Explorations for the First Week of Calculus

By Paul A. Foerster, foerster@idworld.net

Here are four Explorations (including solutions) that I wrote to give my students a taste of the four concepts of calculus (**limits, derivatives, integrals, integrals**) during the first week of the course. The concepts are developed graphically, numerically, and verbally, using accurate graphs on paper, verified by graphing calculator. You are welcome to use these Explorations in various ways (including cooperative groups or flipped classroom) for direct teaching of students in your school. Please do not use them in any way that would infringe on the copyrights**.**

- **Exploration 1-1a: Instantaneous Rate of Change of a Function (first day of calculus)**
  Students investigate the motion of a door swinging open and closed, first by sketching a reasonable graph of number of degrees, $d(t)$, the door is open, versus number of seconds, t, since it was pushed. Then they investigate the function $d(t) = 200t(2^{-t})$ that has a graph similar to the one they have drawn. They estimate the "instantaneous rate" at $t = 1$ by calculating average rates over smaller and smaller time intervals about $t = 1$. They learn that a ***derivative*** is the ***limi****t* of the average rates of change as the time interval approaches zero. They can go home on Day 1 and say, "Hey Mom and Dad! I did *calculus* today!"
- **Exploration 1-2a: Graphs of Functions (Day 2)**
  Students get a brief review of graphs of different kinds of functions from precalculus.
- **Exploration 1-3a: Introduction to Definite Integrals (Day 3)**
  Students investigate the motion of a car as it slows down after passing a truck then continues at a constant velocity. They observe that when the velocity is constant, (distance) = (velocity)(time), and thus the distance traveled is equal to the area of a region under the velocity-time graph. From a given accurately plotted graph on paper (whose equation is not given), they estimate the distance the car goes while slowing down by counting squares under the curved portion of the graph. They learn on Day 3 that a ***definite integral*** gives a way of multiplying the y-variable by the x-variable if y varies with x. (Indefinite integrals and the fundamental theorem come later, after they learn and apply derivative formulas.)
- **Exploration 1-4a: Definite Integrals by the Trapezoidal Rule (Days 4 and 5)**
  Students investigate the motion of a hypothetical spaceship as it speeds up and slows down, first by counting squares under the given accurately-drawn velocity-time graph as in Exploration 1-3a. Then they are given the equation of the graph, divide the region under the graph into four trapezoids, and calculate by hand the sum of the areas. Next they download a program to find the sum of the areas for n trapezoids. They observe numerically that the ***definite integral*** is equal to the ***limit*** of the ***sum*** of the areas as n approaches infinity.

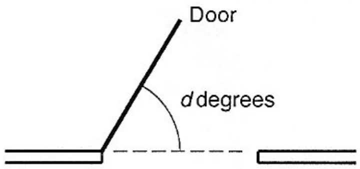
The Explorations accompany my text *Calculus: Concepts and Applications,* originally published by Key Curriculum Press and now with Kendall Hunt. For a 30-day free trial of the text, including the 150+ Explorations, go to www.flourishkh.com, click on "Preview Now," under "Mathematics," click on "Calculus," then click on "Request a trial."

Paul A. Foerster, Teacher Emeritus of Mathematics
Alamo Heights High School, San Antonio

Name: ________________ Group Members: ____

Foerster Calculus Exploration 1-1a: Instantaneous Rate of Change of a Function Date: ____________

Objective: Explore the instantaneous rate of change of a function.



The diagram shows a door with an automatic closer. At time $t = 0$ seconds someone pushes the door. It swings open, slows down, stops, starts closing, then slams shut at time $t = 7$ seconds. As the door is in motion the number of degrees, d, it is from its closed position depends on t.

1. Sketch a reasonable graph of d versus t.

2. Suppose that d is given by the equation

   $d = 200t \cdot 2^{-t}.$

   Plot this graph on your grapher. Sketch the results here.

3. Make a table of values of d for each second from $t = 0$ through $t = 10$. Round to the nearest 0.1°.

t	d
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

4. At time $t = 1$ second, does the door appear to be opening or closing? How do you tell?

5. What is the average rate at which the door is moving for the time interval [1, 1.1]? Based on your answer, does the door seem to be opening or closing at time $t = 1$? Explain.

6. By finding average rates for time intervals [1, 1.01], [1, 1.001], and so on, make a conjecture about the *instantaneous* rate at which the door is moving at time $t = 1$ second.

7. Read Section 1-1. What do you notice?!

8. In calculus you will learn by four methods:
   - algebraically,
   - numerically,
   - graphically,
   - verbally (talking and writing).

   Write a paragraph telling what you have learned as a result of doing this Exploration that you did not know before. Use the back of this sheet.

Name ____________________ Group ____________________ Date ____________________

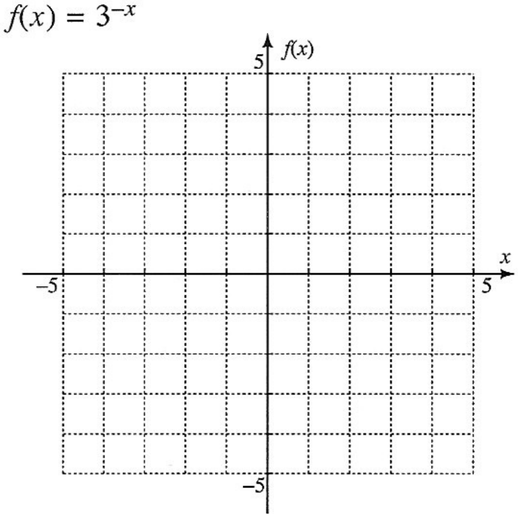
Exploration 1-2a: Graphs of Functions

Objective: Recall the graphs of familiar functions, and tell how fast the function is changing at a particular value of x.

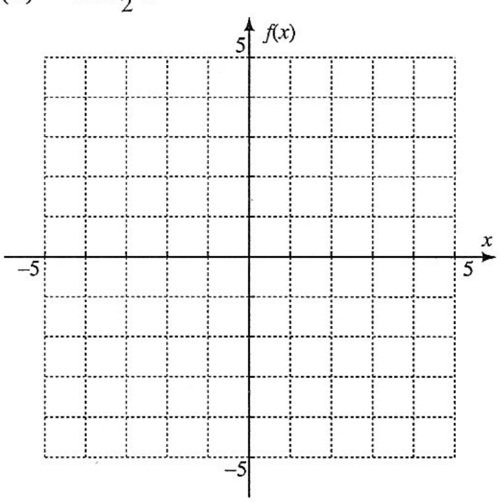
For each function:

a. Without using your grapher, sketch the graph on the axes provided.
b. Confirm by grapher that your sketch is correct.
c. Tell whether the function is increasing, decreasing, or not changing when $x = 1$. If it is increasing or decreasing, tell whether the rate of change is slow or fast.

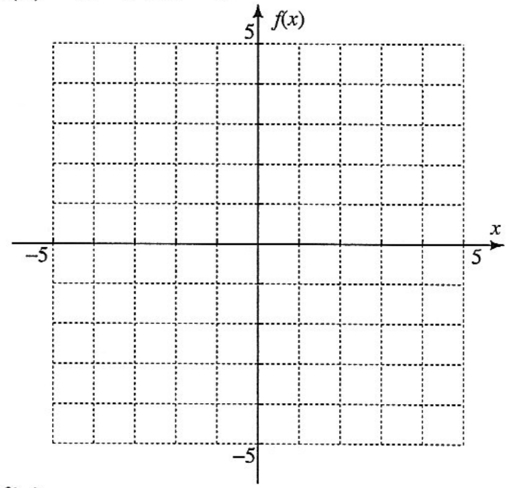
1. $f(x) = 3^{-x}$



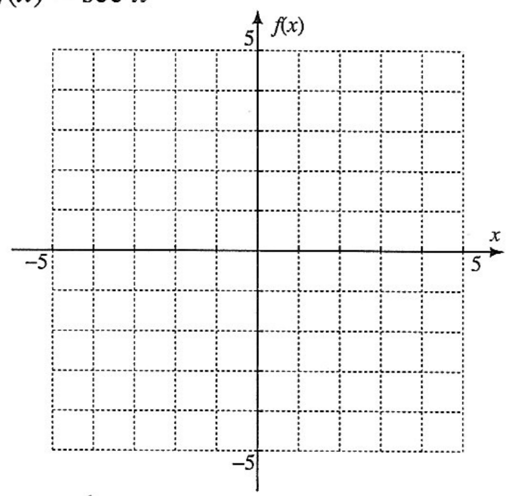
2. $f(x) = \sin \frac{\pi}{2} x$



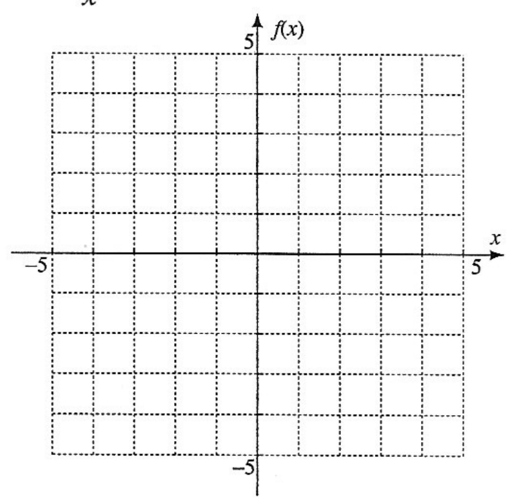
3. $f(x) = x^2 + 2x - 2$



4. $f(x) = \sec x$



5. $f(x) = \dfrac{1}{x}$

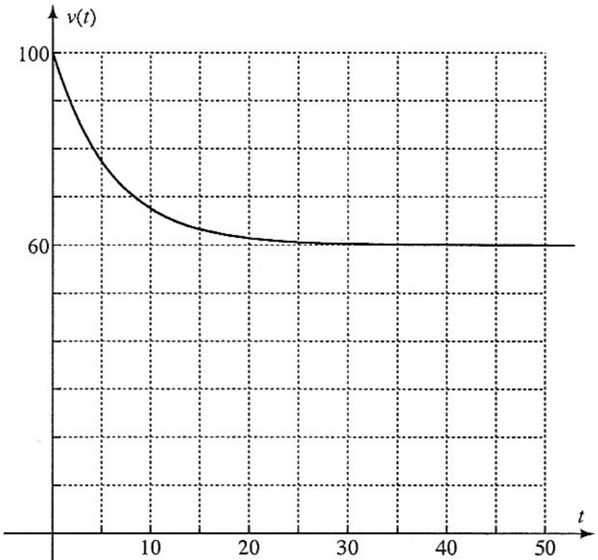


Name: ________________ Group Members: ________________________________

Foerster Calculus — Exploration 1-3a: Introduction to Definite Integrals — Date: ____________

Objective: Find out what a definite integral is by working a real-world problem concerning speed of a car.

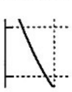
As you drive on the highway you accelerate to 100 feet per second to pass a truck. After you have passed, you slow down to a more moderate 60 ft/sec. The diagram shows the graph of your velocity, $v(t)$, as a function of the number of seconds, t, since you started slowing.



1. What does your velocity seem to be between $t = 30$ and $t = 50$ seconds? How far do you travel in the time interval $[30, 50]$?

2. Explain why the answer to Problem 1 can be represented as the area of a *rectangular* region of the graph. Shade this region.

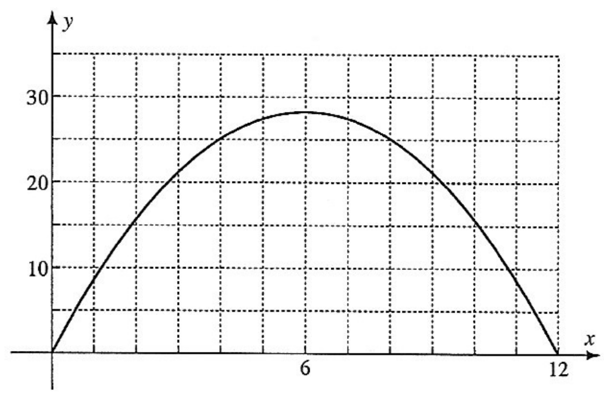
3. The distance you travel between $t = 0$ and $t = 20$ can also be represented as the area of a region bounded by the (curved) graph. Count the number of squares in this region. Estimate the area of parts of squares to the nearest 0.1 square space. For instance, how would you count this partial square?



4. How many feet does each small square on the graph represent? How far, therefore, did you go in the time interval $[0, 20]$?

5. Problems 3 and 4 involve finding the product of the x-value and the y-value for a function where y may *vary* with x. Such a product is called the **definite integral** of y with respect to x. Based on the units of t and $v(t)$, explain why the definite integral of $v(t)$ with respect to t in Problem 4 has feet for its units.

6. The graph shows the cross-sectional area, y square inches, of a football as a function of the distance, x inches, from one of its ends. Estimate the definite integral of y with respect to x.



7. What are the units of the definite integral in Problem 6? What, therefore, do you suppose the definite integral represents?

8. What have you learned as a result of doing this Exploration that you did not know before? (Over.)

Name ______________________ Group ______________________ Date ______________________

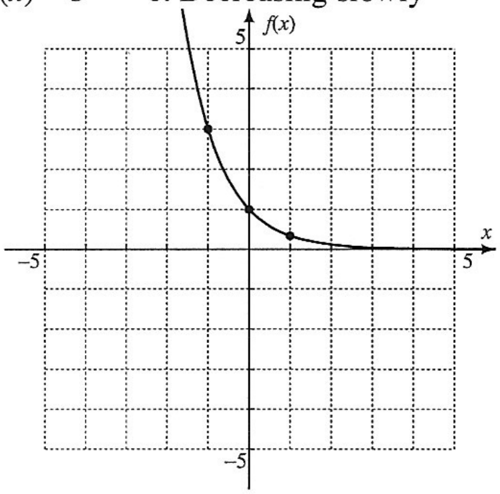
Solutions, Exploration 1-2a: Graphs of Functions

Objective: Recall the graphs of familiar functions, and tell how fast the function is changing at a particular value of x.

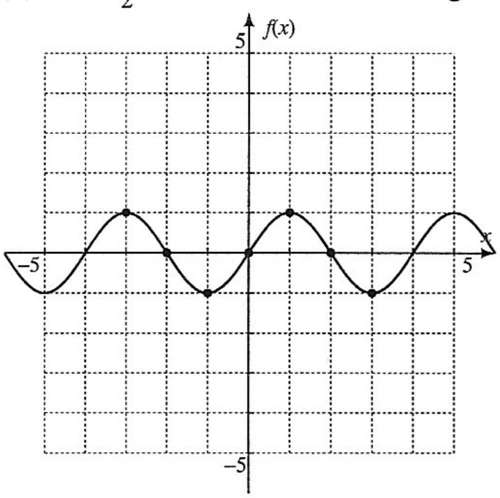
For each function:

a. Without using your grapher, sketch the graph on the axes provided.
b. Confirm by grapher that your sketch is correct.
c. Tell whether the function is increasing, decreasing, or not changing when $x = 1$. If it is increasing or decreasing, tell whether the rate of change is slow or fast.

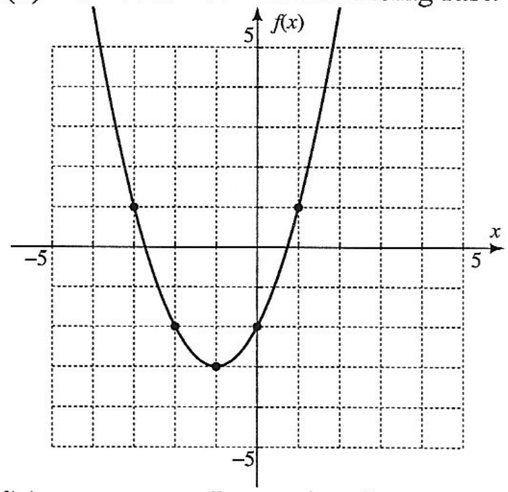
1. $f(x) = 3^{-x}$ • c. Decreasing slowly



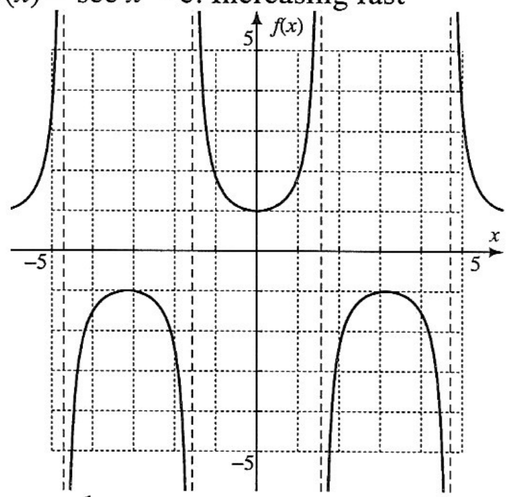
2. $f(x) = \sin \frac{\pi}{2} x$ • c. Neither increasing nor decreasing.



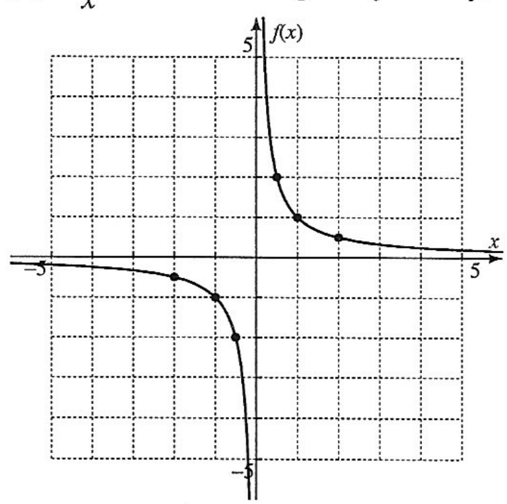
3. $f(x) = x^2 + 2x - 2$ • c. Increasing fast.



4. $f(x) = \sec x$ • c. Increasing fast



5. $f(x) = \dfrac{1}{x}$ • c. Decreasing fairly slowly.

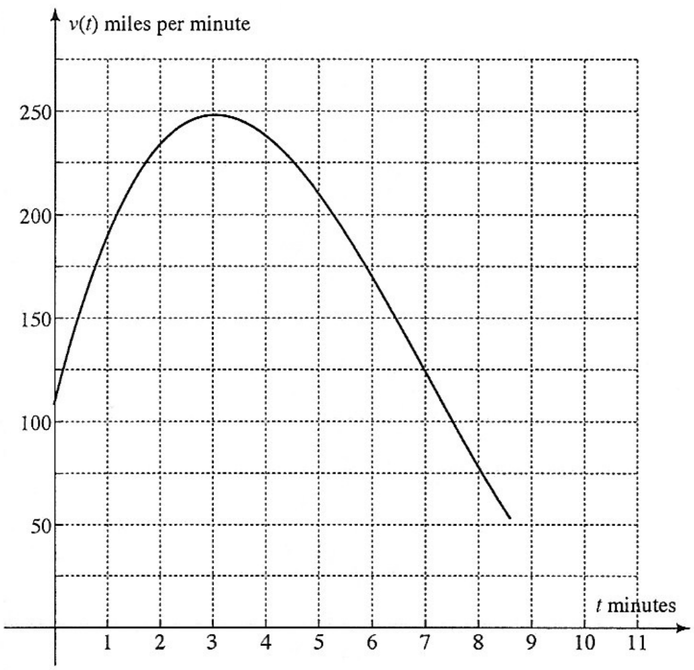


Name: ________________ Group Members: ____

Foerster Calculus Exploration 1-4a: Definite Integrals by Trapezoidal Rule Date: ____________

Objective: Estimate the definite integral of a function numerically instead of graphically by counting squares.

Rocket Problem: Ella Vader (Darth's daughter) is driving in her rocket ship. At time $t = 0$ minutes she fires her rocket engine. The ship speeds up for awhile, then slows down as Alderaan's gravity takes its effect. The graph of her velocity, $v(t)$ miles per minute, is shown below.



1. What mathematical concept would be used to estimate the distance Ella goes between $t = 0$ and $t = 8$?

2. Estimate the distance in Problem 1 geometrically.

3. Ella figures that her velocity is given by

$$v(t) = t^3 - 21t^2 + 100t + 110.$$

Plot this graph on your grapher. Does the graph confirm or refute what Ella figures? Tell how you arrive at your conclusion.

4. Divide the region under the graph from $t = 0$ to $t = 8$, which represents the distance, into four strips of equal width. Draw four trapezoids whose areas approximate the areas of these strips, and whose parallel sides go from the x-axis to the graph. By finding the areas of these trapezoids, estimate the distance Ella goes. Does the answer agree with Problem 2?

5. The technique in Problem 4 is the **trapezoidal rule**. Put a program into your grapher to use this rule. The function equation may be stored as y_1. The input should be the starting time, the ending time, and the number of trapezoids. The output should be the value of the definite integral. Test your program by using it to answer Problem 4.

6. Use the program from Problem 5 to estimate the definite integral using 20 trapezoids.

7. The *exact* value of the definite integral is the *limit* of the estimates by trapezoids as the width of each trapezoid approaches zero. By using the program from Problem 5 make a conjecture about the exact value of the definite integral.

8. What is the fastest Ella went? At what time was that?

9. Approximately what was Ella's rate of change of velocity when $t = 5$? Was she speeding up or slowing down at that time?

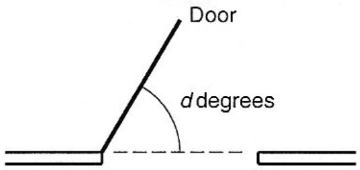
10. At what time does Ella stop? Based on the graph, does she stop abruptly or gradually?

11. What have you learned as a result of doing this Exploration that you did not know before? (Over.)

Name: ________________ Group Members: ____

Foerster Calculus Solutions, Exploration 1-1a: Instantaneous Rate of Change of a Function Date: ________

Objective: Explore the instantaneous rate of change of a function.



The diagram shows a door with an automatic closer. At time $t = 0$ seconds someone pushes the door. It swings open, slows down, stops, starts closing, then slams shut at time $t = 7$ seconds. As the door is in motion the number of degrees, d, it is from its closed position depends on t.

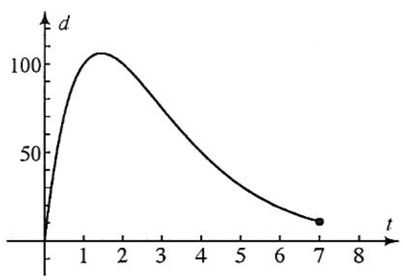
1. Sketch a reasonable graph of d versus t.

   • Any graph is reasonable that starts at the origin, rises, reaches a maximum, drops toward zero degrees, and stops at $t = 7$ seconds.

2. Suppose that d is given by the equation

   $d = 200t \cdot 2^{-t}$.

   Plot this graph on your grapher. Sketch the results here.



3. Make a table of values of d for each second from $t = 0$ through $t = 10$. Round to the nearest 0.1°.

t	d
0	0
1	100
2	100
3	75
4	50
5	31.25
6	18.75
7	10.93...
8	6.25
9	3.51...
10	1.95...

4. At time $t = 1$ second, does the door appear to be opening or closing? How do you tell?

   • Opening.

   • The graph is still increasing at $t = 1$.

5. What is the average rate at which the door is moving for the time interval [1, 1.1]? Based on your answer, does the door seem to be opening or closing at time $t = 1$? Explain.

   • $d() = 100$ and $d(1.1) = 102.63362...$

   • Door opened by 2.63362... degrees in 0.1 s.

   • Av. rate $= \dfrac{2.63362...}{0.1} = 26.3362... \approx 26.34°/s$

6. By finding average rates for time intervals [1, 1.01], [1, 1.001], and so on, make a conjecture about the *instantaneous* rate at which the door is moving at time $t = 1$ second.

   • [1, 1.01]: 30.2342... °/s

   • [1, 1.001]: 30.6400... °/s

   • [1, 1.0001]: 30.6807... °/s

   • Conjecture: Any number slightly *above* 30.6807... is a reasonable conjecture. Some students will conjecture 31°/s, thinking, "All answers in mathematics are whole numbers." The exact answer is 30.6853...°/s.

7. Read Section 1-1. What do you notice?!

   • This problem is the Example in Section 1-1.

8. In calculus you will learn by four methods:
   • algebraically,
   • numerically,
   • graphically,
   • verbally (talking and writing).
   Write a paragraph telling what you have learned as a result of doing this Exploration that you did not know before. Use the back of this sheet.

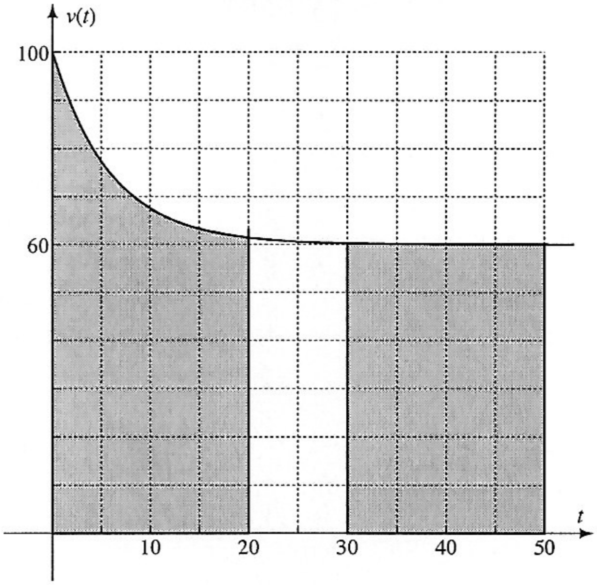
   • Answers will vary.

Name: ________________ Group Members: ____

Foerster Calculus   Solutions, Exploration 1-3a: Introduction to Definite Integrals   Date: ____________

Objective: Find out what a definite integral is by working a real-world problem concerning speed of a car.

As you drive on the highway you accelerate to 100 feet per second to pass a truck. After you have passed, you slow down to a more moderate 60 ft/sec. The diagram shows the graph of your velocity, $v(t)$, as a function of the number of seconds, t, since you started slowing.



1. What does your velocity seem to be between $t = 30$ and $t = 50$ seconds? How far do you travel in the time interval $[30, 50]$?

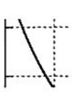
   • From $t = 30$ to $t = 50$ s, the velocity seems to be about 60 ft/s. Distance = rate · time, so the distance traveled is about 60 ft/s·(50 – 30)s = 1200 ft.

2. Explain why the answer to Problem 1 can be represented as the area of a *rectangular* region of the graph. Shade this region.

   • The rectangle on the graph has height = 60 and base from 30 to 50 has area base · height = 1200.

   • See graph above Problem 1.

3. The distance you travel between $t = 0$ and $t = 20$ can also be represented as the area of a region bounded by the (curved) graph. Count the number of squares in this region. Estimate the area of parts of squares to the nearest 0.1 square space. For instance, how would you count this partial square?

    • About 0.7 square.

   • All the squares and partial squares under the graph from $t = 0$ to $t = 20$ have area about 28.6 square spaces.

4. How many feet does each small square on the graph represent? How far, therefore, did you go in the time interval $[0, 20]$?

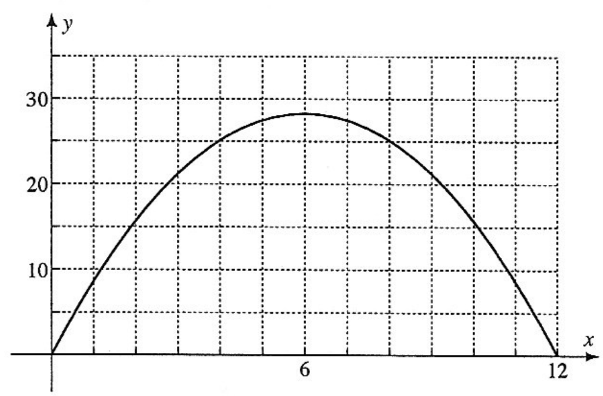
   • Each small square has base representing 5 s and height representing 10 ft/s. So the area of each small square = base · height represents 50 ft. Therefore the distance was about 28.6 · 50 = 1430 ft. (Exact ans. is 1431.3207....)

5. Problems 3 and 4 involve finding the product of the x-value and the y-value for a function where y may *vary* with x. Such a product is called the **definite integral** of y with respect to x. Based on the units of t and $v(t)$, explain why the definite integral of $v(t)$ with respect to t in Problem 4 has units "feet."

   • The x-value is in seconds and the y-value is in feet/second. So their product (i.e., the definite integral) is in

   • seconds $\cdot \dfrac{\text{feet}}{\text{second}}$ = feet

6. The graph shows the cross-sectional area, y square inches, of a football as a function of the distance, x inches, from one of its ends. Estimate the definite integral of y with respect to x.



   • About 45.2 square spaces, each with base 1 and height 5. So each square space represents 5 units of the definite integral. 5 · 45.2 = 226 square units. (Exactly 226.1946...)

7. What are the units of the definite integral in Problem 6? What, therefore, do you suppose the definite integral represents?

   • (x-units)·(y-units) = (in.)·(in.^{2}) = in.^{3}, the *volume* of the football.

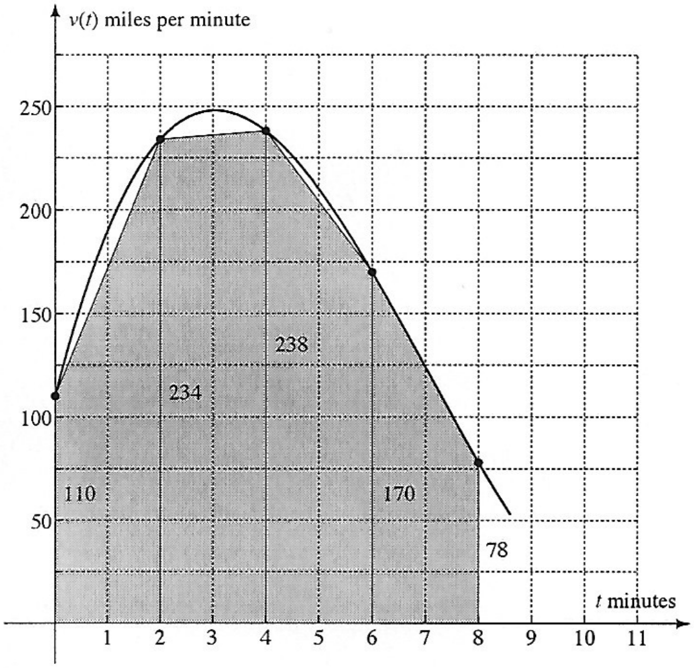
8. What have you learned as a result of doing this Exploration that you did not know before?
   • Answers will vary.

Name: ________________ Group Members: ______________________________________

Foerster Calculus Solutions, Exploration 1-4a: Definite Integrals by Trapezoidal Rule Date: __________

Objective: Estimate the definite integral of a function numerically instead of graphically by counting squares.

Rocket Problem: Ella Vader (Darth's daughter) is driving in her rocket ship. At time $t = 0$ minutes she fires her rocket engine. The ship speeds up for awhile, then slows down as Alderaan's gravity takes its effect. The graph of her velocity, $v(t)$ miles per minute, is shown below.



1. What mathematical concept would be used to estimate the distance Ella goes between $t = 0$ and $t = 8$?

   • Definite integral. (time)(velocity) = distance.

2. Estimate the distance in Problem 1 geometrically.

   • There are about 60.8 squares under the graph, each representing (1)(25) = 25 miles. So the total distance is about (60.8)(25) = 1520 miles.

3. Ella figures that her velocity is given by

   $v(t) = t^3 - 21t^2 + 100t + 110.$

   Plot this graph on your grapher. Does the graph confirm or refute what Ella figures? Tell how you arrive at your conclusion.

   • By trace or by table, the values of $v(t)$ for integer values of x confirm the values shown on the graph.

4. Divide the region under the graph from $t = 0$ to $t = 8$, which represents the distance, into four strips of equal width. Draw four trapezoids whose areas approximate the areas of these strips, and whose parallel sides go from the x-axis to the graph. By finding the areas of these trapezoids, estimate the distance Ella goes. Does the answer agree with Problem 2? • Graph, above Problem 1.

   • Area = 344 + 472 + 408 + 248 = 1472. • Close.

5. The technique in Problem 4 is the **trapezoidal rule**. Put a program into your grapher to use this rule. The function equation may be stored as y_1. The input should be the starting time, the ending time, and the number of trapezoids. The output should be the value of the definite integral. Test your program by using it to answer Problem 4.
   • Answer for $n = 4$ is 1472, which agrees.

6. Use the program from Problem 5 to estimate the definite integral using 20 trapezoids.

   • 20 trapezoids: integral ≈ 1518.08 mi.

7. The *exact* value of the definite integral is the *limit* of the estimates by trapezoids as the width of each trapezoid approaches zero. By using the program from Problem 5 make a conjecture about the exact value of the definite integral.
   • 15519.52 miles for 40 trapezoids
   • 1519.999232 miles for 1000 trapezoids
   • Conjecture: Exact value is 1520 miles.

8. What is the fastest Ella went? At what time was that?

   • By graph, about 248 mi/min at 3 minutes (Exactly 248.0209... at $7 - \sqrt{47/3}$ minutes)

9. Approximately what was Ella's rate of change of velocity when $t = 5$? Was she speeding up or slowing down at that time?

   • $\dfrac{v(5.1) - v(4.9)}{5.1 - 4.9} = -34.99$
   • Slowing at about 35 (mi/min)/min

10. At what time does Ella stop? Based on the graph, does she stop abruptly or gradually?

    • Ella stops at 11 minutes because $v(11) = 0$.
    • The stop is gradual because the graph just touches the horizontal axis, tangent to it.

11. What have you learned as a result of doing this Exploration that you did not know before?
    • Answers will vary.