln 2 \qquad\qquad\qquad 2\ln(x) = x \times \ln(2),\ \text{domain } x > 0$$

$$\ln x^{\frac{1}{x}} = \ln 2^{\frac{1}{2}} \qquad\qquad\qquad\qquad \ln(x)^2 = \ln(2)^x$$

$$x^{\frac{1}{x}} = 2^{\frac{1}{2}} \qquad\qquad\qquad\qquad\qquad x^2 = 2^x$$

$$\sqrt[x]{x} = \sqrt{2} \qquad\qquad\qquad\qquad x = \pm\sqrt{2^x} \ \text{domain } x > 0$$

$$x = \{2, 4\} \text{ by inspection} \qquad\qquad x = \{2, 4\} \text{ by inspection}$$

      Approach #3 If the problem is not contrived to give a clean answer, or if you are unsure of yourself, and if the problem involves the log or ln functions. which are available on any graphing calculator, you can graph your conditions and use the "Calc" button.

window: domain: [0,5]

           range: [0, 0.5]

           curve intercepts: (2 & 4)

           $x = \{2, 4\}$

$y = \dfrac{\ln 2}{2}$

$y = \dfrac{\ln x}{x}$
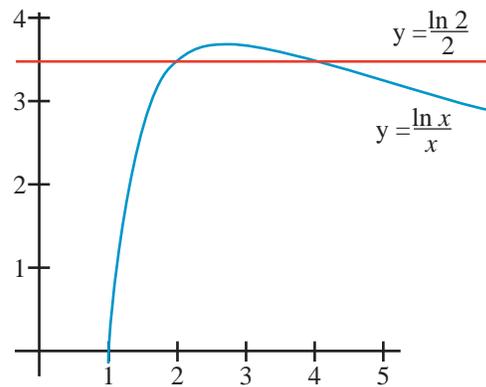
18.) Solve $\log_2 8 - \log_{\frac{1}{2}} 8 =$

    Let $\log_2 8 = x$ and   let $\log_{\frac{1}{2}} 8 = y$

    Then $2^x = 8$ and $(\tfrac{1}{2})^y = 8$

               $x = 3$ $2^{(-y)} = 2^3$

               $x = 3$ $y = -3$

    Substituting x and y back into the original expression

        $3 - (-3) = 6$

With practice you can do these sort of problems in your head. Cover up the work and try.

Chapter 10 Exercises

1.)   Solve for x.
   a.) $\log_2 x = 3$
   b.) $\log_{\frac{1}{3}} x = 4$
   c.) $\log_4 x = -\frac{1}{2}$
   d.) $\log_{(-4)} x = \frac{1}{2}$
   e.) $\log_3 x = -4$

2.)   Solve for x.
   a.) $\log_4 16 = x$
   b.) $\log_{\frac{1}{2}} 8 = x$
   c.) $\log_5 0 = x$
   d.) $\log_{(-2)} 10 = x$
   e.) $\log_3 1 = x$

3.)   Solve for x.
   a.) $\log_x 16 = 4$
   b.) $\log_x 4 = 1$
   c.) $\log_x 16 = -4$
   d.) $\log_x 64 = \frac{3}{4}$
   e.) $\log_x 0 = 2$

4.)   Simplify the expression
   a.) $\log_{51} 51^8$
   b.) $5^{\log_5 10}$
   c.) $3\log_{24} 24^{-\frac{2}{3}}$
   d.) $8 \log_{19} 19^2$
   e.) $4^{\log_4 85}$

5.)   Solve for x.
   a.) $0.196^{3x} = 2.68$
   b.) $242^x = 5{,}240$

6.)   Find x
   a.) $\log_3 79 = x$
   b.) $\log_x 79 = 6$
   c.) $\log_3 x = 7.2$

7.) Simplify

a.) $\log_5 625 + \log_3 (\frac{1}{81}) =$

b.) $4^{2\times\log_4 3} + 3^{2\times\log_3 4} =$

c.) $6^{\frac{1}{2}\times\log_6 36} \times 9^{2\times\log_9 81} =$

d.) $3 \log_{10} 4 - 2 \log_{10} 8 =$

e.) $\log_2 4 + \log_4 2 =$

8.) Evaluate without a calculator

a.) $\ln 1$ e.) $\log 1$

b.) $\ln e$ f.) $\log 10$

c.) $\ln e^y$ g.) $\log 10^y$

d.) $e^{\ln y}$ h.) $10^{\log y}$

9.) Rewrite as an expression using sums and differences using the Log Rules

$$\log\frac{16x^2}{y}$$

10.) The rule $\log (a + b) = \log a + \log b$ seems plausible. Let $a = 10$ and $b = 20$. Investigate using your calculator. What did you find out?

11.) The rule $\log\dfrac{a}{b} = \dfrac{\log a}{\log b}$ looks plausible. Let $a = 10$ and $b = 20$. Investigate using your calculator. What did you find out?

12.) Solve without a calculator. $20\left(\dfrac{1}{2}\right)^{\frac{x}{3}} = 5$

13.) Solve $\ln (3x - 2) + \ln (x - 1) = 2(\ln x)$ in two different ways:

a.) algebraically without a graphing calculator

b.) graphically with a calculator

14.) Solve $3^{(x - 1)} = 5^{(2x + 3)}$ in two different ways:

a.) algebraically without a graphing calculator

b.) graphically with a calculator

15.) State the **Equivalence Symbolism Rule** from memory.

Appendix A:
How Did Briggs Construct His Table of Common Logs?

The first five chapters of this text are pretty well written on the assumption that common logarithms (logs with a base of 10) are somehow magically available. In the "post-scientific calculator world" we live in, that assumption is close to accurate. However, prior to the availability of scientific calculators common logarithms (powers of 10 required for any given numbers) were only available in a table of logarithms that looked like the following:

From the log table, can you see that the log of 2.81 ($\log_{10} 2.81$) is 0.4487?

Check it out on your calculator.

What do you think the log of 28.1 would have been? (see bottom of page) of 281? of 2,810? of 0.00281? If you do not know, use your calculator to find out.

(Beware. Obtaining log table values of numbers

$0 < x < 1$

involve a special case complication. See Appendix D if you are interested.)

Common Logarithms of Numbers

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428

There are sources that detail how Henry Briggs spent decades developing and applying methods used to obtain this logarithmic information. Very little of that history is relevant to today's student. Still, a brief treatment of that process may be of interest to the curious student. Actually, Mr. Briggs documented several techniques that he used. I have chosen what I believe are the main ones to share with you. After developing the necessary background, I will be presenting what I believe to be the spirit of Mr. Briggs' work. Historical and mathematical purists should refer to the following websites: 1.) www.jacques-laporte.org, *Briggs and the HP-35 submenu*, 2.) http://www.jacques-laporte.org/The method of Henry Briggs.htm, and 3.) if you read Danish, www.matematiksider.dk by Erik Vestergaard. All three websites were viable June 2010..

Answers: log 28.1 = 1.4487… where 1 is called the characteristic of the log and 0.4487 is called the mantissa
log 281 = 2.4487… where 2 is called the characteristic of the log and 0.4487 is called the mantissa
log 2,810 = 3.4487… where 3 is called the characteristic of the log and 0.4487 is called the mantissa
log 0.002810 = 0.4487 – 3 = –2.55129   See Appendix D for more discussion of the special case of using a log table to take a log of a number x when 0 < x < 1
Note that the mantissa is the same for numbers 28.1, 281, 2,810, and all numbers x > 1 starting with digits 2,8,1.
The characteristic, however, is dependent on and communicates information about the magnitude of the number.

Step #1:   Finding the square root of a number without using a calculator.

The ability to find a square root was well known in Mr. Briggs' time. Even as late as the 1960s, I was taught an algorithm that was probably very similar to the one he used. The reader is strongly advised to skip over this material in Step #1. It is really tedious and very boring and irrelevant to today's student.

1.) From the decimal in the radical, group the digits in groups of two. Group in twos going to the left and to the right.

2.) Find the extreme left digit (or pair of digits) and mentally approximate the square root of it placing it above the group. Here, the square root of 7 would be approximated to be 2 … place the 2 above the seven. Square the number (2) and place below the seven.

3.) Subtract the 4 from the 7 and bring down the next two digits from the radical … 46.

$$\text{Step 1} \qquad \text{Step 2} \qquad \text{Step 3}$$

$$\sqrt{746.84} \qquad \sqrt{7\ 46\ .\ 84} \qquad \overset{2}{\sqrt{7\ 46\ .\ 84}} \qquad \overset{2}{\sqrt{7\ 46\ .\ 84}}$$

Step 2: $\dfrac{4}{\ }$ under the 7

Step 3: 4 ↓ / 3 46

4.) Double the partial square root at that stage (2) and place the result to the left of the 3 46 with a _ indicating a digit to fill in later. Think: What is the highest digit I can place in the missing digit location that will divide the 346? In this case it is 7.

5.) Locate that 7 above the radical and in the missing digit location. (4 becomes 47)

6.) Multiply 7×47 placing the result below the 3 46. Proceed to subtract and bring down the next group of two digits … 84.

$$\text{Step 4} \qquad \text{Step 5} \qquad \text{Step 6}$$

Step 4:
$$\overset{2}{\sqrt{7\ 46\ .\ 84}}$$
4 ↓
(4_) 3 46

Step 5:
$$\overset{2\ 7}{\sqrt{7\ 46\ .\ 84}}$$
4 ↓
(47) 3 46

Step 6:
$$\overset{2\ 7}{\sqrt{7\ 46\ .\ 84}}$$
4 ↓
(47) 3 46
3 29
17 . 84

7.) Repeat starting in step 4. Double the partial square root at that stage (27) and place the result (54_) to the left of the 17 84. Think: What is the highest digit I can place in the missing digit location that will divide the 1,784? In this case it is 3.

8.) Locate that 3 above the radical and in the missing digit location.

9.) Multiply 3×543 placing the result below the 1,784. Proceed to subtract and bring down the next group of two digits. Repeat step 4–6 (or 7–9) as many times as desired.

$$\text{Step 7} \qquad \text{Step 8} \qquad \text{Step 9}$$

Step 7:
$$\overset{2\ 7\ .}{\sqrt{7\ 46\ .\ 84}}$$
4 ↓
(47) 3 46
3 29
(54_) 17 . 84

Step 8:
$$\overset{2\ 7\ .}{\sqrt{7\ 46\ .\ 84}}$$
4 ↓
(47) 3 46
3 29
(543) 17 . 84

Step 9:
$$\overset{2\ 7\ .\ 3}{\sqrt{7\ 46\ .\ 84\ 00}}$$
4 ↓
(47) 3 46
3 29
(543) 17 . 84
16 . 29
1 . 55 00

ck: $27.3 \times 27.3 = 745.29$

Step #2: Repeated applications of the square root process could result in the following table of information.

$10^{\frac{1}{2}} = 10^{0.5}$ $= 3.16227766$

$10^{\frac{1}{4}} = 10^{0.25}$ $= 1.77827941$

$10^{\frac{1}{8}} = 10^{0.125}$ $= 1.333521432$

$10^{\frac{1}{16}} = 10^{0.0625}$ $= 1.154781985$

$10^{\frac{1}{32}} = 10^{0.03125}$ $= 1.074607828$

$10^{\frac{1}{64}} = 10^{0.0156625}$ $= 1.036632928$

$10^{\frac{1}{128}} = 10^{0.0078125}$ $= 1.018151722$

$10^{\frac{1}{256}} = 10^{0.00390625}$ $= 1.009035045$

etc.

Also, because $b^{\frac{m}{n}} = \sqrt[n]{b^m}$, Mr. Briggs could also have calculated $10^{\frac{3}{4}}$ ($10^{0.75}$) or $10^{\frac{3}{8}}$ ($10^{0.375}$) or $10^{\frac{5}{16}}$ ($10^{0.3125}$) or $10^{\frac{17}{32}}$ ($10^{0.53125}$) or $10^{\frac{63}{128}}$ ($10^{0.4921875}$) or $10^{\frac{11}{256}}$ ($10^{0.04296875}$) or any power whose denominator was a power of two (2).

This technique was used by Mr. Briggs to get many of the values in his log table. More will be discussed.

Step #3: Cube Roots

It is well documented that the mathematicians of the late 1500s could solve a generalized cubic equation. Appendix B documents this fact.

As the ability to solve a cubic equation is predicated on the ability to extract a cube, it would be safe to assume that Mr. Briggs knew how to extract a cube. That would allow him to calculate the following log values in a manner similar to the way he extracted log values using the ability to take a square root (refer to www.nist.gov/dads/HTML/cubeRoot.html, viable June, 2010, if you are really curious.) *Historical purists please forgive. There is nothing I can find that says Mr. Briggs utilized a cube root algorithm in constructing his log table but it was a great excuse to put Cardan's formula into my text (Appendix B). It was available during the time period that Briggs was doing his work.*

$10^{\frac{1}{3}(0.333)}$ $10^{\frac{2}{3}(0.667)}$ $10^{\frac{1}{9}(0.111)}$ $10^{\frac{2}{9}(0.222)}$ $10^{\frac{3}{9}(0.444)}$ $10^{\frac{5}{9}(0.556)}$ $10^{\frac{7}{9}(0.778)}$ $10^{\frac{8}{9}(0.889)}$

or any root that is a composite of the factors of 2 and 3,

e.g., $10^{\frac{1}{6}(0.167)}, 10^{\frac{5}{6}(0.833)}, 10^{\frac{1}{12}(0.83)}, 10^{\frac{5}{12}(0.417)}, 10^{\frac{11}{12}(0.917)}, 10^{\frac{1}{18}(0.056)}, 10^{\frac{17}{18}(0.944)}, 10^{\frac{53}{54}(0.981)}$, etc.

The humble log table started in Step 2 has been greatly expanded!

Step #4: Integer Factoring

The reader has, no doubt, been exposed to the idea of integer factoring. For example, to factor 210 …

$^{210}\!/_2 = 105$ Therefore $210 = 2 \times 105$

$^{105}\!/_7 = 15$ Therefore $105 = 7 \times 15$
                Therefore $210 = 2 \times 7 \times 15$

$^{15}\!/_3 = 5$ Therefore $3 \times 5 = 15$
              Therefore $210 = 2 \times 7 \times 3 \times 5$

It can be proven that integer factoring or *factoring to the primes* always results in a unique set of factors. In other words, there is only one set of factors in a complete integer factorization of an integer.

Step #5: "Non-integer Factoring" For example, "factor" the number 131.

$\dfrac{131}{7} = 18.71428571$ Therefore $131 = 7 \times \underline{18.71428571}$

$\dfrac{18.71428571}{5} = 3.7428571$ Therefore $18.71428571 = 5 \times 3.7428571$

                        Therefore $131 = 7 \times 5 \times \underline{3.7428571}$

$\dfrac{3.7428571}{3} = 1.247619048$ Therefore $3.7428571 = 3 \ \times 1.247619048$

                        Therefore $131 = 7 \times 5 \times 3 \times \ \underline{1.247619048}$

$\dfrac{1.247619048}{1.2} = 1.03968254$ Therefore $1.247619048 = 1.2 \times 1.03968254$

                        Therefore $131 = 7 \times 5 \times 3 \times 1.2 \times \underline{1.03968254}$

$\dfrac{1.03968254}{1.035} = 1.004524193$ Therefore $1.03968254 = 1.035 \times 1.004524193$

                        Therefore $131 = 7 \times 5 \times 3 \times 1.2 \times 1.035 \times \underline{1.004524193}$

What do you notice the sequence of factors that are underlined above?

18.71428571, 3.7428571, 1.247619048, 1.03968254, 1.004524193, …
These numbers are decreasing and approaching 1. In other words 131 can be factored as follows

---

$131 = a \times b \times c \times d \times e \times f \times g \times \ldots \times x \times$ ***some number very close to one*** then for all practical
                        purposes, we could probably just write the expression as

$131 = a \times b \times c \times d \times e \times f \times g \times \ldots \times x$ and eliminate that last factor by applying the multiplicative
                        property of 1, $x \times 1 = x$.

"Non-integer factoring" does not generate a unique set of factors.

Step # 6:

Combine concepts #2 and #5 above to approximate the log of any number to as many places as desired. Replicating the chart of even root logarithms from step #2 above, we proceed to approximate log 5 to several places of accuracy. This process starts with the technique of "non-integer factoring." I learned this technique from Dr. Art Miller of Mount Allison University, Sackville, N.B., Canada, who in turn gives credit for this algorithm to Mr. Henry Briggs. (www.mta.ca/~amiller/)

$$10^{0.5} = 3.16227766 \qquad \#1$$

$$10^{0.25} = 1.77827941 \qquad \#2$$

$$10^{0.125} = 1.333521432 \qquad \#3$$

$$10^{0.0625} = 1.154781985 \qquad \#4$$

$$10^{0.03125} = 1.074607828 \qquad \#5$$

$$10^{0.0156625} = 1.036632928 \qquad \#6$$

$$10^{0.0078125} = 1.018151722 \qquad \#7$$

$$10^{0.00390625} = 1.009035045 \qquad \#8$$

Divide the number 5 by the largest value < 5 shown in the table above.   Likewise divide each of the resulting "non-integer factors" (1.58113881, 1.1856868853, 1.026762541, etc.) by the largest value in the table less than the current "non-integer factor."   This is shown below.

Step 6.a: $\dfrac{5}{3.16227766} =$ 1.58113883   **(dividing by table value #1 above)**

Therefore $5 =$ $3.16227766 \times 1.58113883$

Step 6.b: $\dfrac{1.58113883}{1.333521432} =$ 1.1856868853   **(dividing by table value #3 above)**

 Therefore $1.58113883 = 1.333521432 \times 1.1856868853$

 and therefore $5 = 3.16227766 \times 1.333521432 \times 1.1856868853$

Step 6.c: $\dfrac{1.1856868853}{1.154781985} =$ 1.026762541   **(dividing by table value #4 above)**

Therefore $1.1856868853 = 1.154781985 \times 1.026762541$

and therefore $5 = 3.16227766 \times 1.333521432 \times 1.154781985 \times 1.026762541$

Step 6.d: $\dfrac{1.026762541}{1.018151722} =$ 1.008457304   **(dividing by table value #7 above)**

 Therefore $1.026762541 = 1.018151722 \times 1.008457304$

 and therefore $5 = 3.16227766 \times 1.333521432 \times 1.154781985 \times 1.018151722 \times 1.008457304$

> ****** etc. until the last factor is close enough to 1 to give the desired accuracy *******

Finally $5 = 3.16227766 \times 1.333521432 \times 1.154781985 \times 1.018151722 \times 1.008457304$

take the log of both sides, **iff Log Rule**

Step 6.f: $\log (5) = \log (3.16227766 \times 1.333521432 \times 1.154781985 \times 1.018151722 \times 1.008457304)$

(continuing from the previous page)

Step 6.f: log (5) = log [3.16227766 × 1.333521432 × 1.154781985 × 1.018151722 × 1.008457304]

Step 6.g: log (5) = log (3.16227766) + log (1.333521432) + log (1.154781985) + log(1.018151722)

+log (1.008457304) (omit this term!)

Step 6.h: log (5) = 0.5 + 0.125 + 0.0625 + 0.0078125 (log values taken from table on prior page)

Step 6.i: log (5) = **0.6953125** which compares very favorably with the calculator value

by calculator log (5) = **0.6989700043**

Question: How could you improve the noncalculator log (5) value shown above?

Answer: Repeat the iterative process as many times necessary to obtain the desired accuracy!!

```java
class CalcLogOf5    // Java code… Mr. Briggs could not have imagined such a tool
{
   public static void main(String args[])
   {
      double x = 5;
      double origX = x;
      double logSum = 0;
      double rootOf10 = 10;
      double power = 1;

      while (rootOf10 > 1.0000000000001)
         {
         rootOf10 = Math.sqrt(rootOf10);
         power /= 2.0;

         if (rootOf10 < x)
         {
            logSum +=power;   // summing the "logs"
            System.out.print(power + " + ");
            x = x / rootOf10;
         }// end if

         }// end while

      System.out.println("\n\n  logSum = " + logSum);
      System.out.println("log " + origX + "  = " + Math.log10(origX));
   }// end main
}// end class
```

(output on next page)

--------------------Configuration: <Default>--------------------

0.5 + 0.125 + 0.0625 + 0.0078125 + 0.001953125 + 9.765625E-4 + 4.8828125E-4 + 1.220703125E-4 + 6.103515625E-5 + 3.0517578125E-5 + 1.52587890625E-5 + 7.62939453125E-6 + 1.9073486328125E-6 + 9.5367431640625E-7 + 1.1920928955078125E-7 + 2.9802322387695312E-8 + 7.450580596923828E-9 + 3.725290298461914E-9 + 1.862645149230957E-9 + 9.313225746154785E-10 + 2.3283064365386963E-10 + 1.1641532182693481E-10 + 1.8189894035458565E-12 + 5.6843418860808015E-14 + 2.8421709430404007E-14 +

logSum = 0.6989700043360187

log 5.0   = 0.6989700043360189

We now have a pre-calculator method of expanding the "log table" started in steps 2 & 3 to contain the log of any number we wish. For example it would be helpful to have common log (base 10) values for all the prime numbers up to 100: log 2, log 3, log 5, log 7, log 11, log 13, log 17, …. We now know how to do it, right? Then we can use prime number log values together with the Log of a Product Rule [log ($a \times b$) = log a + log b] to determine unknown logarithms of composite numbers:

log 2 = 0.3010299957
log 3 = 0.4771212547

log (6) = log (2 × 3) = 0.3010299957 + 0.4771212547 = 0.7781512504

calculator check: log 6 = 0.7781512504

The practical ramifications of this "trick" are huge as the factor combinations for each prime number combination are numerous ….

log (2 × 2), log (2 × 3), log (2 × 4), … log (3 × 3), log (3 × 4), … log (5 × 5), log (5 × 6), …

Step 7:

Another way to expand our log table using pre-calculator methods also involves using previously developed log values. By cross multiplying a proportion and applying log rules, Briggs could use numbers with known logarithmic values to obtain new ones. From work done so far in Appendix A, we could set up the following ration.   (Logarithm… logos/ratio… arithmos/number… remember ?)

$$\frac{5 \text{ known log}}{10 \text{ known log}} = \frac{3.16227766 \text{ known log}}{6.32455532 \text{ unknown log}} \qquad (10^{0.5} = 3.16227766 \text{ remember?})$$

$$5 \times 6.32455532 = 10 \times 3.16227766 \qquad \text{cross-multiplying}$$

$$\log (5 \times 6.32455532) = \log (10 \times 3.16227766)$$

$$\log 5 + \log 6.32455532 = \log 10 + \log 3.16227766$$

$$0.6953125 + \log 6.32455532 = 1 + 0.5$$

$$\log 6.32455532 = 0.8046875$$

calculator check: $10^{0.8046875} = 6.3780438$ \qquad\qquad (not too far off)

Assuming we have the log values for 2, 3, 4, 5, 6, 7, 8, and 9 we could proceed to set up and use other ratios:

$$\frac{2}{3} = \frac{\text{value with known log}}{\text{value with unknown log}}, \qquad \frac{4}{9} = \frac{\text{value with known log}}{\text{value with unknown log}}$$

Do not be distracted by the fact that there are infinite numbers and hence infinite ways that they can be multiplied or divided. As daunting as Mr. Briggs' task was, it was finite in its nature. It is important to remember that the way log tables were used only required values for the significant digits of the numbers which he called the "mantissa." All the details associated with magnitude were dealt with using what was called the "characteristic." (This is where working with logarithms is almost exactly like working with scientific notation.) The work involved in multiplying 34.1×802 requires the same log table information as multiplying 0.0341×8.02, etc., etc. To illustrate, let's use a hypothetical 3-digit log table. Compare the following. Notice that the information in bold, the mantissa, would have come from a 3 digit

$$\boxed{\begin{array}{l} \log 2.73 = 0.437 \\ \log 3.41 = 0.533 \\ \log 8.02 = 0.904 \end{array}}$$

log table while the information in parenthesis, the characteristic—which basically keeps track of the order of magnitude—would have been mentally supplied by the "human calculator." (There are complications in determining the log of a value x, $0 < x < 1$, which are discussed in Appendix D.)

$34.1 \times 802 = x$	$0.0341 \times 8.02 = x$
$\log (34.1 \times 802) = \log (x)$	$\log (0.0341 \times 8.02) = \log (x)$
$\log (34.1) + \log (802) = \log (x)$	$\log (0.0341) + \log (8.02) = \log (x)$
(*look up log 3.41 and log 8.02*)	(*again look up log 3.41 and log 8.02!!!!!*)
$(1).\mathbf{533} + (2).\mathbf{904} = \log (x)$	$0.\mathbf{533}{-}2 + (0).\mathbf{904} = \log (x)$
	(*not –2.532, see Appendix D*)
$(3) + 1.437 = \log (x)$	$-0.563 = \log (x)$ ($10^{(-0.563)}$ works in calc.)
$(4).437 = \log (x)$	$0.437 - 1 = \log (x)$
	(*see Appendix D*)
$10^{(4).437} = 10^{\log x}$	$10^{0.437-1} = 10^{\log x}$
$10^4 \times 10^{0.437} = x$	$10^{0.437} \times 10^{-1} = x$
$27{,}300 = x$ (3 sig. digits)	$0.27300 = x$ (3 sig. digits)

By calculator $x = 27{,}348.2$ and 0.273482 respectively

Yes, by using a hypothetical three (3) digit log table to obtain logs and anti-log values, we would have been off a bit in the calculations shown above, but using Mr. Briggs' 13–14-digit log table values, you would hardly notice if you were an engineer, an astronomer, or a scientist.

As stated previously, there is a simple esthetic infinite-series polynomial that will allow a person to compute $\log_e (x)$ ($\ln x$) to as many places as desired.

$$\ln x = \frac{1}{1}\frac{(x-1)^1}{x^1} + \frac{1}{2}\frac{(x-1)^2}{x^2} + \frac{1}{3}\frac{(x-1)^3}{x^3} + \frac{1}{4}\frac{(x-1)^4}{x^4} + \frac{1}{5}\frac{(x-1)^5}{x^5} + \ldots$$

(for x > 1)

Using this series, it would have been much simpler to develop the table of natural logs than it was to develop the table of common logs!! This is yet another reason why (from the 1600s to the era of the calculator) mathematicians and scientists usually use the natural log (*e*-based) logarithm. What are the chances that the engineers at HP and TI that program those nifty scientific calculators know about this series?

It was not long after Mr. Briggs did his work that other mathematicians figured this series out. From that time on, mathematicians could develop common log (base 10) tables by generating the ln value and converting over to base 10 using the **Change of Base Log Rule …** $\log x = \dfrac{\log_e x}{\log_e 10}$

e.g., $\ln (7) = 1.945910149$ by the series above

              $\ln (10) = 2.302585093$ by series above

therefore $\log (7) = \dfrac{\ln 7}{\ln 10} = \dfrac{1.945910149}{2.302585093} = 0.84509804$

By calculator $\log (7) = 0.84509804$

Appendix B:
Cardano's Formula—Solving the Generalized Cubic Equation

Material taken from www.math.vanderbilt.edu/~schectex/courses/cubic
Eric Schechter, Website viable as of June, 2010.

The Cubic Formula
(Solve Any 3rd-Degree Polynomial Equation)

I'm putting this on the web because some students might find it interesting. It could easily be mentioned in many undergraduate math courses, though it doesn't seem to appear in most textbooks used for those courses. None of this material was discovered by me.   − ES

You should know that the solution of $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

There is an analogous formula for polynomials of degree three. The solution of $ax^3 + bx^2 + cx + d = 0$ is

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$

$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}.$$

(A formula like this was first published by Cardano in 1545.) Or, more briefly,

$$x = \left\{ q + \left[q^2 + \left(r - p^2\right)^3 \right]^{1/2} \right\}^{1/3} + \left\{ q - \left[q^2 + \left(r - p^2\right)^3 \right]^{1/2} \right\}^{1/3} + p,$$

where

$$p = \frac{-b}{3a}, \quad q = p^3 + \frac{bc - 3ad}{6a^2}, \quad r = \frac{c}{3a}.$$

Appendix C: Semilog Paper

It is often useful to plot the logarithm curve in such a way that allows for techniques applied to linear functions to also be applied on the log curve. This is particularly true when plotting experimental data.

When $y = a \times x^n$,

$$\log y = \log (a \times x^n)$$
$$\log y = \log a + \log (x^n)$$
$$\log y = \log a + n (\log x)$$
$$\log y = n (\log x) + \log a$$

Compare the following and see if you can make the connection.

$$\log y = n \log x + \log a$$
$$y = mx + b \qquad \text{(equation of a line in slope-intercept form)}$$

In the graph paper at the right below, known as semi-log paper, the y axis markings are not equally spaced, but the markings along the x axis are. (If both axes are spaced the way that the y axis is spaced the paper is called semi-log log paper.) Notice that the spacing along the y axis reflects the growth rate of the log curve … rapid at first and then gradual after that.
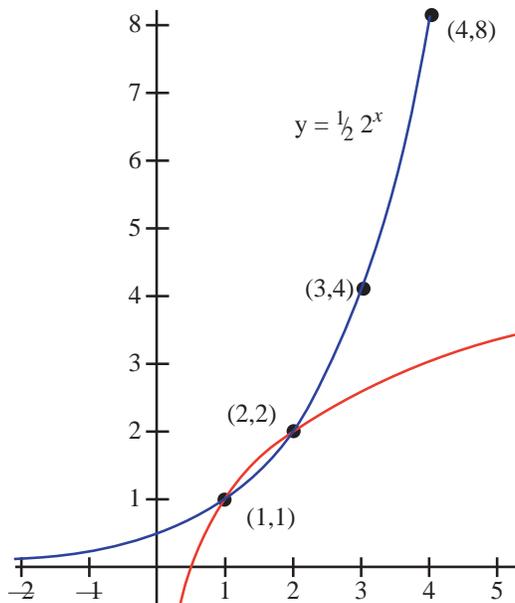
$y = \frac{1}{2} (2^x)$ on a Standard Graph
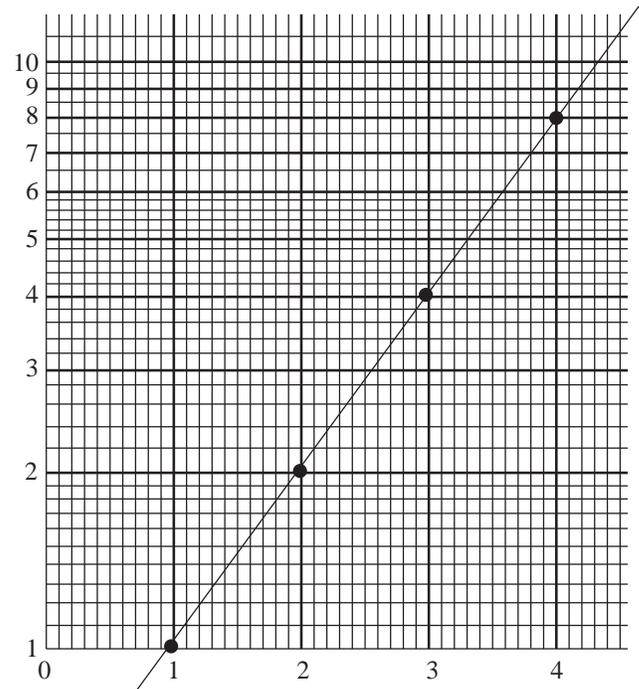
$y = \frac{1}{2} (2^x)$ on a Semilog Graph

Appendix D: Logarithms of Values Less than One

A long, long time ago, in chapter 2, we were introduced to the idea that the log of numbers $x > 1$ was different from the log of numbers $0 < x < 1$. (See the figure at right.) Our scientific calculators distinguish the two cases for us, but b.c. (before calculators) one had to use log tables one way for $x > 1$ and another way for $0 < x < 1$.

For both functions
domain: $x > 0$
range: all real

$y = \log_2 x$

$0 < x < 1$

$y = \log_{1/2} x$
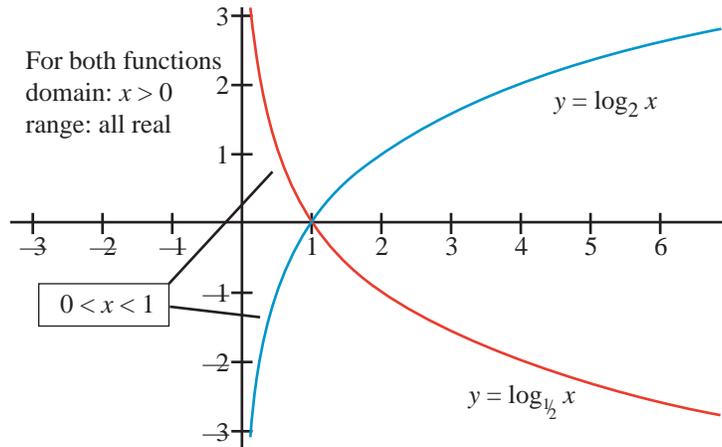
Lets do another experiment. The mantissa is the part of the logarithm that represents the significant digits of the logarithm while the characteristic represents the order of magnitude of the logarithm.

x	Log (x) expected	Log (x) actual from calculator
2,197	3.341830057	3.341830057
219.7	2.341830057	2.341830057
21.97	1.341830057	1.341830057
2.197	0.341830057	0.341830057
*******	*********	**********
0.2197	−1.341830057	**−0.658169943**
0.02197	−2.341830057	**−1.658179943**
0.002197	−3.341830057	**−2.658169943**

Since   1,000   <   2,197   <   10,000
Then   3   <   log 2,197 <   4
              2   <   log 219.7 <   3
              1   <   log 21.97 <   2
              0   <   log 2.197 <   1

What is going on here???
The pattern of mantissas has changed.
Or has it???

Maybe the pattern that should have been anticipated is not the decrease of the characteristic combined with a constant mantissa as shown in the second column, but rather the fact that the argument of log (x) decreases by one order of magnitude in the argument. That pattern matches exactly the correct data shown in the third column.

Let's try again. log 4.59 $(x > 1)$ $=$ 0.661826855.
              log 459 $(x > 1)$ $=$ 2.661826855.
  but log (0.00459) $(0 < x < 1)$ $\neq$ −3.661826855.
      log (0.00459) $(0 < x < 1)$ $=$ log $(4.59) - 3$
                        $= 0.661826855 - 3 = -2.338187314$

This is the way that you would have had to work with log x values $(0 < x < 1)$ when you were working with log tables.

Calculator check: log $(0.00459) = -2.338187314$ check!

Thank you calculator engineers for taking care of this for us so that we do not have to worry about these special case situations—log x for $(0 < x < 1)$—any more!!!

Appendix 2.71818 : Euler's Equation, An Introduction

It has been previously noted (without proof) that

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$$

It is not appropriate to the level of this text to discuss where this magic expression comes from.

A very famous mathematician, Brook Taylor (circa 1712), is given credit for finding a way to approximate any function to any degree of accuracy by adding up a series of smaller functions.   The technique to do this is appropriately called Taylor series. To understand how Mr. Taylor did his magic you would need to take a Calculus class.   That is clearly not possible in the space here.   By Mr. Taylor's work the following, more general formula, can be proved.

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \dots$$

Also by Mr. Taylor's work

$$\cos(x) = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$$

and

$$\sin(x) = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Putting these three equations together yields a remarkable result called "Euler's Equation"

$$e^{\Pi i} + 1 = 0$$

and will be shown on the next 2 pages.

Step 1: Rearranging terms from e^x shown above (Can you anticipate the cos(x) and sin(x) ?)

$$e^x = \frac{x^0}{0!} + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots + \frac{x^1}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

$$e^x = \frac{x^0}{0!} + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots + \frac{x^1}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

Step 2: Arbitrarily substitute x = Π i   (don't ask questions at this stage)

$$e^{\Pi i} = \frac{(\Pi i)^0}{0!} + \frac{(\Pi i)^2}{2!} + \frac{(\Pi i)^4}{4!} + \frac{(\Pi i)^6}{6!} + \frac{(\Pi i)^8}{8!} + \dots$$

$$\frac{(\Pi i)^1}{1!} + \frac{(\Pi i)^3}{3!} + \frac{(\Pi i)^5}{5!} + \frac{(\Pi i)^7}{7!} + \dots$$

Step3:   Now from Algebra we know that $(ab)^m = a^m b^m$ Hence the equation above can be written as:

$$e^{\Pi i} = \frac{\Pi^0 i^0}{0!} + \frac{\Pi^2 i^2}{2!} + \frac{\Pi^4 i^4}{4!} + \frac{\Pi^6 i^6}{6!} + \frac{\Pi^8 i^8}{8!} + \dots$$

$$+ \frac{\Pi^1 i^1}{1!} + \frac{\Pi^3 i^3}{3!} + \frac{\Pi^5 i^5}{5!} + \frac{\Pi^7 i^7}{7!} + \dots$$

Step 4a: Apply Algebra rules $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$ to the first set of terms

$$e^{\Pi i} = \underbrace{\frac{\Pi^0 (1)}{0!} - \frac{\Pi^2 (1)}{2!} + \frac{\Pi^4 (1)}{4!} - \frac{\Pi^6 (1)}{6!} + \frac{\Pi^8 (1)}{8!} + \dots}$$

$$\cos(\Pi)$$

Step 4b: and factoring out an "i" from the second set of terms

$$+ i\left(\frac{\Pi^1}{1!} + \frac{\Pi^3 i^2}{3!} + \frac{\Pi^5 i^4}{5!} + \frac{\Pi^7 i^6}{7!} + \dots \right)$$

Step 5: Substituting cos(Π) into the first set of terms and again applying Algebra rules

$i^0 = 1, \ i^1 = i, \ i^2 = -1, \ i^3 = -i, \ $ and $ \ i^4 = 1 \ $ to the second set of terms.

$$e^{\Pi i} = \cos(\Pi) + i(\underbrace{\frac{\Pi^1}{1!} - \frac{\Pi^3}{3!} + \frac{\Pi^5}{5!} - \frac{\Pi^7}{7!} + ...}_{\sin(\Pi)})$$

Step 6: $e^{\Pi i} = \cos(\Pi) + i * \sin(\Pi)$

From trig we know that cos(Π) = -1   and sin(Π) = 0

Step 7: $e^{\Pi i} = -1 + i * 0$

Step 8: $e^{\Pi i} = (-1) + 0$

Step 9: $e^{\Pi i} + 1 = 0$

There you have it,   Euler's equation…

e, Π, i, 1, and 0 all in the same equation!!!

Nerd heaven!!

The following discussion has nothing to do with logarithms or the number e but if you got this far you might be interested in the following bit of trivia…

i^i **is a real number**

$$e^{\Pi i} + 1 = 0$$

$$e^{\Pi i} = -1$$

$$(e^{\Pi i})^{1/2} = (-1)^{1/2}$$

$$(e^{\Pi i})^{1/2} = i$$

$$(e^{\Pi i/2}) = i$$

$$(e^{\Pi i/2})^i = i^i$$

$$(e^{\Pi i*i/2}) = i^i$$

$$(e^{-\Pi/2}) = i^i$$

i^i **is a real number!**

QED due to closure operations of real numbers

Appendix F : Exponents, Powers, Logarithms…What's the Difference?

Many people use the terms exponents, powers, and logarithms interchangeably.   I have heard people read "$2^3 = 8$" as "2 to the 3rd power is 8" and then say in their next breath that "8 is a power of 2." Well, what is it? Is 3 the power or is 8 the power?   Did they mean "8 is the 3rd power of 2" but just not explicitedly state that? As long as everyone in the room understands from context clues what is meant I guess it really does not matter what term is used.   However, when the terms exponents, powers, and logarithms are used quickly, interchangeably, and esoterically with students trying to learn new ideas and concepts then confusion can result.

I propose that the terms "exponents" and "powers" be used interchangeably whenever repeated multiplication is implied.   That is, when only **two numbers** are involved … a base and an exponent/power then the base is multiplied by itself the number of times indicated by the exponent/power.

Another way to think of this rule is to apply the definition of exponentiation ….

m times

Definition of exponentiation: $b^m = b*b*b* \ldots \qquad * b$ (*b* times itself *m* times, m is an exp/power)

Here there is a **relation between two numbers** *being described.*

The term logarithm should be use whenever **a relation among three numbers** *is indicated.*

$\log_2 8 \ = 3$ Here 3 is the logarithm of the number 8 when 2 is the base.

You can see where confusion can arise. In the equation $\log_2 8 \ = 3$ involving **three numbers** the "3" is clearly the logarithm.

But by the Equivalent Symbolism Rule, $b^y = x$ is equivalent to $y = \log_b x$

$\log_2 8 \ = 3$ is equivalent to $2^3 = 8$ transforming the "logarithm 3" into a power or "exponent of 3."

One can assist students by only using the term logarithm as part of a prepositional phrase.   That is, do not say "log" but say "log of a number" or, better yet, log, base b of a number.

To review: when talking to people who are not "in the know" and cannot interpret changing and imprecise vocabulary use the terms power/exponents when talking about the interaction and relation between two numbers (repeated multiplication shown in exponential form, 2^3 …3 is an exponent/power) and use the term logarithm when talking about the interaction and relation among three numbers (logarithmic form. $\log_2 8 \ = 3$, **3 is a logarithm here but in 2^3 the symbol "3" is a power/exponent.)**

Answers to Exercises

Chapter 1 Answers

1.)
$$100{,}000 < 285{,}962 < 1{,}000{,}000$$
$$10^5 < 285{,}962 < 10^6$$
$$10^{5.\text{something}} = 285{,}962$$
$$\log 285{,}962 = 5.\text{something}$$

2.)
$$0.0001 < 0.000368 < 0.001$$
$$10^{-4} < 0.000368 < 10^{-3}$$
$$10^{-3.\text{something}} = 0.000368$$
$$\log (0.000368) = -3.\text{something}$$

3.) a.)
$$10 < 56 < 100$$
$$10^1 < 10^{1.\text{something}} < 10^2$$

b.)
$$100 < 687 < 1{,}000$$
$$10^2 < 10^{2.\text{something}} < 10^3$$

c.)
$$10{,}000 < 43{,}921 < 100{,}000$$
$$10^4 < 10^{4.\text{something}} < 10^5$$

d.)
$$0.01 < 0.0219 < 0.1$$
$$10^{-2} < 10^{-1.\text{something}} < 10^{-1}$$

e.)
$$0.000001 < 0.0000038 < 0.00001$$
$$10^{-6} < 10^{-5.\text{something}} < 10^{-5}$$

f.)
$$0.00001 < 0.00007871 < 0.0001$$
$$10^{-5} < 10^{-4.\text{something}} < 10^{-4}$$

4.) a.)
$$4{,}526 \times 104{,}264 = 10^{(\log 4{,}526 + \log 104{,}264)}$$
$$= 10^{(3.65571455 + 5.018134382)}$$
$$= 10^{8.673848932}$$
$$= 471{,}898{,}864$$

b.)
$$0.061538 \times 40{,}126.7 = 10^{(\log 0.061538 + \log 40{,}126.7)}$$
$$= 10^{(-1.210856623 + 4.603433445)}$$
$$= 10^{(3.392576823)}$$
$$= 2{,}469.316865$$

c.)
$$0.015872 \times 0.000000183218 = 10^{(\log 0.015872 + \log 0.000000183218)}$$
$$= 10^{(-1.799368345 + -6.7370318621)}$$
$$= 10^{(-8.536400207)}$$
$$= 2.908036096 \times 10^{-9}$$

5.) a.) $x = \log_3 y$ f.) $8^y = x$
b.) $x = \log_2 5$ g.) $3^y = x$
c.) $x = \log_7 y$ h) $7^y = x$
d.) $q = \log_p y$ i.) $2^8 = x$
e.) $3.2 = \log_w g$ j.) $x^9 = 11$

Chapter 2 Answers

1.) $y = 2x + 5$

x	y
−2	1
−1	3
0	5
1	7
2	9

2.) $x = 2y + 5$

x	y
1	−2
3	−1
5	0
7	1
9	2

3.) The graphs $y = 2x + 5$ and $x = 2y + 5$ are symmetric with the line $y = x$.

Chapter 2 Answers
(continued)

4.) $r(s(x)) = s(r(x)) = x$
$s(r(x)) = r(s(x)) = x$

5.) No, not iff. If $f(x) = 3x$ and $g(x) = 2x$, then
$f(g(x)) = f(2x) = 6x$ and $g(f(x)) = g(3x) = 6x$
$f(g(x)) = g(f(x))$, but the two graphs are not
symmetric with the line $y = x$

6.) Inverse Log Rules
$\log_b b^x = x$ and $b^{\log_b x} = x$

7.) iff Antilog Rule … $p = q$ iff $b^p = b^q$

8.) iff Log Rule. $p = q$ iff $\log p = \log q$

9.) Cannot be done
for $y = b^x$, $b > 0$
for $y = \log_b x$, $b > 0$, $b \neq 1$

Chapter 3 Answers

1.) a.)
$$\frac{676}{94283} = 10^{(\log 676 - \log 94283)}$$
$$= 10^{(2.829946696 - 4.974433393)}$$
$$= 10^{-2.144486697}$$
$$= 0.0071699034$$

b.)
$$\frac{0.000000676}{94.283} = 10^{(\log 0.000000676 - \log 94.283)}$$
$$= 10^{(-6.170053304 - 1.974433393)}$$
$$= 10^{-8.144486697}$$
$$= 7.169903376 \times 10^{-9}$$

c.)
$$\frac{6.76}{0.94283} = 10^{(\log 6.76 - \log 0.94283)}$$
$$= 10^{(0.829946696 - (-0.025566607))}$$
$$= 10^{0.855513303}$$
$$= 7.169903376$$

Since each numerator and denominator have the same significant digits then the significant digits of each quotient will be the same. Only the

Chapter 4 Answers

1.)
$$64 < 200 < 256$$
$$4^3 < 4^{3.\text{something}} < 4^4$$
$$\log_4 200 = 3.\text{something}$$

2.)
$$10^x = 14{,}290$$
$$x \approx 4.\text{something}$$
$$\log 10^x = \log 14{,}293$$
$$x = 4.155032229$$

3.)
$$4{,}913 < 14{,}290 < 83{,}521$$
$$17^3 < 17^{3.\text{something}} < 17^4$$
$$\log_{17} 14290 = 3.\text{something}$$

Chapter 4 Answers
(continued)

4.)
$$17^x = 14{,}290$$
$$\log_{10} 17^x = \log_{10} 14{,}290$$
$$\log_{10} 17^x = 4.15503 \text{ ??} \quad \text{Now what??}$$

5.)
$$17^x = 14{,}290$$
$$\log 17^x = \log 14{,}290$$
$$x \log 17 = \log 14{,}290$$
$$x = \frac{\log 14{,}290}{\log 17}$$
$$x \approx 3.376842514$$

6.) $17^{3.376842514} = 14{,}290$

7.) $\sqrt[5]{621}$
$$x = 621^{\frac{1}{5}}$$
$$\log x = \left(\tfrac{1}{5}\right) \log 621$$
$$x = 3.619247808$$

8.) $\sqrt[9]{621}^{\,7}$
$$x = 621^{\frac{7}{9}}$$
$$\log x = \log 621^{\frac{7}{9}}$$
$$x = (7/9)\log 621 = 148.732055$$

Chapter 5 Answers

1.) a.) $25 = 5^2 < 5^{2.7} < 5^3 = 125$
$$25 < 5^{2.7} < 125$$
   b.) $9^2 = 81, \quad y \approx 81$

2.)
$$5^{2.7} = 77.129$$
$$8.64^{2.13} = 98.804$$

3.) a.) $4^2 = 16 < 32.7 < 64 = 4^3$
   Therefore $2 < x < 3$
   b.) $5^2 = 25 < 117 < 125 = 5^3$
   Therefore $2 < x < 3$

4.) a.)
$$32.7 = 4x$$
$$\log 32.7 = \log 4x$$
$$\log 32.7 = x \log 4$$
$$x = 2.515609365$$
   b.)
$$117 = 5^x$$
$$\log 117 = x \log 5$$
$$x = \frac{\log 117}{\log 5}$$
$$x = 2.958905030$$

5.) a.) $9 < 10 < 27$
$$3^2 = 9 \qquad 3^3 = 27$$
$$2 < x < 3$$
   b.) $16 < 62.73 < 81$
$$2^4 = 16 \qquad 3^4 = 81$$
$$2 < x < 3$$

Chapter 5 Answers
(continued)

6.) a.)
$$\log 10 = \log x^{2.6}$$
$$1 = 2.6 \log x$$
$$\tfrac{1}{2.6} = \log x$$
$$\log x = 0.3846153846$$
$$10^{\log x} = 10^{0.3846153846}$$
$$x = 2.424462017$$
   b.)
$$\log 62.73 = \log x^{4.31}$$
$$1.797475288 = 4.31 \log x$$
$$\log x = 0.4170476305$$
$$10^{\log x} = 10^{0.4170476306}$$
$$x = 2.612447855$$

7.) a.) $5^4 = 625, \; 5^5 = 3{,}125, \; 7^4 = 2{,}401, \; x \approx 4$
   b.) $8^2 = 64, \; 8^3 = 512, \; 6^3 = 216, \; x \approx 3$

8.) a.)
$$5^{4.6} = 7^x$$
$$\log 5^{4.6} = \log 7^x$$
$$3.21526202 = x \log 7$$
$$x = 3.80460238$$
   b.)
$$8^{2.7} = 6^x$$
$$\log 8^{2.7} = \log 6^x$$
$$2.438342965 = x \log 6$$
$$x = 3.133507739$$

9.) a.) $2^5 = 32 < 50 < 64 = 2^6$
$$5 < y < 6$$
   b.) $3^3 = 27 < 28 < 81 = 3^4$
$$3 < y < 4$$

10.) a.) $\log_2 50 = \dfrac{\log 50}{\log 2}$
$$= 5.64385619$$
   b.)
$$3^y = 28$$
$$\log 3^y = \log 28$$
$$y = \frac{\log 28}{\log 3}$$
$$y = 3.033103256$$

11.) a.) $2^6 = 64, \; x \approx 64$
   b.) $9^5 = 59{,}049, \; x \approx 59{,}049$

12.) a.) $2^{6.1} = 68.5935016$
   b.) $9^{5.1} = 73{,}559.16625$

13.) a.) $2^5 = 32, \; x \approx 2$
   b.) $5^3 = 125, \; x \approx 5$

14.) a.)
$$x^{4.9} = 37.1$$
$$\log x^{4.9} = \log 37.1$$
$$4.9 \log x = \log 37.1$$
$$\log x = \frac{\log 37.1}{4.9}$$
$$10^{\log x} = 10^{0.3202803897}$$
$$x = 2.09064546$$
   b.)
$$x^{3.207} = 126.21$$
$$\log x^{3.207} = \log 126.21$$
$$3.207 \log x = \log 126.21$$
$$\log x = \frac{\log 126.21}{3.207}$$
$$10^{\log x} = 10^{0.6551586426} = 4.520210321$$

15.)
$$0.2 = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$
$$0.2\sqrt{2\pi} = e^{-x^2/2}$$
$$\ln 0.5013256549 = -\tfrac{1}{2}x^2 \ln e$$
(not to worry, $\ln 0 < x < 1$ will be negative)
$$-0.6904993792 = -\tfrac{1}{2}x^2$$
$$1.380998758 = x^2$$
$$x = \pm 1.175159035$$

16.) a.)
$$T(t) = T_m + (T_i - T_m)e^{-kt}$$
$$60 = 20 + (90 - 20)e^{-3k}$$
$$40 = 70e^{-3k}$$
$$\tfrac{4}{7} = e^{-3k}$$
$$\ln 0.5714285714 = \ln e^{-3k}$$
$$-0.559615788 = -3k \ln e$$
$$-0.559615788 = -3k$$
$$k = 0.186538596$$
b.)
$$30 = 20 + (90 - 20)e^{-0.186538596t}$$
$$10 = 70\, e^{-0.186538596t}$$
$$\tfrac{1}{7} = e^{-0.186538596t}$$
$$\ln (\tfrac{1}{7}) = \ln e^{-0.186538596t}$$
$$\ln (\tfrac{1}{7}) = -0.186538596t \ln e$$
$$\ln (\tfrac{1}{7}) = -0.186538596t$$
$$t = 10.43167575 \text{ min}$$

17.)
$$r = e^{at}$$
estimate $360° + 360° + 90° = 810°$
Solve
$$4 = e^{at}$$
$$4 = e^{0.1t}$$
$$\ln 4 = \ln e^{0.1t}$$
$$\ln 4 = \ln e^{0.1t}$$
$$\ln 4 = 0.1t \ln e$$
$$t = \frac{\ln 4}{0.1}$$
$$t = 13.86294361 \text{ radians} = 794.288°$$

18.) a.)
$$\text{pH} = -\log[\text{H+}]$$
$$\text{pH} = -\log [9.2 \times 10^{-12}]$$
$$\text{pH} = 11.0362$$
b.)
$$4.2 = -\log x$$
$$-4.2 = \log x$$
$$10^{-4.2} = x$$
$$x = 6.30957 \times 10^{-5}$$

1.)
$$P_r = P_o [(1 + \tfrac{r}{k})^k]^y$$
$$3x = x[(1 + \tfrac{7\%}{12})^{12}]^y$$
$$3 = [(1 + \tfrac{7}{1200})^{12}]^y$$
$$3 = 1.072290081^y$$
$$\log 3 = \log 1.072290081^y$$
$$\log 3 = y \log 1.072290081$$
$$y = 15.74 \text{ yrs.}$$

2.)
$$P_f = P_o e^{ry}$$
$$1{,}500 = 900e^{10r}$$
$$\tfrac{15}{9} = e^{10r}$$
$$\ln (\tfrac{5}{3}) = 10r (\ln e)$$
$$0.5108256238 = 10r$$
$$r = 0.05108256238$$
$$r = 5.1\%$$

3.)
$$P_r = P_o [(1 + \tfrac{r}{k})^k]^y$$
$$2{,}000 = 1{,}000[(1 + \tfrac{20\%}{1})^1]^y$$
$$2 = 1.20^y$$
$$y = \frac{\log 2}{\log 1.2}$$
$$y = 3.801784017 \text{ yrs.}$$
$$0.801784017 \times 365 = 292$$
On the 292nd day of the third year after the money was invested.

4.)
$$P_r = P_o [(1 + \tfrac{r}{k})^k]^y$$
$$10{,}000 = P_o [(1 + \tfrac{5\%}{2})^2]^{20}$$
$$10{,}000 = P_o[(1.025)^2]^{20}$$
$$10{,}000 = P_o \times 2.6850638384$$
original principal = $3,724.31

5.)
$$Q_f = Q_i \times 10^{-kt}$$
$$400 = 500 \times 10^{-1{,}000k}$$
$$\tfrac{4}{5} = 10^{-1{,}000k}$$
$$\log 0.8 = \log 10^{-1{,}000k}$$
$$\log 0.8 = -1{,}000k \log 10$$
$$\log 0.8 = -1{,}000k$$
$$k = 9.6910013 \times 10^{-5}$$

6.)
$$Q_f = Q_i \times 10^{-kt}$$
$$Q_f = 500 \times 10^{-9.651001301 \times 10^{-5} \times 2{,}000}$$
$$Q_f = 500 \times 0.64_r$$
$$Q_f = 320$$

7.)
$$I_f = I_i\, 10^{-kt}$$
$$100{,}000 = 1{,}000{,}000 \times 10^{-9.4k}$$
$$0.10 = 10^{-9.4k}$$
$$\log 0.1 = -9.4k (\log 10)$$
$$k = 0.1063829787$$

8.)
$$\tfrac{1}{2}x = x\, 10^{-0.1063829787t}$$
$$\log 0.5 = \log 10^{-0.1063829787t}$$
$$t = 2.82968196 \text{ cm.}$$

9.)
$$\text{Db} = 10\log \frac{power_a}{power_b}$$
$$65 = 10\log \frac{x}{10^{-12} \text{ W}/\text{ft}^2}$$
$$6.5 = \log \frac{x}{10^{-12} \text{ W}/\text{ft}^2}$$
$$10^{\text{left}} = 10^{\text{right}}$$
$$3{,}162{,}277.66 = \frac{x}{10^{-12}}$$
$$x = 3{,}162{,}277.66 \times 10^{-12}$$
$$x = 0.00000316227766 \text{ W}/\text{ft}^2$$

10.) $12{,}024 \text{ m} = 2^{(y-2{,}007)/2} \, 376 \text{ m}$
$31.9787234 = 2^{(y-2{,}007)/2}$
$\log 31.9787234 = \dfrac{y-2{,}007}{2} \log 2$
$4.999040442 = \dfrac{y-2{,}007}{2}$
$9.998080884 = y - 2{,}007$
$y \approx 2{,}017$

11.) $\dfrac{1{,}050{,}000}{200} = 5{,}250 \text{ pages}$
$2^x = 5{,}250$
$\log 2^x = \log 5{,}250$
$x \log 2 = \log 5{,}250$
$x = \dfrac{\log 5{,}250}{\log 2}$
$x = 12.358$
$x = 13 \text{ bisections}$

12.) $2 < \text{dimension} < 3$

13.) $f = I \times e^{-0.0001205473358t}$
$0.80\,x = x \times e^{-0.0001205473358t}$
$0.80 = e^{-0.0001205473358t}$
$\ln 0.80 = \ln e^{-0.0001205473358t}$
$\ln 0.80 = -0.00012054733t \ln e$
$t = 1{,}851.086 \text{ years ago}$

14.) $2^x = 64$
$x = 6$

15.) $2^x = 8{,}192$
$\log 2^x = \log 8{,}192$
$x = \dfrac{\log 8{,}192}{\log 2}$
$x = 13 \ (13^{\text{th}} \text{ generation})$
$13 - 2 = 11 \text{ Gs}$

16.) $V_f = v_i e^{-t/RC}$
let $RC = (120\,\Omega) \times (35\,\mu F) = 4.2 \times 10^{-3} \text{ s}$
$10\%\,x = x e^{-t/4.2 \times 10^{-3}\,s}$
$\ln(0.10) = \ln e^{-t/4.2 \times 10^{-3}\,s}$
$\ln(0.10) = -t/4.2 \times 10^{-3}\,s \ln e$
$-t = \ln 0.10 \times 4.2 \times 10^{-3} \text{ s}$

1.) m = 4, 6, 7, 7.75, 7.875, ... , 8. Instantaneous rate of speed will be the same value as the slope of the tangent line which is suggested by the approaching secant line slopes.

2.) $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, $m_{sec1} = 0.8610$, $m_{sec2} = 1.1752$, $m_{sec3} = 1.7183$, $m_{sec4} = 2.1391$. The slopes of the secant lines approach the slope of the tangent at $(1, e) = e$.

3.) area rec$_1$ + area rec$_2$ + area rec$_3$ + area rec$_4$ =
$b_1 \times h_1 + b_2 \times h_2 + b_3 \times h_3 + b_4 \times h_4 =$
$\tfrac{1}{2} \times 1 + \tfrac{1}{2} \times \tfrac{2}{3} + \tfrac{1}{2} \times \tfrac{1}{2} + 0.21828 \times \tfrac{2}{5} =$
$(\tfrac{1}{2}) + (\tfrac{1}{3}) + (\tfrac{1}{4}) + 0.087313 = 1.17$

4.) area rec$_1$ + area rec$_2$ + area rec$_3$ + area rec$_4$ =
$b_1 \times h_1 + b_2 \times h_2 + b_3 \times h_3 + b_4 \times h_4 =$
$\tfrac{1}{2} \times \tfrac{2}{3} + \tfrac{1}{2} \times \tfrac{1}{2} + \tfrac{1}{2} \times \tfrac{2}{5} + 0.21828 \times 0.36788$
$=$
$\tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + 0.0803 = 0.86$

5.) $\tfrac{1}{2}(1.170645 + 0.863634) = 1.01714$
As the number of rectangular partitions increases, the average of the sum of rectangles will get closer and closer to *1*.

6.)
$\log 0.9 = -0.0457574906$
$\log 0.99 = -0.00\mathbf{43}648054$
$\log 0.999 = -0.000\mathbf{434}51177$
$\log 0.9999 = -0.0000\mathbf{434}316$
$\log 0.99999 = -0.0000\mathbf{43429}665$
$\log 0.999999 = -0.000000\mathbf{43429}47$
etc.
$\log e = 0.4342944819$

$\log_{b^n} x^n = \log_b x \quad ?????$
arbitrarily let $b = 2$, $x = 3$, and $n = 4$.
$\log_{2^4} 3^4 = \log_2 3 \quad ?????$
$\log_{16} 81 = \log_2 3 \quad ????$
$\log_{16} 81 = \dfrac{\log 81}{\log 16} = 1.584962501$
$\log_2 3 = \dfrac{\log 3}{\log 2} = 1.584962501$

Seems to be an identity.
$\log_{b^n} x^n = \log_b x$

Applying Equivalent Symbolism Rule
$\log_b x = y \qquad$ is equivalent to
$b^y = x$
$b^{n \log_b x} = x^n$

Applying Power of a Power Rule
$(b^m)^n = b^{mn}$
$b^{n \log_b x} = x^n$
$b^{\log_b x^n} = x^n$

1.) Exponential functions have a horizontal asymptote—y = 0. Horizontal asymptotes are therefore not affected by a horizontal shift. Log functions have a vertical asymptote—x = 0. Vertical asymptotes are affected by a horizontal shift.

2.) Graph $y = -\frac{1}{3}\log_{(1/2)}(x+4) + 2$. There are several correct approaches.

   a.) Graph $y = \log_{(1/2)} x$. Use the change of base formula: $\frac{\log x}{\log \frac{1}{2}}$. This should look like $y = \log_b x$ with $0 < b < 1$

   b.) Graph $y = \log_{(1/2)}(x+4)$. This will cause a horizontal shift to the left for the graph in part a.

   c.) Graph $y = \log_{(1/2)}(x+4) + 2$. This will cause a vertical shift up for the graph in part b.

   d.) Graph $y = \frac{1}{3}\log_{(1/2)}(x+4) + 2$. This will cause a flattening of the graph in part c.

   e.) Graph $y = -\frac{1}{3}\log_{(1/2)}(x+4) + 2$. This will cause a reflection about the x-axis for the graph in part d.

3.) $y = -\log_b x$

   a.) Graph $y = \log_b x$

   b.) Reflect the graph in part a about the x–axis.

1.) a.) $2^3 = x$, $x = 8$
   b.) $\frac{1}{3}^4 = x$, $x = \frac{1}{81}$
   c.) $4^{-1/2} = x$, $x = \frac{1}{2}$
   d.) $x = \emptyset$ for $\log_b x = y$, $b > 0$, $b \neq 1$
   e.) $3^{-4} = x$, $x = \frac{1}{81}$

2.) a.) $4^x = 16$, $x = 2$
   b.) $\frac{1}{2}^x = 8$, $x = -3$
   c.) $x = \emptyset$, $5^x = 0$, $x = $??
   d.) $x = \emptyset$, for $\log_b x = y$, $b > 0$, $b \neq 1$
   e.) $3^x = 1$, $x = 0$

3.) a.) $x^4 = 16$, $x = 2$
   b.) $x^1 = 4$, $x = 4$
   c.) $x^{-4} = 16$, $x = \frac{1}{2}$
   d.) $x^{3/4} = 64$, $x^{3/4} = 2^6$, $\left(x^{3/4}\right)^{4/3} = \left(2^6\right)^{4/3}$, $x = 2^8$, $x = 256$
   e.) $x^2 = 0$, $x = \emptyset$

4.) All problems in 4 are best solved by Inverse Log Rules #1 and #2
   a.) 8   b.) 10   c.) –2   d.) 16   e.) 85

5.) a.) $3x = \dfrac{\log 2.68}{\log 0.196}$, $x = -0.2016429845$
   b.) $x = \dfrac{\log 5,240}{\log 242}$, $x = 1.560243021$

6.) a.) $3^x = 79$
      $x = \dfrac{\log 79}{\log 3}$
      $x = 3.977242834$
   b.) $x^6 = 79$
      $x^{6 \cdot \frac{1}{6}} = 79^{\frac{1}{6}}$
      $x = 2.071434389$
   c.) $3^{7.2} = x$
      $x = 2,724.413565$

7.) a.) $4 - 4 = 0$
   b.) $4^{\log_4 3^2} + 3^{\log_3 4^2} = 9 + 16 = 25$
   c.) $6^{\log_6 36^{1/2}} \times 9^{\log_9 81^2} = 6 \times 6,561 = 39,366$
   d.) $\log 4^3 - \log 8^2 = \log \frac{64}{64} = 0$
   e.) $2 + \frac{1}{2} = 2.5$

8.) a.) $\ln 1 = 0$ e.) $\log 1 = 0$
   b.) $\ln e = 1$ f.) $\log 10 = 1$
   c.) $\ln e^y = y$ g.) $\log 10^y = y$
   d.) $e^{\ln y} = y$ h.) $10^{\log y} = y$

9.) $\log \dfrac{16x^2}{y} = \log 16 + 2\log x - \log y$
      Or $2\log 4 + 2\log x - \log y$

10.) $\log(10 + 20) = \log 10 + \log 20$???
       $\log 30 = 1 + 1.3$???
       $1.48 \neq 1.3$

11.) $\log \frac{10}{20} = \dfrac{\log 10}{\log 20}$???
      $\log 0.5 = \frac{1}{1.3}$???
      negative $\neq$ positive
(The actual values are not really important.)

12.) $\left(\dfrac{1}{2}\right)^{2/3} = \frac{1}{4}$

     $\left[\left(\dfrac{1}{2}\right)^{2/3}\right]^3 = \frac{1}{4}^3$
     $\frac{1}{2}^x = \frac{1}{64}$
     $\frac{1}{2}^x = \frac{1}{2}^6$

13.) $\ln [(3x - 2)(x - 1)] = \ln x^2$

$\quad\quad\quad\quad 3x^2 - 5x + 2 = x^2$

$\quad\quad\quad\quad 2x^2 - 5x + 2 = 0$

$\quad\quad\quad (2x - 1)(x - 2) = 0$

$\quad\quad\quad\quad\quad x = \frac{1}{2}, x = 2$

$\quad\quad$ Recall that for $\ln p = q$, $p > 0$ (domain for log curves is positive). Hence, $3x - 2 > 0$ and $x - 1 > 0$ individually. Therefore, reject $x = \frac{1}{2}$.

14.)

$\quad\quad\quad\quad\quad\quad\quad \log (3^{x - 1}) = \log(5^{2x + 3})$

$\quad\quad\quad\quad\quad\quad\quad \log 3^{(x - 1)} = \log 5^{(2x + 3)}$

$\quad\quad\quad\quad\quad\quad (x - 1) \log 3 = (2x + 3) \log 5$

$\quad\quad\quad\quad x \log 3 - \log 3 = 2x \log 5 + 3\log 5$

$\quad\quad\quad x \log 3 - 2x \log 5 = \log 3 + 3\log 5$

$0.4771212547x - 2x\, 0.6989700043 = 2.574031268$

$\quad\quad\quad\quad -0.9208187540x = 2.574031268$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad x = -2.795372333$

15.) $x = b^y$ is equivalent to $y = \log_b x$

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

For over 350 years, from the early 1600s until the widespread availability of calculators in the 1970s, most of the mathematics done by scientists, engineers, and astronomers was assisted by logarithms. The logarithmic technique was developed to aid in the drudgery of simplifying long and tedious arithmetic expressions. Logarithms worked by reducing arithmetic expressions of one level of difficulty to a lesser level of difficulty. Scientific calculators have made much of the pre-1970 precalculus curriculum obsolete. But the use of calculators has also come with a price. The instruction of logarithms today is much, much more condensed and abstract than it used to be. As a result, many of today's students do not achieve the same level of understanding and "internalization" of logarithmic concepts and ideas. Many of them do not understand the "magic" formulas they are taught and asked to manipulate.

The website **The Math Forum**, "Ask Dr. Math," has the following request for help.
"I understand what logs are … but I don't understand why they are what they are. Please help me."

This is a plea for help from a student who, at the time of his plea, was enrolled in a calculus class!

Explaining Logarithms, A Progression of Ideas Illuminating an Important Mathematical Concept, *does not* advocate a return to the precalculator "good old days." The author lived through them. They were not so good!! However, this book is written under the belief that a quick review of mathematics as it was practiced for hundreds of years would be helpful for many students in understanding logarithms as they are still used today. The student quoted above was not instructed in a way that he internalized what logarithms are all about. It is a "readiness issue" which this book attempts to remedy.

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