Steve Keith
http://www.baselines.com
steve@baselines.com

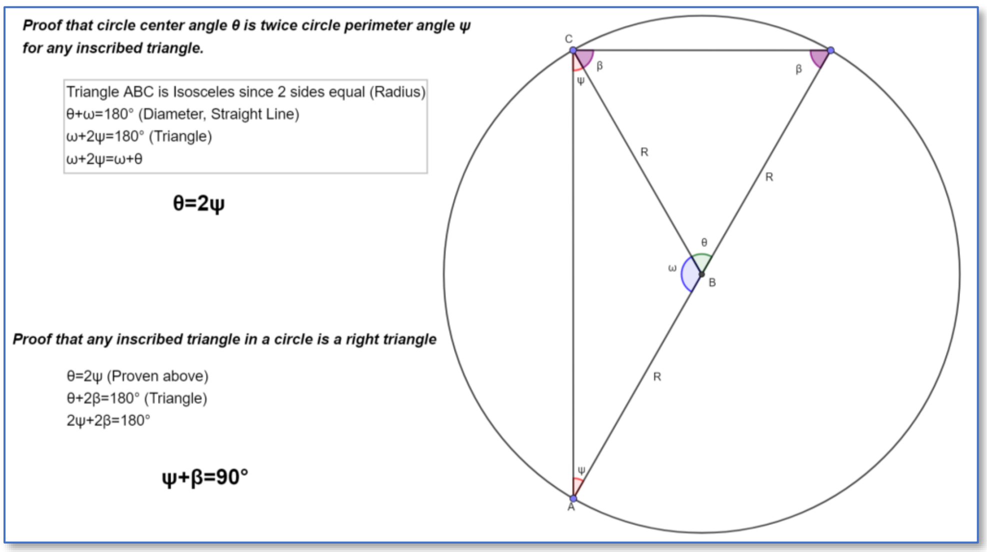
Purpose

The overall goal is to show how far the Moon 'falls' towards Earth in one second of its orbit. In going through this exercise, we will derive many of the concepts needed from scratch. This includes some geometric proofs, and an explanation of the 'least squares' method of curve fitting.

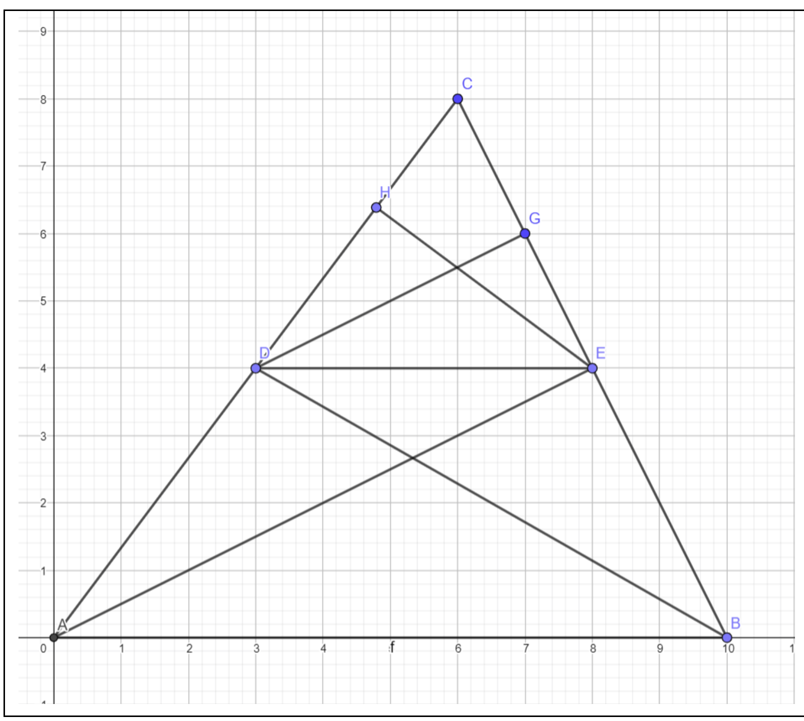
Definitions

We will define a *right angle* as being 90 degrees and a *circle* as having 360 degrees.
We also estimate by assuming orbits are perfectly circular and bodies are spheres.

Inscribed Triangle Proofs



Similar Triangle Proof



Two triangles are similar if at least two angles are the same.

In similar triangles, each side of one triangle is proportional to a side of the other triangle.

Here it is shown that triangle ABC is similar to CDE. We prove that CD is proportional to DA. The same method can be used to prove proportionality of the other two sides.

1 – Draw perpendicular lines to sides BC and AC as shown. (DG and EH)

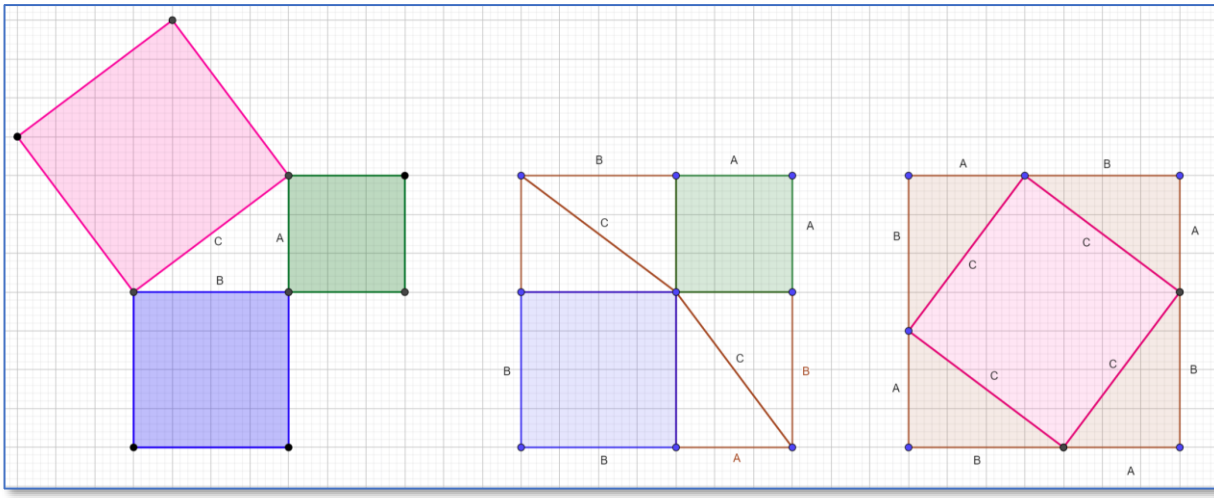
2 – Draw lines connecting AE and BD

3 – Area of CDE / Area of ADE = $\frac{\frac{1}{2} * CD * EH}{\frac{1}{2} * DA * EH} = \frac{CD}{DA}$

(EH is the height of both triangles)

Derivation of Pythagorean Theorem

We are definitely going to need this. Everybody knows it, but to be more thorough, I'll develop it from scratch. The theorem says that the square of the two sides of a right triangle equal the square of the hypotenuse.



The middle figure and the one on the far right are equal in area by inspection (A+B) * (A+B). The square in the middle is constructed with four white triangles of the same shape and dimension as the white triangle in the left figure. The A and B areas in the middle figure (green and blue shaded) are also the same as the figure on the left.

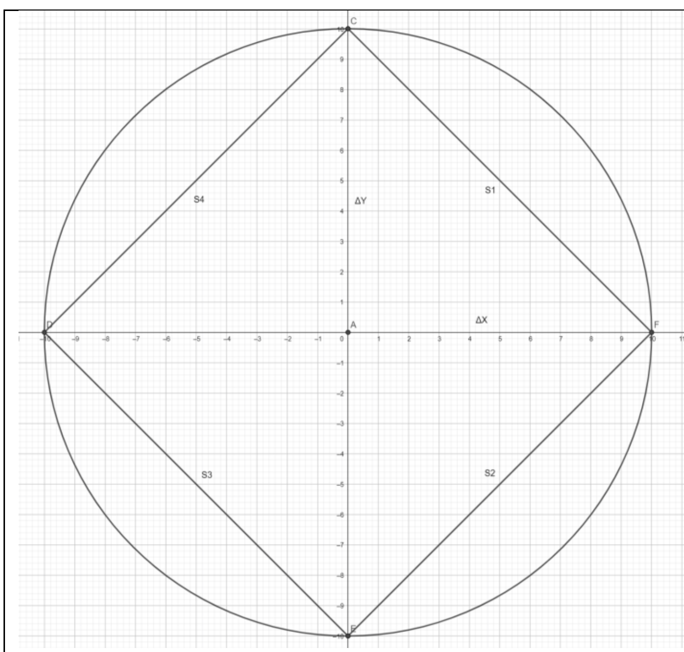
The area for the figure in the middle is $A^2 + B^2 + 2AB$. [(A+B) * (A+B) expanded]

The area for the figure on the right is $4(AB/2) + C^2$ (four brown triangle areas and the pink square in the middle).

We can equate these two: $A^2 + B^2 + 2AB = 2AB + C^2$

Eliminate the like term and you get the Pythagorean Theorem: $A^2 + B^2 = C^2$

Getting the circumference from the diameter of a circle



Given the Pythagorean Theorem, we can see that the formula of a circle centered on 0,0 will be the following, where r is the radius.

$$x^2 + y^2 = r^2$$

…because every x and y triangle that produces a point on the radius will be a right triangle.

We can rewrite this equation

$$f(y) = \sqrt{r^2 - x^2}$$

To get a rough estimate of the circumference, we inscribe a square as shown. (Each segment S is the hypotenuse of a triangle with sides Δx and Δy) Then the estimate will be

$$C_{est} = \sum_{i=1}^{4} S_i = \sum_{i=1}^{4} \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sum_{i=1}^{4} \sqrt{1 + \frac{\Delta y_i^2}{\Delta x_i^2}} \Delta x$$

In the limit,

$$C_{exact} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Using the formula for the circle [f(y)], we differentiate it and square it and substitute this result in the equation above.

$$Let\ u = r^2 - x^2$$

$$Let\ f(u) = \sqrt{u}$$

$$\frac{du}{dx} = -2x$$

$$\frac{df(u)}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{df(u)}{du} * \frac{du}{dx} = -xu^{-\frac{1}{2}} = -\frac{x}{\sqrt{r^2 - x^2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{r^2 - x^2}$$

Substituting and simplifying (You can use the Pythagorean Theorem to show that $\sin^2\theta + \cos^2\theta = 1$.)

$$C_{exact} = \int \sqrt{\frac{r^2}{r^2 - x^2}}\, dx = \int \sqrt{\frac{1}{1 - \frac{x^2}{r^2}}}\, dx$$

$$Let\ \frac{x}{r} = \sin\theta$$

$$C_{exact} = \int \sqrt{\frac{1}{1 - \sin^2\theta}}\, dx = \int \sqrt{\frac{1}{\cos^2\theta}}\, dx$$

We need to find *dx in terms of* θ in order to perform the integration. From the definition above, we know that $x = \mathrm{r} * \sin\theta$. Taking the derivative:

$$\frac{dx}{d\theta} = r\cos\theta \quad so\ dx = r\cos\theta\, d\theta$$

The square term can come out of the radical, the cosine terms cancel out.

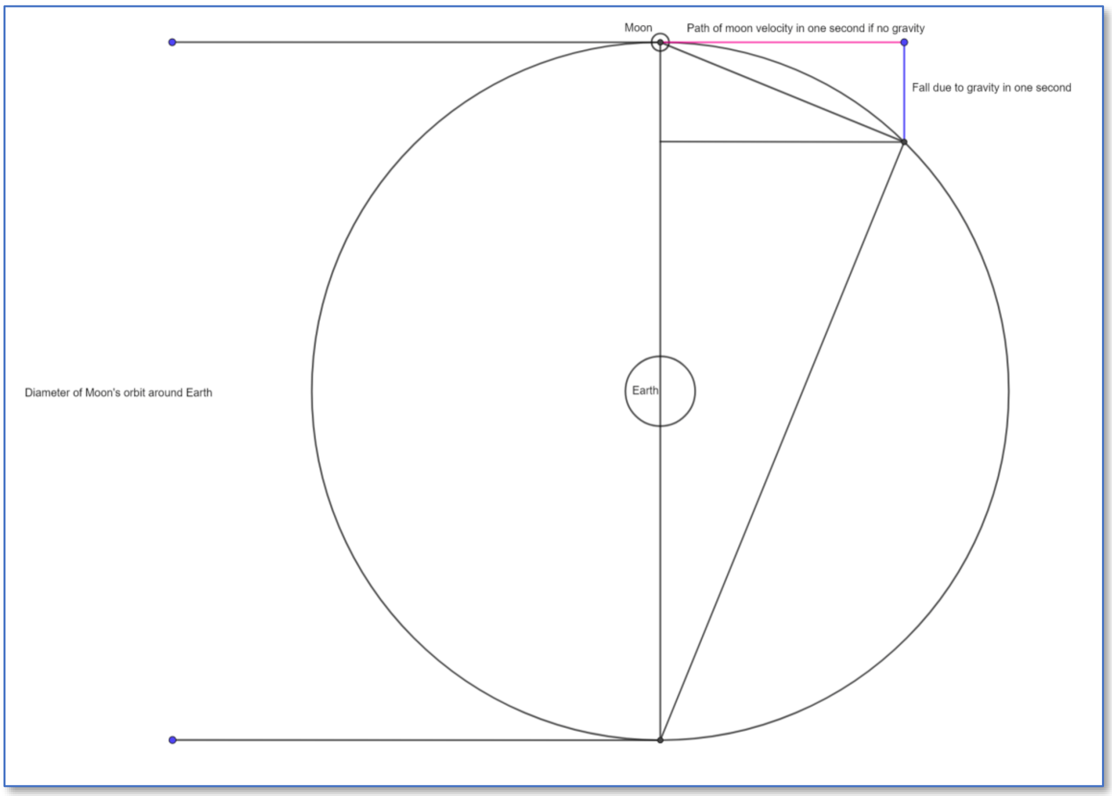
$$C_{exact} = \int \sqrt{\frac{1}{\cos^2\theta}}\, r\cos\theta\, d\theta = \int \frac{1}{\cos\theta} r\cos\theta\, d\theta = \int r d\theta$$

We want to integrate the angle θ over the whole circle. Doing so gives us the formula for the circumference.

$$C_{exact} = \int_0^{2\pi} r d\theta = 2\pi r$$

Graphical representation of the goal of this paper

The diagram presented in this section depicts the main schematic that will be used. Several of the needed values (like the distance from the Earth to the Moon) will be derived in the following sections. Shown below you will see the Earth, the Moon and the ideal path the Moon takes when it revolves around the Earth. Actually, the path is elliptical, but we'll approximate as a circle. It also shows an inscribed triangle and a smaller triangle within. The red line is the straight-line path the Moon would take in one second if it were not pulled into orbit by the Earth. (not to scale). The blue line indicates the amount the Moon 'falls' in that one second to maintain its orbit. This fall is the length we are trying to calculate.



Several values need to be known in order to do this.

- Diameter/Circumference of the Earth and Moon
- Distance from the Earth to the Moon (yielding circumference for the revolution of the Moon)
- Time it takes for the Moon to revolve around the Earth

If we know the velocity of the Moon, we can calculate how far it would go in one second, giving us the length of the red line.

With some trigonometry, we can calculate the length of the blue line, which is what we are trying to find.

In the next few sections, I will go back to the methods used in times past when things such as lasers were not available to obtain values necessary to complete this task.

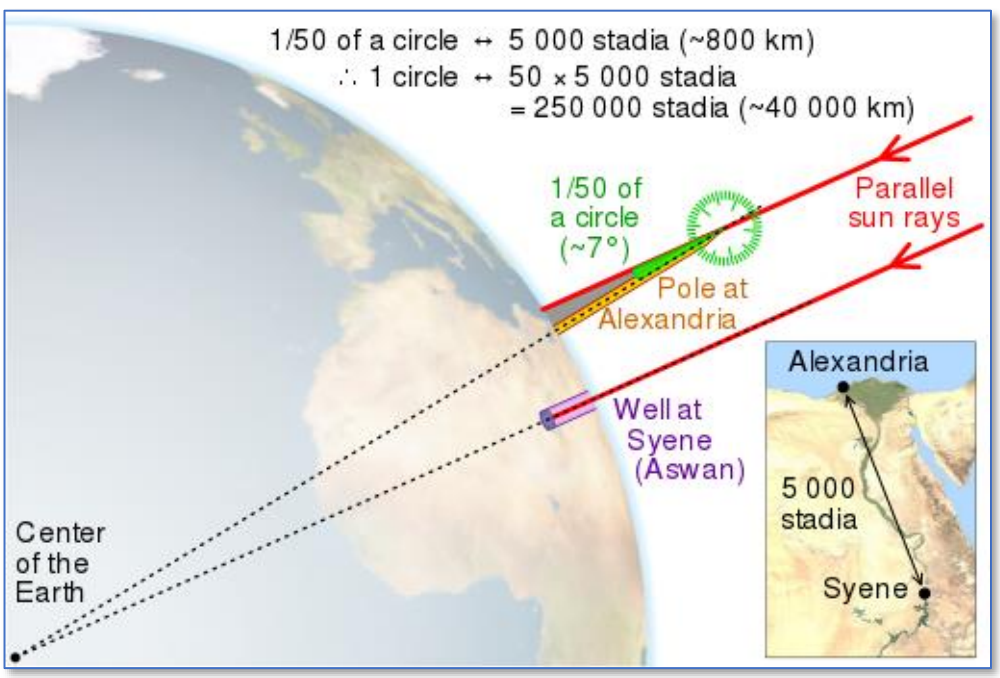
Calculating the circumference and diameter of the Earth (some text from Wikipedia)

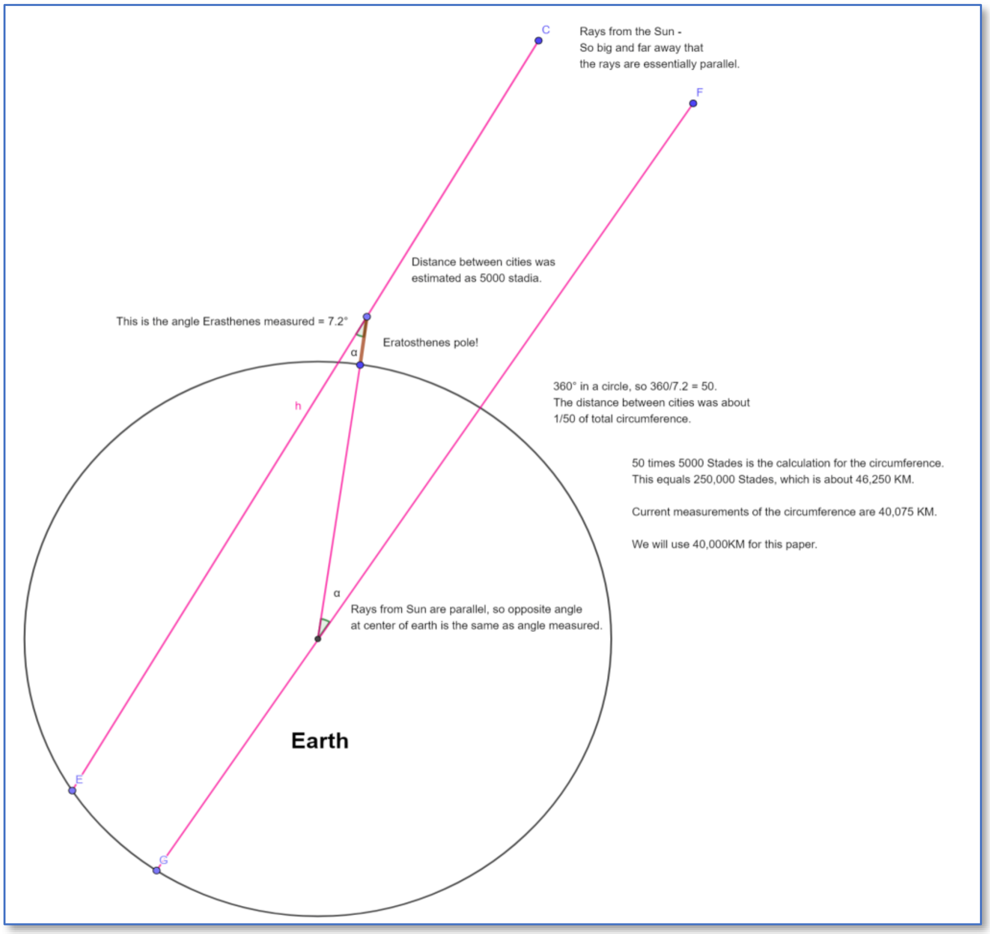
Eratosthenes calculated the circumference of the Earth without leaving Egypt. He knew that at local noon on the summer solstice in Syene (modern Aswan, Egypt), the Sun was directly overhead. (Syene is at latitude 24°05′ North, near to the Tropic of Cancer, which was 23°42′ North in 100 BC[16]) He knew this because the shadow of someone looking down a deep well at that time in Syene blocked the reflection of the Sun on the water. He then measured the Sun's angle of elevation at noon in Alexandria by using a vertical rod, known as a gnomon, and measuring the length of its shadow on the ground. Using the length of the rod, and the length of the shadow, as the legs of a triangle, he calculated the angle of the sun's rays. This turned out to be about 7°, or 1/50th the circumference of a circle. Taking the Earth as spherical, and knowing both the distance and direction of Syene, he concluded that the Earth's circumference was fifty times that distance.

His knowledge of the size of Egypt was founded on the work of many generations of surveying trips. Pharaonic bookkeepers gave a distance between Syene and Alexandria of 5,000 stadia (a figure that was checked yearly). Some historians say that the distance was corroborated by inquiring about the time that it took to travel from Syene to Alexandria by camel. Some claim Eratosthenes used the Olympic stade of 176.4 m, which would imply a circumference of 44,100 km, an error of 10%, but the 184.8 m Italian stade became (300 years later) the most commonly accepted value for the length of the stade, which implies a circumference of 46,100 km, an error of 15%. It was unlikely that Eratosthenes could have calculated an accurate measurement for the circumference of the Earth. He made five important assumptions (none of which is perfectly accurate):

1. That the distance between Alexandria and Syene was 5000 stadia,
2. That Alexandria is due north of Syene
3. That Syene is on the Tropic of Cancer
4. That the Earth is a perfect sphere.
5. That light rays emanating from the Sun are parallel.

Eratosthenes later rounded the result to a final value of 700 stadia per degree, which implies a circumference of 252,000 stadia, likely for reasons of calculation simplicity as the larger number is evenly divisible by 60. In 2012, Anthony Abreu Mora repeated Eratosthenes's calculation with more accurate data; the result was 40,074 km, which is 66 km different (0.16%) from the currently accepted polar circumference of the Earth.[19]

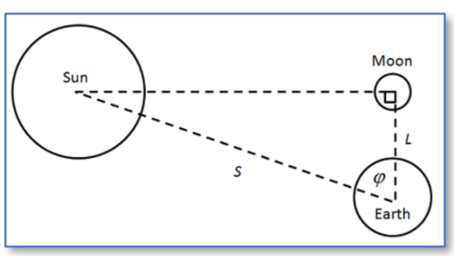




So, the takeaway here is that the circumference we will use for the Earth is 40,000km. Knowing this, we know that the diameter of the Earth is 40,000km/π (Circumference is 2* π *R). We'll use 12,700km for the diameter.

What about the Moon?

How far away is the Moon? Without any modern equipment, it took a lot of brainpower to come up with an estimate for this. One almost unbelievable stroke of luck made it possible to figure this out. Even though it is much smaller than the Sun, the Moon is at the perfect distance from the Earth so that it appears almost exactly the same size as the Sun to an observer on Earth (angular size in the sky is about .5 degree). That is why total Solar eclipses look so cool. Aristarchus was one of the first to try and calculate distances and sizes for the Moon and the Sun, but his estimates were off by quite a bit. His initial step was to use the triangle formed between the Earth, Moon and Sun when the moon was half full, estimating the angle made in order to determine proportionally how much further the Sun is away from the Earth than the Moon is.

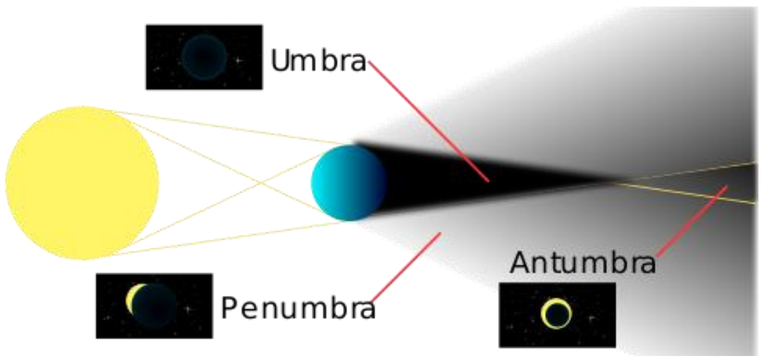


However, his calculation was thrown off by a bad estimate of the angle shown here (Earth angle 😊). He used 87 when it is much closer to 89 degrees.

In the last section, we found the circumference (hence the diameter and radius) of the Earth, which we can use to find the distance to the moon. (Credit for the information below is given to Esther Inglis-Arkell.)

Since the Sun is not a point source of light, the rays are not perfectly parallel and the shadow it casts is a cone. If you hold up a penny so that it just covers the Sun completely and measure the length of the shadow cone, then you will find it is 108 penny diameters away from you. This proportion is true for any sized circle that you use…a quarter would be 108 quarter lengths away when it fully covers the Sun. The Earth itself casts a cone that is 108 Earth diameters long (from Earth center to point of cone).

One important fact to remember is that since the Moon occupies the same angular size in the sky as the Sun does to an observer on Earth, then the same formula applies to the cone. The shadow cone produced by the Moon during a solar eclipse is 108 Moon diameters long. We will use this observation a little later in the document.



During a lunar eclipse, the moon travels through the shadow that is caused by the Earth, so we know that the Moon must be at least as close as 108 earth diameters, or else the Earth's shadow would not fall upon the Moon.

If you watch really closely during a full lunar eclipse, you can see that the width of the shadow cast by the Earth upon the moon is about 2 and one-half times the width of the moon. (Following images are from https://erikras.com/2011/12/16/how-big-is-the-earths-shadow-on-the-moon/)

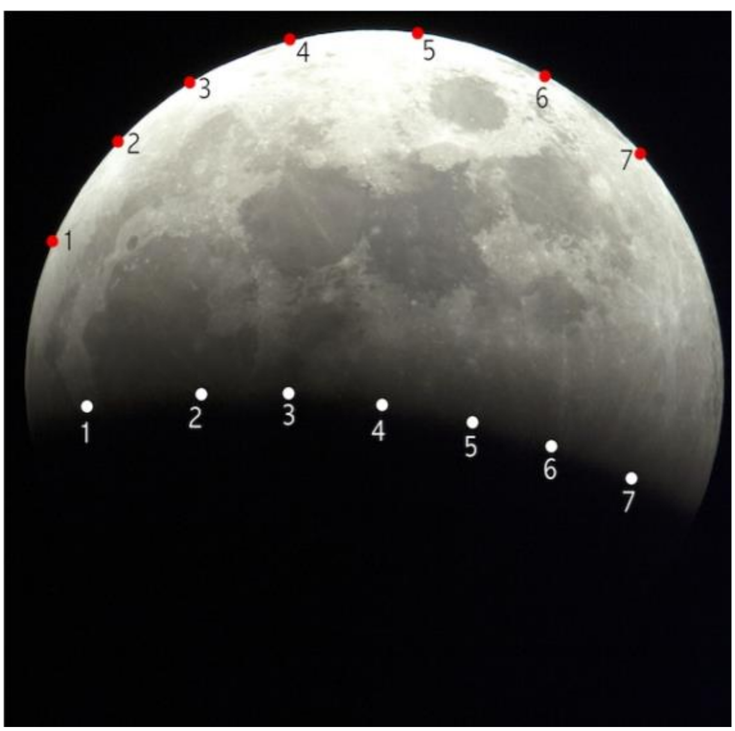
You will notice that there is a lighter shadow with a darker shadow inside. This is because the Sun is not a point source, and will cast both an umbra and a penumbra as shown earlier. We are going to try and measure the umbra portion, as that will give us the closest approximation to the Earth's shadow cone size.

The following information shows how to find the size of the Earth's shadow relative to the size of the Moon by observation and calculation alone. It comes in part from https://cordis.europa.eu/docs/projects/cnect/3/283783/080/deliverables/001-D29lunareclipseactivity1v02A.pdf.

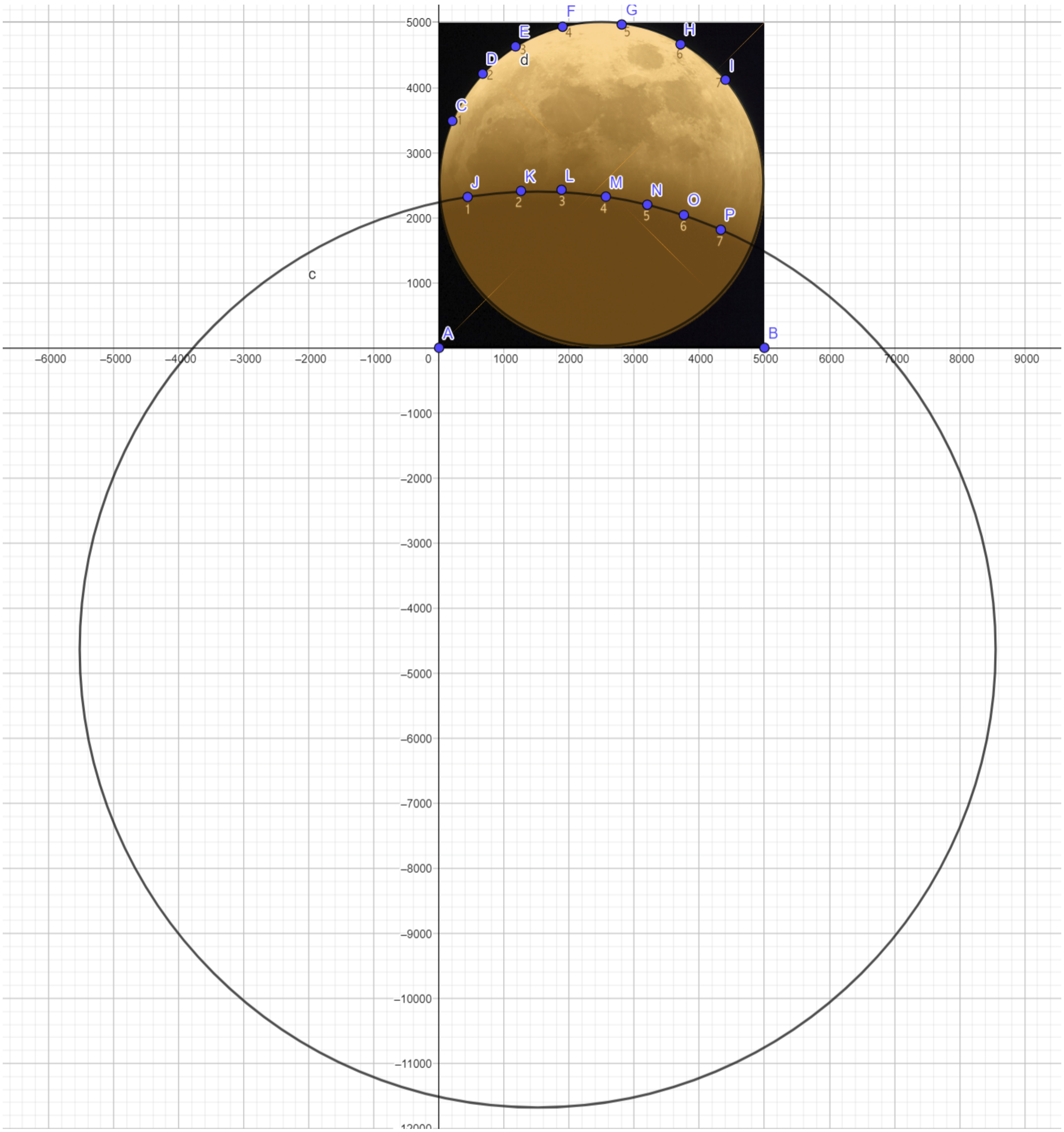
In order to do this, we need to perform some math. We will measure and determine a number of point coordinates on both the Earth's umbra shadow and the Moon's circumference. Then we will use 'Least Squares' curve fitting to determine the radius ratio of the Moon and Earth Shadow. This will allow us to determine the ratio between the Moon's diameter and the Earth's shadow's diameter. These are important to know if we want to calculate the distance from the Earth to the Moon.

Presented here is a picture of the Moon with the Earth's shadow upon it, along with a number of points. The picture is brought into a program so that the X and Y coordinates can be found. (You can do this by hand, but it is an intensive process.) You can see both the umbra and penumbra. We choose the points on the umbra shadow.

Following these diagrams, you will see a derivation on how the 'Least Squares' method is used.



The picture is imported into a graphing tool that helps to visualize the results. A circle is drawn through the points to show the size of the Earth's shadow. I use an online tool (Geogebra) to draw this:
https://www.math10.com/en/geometry/geogebra/fullscreen.html



Geogebra allowed me to draw the circle for the Earth's shadow by specifying three points. It even gives the formula of the circle, but it is good to know the least squares method of curve fitting, so I will include that.

For reference, here are the point coordinates and the circle formula so we can verify our calculations.

$A = (0, 0)$

$B = (5000, 0)$

$C = (203.61, 3488.25)$

$D = (670.54, 4211.79)$

$E = (1175.33, 4632.45)$

$F = (1898.87, 4935.32)$

$G = (2807.5, 4968.97)$

$H = (3711.92, 4666.1)$

$I = (4401.8, 4119.24)$

$J = (439.18, 2318.81)$

$K = (1259.46, 2411.36)$

$L = (1882.04, 2428.18)$

$M = (2563.51, 2323.02)$

$N = (3200, 2200)$

$O = (3762.4, 2041.18)$

$P = (4330.29, 1814.02)$

c : Circle (J, M, P)

$\rightarrow (x - 1515.12)^2 + (y + 4636.83)^2 = 49538584.79$

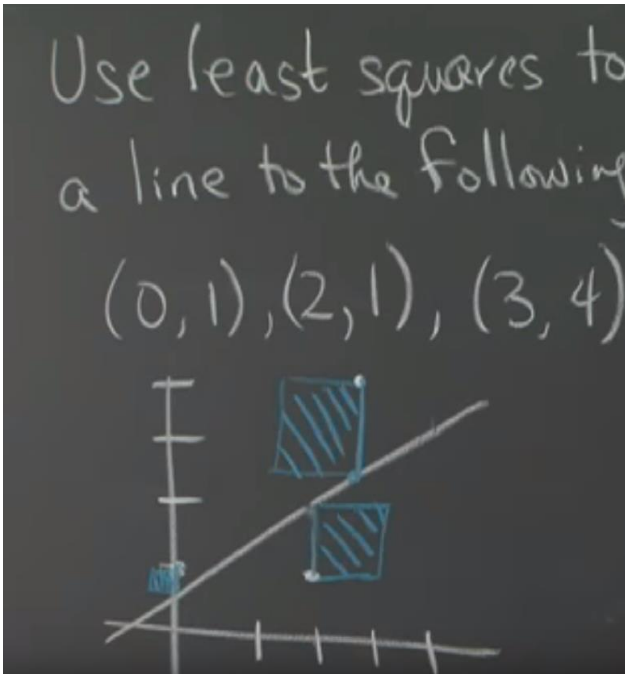
d : Circle (C, F, I)

$\rightarrow (x - 2495.72)^2 + (y - 2519.56)^2 = 6192128.33$

Least Squares Method

The least squares method is used to fit collected data points to a straight line, a polynomial or another type of function as well. For simple two-dimensional line fitting, the method leads to two simultaneous equations that need to be solved, the two variables being the slope and the y-intercept. As the function becomes more complicated (non-linear) the number of equations will increase, and some tricks are needed to put the problem in the right form. Least squares can be referred to as root mean square, because this method squares the sum and divides by the number of points and then takes the square root of the result. This becomes obvious in the second video link below.

Basically, you are trying to evaluate all the experimental points and then determine how far away they are from the ideal function that you want to fit, and then choose variables so that the error is the smallest. In this picture, you see three points and a line of best fit. A square is drawn of the error distance of each point from the line. Least squares will minimize the area in these boxes.



There are two things you can vary here to do that, the slope of the line and the Y intercept point.

The formula for a line is y=mx+b, where m is the slope and b is the point on the Y axis that the line goes through.

There is a good video here:
https://www.youtube.com/watch?v=YwZYSTQs-Hk

One thing to remember from calculus is that to find a minimum (or maximum), you take the derivative and set it equal to zero. Since there are two variables we are looking for, we will need to do partial derivatives.

Here is another video:
https://www.youtube.com/watch?v=3hz6Tb1i2FY

We will not go through the method here, because we are looking for a circle best fit, but the videos explain the line best fit described here.

Finding the best circle to fit a set of points

Many thanks to Randy Bullock (bullock@ucar.edu) who supplied the derivation used here.

Given a finite set of points we want to find the circle and radius that best fits the points. We will define x-bar as the average of all x coordinates and y-bar as the average of all y coordinates that define the circle. (for N points, which in our case is 7)

$$\overline{x} = \frac{1}{N}\sum_i x_i \quad \text{and} \quad \overline{y} = \frac{1}{N}\sum_i y_i$$

We also will use a transform to make things easier. We define 'u' and 'v' below for each point.

$$\text{let } u_i = x_i - \overline{x}, \quad v_i = y_i - \overline{y}$$

We'll solve using (u, v) coordinates and then convert back to (x, y).

We'll use the Moon's circumference points to go through the derivation. **We define the center of the best fit circle as $(\mathbf{u_c}, \mathbf{v_c})$ and the radius as R.** As mentioned in the previous section, we want to minimize the Error (**S**) between the circumference and the points measured. We will define a function below that will evaluate to zero...it is the equation for a circle, with all variables on one side, so function g should be zero for a true circle. (Here, $\alpha = R^2$)

$$g(u,v) = (u - u_c)^2 + (v - v_c)^2 - \alpha$$

We want to minimize the error S, so we square this function, sum up all the point components and find the minimum, because we are using root-mean-square method.

$$S = \sum_{i=1}^{N} (g(u_i, v_i))^2$$

There are three values to be determined (u_c, v_c, α) so we need to use partial differentiation on all three and set the results equal to zero to find the minimum. Using the chain rule, varying the radius:

$$\frac{\partial S}{\partial \alpha} = 2\sum_{i}^{N} g(u_i, v_i)\, \frac{\partial g}{\partial \alpha}(u_i\,, v_i)$$

$$= -2\sum_{i}^{N} g(u_i\,, v_i)$$

So, in order for this to be zero:

$\sum_i^N g(u_i\,, v_i) = 0$ **Equation 1**

Let's move on to the partial differentials for u and v for the circle center coordinates:

$$\frac{\partial S}{\partial u_c} = 2\sum_{i}^{N} g(u_i, v_i)\, \frac{\partial g}{\partial u}(u_i\,, v_i)$$

$$= 2\sum_{i}^{N} g(u_i\,, v_i) 2(u_i - u_c)\,(-1)$$

$$= -4\sum_{i}^{N} (u_i - u_c)\, g(u_i\,, v_i)$$

$$= -4\sum_{i}^{N} u_i\, g(u_i\,, v_i) + 4u_c \sum_{i}^{N} g(u_i\,, v_i)$$

Since $\sum_i^N g(u_i, v_i)$=0 (First equation), $\frac{\partial S}{\partial u_c} = 0$ if:

$\sum_i^N u_i\, g(u_i\,, v_i) = 0$ **Equation 2**

A similar method is used to obtain the partial differential for v_c:

$\sum_i^N v_i\, g(u_i\,, v_i) = 0$ **Equation 3**

Let's expand equation 2:

$$\sum_{i}^{N} u_i\, [(u_i - u_c)^2 + (v_i - v_c)^2 - \alpha] = 0$$

$$\sum_{i}^{N} u_i\, \left[u_i^2 - 2u_i u_c + u_c^{\,2} + v_i^2 - 2v_i v_c + v_c^2 - \alpha\right] = 0$$

$$\frac{1}{2}\sum_{i}^{N} u_i^3 + u_i u_c^2 + v_i^2 u_i + v_c^2 u_i - \alpha u_i = \sum_{i}^{N} (u_i^2 u_c + u_i v_i v_c)$$

Remember our definition for $u_i = x_i + \bar{x}$. Since we are summing differences from each point to the average, the sum of all of these has to be zero. Any term above that has just $\sum u_i$ or $\sum u_i$ multiplied by a constant has to go to zero and can be removed.

$$\sum_{i}^{N}(u_i^2 u_c + u_i v_i v_c) = \frac{1}{2}\sum_{i}^{N}(u_i^3 + v_i^2 u_i)$$

The same goes for v_i in equation 3.

$$\sum_{i}^{N}(v_i^2 v_c + u_i v_i u_c) = \frac{1}{2}\sum_{i}^{N}(v_i^3 + u_i^2 v_i)$$

These are simultaneous equations that can be solved for (u_c, v_c). To simplify the solution, let $S_u = \sum_i u_i,\ S_{uu} = \sum_i u_i^2$, etc., then:

$$u_c S_{uu} + v_c S_{uv} = \frac{1}{2}(S_{uuu} + S_{uvv})$$

$$u_c S_{uv} + v_c S_{vv} = \frac{1}{2}(S_{vvv} + S_{vuu})$$

To find the radius R, expand equation one.

$$\sum_{i}^{N}\left[u_i^2 - 2u_i u_c + u_c^2 + v_i^2 - 2v_i v_c + v_c^2 - \alpha\right] = 0$$

Remember that $S_u = S_v = 0$, and substitute/reduce the above equation:

$$N(u_c^2 + v_c^2 - \alpha) + S_{uu} + S_{vv} = 0$$

$$\alpha = u_c^2 + v_c^2 + \frac{S_{uu} + S_{vv}}{N}$$

Remember also that $R = \sqrt{\alpha}$.

We now have enough information to calculate the ratio of the Moon's radius to the Earth's Shadow's radius, so let's plug in some numbers and see what we get.

These point location (coordinates) are based on pixel position in the image of the moon with Earth's shadow displayed earlier in this document. As well, u and v have been calculated along with the averages of the x and y points.

Excel spreadsheet available on request from steve@baselines.com

Moon														
i	x	y	u	v	S_{uu}	S_{vv}	S_{uuu}	S_{vvv}	S_{uv}	S_{uvv}	S_{vuu}	$S_{uuu}+S_{uvv}$	$S_{vvv}+S_{vuu}$	$S_{uu}+S_{vv}$
1	204	3,488	-1,921	-946	3,688,759	895,016	-7,084,683,682	-846,732,452	1,817,003	-1,718,980,532	-3,489,761,213			
2	671	4,212	-1,454	-223	2,113,198	49,512	-3,071,922,729	-11,017,050	323,463	-71,974,775	-470,213,725			
3	1,175	4,632	-949	198	900,400	39,262	-854,384,762	7,779,711	-188,021	-37,255,763	178,411,760			
4	1,899	4,953	-225	519	50,785	269,379	-11,444,517	139,812,212	-116,963	-60,705,666	26,358,054			
5	2,808	4,969	683	535	466,866	285,869	318,997,996	152,844,737	365,325	195,327,313	249,617,751			
6	3,712	4,666	1,588	232	2,520,778	53,730	4,002,227,921	12,454,441	368,023	85,306,756	584,309,064			
7	4,402	4,119	2,278	-315	5,187,351	99,265	11,814,584,965	-31,274,590	-717,580	226,082,651	-1,634,341,669			
AVG	2,124	4,434												
SUMS			0	0	14,928,137	1,692,033	5,113,375,192	-576,132,992	1,851,252	-1,382,200,014	-4,555,619,979	3,731,175,178	-5,131,752,970	16,620,169
Earth's Shadow														
i	x	y	u	v	S_{uu}	S_{vv}	S_{uuu}	S_{vvv}	S_{uv}	S_{uvv}	S_{vuu}			
1	439	2,319	-2,052	99	4,209,895	9,860	-8,637,874,517	979,147	-203,744	-20,231,782	418,042,570			
2	1,259	2,411	-1,232	192	1,516,649	36,806	-1,867,787,353	7,061,312	-236,268	-45,327,951	290,969,024			
3	1,882	2,428	-609	209	370,811	43,543	-225,802,955	9,086,153	-127,068	-26,515,302	77,377,216			
4	2,564	2,323	73	104	5,260	10,714	381,506	1,109,039	7,507	777,079	544,482			
5	3,200	2,200	709	-20	502,705	381	356,426,682	-7,426	-13,833	269,880	-9,807,781			
6	3,762	2,041	1,271	-178	1,616,502	31,802	2,055,247,784	-5,671,177	-226,732	40,433,085	-288,270,722			
7	4,330	1,814	1,839	-405	3,383,051	164,422	6,222,469,438	-66,671,534	-745,821	302,422,817	-1,371,793,255			
AVG	2,491	2,220												
SUMS			0	0	11,604,873	297,529	-2,096,939,415	-54,114,486	-1,545,958	251,827,828	-882,938,466	-1,845,111,587	-937,052,952	11,902,401

Solving the simultaneous equations (https://www.youtube.com/watch?v=7sqDS-PvGEI)

$$u_c S_{uu} + v_c S_{uv} = \frac{1}{2}(S_{uuu} + S_{uvv})$$

$$u_c S_{uv} + v_c S_{vv} = \frac{1}{2}(S_{vvv} + S_{vuu})$$

$$\alpha = u_c^2 + v_c^2 + \frac{S_{uu} + S_{vv}}{N}$$

MOON		u	v	VALUE
	Eq1	14,928,137	1,851,252	1,865,587,589
	Eq2	1,851,252	1,692,033	-2,565,876,485
	xbar	2,124	u-center	362
	ybar	4,434	v-center	-1,913
	Transform uv to xy coordinates		x-center	2,486
			y-center	2,522
			R=	2,483

$$u_c S_{uu} + v_c S_{uv} = \frac{1}{2}(S_{uuu} + S_{uvv})$$

$$u_c S_{uv} + v_c S_{vv} = \frac{1}{2}(S_{vvv} + S_{vuu})$$

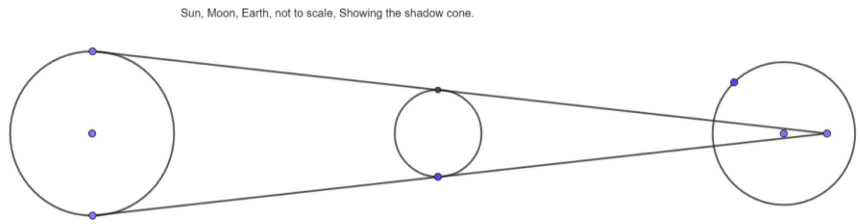
$$\alpha = u_c^2 + v_c^2 + \frac{S_{uu} + S_{vv}}{N}$$

EARTH		u	v	VALUE
	Eq1	11,604,873	-1,545,958	-922,555,794
	Eq2	-1,545,958	297,529	-468,526,476
	xbar	2,491	u=	-940
	ybar	2,220	v=	-6,458
	Transform uv to xy coordinates		x-center	1,551
			y-center	-4,238
			R=	6,655

Ratio of Earth Shadow to Moon (Radius)	2.68

Given this result, we know that the Earth's shadow radius is about 2.7 times as big as the Moon's radius. We also know that since the shadow falls on the moon, that the moon is closer than 108 Earth diameters (108* 12,700KM = 1,371,600KM). Now let's work on determining a more accurate distance from the Earth to the Moon.

Note: We make an assumption that the shadow cone that the Moon makes on the Earth during a solar eclipse comes to a point at the center of the Earth. This is a good approximation. Considering that the observed shadow diameter on the Earth during the solar eclipse is about 110KM, the cone point is actually about 5,500KM past the center, which will add an error of only .014%, negligible in our current methods of determining distances.



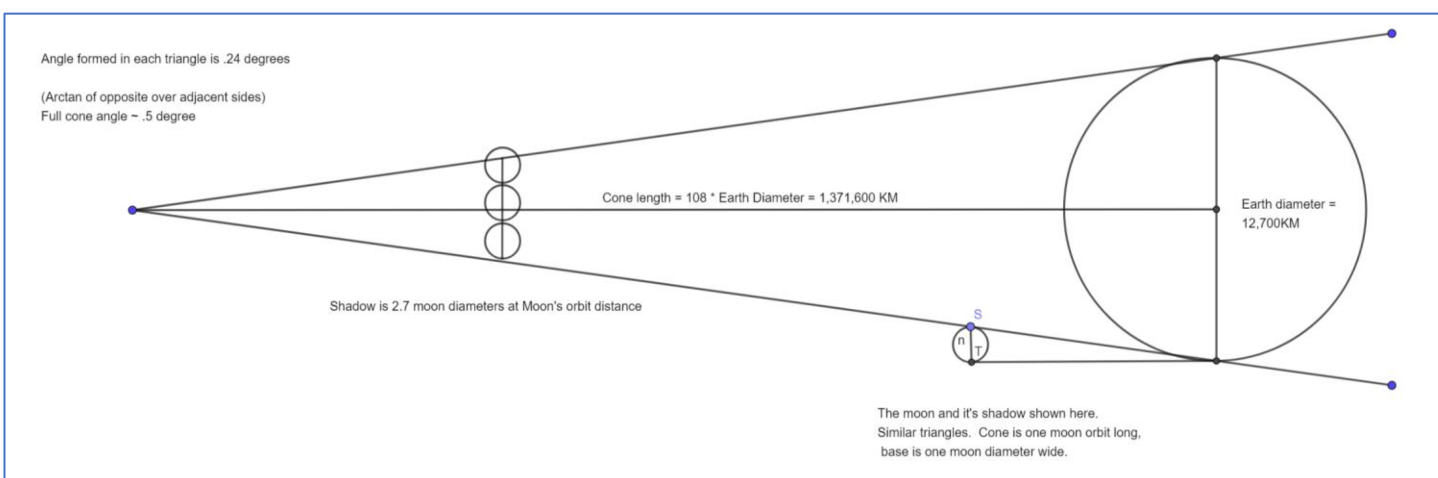
Remember how we mentioned similar triangles at the beginning of this document? Well here is where they come in handy. Below is a drawing (not to scale) showing the Earth, the Earth's shadow cone and the Moon's path through the shadow (showing 3 Moons to indicate the 2.7 shadow size). An inset also shows the Moon and the shadow cone it would make if the sun was shining on it from the left (it's the small triangle at the bottom, not to scale of the rest of the picture).

The drawing shows three triangles. The first and largest is the full shadow cone the Earth throws on the moon (and past). The second shows the part of the cone from where the Moon orbits the Earth to the tip of the Earth's shadow cone (with 2.7 Moon diameter base). The third is an insert of the Moon and the shadow cone it would throw.

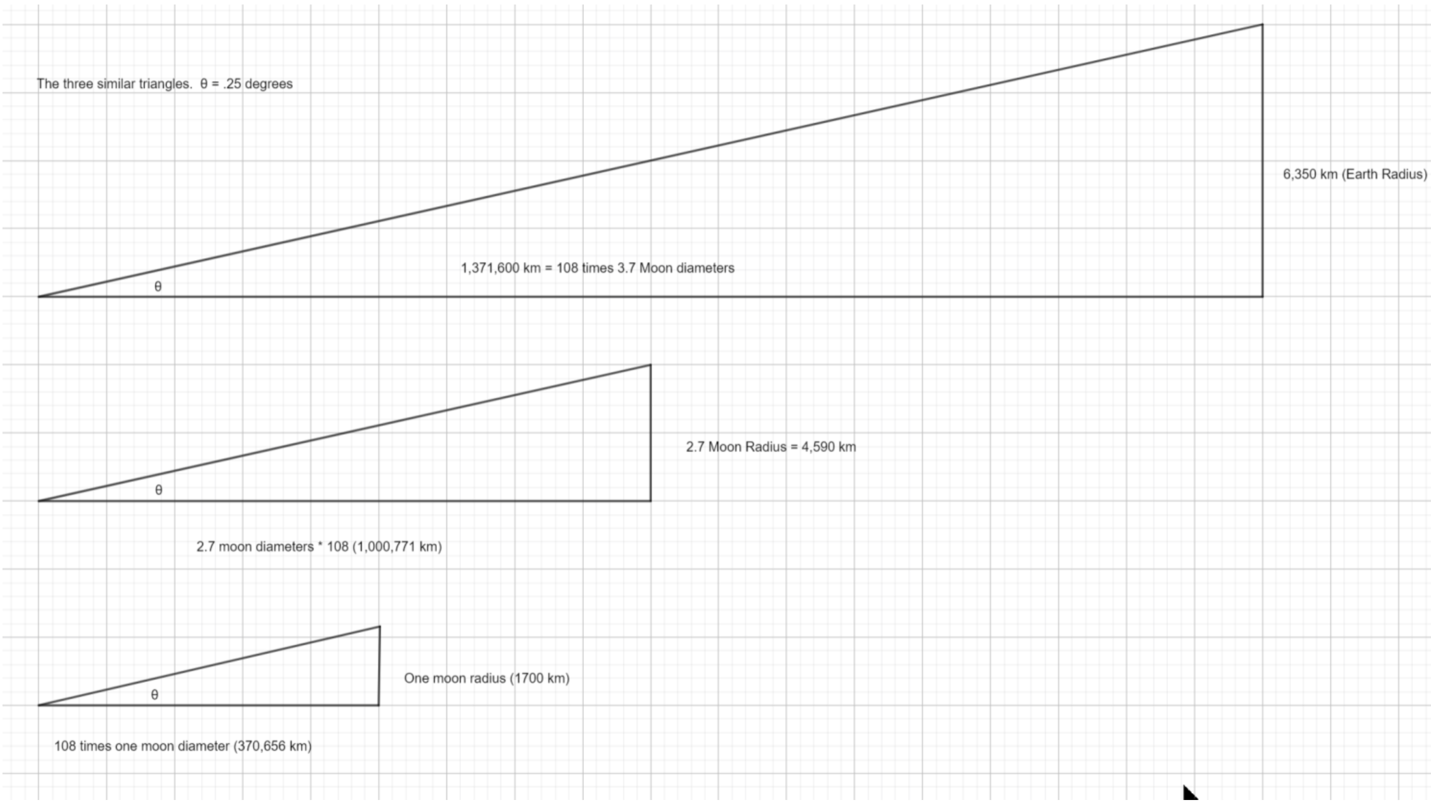
It is obvious that the two larger triangles are similar, because the medium size triangle has the same three angles as the larger triangle.

Since the Moon and the Sun appear the same size in the sky to an observer on Earth, that means they take up the same angular size in the sky, which we determined was .5 degrees (tangent of half diameter of Earth over height of cone * 2). This means that this small triangle is also similar to the two larger triangles.

The smaller triangle has a base of one moon diameter and a length of 108 times one moon diameter. By proportion for similar triangles, that means the middle triangle which has a base of 2.7 Moon diameters has a height of 108 times 2.7 moon diameters. We also now know that the overall length of the large cone is 108 times 3.7 Moon diameters. (2.7 for the top cone, and 1 for the rest of the large cone since the Moon is one times 108 Moon diameters away from the Earth.



This gives us the ability to calculate the actual distance of the Moon from the Earth.



D_m is the diameter of the Moon. Units are km.

$$108 * 3.7 * D_m = 1{,}371{,}600\ km$$

D_m= 3432.4 km (Real value we know today is 3,474)

L_m is the distance to the moon.

$$L_m = 1{,}371{,}600 - 1{,}000{,}771 = 370{,}829\ km$$

$L_m = 370{,}829$ KM which is 230,422 miles (Today's known value is 238,900 miles)

Well … we've come a LONG way. We now know with a fair amount of accuracy:

The diameter of the Earth (12,700 km)
The diameter of the Moon (3,400 km)
The distance of the Moon from the Earth (370,829 km)

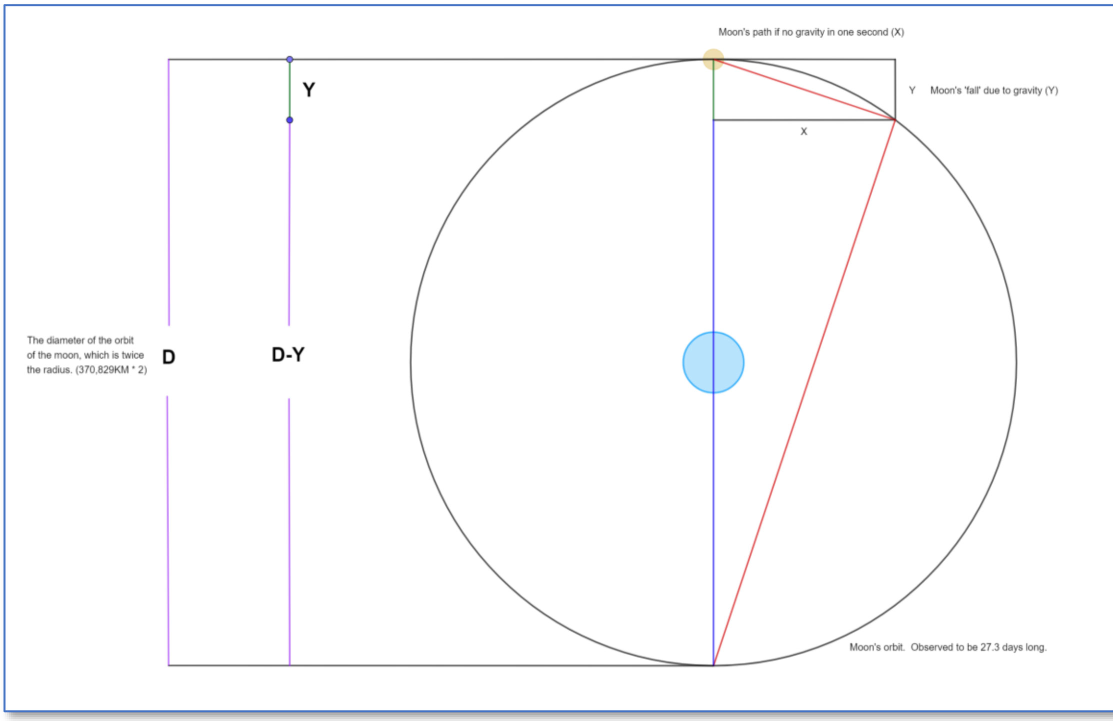
Along with many other factors of our investigation.

By observation, we also know that the Moon revolves around the Earth in 27.3 days.

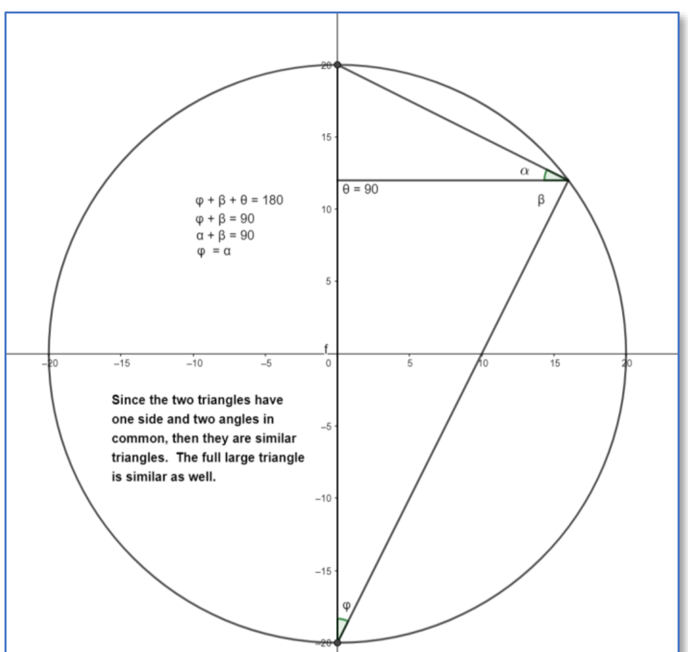
Since we now know the distance to the moon, we can calculate how fast it is revolving. The total path is the circumference, 2*pi*L_m

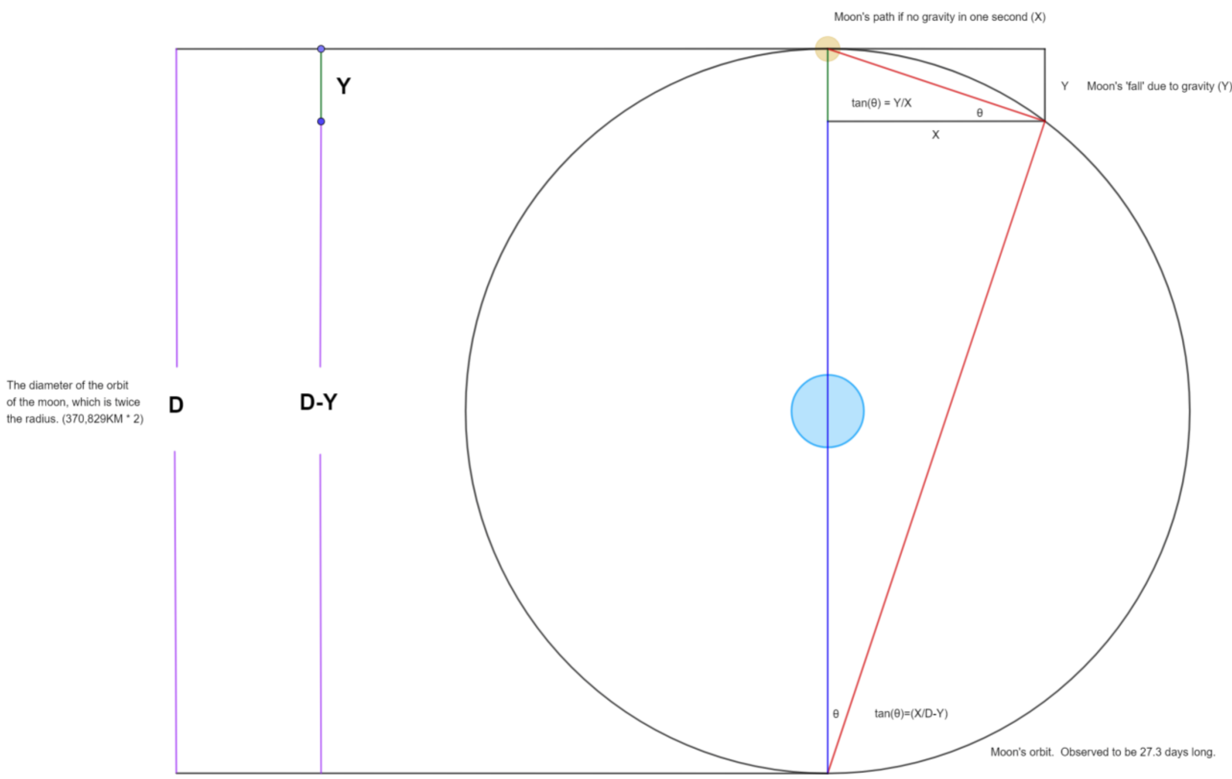
$$V_m = \frac{2\pi L_m}{27.32} = 85,285 \text{ km/Day} = 3{,}554\text{km/Hour} = 59\text{km/Minute} = .987\text{km/second}$$

This diagram below shows the blue Earth and the yellow Moon's orbit around the Earth. We assume it is round, but it is really elliptical, so we will add a small error here. The path with length X shows how the moon would move in one second if there were no gravity pulling on it. The path Y shows the amount of fall due to gravity.



We know the Moon moves .987km in one second, so that is our value of X. We have already proven that any triangle inscribed in a circle will be a right triangle, so we know that the large triangle is a right triangle and the smaller (XY) triangle is drawn as a right triangle. In order to show that the triangles are similar, please review the following graphic.





Shown here are the two tangent functions. Since the angle is the same, the arguments are equal.

$$\frac{Y}{X} = \frac{X}{D - Y}$$

Here, Y is negligibly small compared to D, so we can rewrite this equation to get an good approximation.

$$\frac{Y}{X} = \frac{X}{D}$$

$$Y = \frac{X^2}{D}$$

$$Y = \frac{987\hat{}2}{2 * 370{,}829{,}000} = 0.00131\ meters \text{ (about .05 inches)}$$

Wrapup

We have covered a lot of ground here. We proved statements about inscribed and similar triangles, derived geometrically the Pythagorean Theorem, derived the formula for circumference of a circle, calculated the circumference of the Earth, demonstrated the least squares method of curve and line fitting, determined the diameter and distance of the Moon from the Earth and found out how far away the moon is from the earth.

This information allowed us to arrive at an answer to our original question. **The Moon falls about 1/20th of an inch every second due to the gravitational pull on it from the Earth**.

If you notice mistakes or the need for more clarity, email me at steve@baselines.com.