Exterior algebra representations of commutative Moufang loops

By

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1. Introduction. Exterior algebra or associator calculus? The main theorem on commutative Moufang loops is that of Bruck-Slaby [2, Theorem VIII.10.1] stating the free commutative Moufang loop on a finite number n of generators is nilpotent of class $k(n)$, with the positive integer $k(n)$ strictly less than n. Bruck gave the following example [2, VIII.1], [4, I.10.5], [5, 10.3] to show that $k(n) \geqq 1 + [n/2]$: letting V be an infinite-dimensional vector space over the 3-element Galois field F, ΛV its exterior algebra and $E_B = V \times \Lambda V$, define the composition $\circ$ on E_B by

$$(a, x) \circ (b, y) = (a + b, x + y + (x - y)\, ab)\,. \tag{1.1}$$

(Throughout this paper, the wedge-product $a \wedge b$ in ΛV is simply denoted ab.) $(E_B, \circ)$ becomes a commutative Moufang loop, and for a linearly independent subset $\{a_1, \ldots, a_n\}$ of V $(n > 2)$ the n-element subset $\{(a_1, 1), \ldots, (a_n, 1)\}$ of E_B generates a nilpotent subloop of $(E_B, \circ)$ of class $1 + [n/2]$.

Bruck's proof of the Bruck-Slaby Theorem involved a complex triple induction process working in what has since become known as the *Bruck associator calculus*, which defines repeated associators and deals with relationships between them. In the example $(E_B, \circ)$ the associator calculus becomes much simpler, because the associator f_2 there [2, VIII.7] reduces to zero, i.e. $f_2(x, y, z; a, b) = 1$ is identically satisfied. In [1] Bruck showed that if this identity were satisfied by all commutative Moufang loops then $k(n) = 1 + [n/2]$ for $n \geqq 3$ and the full intricacies of the associator calculus in the theory of commutative Moufang loops would be superfluous, in particular for the proof of the Bruck-Slaby Theorem, the theory essentially reducing to the exterior algebra of $(E_B, \circ)$. It thus became interesting to decide the exact value of $k(n)$, at the very least for $n = 5$. This was the "open question" raised in [1].

Two methods of answering this open question were published in 1978. The author [6], [7] applied the Macdonald-Wamsley technique of nilpotent group theory to the multiplication group of a commutative Moufang loop by making long calculations in the Bruck associator calculus. Quite independently Malbos [3] produced a wonderful exterior algebra construction: with V as above, take $\Lambda^* V$ to be the odd part $\bigoplus_{k=0}^{\infty} \Lambda^{2k+1} V$ of the exterior algebra ΛV, and set $E_M = \Lambda^* V \times \Lambda^* V$. Define a product $\circ$ on E_M by

(1.2) $$(a, b) \circ (c, d) = (a + c - ac(b - d), b + d - (ad + bc)(b - d)).$$

$(E_M, \circ)$ becomes a commutative Moufang loop, and for a linearly independent subset $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ of V $(n > 1)$ the n-element subset $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ of E_M generates a nilpotent subloop of $(E_M, \circ)$ of class $n-1$.

Malbos's example raises afresh the question as to whether the Bruck associator calculus is superfluous: does the theory of commutative Moufang loops reduce to exterior algebra, now that of $(E_M, \circ)$? The third section of the current note aims to indicate how one can deduce from the work of [7] that this is not the case. To this end the next section streamlines Malbos's construction into the notation adumbrated under [6, (6.12)].

2. The wedge-dash notation. For an element $x = (a, b)$ of Malbos's F-algebra E_M, define

(2.1) $$x' = (-b, a).$$

One readily checks

(2.2) $$': E_M \to E_M \quad \text{is linear},$$

(2.3) $$x'' = -x,$$

and for x, y, z in E_M

(2.4) $$(xyz)' = x'y'z',$$

indeed for $x_1, \ldots, x_{2r+1}$ in E_M

(2.5) $$(x_1 \ldots x_{2r+1})' = x_1' \ldots x_{2r+1}'.$$

The surprising feature of this *dash-mapping* is that it suffices for the definition of Malbos's commutative Moufang loop product on E_M: (1.2) becomes

(2.6) $$x \circ y = x + y + xyy' + yxx'.$$

The commutativity of $\circ$ is of course immediate from this, and the commutative Moufang loop law follows by checking that both $(x \circ x) \circ (y \circ z)$ and $(x \circ y) \circ (x \circ z)$ reduce to

(2.7) $$\begin{aligned} &-x + y + z + yxx' + zxx' + yzz' + zyy' \\ &- xyy' - xzz' - xyz' - xzy' - xyy'zz' + yzz'xx' + zyy'xx'. \end{aligned}$$

The dash mapping is an automorphism of $(E_M, \circ)$. Now (2.3) and the fact that -1 is not a square in F imply that the dash-mapping has no eigenvectors, so that

(2.8) $$xx' = 0 \quad \text{implies} \quad x = 0.$$

It is this property that gives the wedge-dash notation its power for checking commutative Moufang loop identities.

By (2.6), $x \circ x = -x$ and $x \circ (-x) = 0$, so inverses in $(E_M, +)$ and $(E_M, \circ)$ coincide, and $(E_M, \circ)$ has exponent 3. Replacing x in (2.7) with $-x$ yields

$$
\begin{aligned}
x \circ (y \circ z) = x &+ y + z \\
&+ xyy' + yzz' + zxx' \\
&+ xzz' + yxx' + zyy' \\
&+ xyy'zz' + yzz'xx' + zyy'xx' \\
&+ xyz' + xzy' .
\end{aligned} \tag{2.9}
$$

The only part of this expression not cyclically symmetric in x, y, z is $xyz' + xzy'$. Now the associator (x, y, z) is defined [2, I.(2.1)] by

$$(x, y, z) = (z \circ (x \circ y)) \circ (x \circ (y \circ z))^{-1} \tag{2.10}$$

and thus here

$$(x, y, z) = -x'yz - xy'z - xyz' . \tag{2.11}$$

This formula for (x, y, z) may be extended by induction to an explicit formula for the general associator f_P (notation as in [6]) in the wedge-dash notation. Some preparation is necessary. Let 2^k denote the k-th direct power of the partially ordered group $2 = (\{0 < 1\}, +)$, elements of 2^k being denoted as concatenations of components such as $0^{k-2}10 = 0 \dots 010$ or $b = b_1 \dots b_k$. Let $|b|$ denote the sum $b_1 + \dots + b_k$ as a natural number, and $\bar{b}$ the complement $(1 - b_1) \dots (1 - b_k)$ of b, so in particular $|\bar{b}| = k - |b|$. For z, x, y in E_M, define

$$
\begin{cases}
(zxy)^0 = z'xy + zx'y + zxy' , \\
(zxy)^1 = z'x'y + z'xy' + zx'y' , \\
(xy)^{00} = (xy)^{11} = x'y + xy' , \\
(xy)^{10} = xy , \\
(xy)^{01} = x'y' .
\end{cases} \tag{2.12}
$$

For b in 2^k, z, x_i, y_i in E_M, define

$$f^b(z, x_1, y_1 ! \, x_2, y_2 ! \dots ! \, x_k, y_k) = (-1)^{|b|} (zx_1y_1)^{b_1} \prod_{1 < i \leq k} (x_i y_i)^{b_{i-1} b_i} . \tag{2.13}$$

The right hand side of (2.13) expands under the distributivity of the wedge product over addition in E_M to a sum of products of $z, x_1, y_1, \dots, x_k, y_k, z', x_1', \dots, y_k'$. A straightforward induction over k shows that the number of dashed factors in such a product is $k + b_k$. In particular, using (2.12):

$$\text{for} \quad b_k = 0 , \quad f^b(z, x_1, y_1 !)' = f^{\bar{b}}(z, x_1, y_1 !) \tag{2.14}$$

since $(-1)^{|\bar{b}|} = (-1)^{k - |b|} = (-1)^k (-1)^{|b|}$ and the effect of the k dashed factors in each term of the expression of the left hand side is to contribute a sign change of $(-1)^k$ on application of the dash mapping (by virtue of (2.3)).

The general associator f_P in the wedge-dash notation may now be presented as

$$(2.15)\qquad \begin{aligned} &f_P(z, x_1, y_1; a_{11}, \ldots, a_{1p_1}! \ldots ! x_k, y_k; a_{k1}, \ldots, a_{kp_k}) \\ &= (-1)^k a_{11} a_{11}' \ldots a_{kp_k} a_{kp_k}' \sum_{0^k \leqq b \leqq 1^{k-1}0} f^b(z, x_1, y_1! \ldots ! x_k, y_k). \end{aligned}$$

The verification of this formula takes two stages. Assume firstly that it holds for an associator f_P without symmetric arguments, i.e.

$$f_P(z, x_1, y_1!! x_k, y_k) = (-1)^k \sum_{0^k \leqq b \leqq 1^{k-1}0} f^b(z, x_1, y_1!! x_k, y_k).$$

Then by (2.11), (2.12), (2.14), and (2.13),

$$\begin{aligned} &f_{P,0}(z, x_1, y_1!! x_k, y_k! x_{k+1}, y_{k+1}) \\ &= -f_P(z, x_1, y_1!! x_k, y_k)'(x_{k+1} y_{k+1})^{10} \\ &\quad - f_P(z, x_1, y_1!! x_k, y_k)(x_{k+1} y_{k+1})^{00} \\ &= (-1)^{k+1} \sum_{0^k \leqq b \leqq 1^{k-1}0} f^b(z, x_1, y_1!! x_k, y_k)'(x_{k+1} y_{k+1})^{10} \\ &\quad + (-1)^{k+1} \sum_{0^k \leqq b \leqq 1^{k-1}0} f^b(z, x_1, y_1!! x_k, y_k)(x_{k+1} y_{k+1})^{00} \\ &= (-1)^{k+1} \sum_{0^{k-1}1 \leqq \tilde{b} \leqq 1^k} f^{\tilde{b}}(z, x_1, y_1!! x_k, y_k)(x_{k+1} y_{k+1})^{10} \\ &\quad + (-1)^{k+1} \sum_{0^k \leqq b \leqq 1^{k-1}0} f^b(z, x_1, y_1!! x_k, y_k)(x_{k+1} y_{k+1})^{00} \\ &= (-1)^{k+1} \sum_{0^{k+1} \leqq b \leqq 1^k 0} f^b(z, x_1, y_1!! x_k, y_k! x_{k+1}, y_{k+1}), \end{aligned}$$

so that the formula also holds for $f_{P,0}$. It holds in the somewhat degenerate form $f_0(z, x, y) = -f^0(z, x, y)$ for $k = 1$, and thus by induction for all associators without symmetric arguments. The second stage of the verification of (2.15), for general associators, follows immediately by an induction with the remark that

$$(2.16)\qquad f_1(z, x, y; a) = -a a' f^0(z, x, y),$$

a direct consequence of (2.15) for $P = (0, 0)$.

3. Limitations of the wedge-dash notation. In this section the notation of [6, §§ 4, 5] will be used: let $X = \{a, b, c, d, e\}$, let L be the free commutative Moufang loop (under the Triple Argument Hypothesis) on $X \cup \{f\}$, and A, B the abelian subloops of L generated respectively by $(a, b, c! d, e)$, $(a, b, c; f! d, e)$ as $FX!$-modules. A was shown in [6, § 4] to be a four-dimensional F-space. It follows from [6] and [7] that B is a five-dimensional F-space, for by applying the multiplication formulae of [6, § 10] to the identity $1 = (d, eb, ac; f! eb, ac)$ one deduces

$$(3.1)\qquad (a, b, d; f! c, e) = (b, c, d; f! e, a)^{-1} (c, d, e; f! a, b)(d, e, a; f! b, c)^{-1},$$

so that B is spanned by the five images of $(a, b, c; f! d, e)$ under the cyclic subgroup $\langle(a, b, c, d, e)\rangle$ of $X!$ On the other hand all the relevant requirements of the Bruck associator calculus, in particular those of [6, § 5] (such as $abc|de$) and [6, § 10], are satisfied if one takes these five images of $(a, b, c; f! d, e)$ as a basis for B in the sense of [7, § 2], and the argument of [7] then shows that B is indeed of dimension 5.

In the wedge-dash notation (i.e. in the loop $(E_M, \circ)$), however, one has

$$(3.2) \qquad (a, b, c; f!\, d, e) = ff'(a, b, c!\, d, e)$$

as a consequence of the formula (2.15). Let $\{a, b, c, d, e, f\}$ here be a linearly independent subset of E_M, and let $\bar{A}$, $\bar{B}$ be the subloops of $(E_M, \circ)$ corresponding to A and B in L. Then the mapping $\bar{A} \to \bar{B}$; $x \mapsto ff'x$ is an isomorphism, so that $\bar{B}$ is only of dimension 4. In particular, the image $\pi(a, b, c, d, e; f)$ of $(a, b, c; f!\, d, e)$ under $\sum \langle(a, b, c, d, e)\rangle$ vanishes in $\bar{B}$, but not in B. This is the source of the limitations of the wedge-dash notation. One may deduce from [6, (5.6)] that

$$(3.3) \qquad (a, b, c!\, d, e; f) = (a, b, c; f!\, d, e)\, \pi(a, b, c, d, e; f)$$

in L, and the vanishing of $\pi(a, b, c, e; f)$ in the wedge-dash notation is equivalent to the latter's inability (apparent from (2.15)) to distinguish between the associators $f_{(1,0)}$ and $f_{(0,1)}$, or more generally between all the various associators f_P for fixed parameters p and k.

In conclusion, the current state of affairs suggests two problems. One should in passing remark that the image of identity (3.1) under $\sum \langle(a, b, c, d, e)\rangle$ yields that $\pi(a, b, c, d, e; f)$ is skew-symmetric in $\{a, b, c, d, e\}$.

Problem 1. Give a simplified proof for the Bruck-Slaby Theorem under the assumption that $\pi(a, b, c, d, e; f) = 1$ holds identically.

This problem is related to the example $(E_M, \circ)$ in the way that [1, Theorem 1] is related to $(E_B, \circ)$.

Problem 2. Develop a faithful exterior algebra representation for the free commutative Moufang loop of exponent 3.

Towards Problem 2 one may remark that in $(E_B, \circ)$

$$(3.4) \qquad ((a, x), (b, y), (c, z); (f, v)!\, (d, t), (e, u)) = (0, vfabcde)$$

while

$$(3.5) \qquad ((a, x), (b, y), (c, z)!\, (d, t), (e, u); (f, v)) = (0, 0),$$

so the direct product $(E_B, \circ) \times (E_M, \circ)$ of Bruck's and Malbos's examples is capable both of distinguishing $f_{(1,0)}$ from $f_{(0,1)}$ and of demonstrating that $k(n) = n - 1$. Having constructed a putative faithful representation, one may check its effectiveness with [7] in the way that this section checked Malbos's example.

References

[1] R. H. Bruck, An open question concerning Moufang loops. Arch. Math. **10**, 419–421 (1959).
[2] R. H. Bruck, A Survey of Binary Systems. Berlin 1971.
[3] J.-P. Malbos, Sur la classe de nilpotence des boucles commutatives de Moufang et des espaces médiaux. C. R. Acad. Sci. Paris Ser. A **287**, 691–693 (1978).
[4] Ju. I. Manin, Kubičeskie Formy. Moskva 1972.
[5] Ju. I. Manin (trans. M. Hazewinkel), Cubic Forms. Amsterdam 1974.

[6] J. D. H. SMITH, A second grammar of associators. Math. Proc. Cambridge Phil. Soc. 84, 405—415 (1978).
[7] J. D. H. SMITH, On the nilpotence class of commutative Moufang loops. Math. Proc. Cambridge Phil. Soc. 84, 387—404 (1978).

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