Exploring Volatility Derivatives: New Advances in Modelling

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1. Volatility Products

Historical Volatility Products

- Historical variance: $\frac{1}{n}\sum_{i=1}^{n} ln(\frac{S_i}{S_{i-1}})^2$
- OTC products:
  - Volatility swap
  - Variance swap
  - Corridor variance swap
  - Options on volatility/variance
  - Volatility swap again
- Listed Products:
  - Futures on realized variance

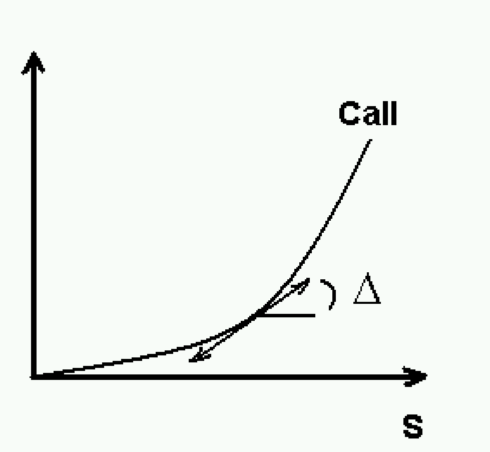
Implied Volatility Products

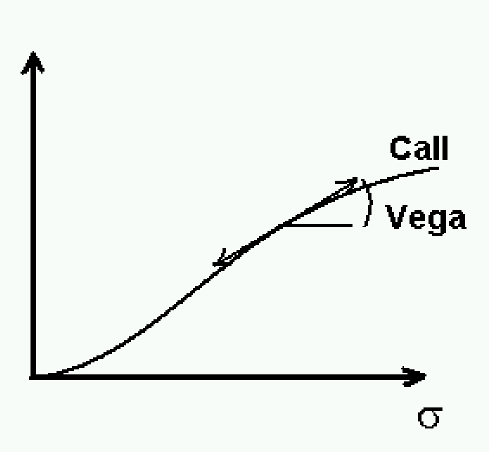
- Definition
  - Implied volatility: input in Black-Scholes formula to recover market price:
  - Old VIX: proxy for ATM implied vol
  - New VIX: proxy for variance swap rate
- OTC products
  - Swaps and options
- Listed products
  - VIX Futures contract
  - Volax

1. Volatility Products: VIX Futures Pricing

Vanilla Options

Simple product, but complex mix of underlying and volatility :

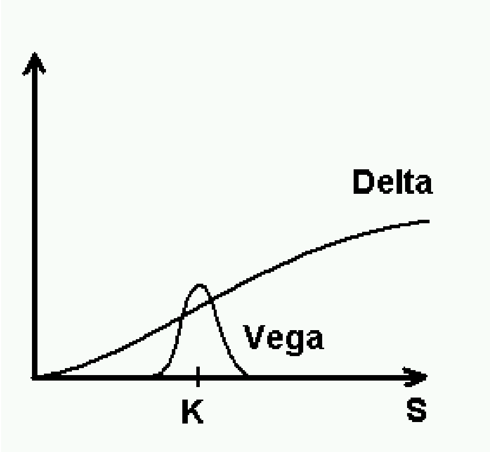




Call option has:

- Sensitivity to S: Δ
- Sensitivity to σ : Vega

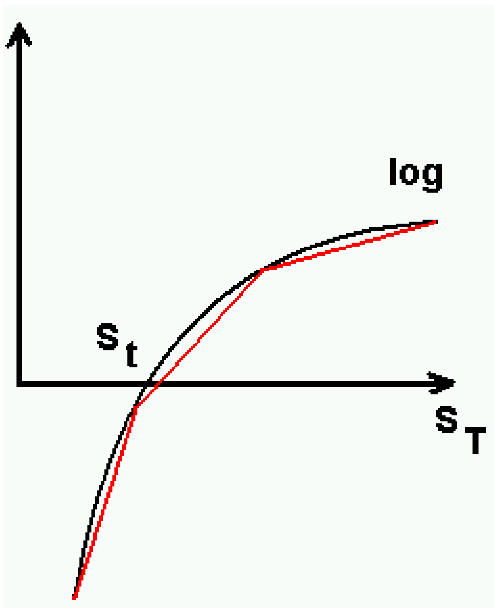
These sensitivities vary through time and spot, and vol:



Volatility Games

To play pure volatility games (eg bet that S&P vol goes up, no view on the S&P itself)

- Need of constant sensitivity to vol
- Achieved by combining several strikes;
- Ideally achieved by a log profile: (variance swaps)



Log Profile

- Under BS: $dS = \sigma S dW$, $\mathbb{E}[ln\frac{S_T}{S_0}] = -\frac{\sigma^2}{2}T$
- For all S,

$$ln\frac{S}{S_0} = \frac{S - S_0}{S_0} - \int_0^{S_0} \frac{(K-S)^+}{K^2}dK - \int_{S_0}^{\infty} \frac{(S-K)^+}{K^2}dK$$

- The log profile is decomposed as:

$$\frac{1}{S_0}\text{Futures} - \int_0^{S_0} \frac{P_{K,T}}{K^2}dK - \int_{S_0}^{\infty} \frac{C_{K,T}}{K^2}dK$$

- In practice, finite number of strikes $\Rightarrow$ CBOE definition:

$$\boxed{VIX_t^2 \equiv \frac{2}{T}\sum \frac{K_{i+1} - K_{i-1}}{2K_i^2} e^{rT} X(K_i, T) - \frac{1}{T}(\frac{F}{K_0} - 1)^2}$$

where X is a Put if $K_i < F$, a Call otherwise

Option prices for one maturity

SPX ↓1095.45 +8.33 1094.97/1095.68 Index OMON

At 16:59 Op 1087.12 Hi 1095.69 Lo 1087.12

Template List | Edit | Contract Months | Security List | SPX Index | Go

Option Monitor: S&P 500 INDEX

Center 1095.45 Number of Strikes 18 -or- % from Center Exchange C (Composite)

CALLS

SPX 22 MAY 04 (Contract Size: 100)

Ticker	Strike	Bid	Ask	Last	Volume
1) SPQ+EH	1040	53.00	55.00		
2) SPQ+EJ	1050	43.80	45.80	47.00	4385
3) SPQ+EL	1060	34.90	36.90	36.00	1
4) SPQ+EM	1065	30.60	32.60	30.10	1
5) SPQ+EN	1070	26.60	28.60	28.00	10
6) SPQ+EO	1075	22.70	24.70	23.80	111
7) SPQ+EP	1080	19.50	20.00	19.20	117
8) SPQ+ER	1090	12.90	14.40	13.00	783
9) SPT+ET	1100	8.10	9.10	8.70	11438
10) SPT+EB	1110	4.60	5.00	4.90	683
11) SPT+EC	1115	3.30	3.60	3.20	738
12) SPT+ED	1120	2.25	2.95	3.00	1239
13) SPT+EE	1125	1.65	2.10	1.90	3978
14) SPT+EF	1130	1.15	1.40	1.35	461
15) SPT+EG	1135	.65	1.05	.90	1521
16) SPT+EH	1140	.50	.60	.65	1548
17) SPT+EI	1145	.30	.50	.50	1
18) SPT+EJ	1150	.30	.40	.30	6754

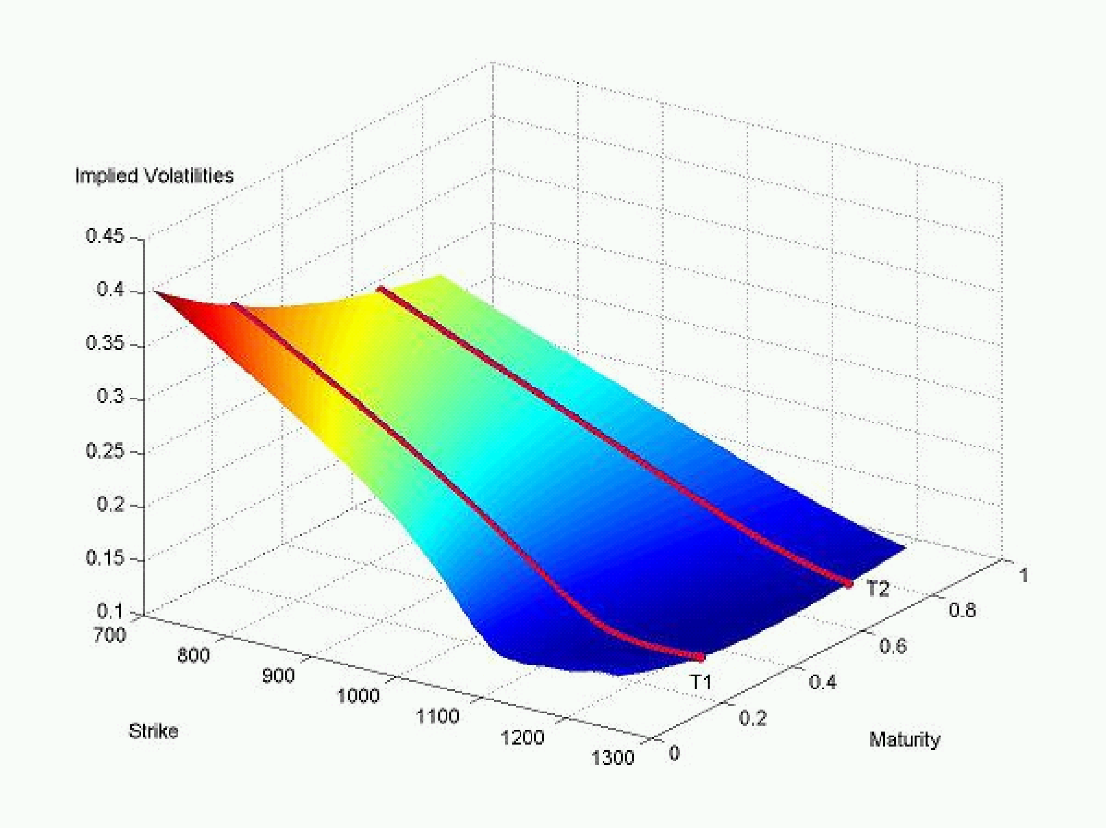
PUTS

SPX 22 MAY 04 (Contract Size: 100)

Ticker	Strike	Bid	Ask	Last	Volume
19) SPQ+QH	1040	1.25	1.75	1.35	3078
20) SPQ+QJ	1050	2.00	2.50	2.40	3283
21) SPQ+QL	1060	3.00	3.70	3.20	2382
22) SPQ+QM	1065	3.80	4.50	4.00	795
23) SPQ+QN	1070	4.70	5.40	4.80	2023
24) SPQ+QO	1075	6.00	6.30	6.30	6859
25) SPQ+QP	1080	7.50	8.00	8.20	1468
26) SPQ+QR	1090	11.00	11.90	11.70	1923
27) SPT+QT	1100	15.30	16.00	16.90	15701
28) SPT+QB	1110	21.40	23.40	22.10	1266
29) SPT+QC	1115	25.10	27.10	26.00	24
30) SPT+QD	1120	29.10	31.10	30.00	131
31) SPT+QE	1125	33.30	35.20	31.50	1532
32) SPT+QF	1130	37.70	39.70	40.00	34
33) SPT+QG	1135	42.30	44.30	43.50	12
34) SPT+QH	1140	47.00	49.00	48.30	85
35) SPT+QI	1145	51.80	53.80	52.50	25
36) SPT+QJ	1150	56.70	58.70	54.20	27

Perfect Replication of $VIX^2_{T_1}$

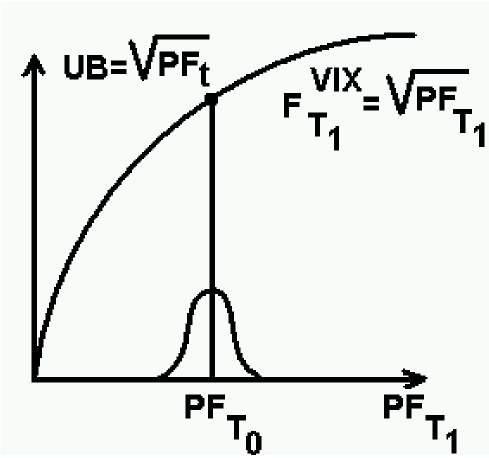
$$
\begin{aligned}
VIX^2_{T_1} &= -\frac{2}{\delta T}\text{price}_t\left(ln\frac{S_{T_1+\delta T}}{S_{T_1}}\right) \\
&= \text{price}_t\left[\frac{2}{\delta T}ln\frac{S_{T_1}}{S_{T_0}} - 2ln\frac{S_{T_1+\delta T}}{S_{T_0}}\right] \\
&= \text{price}_t[PF]
\end{aligned}
$$



We can buy today a PF which gives $VIX^2_{T_1}$ at T_1 : buy T_2 options and sell T_1 options.

Theoretical Pricing of VIX Futures F^{VIX} before launch

- F_t^{VIX}: price at t of receiving $\sqrt{PF_{T_1}} = VIX_{T_1} = F_{T_1}^{VIX}$ at T_1



$$Ft^{VIX} = \mathbb{E}[\sqrt{PF_T}] \leq \sqrt{\mathbb{E}_t[PF_T]} = \sqrt{PF_t} = \text{Upper Bound (UB)}$$

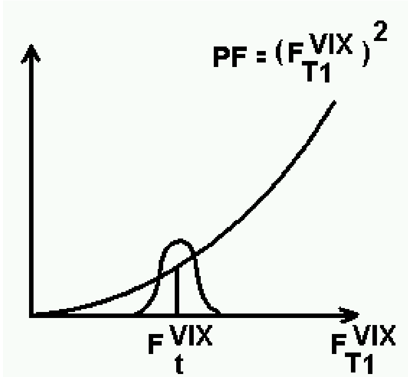
- The difference between both sides depends on the variance of PF (vol vol), which is difficult to estimate.

Pricing of F^{VIX} after launch

Much less transaction costs on F than on PF (by a factor of at least 20)

$\longrightarrow$ Replicate PF by F

instead of F by PF!



$$PF_{T_1} = (F_{T_1}^{VIX})^2 = (F_t^{VIX})^2 + 2\int_t^{T_1} (F_s^{VIX} - F_t^{VIX}) dF_s^{VIX} + QV_{t,T_1}^{F^{VIX}}$$

$$PF_t = \mathbb{E}_t[(F_{t_1}^{VIX})^2] = \mathbb{E}_t[F_{T_1}^{VIX}]^2 + \mathit{Var}_t[F_{T_1}^{VIX}]$$

$$\Longrightarrow F_t^{VIX} = \mathbb{E}_t[F_{T_1}^{VIX}] = \sqrt{PF_t - \mathit{Var}_t[F_{T_1}^{VIX}]}(\leq \sqrt{PF_t} = UB)$$

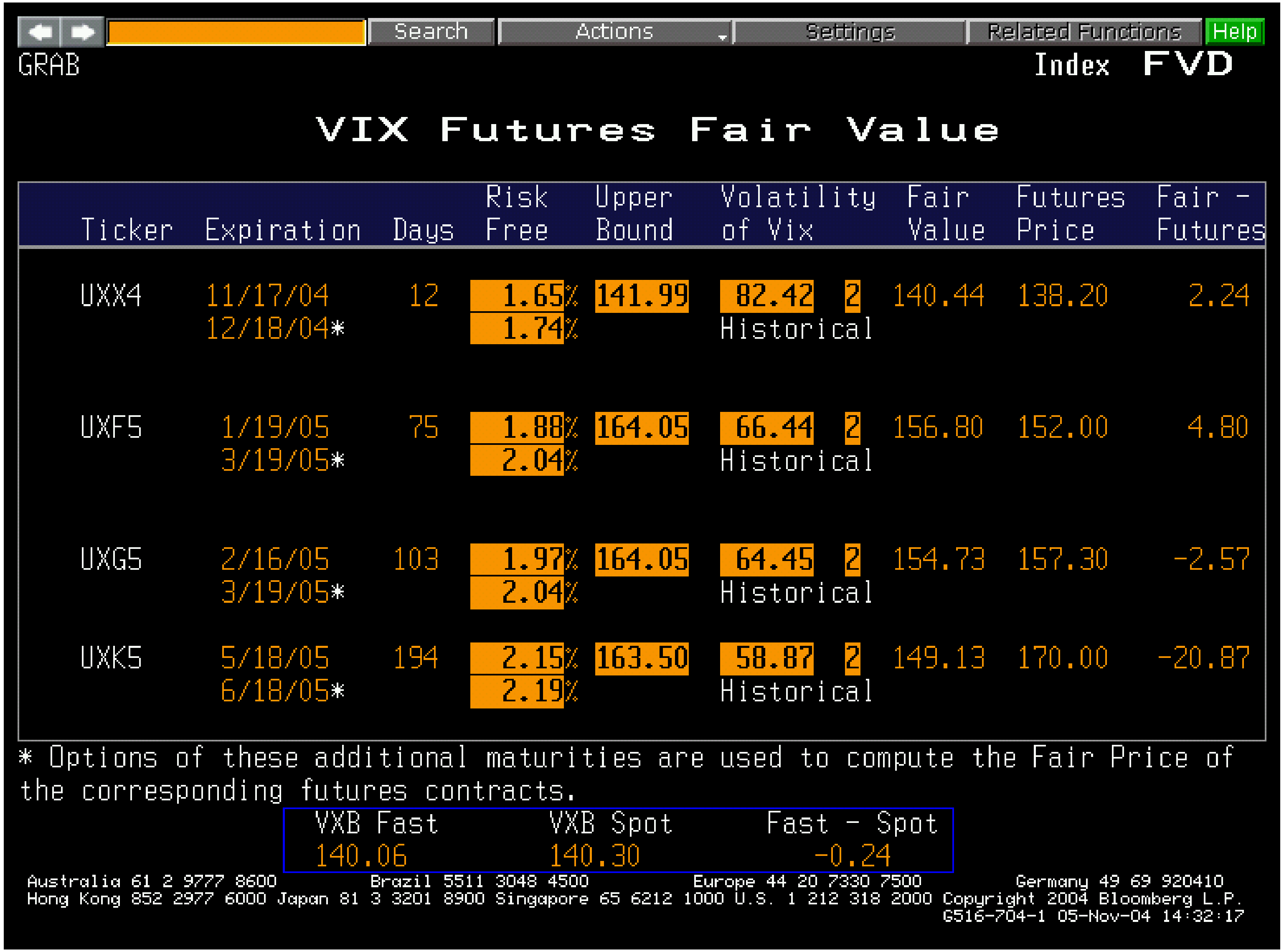
Bias estimation

$$F_t^{VIX} = \sqrt{UB^2 - \mathit{Var}_t[F_{T_1}^{VIX}]}$$

- $\mathit{Var}[F_{T_1}]$ can be estimated by combining the historical volatilities of F and Spot VIX.
- Seemingly circular analysis: F is estimated through its own volatility!

Example : $192 = \sqrt{200^2 - 56^2}$

VIX Fair Value Page



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GRAB Index FVD

VIX Futures Fair Value

Ticker	Expiration	Days	Risk Free	Upper Bound	Volatility of Vix		Fair Value	Futures Price	Fair - Futures
UXX4	11/17/04	12	1.65%	141.99	82.42	2	140.44	138.20	2.24
	12/18/04*		1.74%		Historical				
UXF5	1/19/05	75	1.88%	164.05	66.44	2	156.80	152.00	4.80
	3/19/05*		2.04%		Historical				
UXG5	2/16/05	103	1.97%	164.05	64.45	2	154.73	157.30	-2.57
	3/19/05*		2.04%		Historical				
UXK5	5/18/05	194	2.15%	163.50	58.87	2	149.13	170.00	-20.87
	6/18/05*		2.19%		Historical				

* Options of these additional maturities are used to compute the Fair Price of the corresponding futures contracts.

VXB Fast	VXB Spot	Fast - Spot
140.06	140.30	-0.24

Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 920410
Hong Kong 852 2977 6000 Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2004 Bloomberg L.P.
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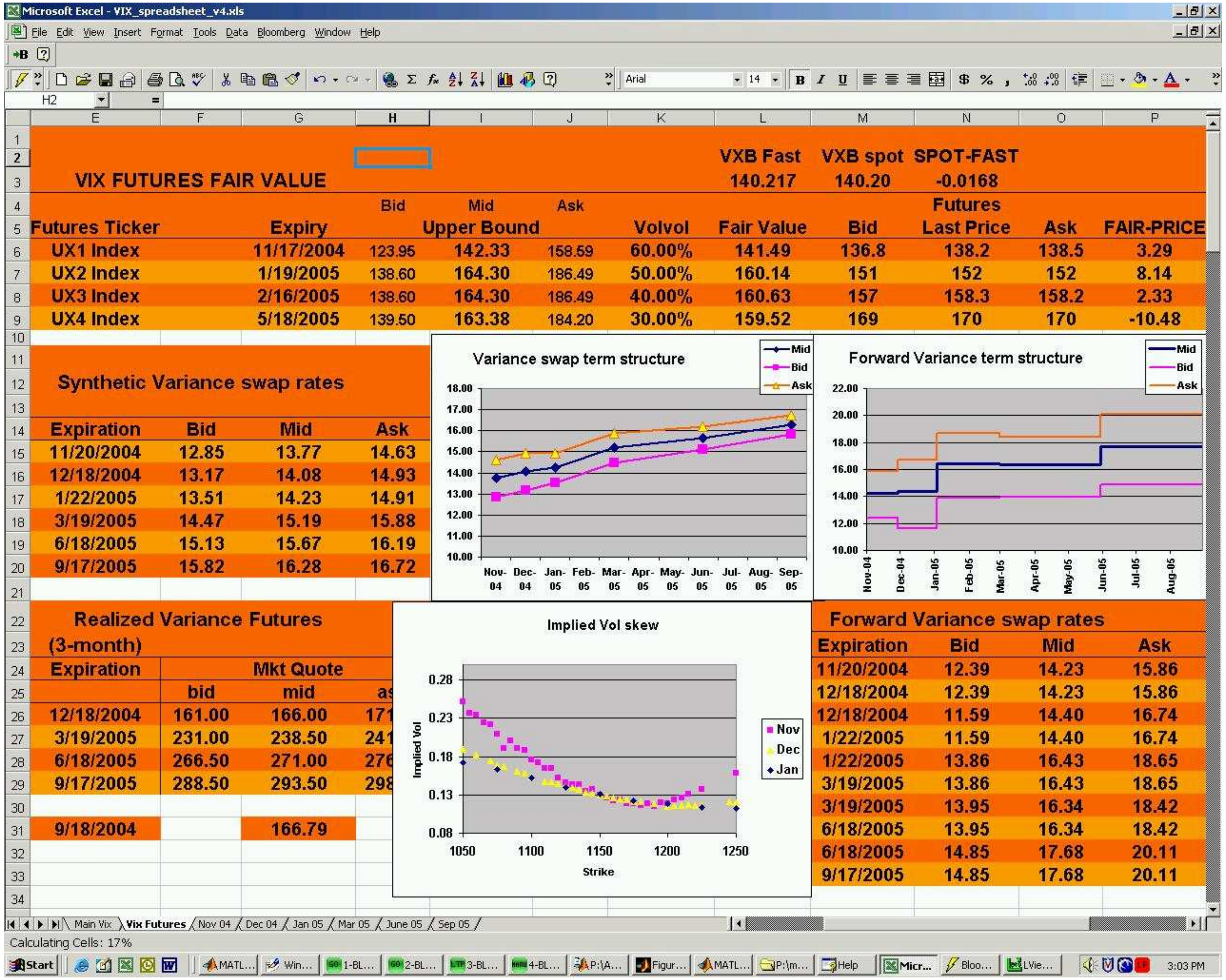
Behind The Scene



VIX FUTURES FAIR VALUE

VXB Fast	VXB spot	SPOT-FAST
140.217	140.20	-0.0168

Futures Ticker	Expiry	Bid	Mid Upper Bound	Ask	Volvol	Fair Value	Bid	Futures Last Price	Ask	FAIR-PRICE
UX1 Index	11/17/2004	123.95	142.33	158.59	60.00%	141.49	136.8	138.2	138.5	3.29
UX2 Index	1/19/2005	138.60	164.30	186.49	50.00%	160.14	151	152	152	8.14
UX3 Index	2/16/2005	138.60	164.30	186.49	40.00%	160.63	157	158.3	158.2	2.33
UX4 Index	5/18/2005	139.50	163.38	184.20	30.00%	159.52	169	170	170	-10.48

Synthetic Variance swap rates

Expiration	Bid	Mid	Ask
11/20/2004	12.85	13.77	14.63
12/18/2004	13.17	14.08	14.93
1/22/2005	13.51	14.23	14.91
3/19/2005	14.47	15.19	15.88
6/18/2005	15.13	15.67	16.19
9/17/2005	15.82	16.28	16.72

Realized Variance Futures (3-month)

Expiration	Mkt Quote bid	mid	as
12/18/2004	161.00	166.00	171
3/19/2005	231.00	238.50	241
6/18/2005	266.50	271.00	276
9/17/2005	288.50	293.50	298
9/18/2004		166.79	

Forward Variance swap rates

Expiration	Bid	Mid	Ask
11/20/2004	12.39	14.23	15.86
12/18/2004	12.39	14.23	15.86
12/18/2004	11.59	14.40	16.74
1/22/2005	11.59	14.40	16.74
1/22/2005	13.86	16.43	18.65
3/19/2005	13.86	16.43	18.65
3/19/2005	13.95	16.34	18.42
6/18/2005	13.95	16.34	18.42
6/18/2005	14.85	17.68	20.11
9/17/2005	14.85	17.68	20.11

VIX Summary

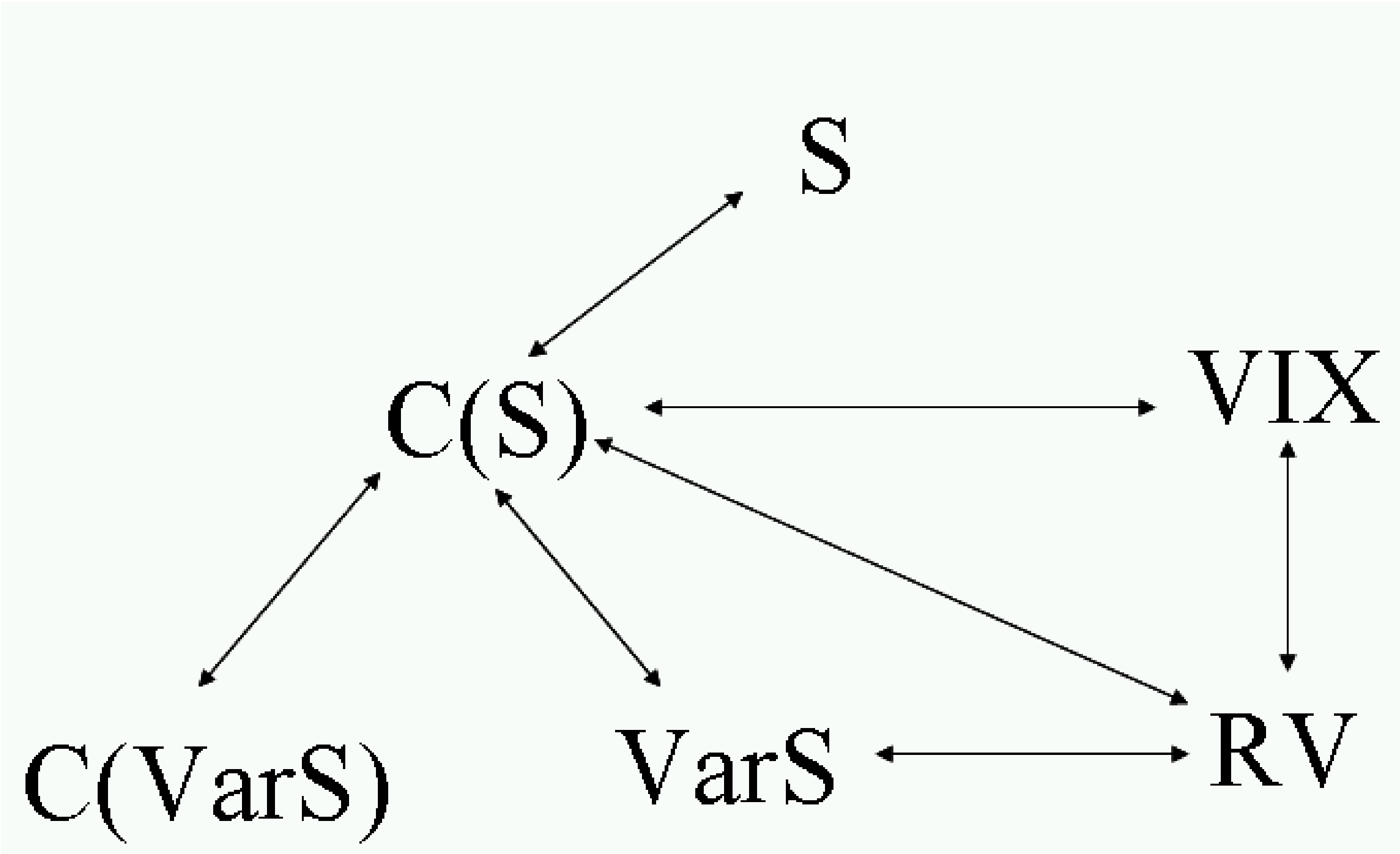
- VIX Futures is a FWD volatility between future dates T_1 and T_2.
- Depends on volatilities over T_1 and T_2.
- Can be locked in by trading options maturities T_1 and T_2.
- 2 problems :
  - Need to use all strikes (log profile)
  - Locks in σ^2, not σ $\longrightarrow$ need for convexity adjustment and dynamic hedging.

2. Linking Various Volatility Products

Volatility as an Asset Class: A Rich Playfield

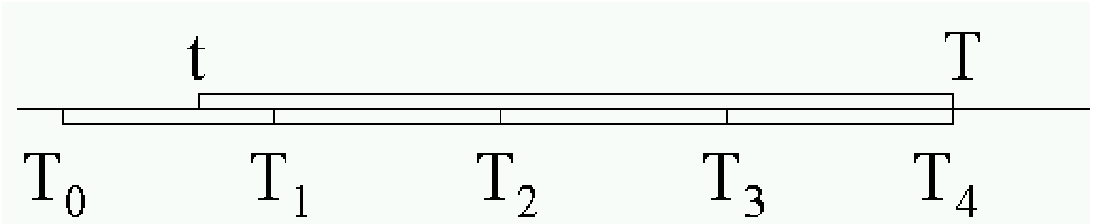
- Options on S (C(S))
- OTC Variance/Vol Swaps (VarS/VolS)
  - (Square of) historical vol up to maturity
- Futures on Realised Variance (RV)
  - Square of historical vol over a future quarter
- Futures on Implied (VIX)
- Options on Variance/Vol Swaps (C(Var S))

Plentiful of Links



RV / Var$S$

- The pay-off of an OTC Variance Swap can be replicated by a string of Realized Variance Futures:



- From 12/02/04 to maturity 09/17/05, bid-ask in vol: 15.03/15.33
- Spread=.30% in vol, much tighter than the typical 1% from the OTC market

RV / VIX

- Assume that RV and VIX, with prices RV and F are defined on the same future period $[T_1\ ,T_2]$
- If at T_0 , $RV_0 < F_0^2$ then buy 1 RV Futures and sell 2 F_0 VIX Futures
- At T_1,

$$\begin{aligned} PL_1 &= RV_1 - RV_0 - 2F_0(F_1 - F_0) \\ &> RV_1 - F_0^2 - 2F_0(F_1 - F_0) \\ &= RV_1 - F_1^2 + (F_1 - F_0)^2 \end{aligned}$$

- If $RV_1 < F_1^2$ sell the PF of options for F_1^2 and Delta hedge in S until maturity to replicate RV.
- In practice, maturities differ: conduct the same approach with a string of VIX Futures

3. Volatility Modeling

Volatility Modeling

- Neuberger (90): Quadratic variation can be replicated by delta hedging Log profiles
- Dupire (92): Forward variance synthesized from European options. Risk neutral dynamics of volatility to fit the implied vol term structure. Arbitrage pricing of claims on Spot and on vol
- Heston (93): Parametric stochastic volatility model with quasi closed form solution
- Dupire (96), Derman-Kani (97): non parametric stochastic volatility model with perfect fit to the market (HJM approach)

Volatility Modeling 2

- Matytsin (99): Parametric stochastic volatility model with jumps to price vol derivatives
- Carr-Lee (03), Friz-Gatheral (04): price and hedge of vol derivatives under assumption of uncorrelated spot and vol increments
- Duanmu (04): price and hedge of vol derivatives under assumption of volatility of variance swap
- Dupire (04): Universal arbitrage bounds for vol derivatives under the sole assumption of continuity

Variance swap based approach (Dupire (92), Duanmu (04))

- $V = QV(0,T)$ is replicable with a delta hedged log profile (parabola profile for absolute quadratic variation)
  - Delta hedge removes first order risk
  - Second order risk is unhedged. It gives the quadratic variation
- V is tradable and is the underlying of the vol derivative, which can be hedged with a position in V
- Hedge in V is dynamic and requires assumptions on

$$V_t = \mathbb{E}[V] = QV_{0,t} + \mathbb{E}_t[QV_{t,T}]$$

Stochastic Volatility Models

- Typically model the volatility of volatility (volvol). Popular example: Heston (93)

$$\frac{dS_t}{S_t} = \sqrt{\nu_t} dW_t$$

$$d\nu_t = \kappa(\nu_\infty - \nu_t)dt + \alpha\sqrt{\nu_t} dZ_t$$

- Theoretically: gives unique price of vol derivatives (1st equation can be discarded), but does not provide a natural unique hedge

- Problem: even for a market calibrated model, disconnection between volvol and real cost of hedge.

Link Skew / Volvol

- A pronounced skew imposes a high spot/vol correlation and hence a high volvol if the vol is high
- As will be seen later, non flat smiles impose a lower bound on the variability of the quadratic variation
- High spot/vol correlation means that options on S are related to options on vol: do not discard 1^{st} equation anymore

From now on, we assume 0 interest rates, no dividends and V is the quadratic variation of the price process (not of its log anymore)

Carr-Lee approach

- Assumes
  - Continuous price
  - Uncorrelated increments of spot and of vol
- Conditionally to a path of vol, $X(T)$ is normally distributed, $= X_0 + \sqrt{V}g$ (g: normal sample)
- Then it is possible to recover from the risk neutral density of $X(T)$ the risk neutral density of V
- Example: $\mathbb{E}[(X_T - X_0)^{2n}] = \mathbb{E}[V^n g^{2n}] = \mu_{2n}\mathbb{E}[V^n]$

4. Lower Bound

Densities of X and V

- How can we link the densities of the spot and of the quadratic variation V? What information do the prices of vanillas give us on the price of vol derivatives?
- Variance swap based approach: no direct link
- Stochastic vol approach: the calibration to the market gives parameters that determines the dynamics of V
- Carr-Lee approach: uncorrelated increments of spot and vol gives perfect reading of density of X from density of V

Spot Conditioning

- Claims can be on the forward quadratic variation
- Extreme case: $f(\nu_T)$ where ν_T is the instantaneous variance at T
- If f is convex,

$$\mathbb{E}[f(\nu_T)] = \mathbb{E}[\mathbb{E}[f(\nu_T|X_T = K)]] \geq \mathbb{E}[f(\mathbb{E}[\nu_T|X_T = K])] = \mathbb{E}[f(\nu_{loc}(K,T))]$$

Which is a quantity observable from current option prices

$X(T)$ not normal $\Rightarrow$ V not constant

- Main point: departure from normality for $X(T)$ enforces departure from constancy for V, or smile non flat $\Rightarrow$ variability of V
- Carr-Lee: conditionally to a path of vol, $X(T)$ is gaussian
- Actually, in general, if X is a continuous local martingale
  - $QV(T) =$ constant $\Rightarrow X(T)$ is gaussian
  - **Not**: conditional to $QV(T) =$ constant, $X(T)$ is gaussian
  - **Not**: $X(T)$ is gaussian $\Rightarrow QV(T) =$ constant

The Main Argument

- If you sell a convex claim on X and delta hedge it, the risk is mostly on excessive realized quadratic variation
- Hedge: buy a Call on V!
- Classical delta hedge (at a constant implied vol) gives a final P&L that depends on the Gammas encountered
- Perform instead a "business time" delta hedge: the payoff is replicated as long as the quadratic variation is not exhausted

Delta Hedging

- Extend $f(x)$ to $f(x,\nu)$ as the Bachelier (normal BS) price of f for start price x and variance ν:

$$f(x,\nu) \equiv \mathbb{E}^{x,\nu}[f(X)] \equiv \frac{1}{\sqrt{(2\pi\nu)}} \int f(y) e^{-\frac{(y-x)^2}{2\nu}} dy$$

  with $f(x,0) = f(x)$

- Then, $f_\nu(x,\nu) = \frac{1}{2} f_{xx}(x,\nu)$

- We explore various delta hedging strategies

Calendar Time Delta Hedging

- Delta hedging with constant vol: P&L depends on the path of the volatility and on the path of the spot price.

$$df(X_t, \sigma(T-t)) = f_x dX_t - \sigma^2 f_\nu dt + \frac{1}{2} f_{xx} dQV_{0,t}$$
$$= f_x dX_t + \frac{1}{2} f_{xx}(dQV_{0,t} - \sigma^2 dt)$$

- Calendar time delta hedge: replication cost of

$$f(X_0, \sigma^2 T) + \frac{1}{2}\int_0^T f_{xx}(dQV_{0,u} - \sigma^2 du)$$

- In particular, for $\sigma = 0$, replication cost of $f(X_t)$

$$f(X_0) + \frac{1}{2}\int_0^T f_{xx} dQV_{0,u}$$

Business Time Delta Hedging

- Delta hedging according to the quadratic variation: P&L that depends **only** on quadratic variation and spot price

$$df(X_t, L - QV_{0,t}) = f_x dX_t - f_\nu dQV_{0,t} + \frac{1}{2} f_{xx} dQV_{0,t} = f_x dX_t$$

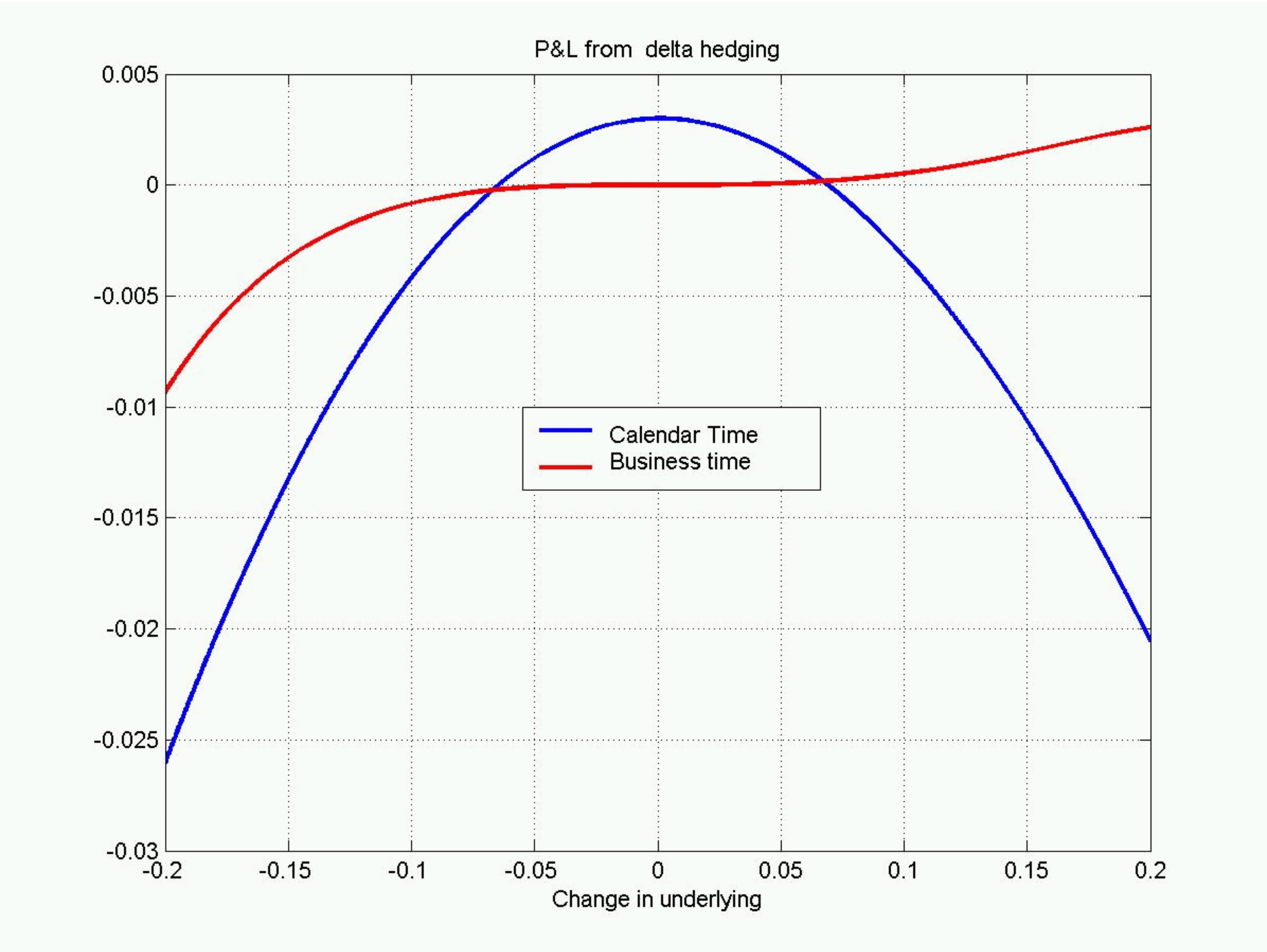
- Hence, for $QV_{0,T} \leq L$

$$f(X_t, L - QV_{0,t}) = f(X_0, L) + \int_0^t f_x(X_u, L - QV_{0,u}) dX_t$$

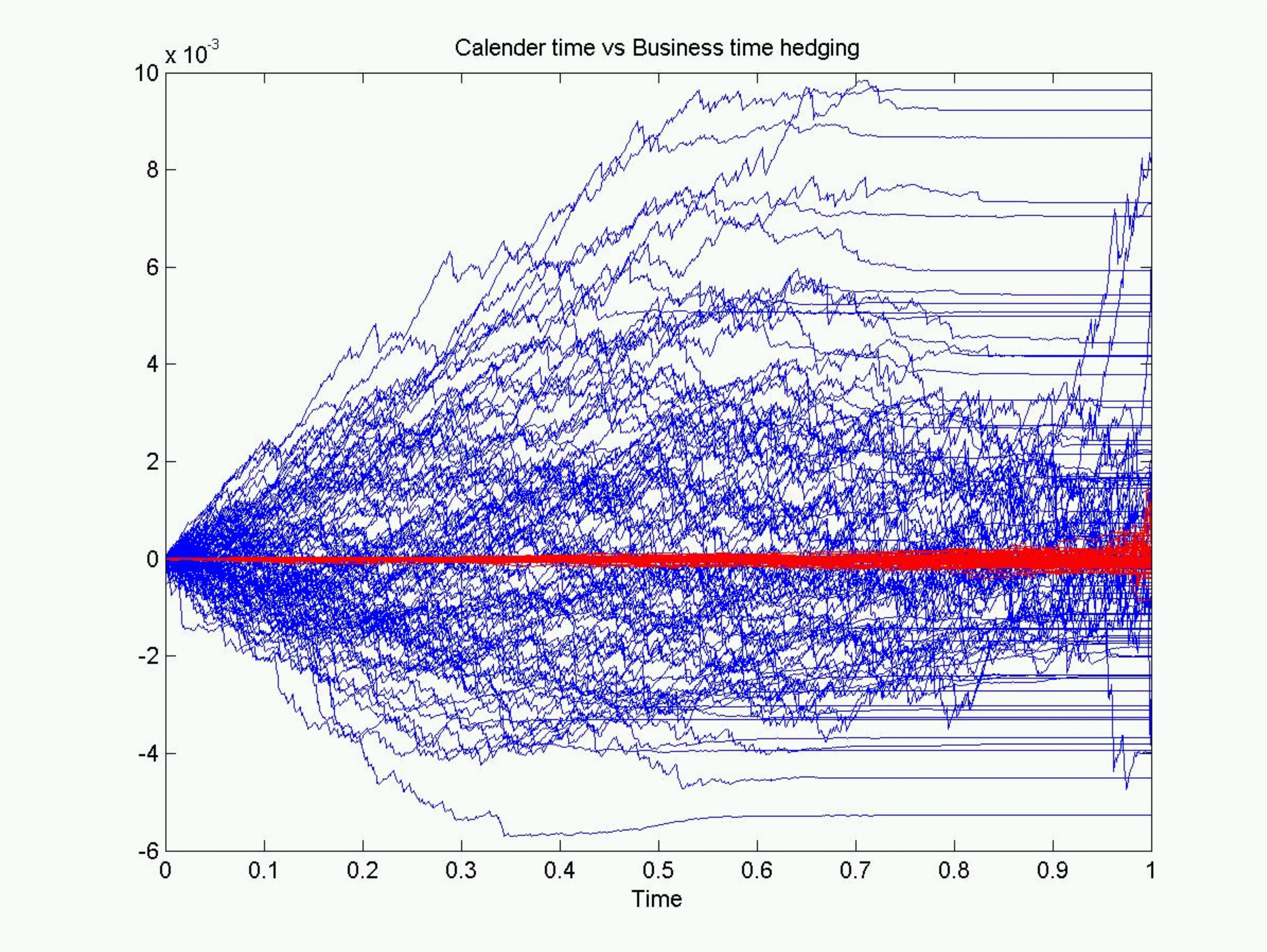
And the replicating cost of $f(X_t, L - QV_{0,t})$ is $f(X_0, L)$

$f(X_0, L)$ finances exactly the replication of f until $\tau : QV_{0,\tau} = L$

Daily P&L Variation



Tracking Error Comparison



Hedge with Variance Call

- Start from $f(X_0, L)$ and delta hedge f in "business time"
- If $V < L$, you have been able to conduct the replication until T and your wealth is $f(X_T, L - V) \geq f(X_T)$
- If $V > L$, you "run out of quadratic variation" at $\tau < T$. If you then replicate f with 0 vol until T, extra cost:

$$\frac{1}{2}\int_{\tau}^{T} f''(X_T)dQV_t \leq \frac{M_f}{2}\int_{\tau}^{T} dQV_t = \frac{M_f}{2}(V - L)$$

where $M_f \equiv \sup f''(x)$

- After appropriate delta hedge, $f(X_0, L) + \frac{M}{2}Call_L^V$ dominates $f(X_T)$ which has a market price $f(X_0, L^f)$

Lower Bound for Variance Call

- C_L^V: price of a variance call of strike L. For all f,

$$C_L^V \geq \frac{2}{M_f}(f(X_0, L^f) - f(X_0, L))$$

- We maximize the RHS for, say, $M_f \leq 2$
- We decompose f as

$$f(x) = f(X_0) + (x - X_0)f'(X_0) + \int f''(K)Vanilla_K(x)dK$$

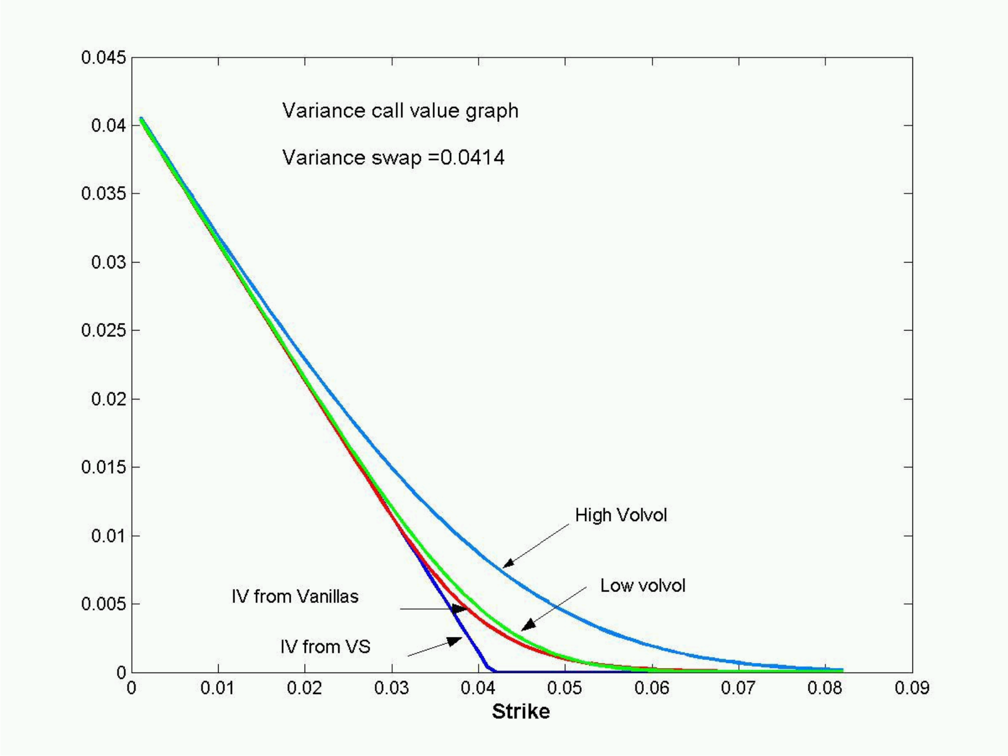
Where $Vanilla_K(x) \equiv K - x$ if $K \leq X_0$ and $x - K$ otherwise.

Then, $C_L^V \geq \int f''(K)(Van_K(L^K) - Van_K(L))dK$

where C_L^V is the price of $Vanilla_K(x)$ for variance V and L^K is the market implied variance for strike K

Lower Bound Strategy

- Maximum when $f'' = 2$ on $A \equiv K : L^K \geq L$, 0 elsewhere
- then, $f(x) = 2\int_A Vanilla_K(x)dK$ (truncated parabola) and $C_L^V \geq 2\int_A (Van_K(L^K) - Van_K(L))dK$



Arbitrage Summary

- If a Variance Call of strike L and maturity T is below its lower bound:
- 1) at t=0,
  - Buy the variance call
  - Sell all options with implied vol $\leq \sqrt{\frac{L}{T}}$
- 2) between 0 and T,
  - Delta hedge the options in business time
  - If $\tau < T$, then carry on the hedge with 0 vol
- 3) at T, sure again

5. Conclusion

Conclusion

- Skew denotes a correlation between price and vol, which links options on prices and on vol
- Business time delta hedge links P&L to quadratic variation
- We obtain a lower bound which can be seen as the real intrinsic value of the option
- Uncertainty on V comes from a spot correlated component (IV) and an uncorrelated one (TV)
- It is important to use a model calibrated to the whole smile, to get IV right and to hedge it properly to lock it in