5.1 Extrema and the Mean Value Theorem

Learning Objectives

A student will be able to:

- Solve problems that involve extrema.
- Study Rolle's Theorem.
- Use the Mean Value Theorem to solve problems.

Introduction

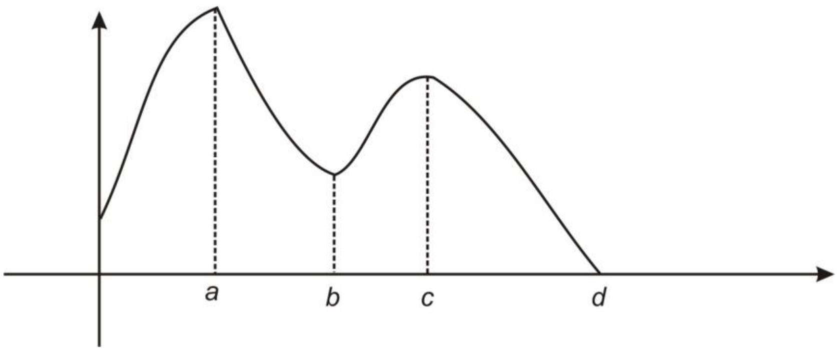
In this lesson we will discuss a second application of derivatives, as a means to study extreme (maximum and minimum) values of functions. We will learn how the maximum and minimum values of functions relate to derivatives.

Let's start our discussion with some formal working definitions of the maximum and minimum values of a function.

Here is an example of a function that has a maximum at $x = a$ and a minimum at $x = d$:

Definition

A function f has a **maximum** at $x = a$ if $f(a) \geq f(x)$ for all x in the domain of f. Similarly, f has a ***minimum*** at $x = a$ if $f(a) \leq f(x)$ for all x in the domain of f. The values of the function for these $x-$values are called ***extreme*** values or ***extrema.***



Let's recall the Min-Max Theorem that we discussed in lesson on Continuity.

Observe the graph at $x = b$. While we do not have a minimum at $x = b$, we note that $f(b) \leq f(x)$ for all x near b. We say that the function has a **local minimum** at $x = b$. Similarly, we say that the function has a ***local maximum*** at $x = c$ since $f(c) \geq f(x)$ for some x contained in open intervals of c.

Min-Max Theorem: If a function $f(x)$ is continuous in a closed interval I, then $f(x)$ has both a maximum value and a minimum value in I. In order to understand the proof for the Min-Max Theorem conceptually, attempt to draw a function on a closed interval (including the endpoints) so that no point is at the highest part of the graph. No matter how the function is sketched, there will be at least one point that is highest.

We can now relate extreme values to derivatives in the following Theorem by the French mathematician Fermat.

Theorem: If $f(c)$ is an extreme value of f for some open interval of c, and if $f'(c)$ exists, then $f'(c) = 0.$

Proof: The theorem states that if we have a local max or local min, and if $f'(c)$ exists, then we must have $f'(c) = 0.$

Suppose that f has a local max at $x = c$. Then we have $f(c) \geq f(x)$ for some open interval $(c - h, c + h)$ with $h > 0.$

So $f(c+h)-f(c)\leq 0.$

Consider $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$.

Since $f(c+h)-f(c)\leq 0$, we have $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}\leq \lim_{h\to 0^+}0=0.$

Since $f'(c)$ exists, we have $f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$, and so $f'(c)\leq 0.$

If we take the left-hand limit, we get $f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}\geq 0.$

Hence $f'(c)\geq 0$ and $f'(c)\leq 0$ it must be that $f'(c)=0.$

If $x=c$ is a local minimum, the same argument follows.

We can now state the Extreme Value Theorem.

Definition

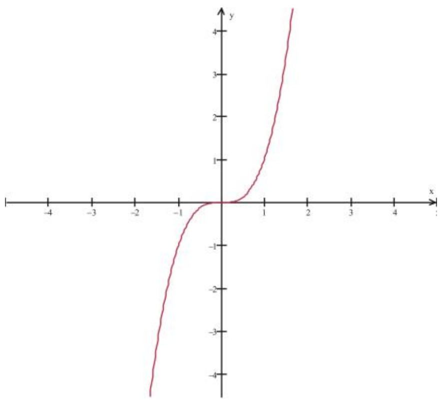
We will call $x=c$ a ***critical value*** in $[a,b]$ if $f'(c)=0$ or $f'(c)$ does not exist, or if $x=c$ is an endpoint of the interval.

Extreme Value Theorem: If a function $f(x)$ is continuous in a closed interval $[a,b]$, with the maximum of f at $x=c_1$ and the minimum of f at $x=c_2$, then c_1 and c_2 are critical values of f.

Proof: The proof follows from Fermat's theorem and is left as an exercise for the student.

Example 1:

Let's observe that the converse of the last theorem is not necessarily true: If we consider $f(x)=x^3$ and its graph, then we see that while $f'(0)=0$ at $x=0$, $x=0$ is not an extreme point of the function.

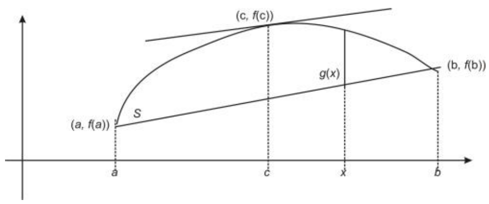


Rolle's Theorem: If f is continuous and differentiable on a closed interval $[a,b]$ and if $f(a)=f(b)$, then f has at least one value c in the open interval (a,b) such that $f'(c)=0$.

The proof of Rolle's Theorem can be found at http://en.wikipedia.org/wiki/Rolle's_theorem.

Mean Value Theorem: If f is a continuous function on a closed interval $[a,b]$ and if f' contains the open interval (a,b) in its domain, then there exists a number c in the interval (a,b) such that $f(b)-f(a)=(b-a)f'(c)$.

Proof: Consider the graph of f and secant line s as indicated in the figure.



By the Point-Slope form of line s we have

$y - f(a) = m(x - a)$ and $y = m(x - a) + f(a)$.

For each x in the interval (a, b), let $g(x)$ be the vertical distance from line S to the graph of f. Then we have

$g(x) = f(x) - [m(x - a) + f(a)]$ for every x in (a, b).

Note that $g(a) = g(b) = 0$. Since g is continuous in $[a, b]$ and g' exists in (a, b), then Rolle's Theorem applies. Hence there exists c in (a, b) with $g'(c) = 0$.

So $g'(x) = f'(x) - m$ for every x in (a, b).

In particular,

$g'(c) = f'(c) - m = 0$ and

$$f'(c) = m$$
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
$$f(b) - f(a) = (b - a)f'(c).$$

The proof is complete.

Example 2:

Verify that the Mean Value Theorem applies for the function $f(x) = x^3 + 3x^2 - 24x$ on the interval $[1, 4]$.

Solution:

We need to find c in the interval $(1, 4)$ such that $f(4) - f(1) = (4 - 1)f'(c)$.

Note that $f'(x) = 3x^2 + 6x - 24$, and $f(4) = 16$, $f(1) = -20$. Hence, we must solve the following equation:

$$36 = 3f'(c)$$
$$12 = f'(c).$$

By substitution, we have

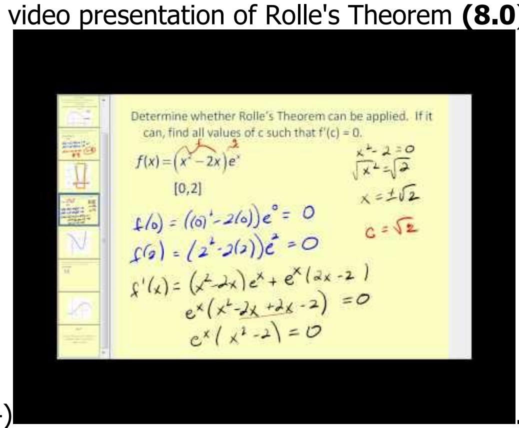
$$12 = 3c^2 + 6c - 24$$
$$3c^2 + 6c - 36 = 0$$
$$c^2 + 2c - 12 = 0$$
$$c = \frac{-2 \pm \sqrt{52}}{2} \approx -4.61, 2.61.$$

Since we need to have c in the interval $(1, 4)$, the positive root is the solution, $c = \frac{-2+\sqrt{52}}{2} \approx 2.61$.

Lesson Summary

1. We learned to solve problems that involve extrema.
2. We learned about Rolle's Theorem.
3. We used the Mean Value Theorem to solve problems.

Multimedia Links

For a video presentation of Rolle's Theorem **(8.0)**, see Math Video Tutorials by James Sousa, Rolle's Theorem (7:54)

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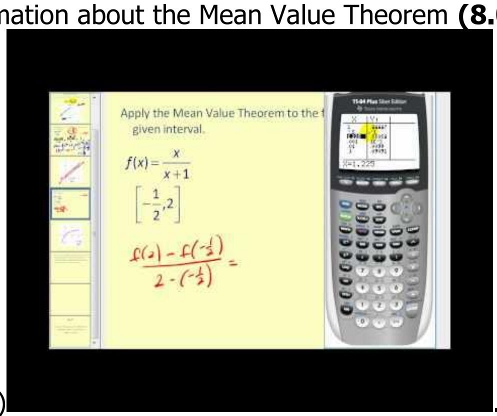
For more information about the Mean Value Theorem **(8.0)**, see Math Video Tutorials by James Sousa, Mean Value Theorem (9:52)

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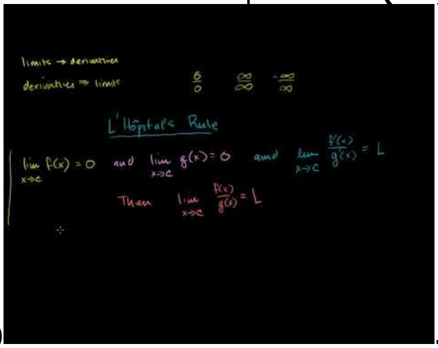
For an introduction to L'Hôpital's Rule **(8.0)**, see [Khan Academy, L'Hôpital's Rule](#)

(8:51) 

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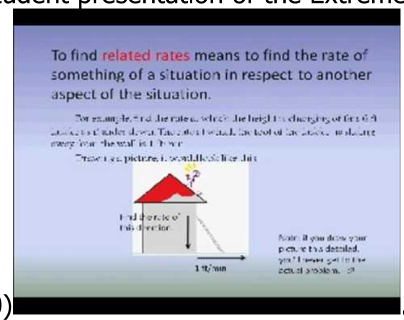
For a well-done, but unorthodox, student presentation of the Extreme Value Theorem and Related Rates **(3.0)(12.0)**,

see [Extreme Value Theorem](#) (10:00) 

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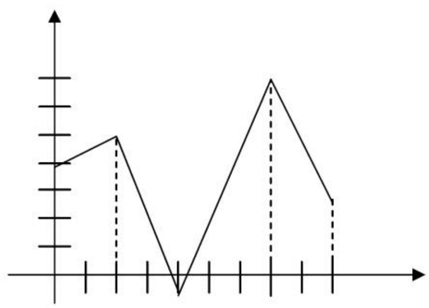
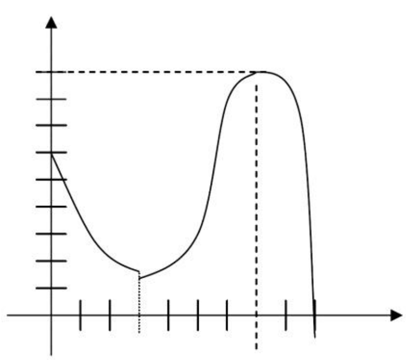
Review Questions

In problems #1–3, find the coordinates of all absolute and relative extrema. (Units on the axes indicate 1 unit).

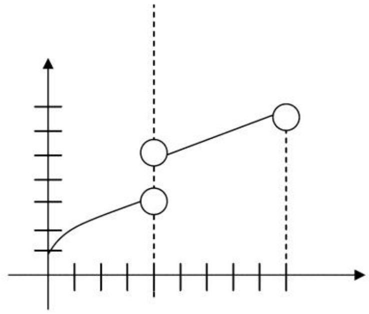
1. Continuous on $[0, 9]$



2. Continuous on $[0, 9]$



3. Continuous on [0, 4) U (4, 9)

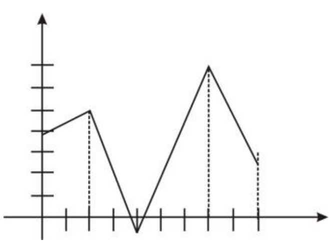


In problems #4–6, find the absolute extrema on the closed interval indicated, and sketch the graph.

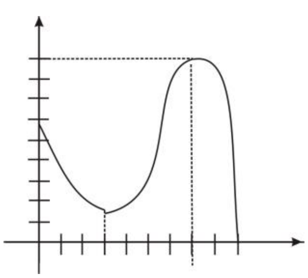
4. $f(x) = -x^2 - 6x + 4$, $[-4, 1]$
5. $f(x) = x^3 - x^4$, $[0, 2]$
6. $f(x) = x^2 + \frac{4}{x^2}$, $[-2, 0]$
7. Verify Rolle's Theorem for $f(x) = 3x^3 - 12x$ by finding values of x for which $f(x) = 0$ and $f'(x) = 0$.
8. Verify Rolle's Theorem for $f(x) = x^2 - \frac{2}{x-1}$.
9. Verify that the Mean Value Theorem works for $f(x) = \frac{x+2}{x}$ on the interval [1, 2].
10. Prove that the equation $x^3 + a_1x^2 + a_2x = 0$ has a positive root at x = r, and that the equation $3x^2 + 2a_1x + a_2 = 0$ has a positive root less than r.

Review Answers

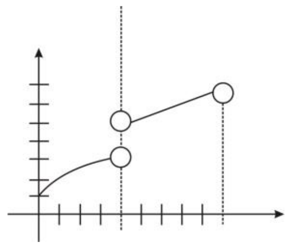
1. Absolute maximum (7, 7); absolute minimum (4, -1); relative maximum (2, 5) and (7, 7); relative minimum (0, 4), (4, -1), and (9, 3).



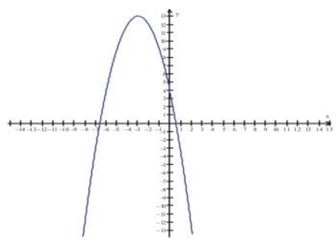
2. Absolute maximum (7, 9); absolute minimum (9, 0); relative maximum (0, 6) and (7, 9); relative minimum (3, 1.5) and (9, 0).



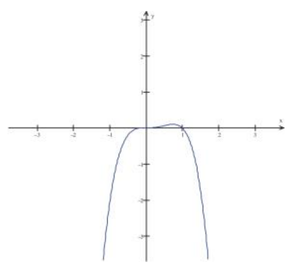
3. Absolute minimum (0, 1); relative minimum (0, 1); there is no max since the function is not continuous on a closed interval.



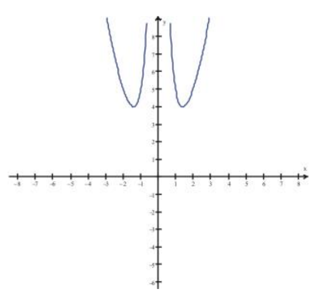
4. Absolute maximum (-3, 13); absolute minimum (1, -3)



5. Absolute maximum $(\frac{3}{4}, f(\frac{3}{4}) \approx .1055)$, absolute minimum (2, -8)



6. Absolute minimum at $(-\sqrt{2}, 4)$



7. $f(x) = 0$ at $x = 0, \pm 2$. $f'(x) = 0, x = \pm\frac{2\sqrt{3}}{3}$; by Rolle's Theorem, there is a critical value in each of the intervals (-2, 0) and (0, 2), and we found those to be $x = \pm\sqrt{2}$.
8. $f(x) = 2$ at $x = -1, 0$. $f'(x) = 0$ at $x \approx -0.84$; by Rolle's Theorem, there is a critical value in the interval (-1, 0) and we found it to be $x \approx -0.48$.
9. Need to find $c \in (1, 2)$ such that $f(2) - f(1) = (2-1)f'(c)$; $c = \sqrt{2}$.
10. Let $f(x) = x^3 + a_1x^2 + a_2x$. Observe that $f(x) = f(r) = 0$. By Rolle's Theorem, there must exist $c \in (0, r)$ such that $f'(0) = 0$.

Rolle's and MVT Practice

Determine whether Rolle's Theorem can be applied to f on the interval $[a,b]$. If Rolle's Theorem can be applied, find all values of c in the interval (a,b) such that $f'(c)=0$.

1. $f(x)=x^2-2x, [0,2]$
2. $f(x)=(x-1)(x-2)(x-3), [1,3]$
3. $f(x)=x^{2/3}-1, [-8,8]$
4. $f(x)=\dfrac{x^2-2x-3}{x+2}, [-1,3]$
5. $f(x)=\sin x, [0,2\pi]$
6. $f(x)=\tan x, [0,\pi]$
7. $f(x)=\sin x, \left[0,\dfrac{\pi}{2}\right]$

Determine whether the Mean Value Theorem can be applied to f on the interval $[a,b]$. If MVT can be applied, find all values of c in (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

8. $f(x)=x^2, [-2,1]$
9. $f(x)=x^{2/3}, [0,1]$
10. $f(x)=\sqrt{2-x}, [-7,2]$
11. $f(x)=\sin x, [0,\pi]$
12. $f(x)=\dfrac{1}{x}, [-1,1]$

Getting at the Concept:

13. Let f be continuous on $[a,b]$ and differentiable on (a,b). If there exists c in (a,b) such that $f'(c)=0$, does it follow that $f(a)=f(b)$? Explain.

14. When an object is removed from a furnace and placed in an environment with a constant temperature of 90ºF, its core temperature is 1500ºF. Five hours later the core temperature is 390ºF. Explain why there must exist a time in the interval when the temperature is decreasing at a rate of 222ºF per hour.

Answers:

1. $c=1$
2. $c=\dfrac{6\pm\sqrt{3}}{3}$
3. Rolle's Theorem cannot be applied to f since f is not differentiable at $x=0$ which is on $(-8,8)$
4. $c=-2+\sqrt{5}$
5. $c=\dfrac{\pi}{2},\dfrac{3\pi}{2}$
6. Rolle's Theorem cannot be applied since f is not continuous at $x=\dfrac{\pi}{2}$ which is on $[0,\pi]$.
7. Rolle's Thrm cannot be applied since $f(0)\neq f\left(\dfrac{\pi}{2}\right)$
8. $c=\dfrac{-1}{2}$ 9. $c=\dfrac{8}{27}$ 10. $c=\dfrac{-1}{4}$ 11. $c=\dfrac{\pi}{2}$
12. MVT cannot be applied since f is not continuous @ $x=0$ which is on $[-1,1]$
13. No. Ex: $f(x)=x^2, [-2,3]$ $f'(0)=0$ *but* $f(-2)\neq f(3)$
14. MVT, $f'(c)=\dfrac{390-1500}{5-0}=-222$

Rolle's Theorem & Mean Value Theorem HW

Determine if Rolle's Theorem can be applied to $f(x)$ on $[a, b]$. If it can, then find all values of $c \in (a,b)$ such that $f'(c) = 0$.

1.) $f(x) = x^2 - 2x,\ [0,2]$

2.) $f(x) = (x-1)(x-2)(x-3),\ [1,3]$

3.) $f(x) = x^{\frac{2}{3}} - 1,\ [-8,8]$

4.) $f(x) = \dfrac{x^2 - 2x - 3}{x+2},\ [-1,3]$

Determine if the MVT can be applied to $f(x)$ on $[a, b]$. If it can, then find all values of $c \in (a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$.

5.) $f(x) = x^2,\ [-2,1]$

6.) $f(x) = x^{\frac{2}{3}},\ [0,1]$

7.) $f(x) = \sqrt{2-x},\ [-7,2]$

Answers! 1.) $x = 1$ 2.) $x = \dfrac{6 \pm \sqrt{3}}{3} \approx 1.423, 2.577$ 3.) Cannot apply Rolle's Thm. (not diff.)

4.) $x = -2 + \sqrt{5} \approx .236$ 5.) $x = -\frac{1}{2}$ 6.) $x = \frac{8}{27}$ 7.) $x = -\frac{1}{4}$

5.2 The First Derivative Test

Learning Objectives

A student will be able to:

- Find intervals where a function is increasing and decreasing.
- Apply the First Derivative Test to find extrema and sketch graphs.

Introduction

In this lesson we will discuss increasing and decreasing properties of functions, and introduce a method with which to study these phenomena, the First Derivative Test. This method will enable us to identify precisely the intervals where a function is either increasing or decreasing, and also help us to sketch the graph. Note on notation: The symbol ϵ and $\in$ are equivalent and denote that a particular element is contained within a particular set.

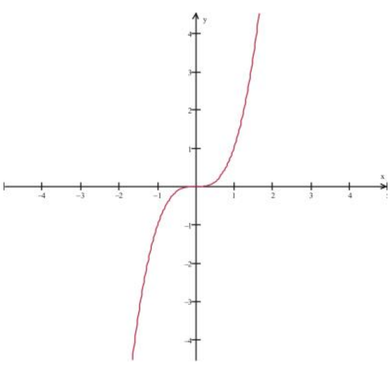
We saw several examples in the Lesson on Extreme and the Mean Value Theorem of functions that had these properties.
If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ for all $x_1, x_2 \in [a,b]$, then we say that f is ***strictly increasing*** on $[a,b]$.
If $f(x_1) > f(x_2)$ whenever $x_1 > x_2$ for all $x_1, x_2 \in [a,b]$, then we say that f is ***strictly decreasing*** on $[a,b]$.

Definition

A function f is said to be ***increasing*** on $[a,b]$ contained in the domain of f if $f(x_1) \le f(x_2)$ whenever $x_1 \le x_2$ for all $x_1, x_2 \in [a,b]$. A function f is said to be ***decreasing*** on $[a,b]$ contained in the domain of f if $f(x_1) \ge f(x_2)$ whenever $x_1 \ge x_2$ for all $x_1, x_2 \in [a,b]$.

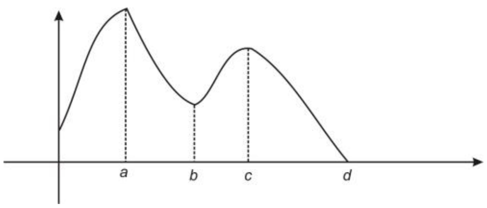
Example 1:

The function $f(x) = x^3$ is strictly increasing on $(-\infty, +\infty)$:



Example 2:

The function indicated here is strictly increasing on $(0, a)$ and (b, c), and strictly decreasing on (a, b) and (c, d).



We can now state the theorems that relate derivatives of functions to the increasing/decreasing properties of functions.

Theorem: If f is continuous on interval $[a, b]$, then:

1. If $f'(x) > 0$ for every $x \in [a, b]$, then f is strictly increasing in $[a, b]$.
2. If $f'(x) < 0$ for every $x \in [a, b]$, then f is strictly decreasing in $[a, b]$.

Proof: We will prove the first statement. A similar method can be used to prove the second statement and is left as an exercise to the student.

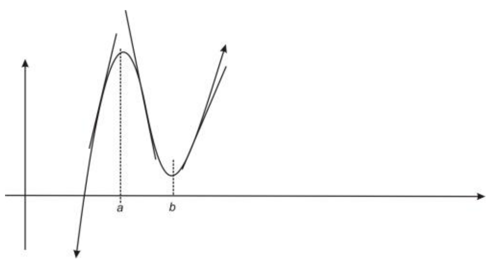
Consider $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. By the Mean Value Theorem, there exists $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c).$$

By assumption, $f'(x) > 0$ for every $x \in [a, b]$; hence $f'(c) > 0$. Also, note that $x_2 - x_1 > 0.$

Hence $f(x_2) - f(x_1) > 0$ and $f(x_2) > f(x_1).$

We can observe the consequences of this theorem by observing the tangent lines of the following graph. Note the tangent lines to the graph, one in each of the intervals $(0, a), (a, b), (b, +\infty)$.



Note first that we have a relative maximum at $x = a$ and a relative minimum at $x = b$. The slopes of the tangent lines change from positive for $x \in (0, a)$ to negative for $x \in (a, b)$ and then back to positive for $x \in (b, +\infty)$. From this we example infer the following theorem:

First Derivative Test

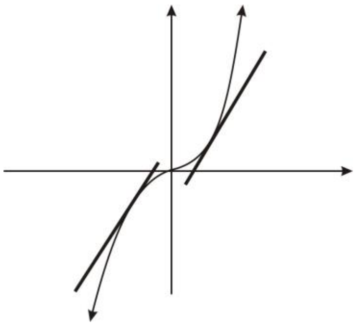
Suppose that f is a continuous function and that $x = c$ is a critical value of f. Then:

1. If f' changes from positive to negative at $x = c$, then f has a local maximum at $x = c$.
2. If f' changes from negative to positive at $x = c$, then f has a local minimum at $x = c$.
3. If f' does not change sign at $x = c$, then f has neither a local maximum nor minimum at $x = c$.

Proof of these three conclusions is left to the reader.

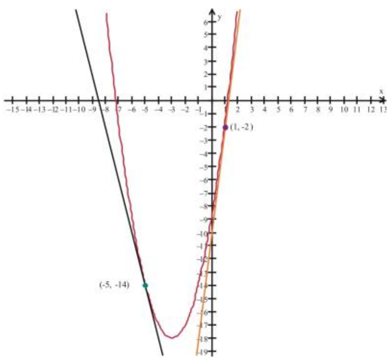
Example 3:

Our previous example showed a graph that had both a local maximum and minimum. Let's reconsider $f(x) = x^3$ and observe the graph around $x = 0$. What happens to the first derivative near this value?



Example 4:

Let's consider the function $f(x) = x^2 + 6x - 9$ and observe the graph around $x = -3$. What happens to the first derivative near this value?



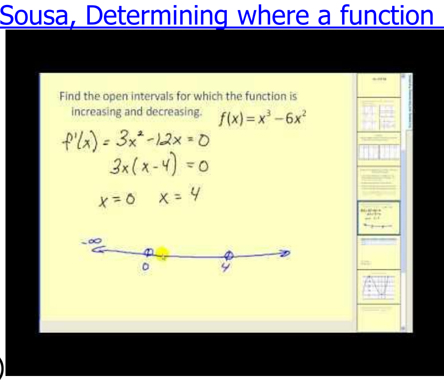
We observe that the slopes of the tangent lines to the graph change from negative to positive at $x = -3$. The first derivative test verifies this fact. Note that the slopes of the tangent lines to the graph are negative for $x \in (-\infty, -3)$ and positive for $x \in (-3, -\infty)$.

Lesson Summary

1. We found intervals where a function is increasing and decreasing.
2. We applied the First Derivative Test to find extrema and sketch graphs.

Multimedia Links

For more examples on determining whether a function is increasing or decreasing **(9.0)**, see Math Video Tutorials by James Sousa, Determining where a function is increasing and decreasing using the first derivative

(10:05) 

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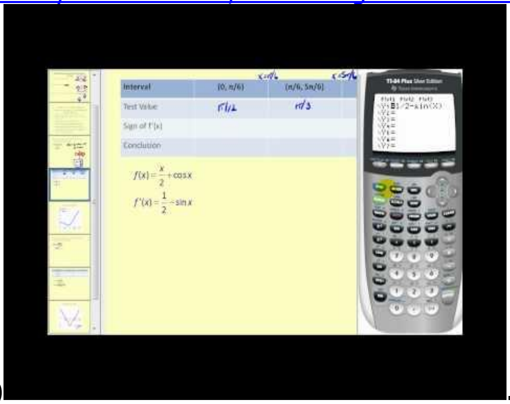
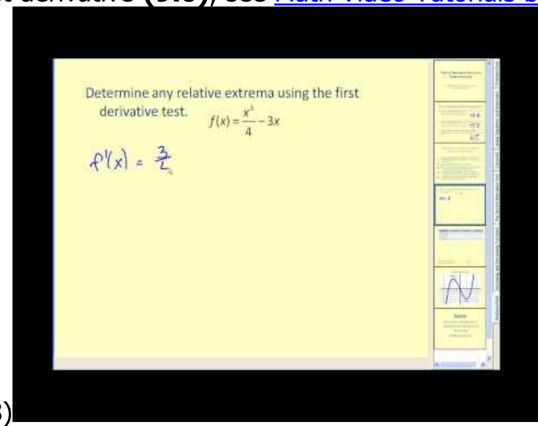
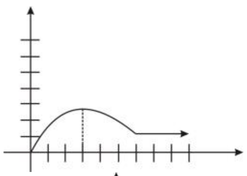
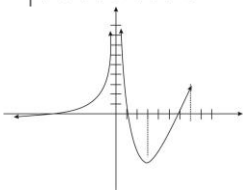
For a video presentation of increasing and decreasing trigonometric functions and relative extrema **(9.0)**, see Math Video Tutorials by James Sousa, Increasing and decreasing trig functions, relative extrema (6:02).

For more information on finding relative extrema using the first derivative **(9.0)**, see Math Video Tutorials by James Sousa, Finding relative extrema using the first derivative (6:18).

Review Questions

In problems #1–2, identify the intervals where the function is increasing, decreasing, or is constant. (Units on the axes indicate single units).

1. 
2. 
3. Give the sign of the following quantities for the graph in #2.
   a. $f'(-3)$
   b. $f'(1)$
   c. $f'(3)$
   d. $f'(4)$

For problems #4–6, determine the intervals in which the function is increasing and those in which it is decreasing. Sketch the graph.

4. $f(x) = x^2 - \frac{1}{x}$

5. $f(x) = (x^2 - 1)^5$
6. $f(x) = (x^2 - 1)^4$

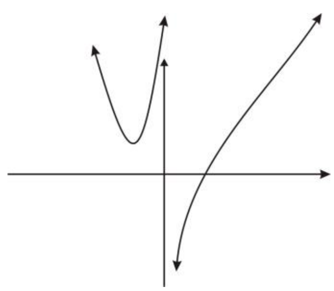
For problems #7–10, do the following:

a. Use the First Derivative Test to find the intervals where the function increases and/or decreases
b. Identify all absolute and relative max and mins
c. Sketch the graph

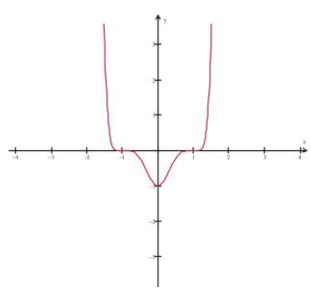
7. $f(x) = -x^2 - 4x - 1$
8. $f(x) = x^3 + 3x^2 - 9x + 1$
9. $f(x) = x^{\frac{2}{3}}(x - 5)$
10. $f(x) = 2x\sqrt{x^2 + 1}$

Review Answers

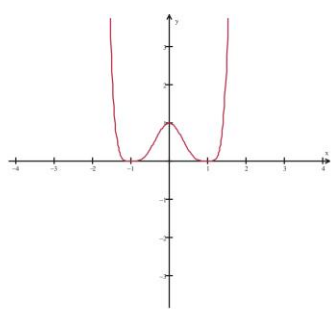
1. Increasing on (0, 3), decreasing on (3, 6), constant on (6, ∞).
2. Increasing on (-∞, 0) and (3, 7), decreasing on (0, 3).
3. $f'(-3) > 0, f'(1) < 0, f'(3) = 0, f'(4) > 0$
4. Relative minimum at $x = -\sqrt[3]{0.5}$; increasing on $(-\sqrt[3]{0.5}, 0)$ and $(0, \infty)$, decreasing on $(-\infty, -\sqrt[3]{0.5})$.



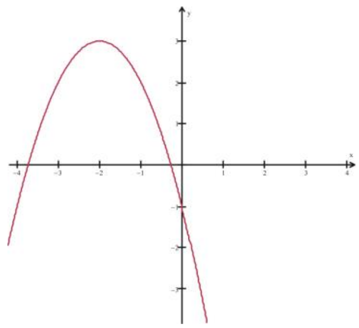
5. Absolute minimum at $x = 0$; decreasing on $(-\infty, -1) \cup (-1, 0)$, increasing on $(0, 1) \cup (1, \infty)$



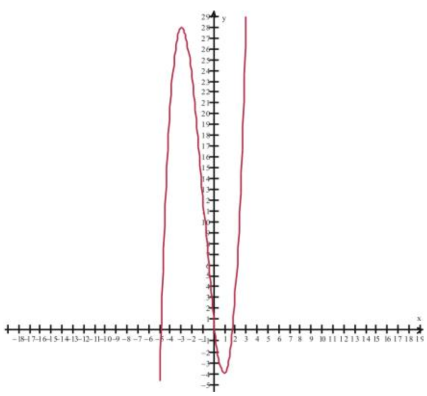
6. Absolute minimum at $x = \pm 1$; relative maximum at $x = 0$; decreasing on $(-\infty, -1), (0, 1),$ increasing on $(-1, 0), (1, +\infty)$.



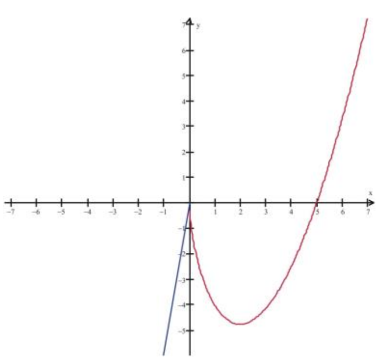
7. Absolute maximum at $x = -2$; increasing on $(-\infty, -2)$, decreasing on $(-2, +\infty)$.



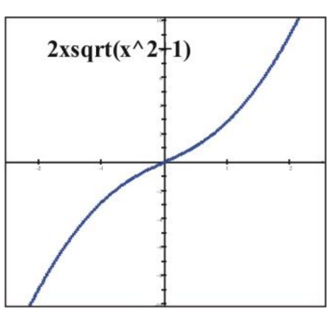
8. Relative maximum at $x = -3, f(-3) = 28$; relative minimum at $x = 1, f(1) = -4$; increasing on $(-\infty, -3)$ and $(1, +\infty)$, decreasing on $(-3, 1)$.



9. Relative maximum at x = 0, f(0) = 0; relative minimum at $x = 2,\ f(2) = -3\left(2^{\frac{2}{3}}\right) = -3\sqrt[3]{4}$, increasing on $(-\infty, 0)$ and $(2, \infty)$, decreasing on (0, 2).



10. There are no maxima or minima; no relative maxima or minima.



5.3 The Second Derivative Test

Learning Objectives

A student will be able to:

- Find intervals where a function is concave upward or downward.
- Apply the Second Derivative Test to determine concavity and sketch graphs.

Introduction

In this lesson we will discuss a property about the shapes of graphs called concavity, and introduce a method with which to study this phenomenon, the Second Derivative Test. This method will enable us to identify precisely the intervals where a function is either increasing or decreasing, and also help us to sketch the graph.

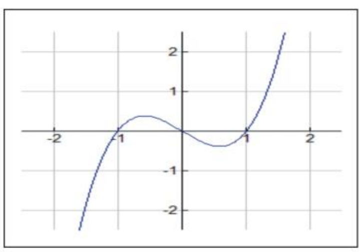
Here is an example that illustrates these properties.

Definition

A function f is said to be ***concave upward*** on $[a, b]$ contained in the domain of f if f' is an increasing function on $[a, b]$ and ***concave downward*** on $[a, b]$ if f' is a decreasing function on $[a, b]$.

Example 1:

Consider the function $f(x) = x^3 - x$:



The function has zeros at $x = \pm 1, 0$ and has a relative maximum at $x = -\frac{\sqrt{3}}{3}$ and a relative minimum at $x = \frac{\sqrt{3}}{3}$. Note that the graph appears to be concave down for all intervals in $(-\infty, 0)$ and concave up for all intervals in $(0, +\infty)$. Where do you think the concavity of the graph changed from concave down to concave up? If you answered at $x = 0$ you would be correct. In general, we wish to identify both the extrema of a function and also points, the graph changes concavity. The following definition provides a formal characterization of such points.

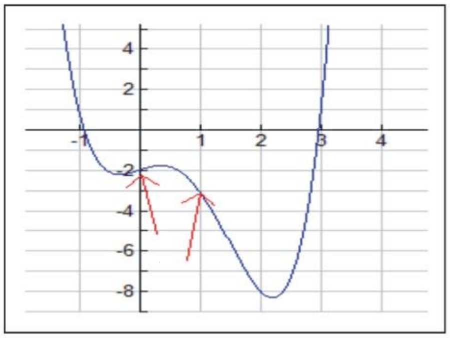
The example above had only one inflection point. But we can easily come up with examples of functions where there are more than one point of inflection.

Definition

A point on a graph of a function f where the concavity changes is called an ***inflection point***.

Example 2:

Consider the function $f(x) = x^4 - 3x^3 + x - 2.$



We can see that the graph has two relative minimums, one relative maximum, and two inflection points (as indicated by arrows).

In general we can use the following two tests for concavity and determining where we have relative maximums, minimums, and inflection points.

Test for Concavity

Suppose that f is continuous on $[a, b]$ and that I is some open interval in the domain of f.

1. If $f''(x) > 0$ for all $x \in I$, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all $x \in I$, then the graph of f is concave downward on I.

A consequence of this concavity test is the following test to identify extreme values of f.

Second Derivative Test for Extrema

Suppose that f is a continuous function near c and that c is a critical value of f. Then

1. If $f''(c) > 0$, then f has a relative maximum at $x = c$.
2. If $f''(c) < 0$, then f has a relative minimum at $x = c$.
3. If $f''(c) = 0$, then the test is inconclusive and $x = c$ may be a point of inflection.

Recall the graph $f(x) = x^3$. We observed that $x = 0$, and that there was neither a maximum nor minimum. The Second Derivative Test cautions us that this may be the case since at $f''(0) = 0$ at $x = 0$.

So now we wish to use all that we have learned from the First and Second Derivative Tests to sketch graphs of functions. The following table provides a summary of the tests and can be a useful guide in sketching graphs.

Signs of first and second derivatives	Information from applying First and Second Derivative Tests	Shape of the graphs
$f'(x) > 0$ $f''(x) > 0$	f is increasing f is concave upward	

Signs of first and second derivatives	Information from applying First and Second Derivative Tests	Shape of the graphs
$f'(x) > 0$ $f''(x) < 0$	f is increasing f is concave downward	
$f'(x) < 0$ $f''(x) > 0$	f is decreasing f is concave upward	
$f'(x) < 0$ $f''(x) < 0$	f is decreasing f is concave downward	

Lets' look at an example where we can use both the First and Second Derivative Tests to find out information that will enable us to sketch the graph.

Example 3:

Let's examine the function $f(x) = x^5 - 5x + 2.$

1. Find the critical values for which $f'(c) = 0.$

$f'(x) = 5x^4 - 5 = 0,$ or

$x^4 - 1 = 0$ at $x = \pm 1.$

Note that $f''(x) = 20x^3 = 0$ when $x = 0.$

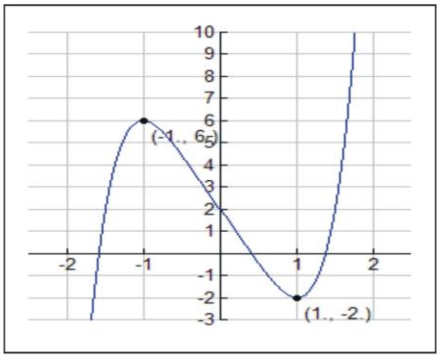
2. Apply the First and Second Derivative Tests to determine extrema and points of inflection.

We can note the signs of f' and f'' in the intervals partitioned by $x = \pm 1, 0.$

Key intervals	$f'(x)$	$f''(x)$	*Shape of graph*
$x < -1$	$+$	$-$	Increasing, concave down
$-1 < x < 0$	$-$	$-$	Decreasing, concave down
$0 < x < 1$	$-$	$+$	Decreasing, concave up
$x > 1$	$+$	$+$	Increasing, concave up

Also note that $f''(-1) = -20 < 0.$ By the Second Derivative Test we have a relative maximum at $x = -1,$ or the point $(-1, 6).$

In addition, $f''(1) = 20 > 0$. By the Second Derivative Test we have a relative minimum at $x = 1$, or the point $(1, -2)$. Now we can sketch the graph.

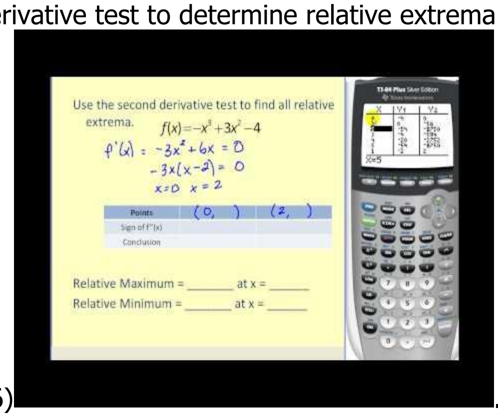


Lesson Summary

1. We learned to identify intervals where a function is concave upward or downward.
2. We applied the First and Second Derivative Tests to determine concavity and sketch graphs.

Multimedia Links

For a video presentation of the second derivative test to determine relative extrema **(9.0)**, see Math Video Tutorials by James Sousa, Introduction to Limits (8:46)



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Review Questions

1. Find all extrema using the Second Derivative Test. $f(x) = \frac{x^2}{4} + \frac{4}{x}$
2. Consider $f(x) = x^2 + ax + b$, with $f(1) = 3$.
   a. Determine a and b so that $x = 1$ is a critical value of the function f.
   b. Is the point (1, 3) a maximum, a minimum, or neither?

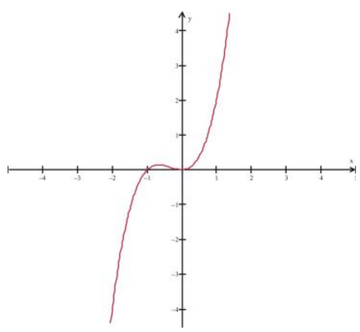
In problems #3–6, find all extrema and inflection points. Sketch the graph.

3. $f(x) = x^3 + x^2$
4. $f(x) = \frac{x^2+3}{x}$
5. $f(x) = x^3 - 12x$
6. $f(x) = -\frac{1}{4}x^4 + 2x^2$
7. Use your graphing calculator to examine the graph of $f(x) = x(x-1)^3$ (Hint: you will need to change the y range in the viewing window)
   a. Discuss the concavity of the graph in the interval $(0, \frac{1}{2})$.
   b. Use your calculator to find the minimum value of the function in that interval.

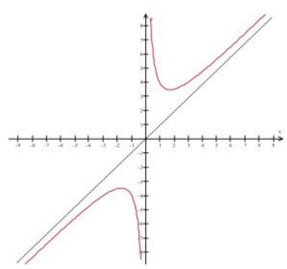
8. True or False: $f(x) = x^4 + 4x^3$ has a relative minimum at $x = -2$ and a relative maximum at $x = 0$.
9. If possible, provide an example of a non-polynomial function that has exactly one relative minimum.
10. If possible, provide an example of a non-polynomial function that is concave downward everywhere in its domain.

Review Answers

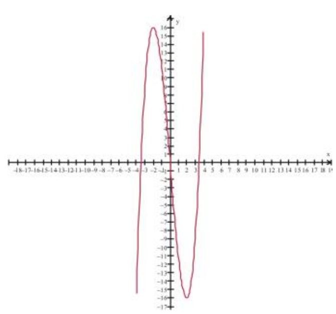
1. There is a relative minimum at x = 2, and it is located at (2, 3).
2. f(1) = 3 suggests that $a+b=2$ and $f'(1)=0=2+a$; solving this system we have that a = -2 and b = 4. The point (1, 3) is an absolute min of f.
3. Relative maximum at $x=\frac{2}{3}$, relative minimum at x = 0; the relative maximum is located at $\left(-\frac{2}{3}, 0.15\right)$; the relative minimum is located at (0, 0). There is a point of inflection at $\left(-\frac{1}{3}, 0.07\right)$.



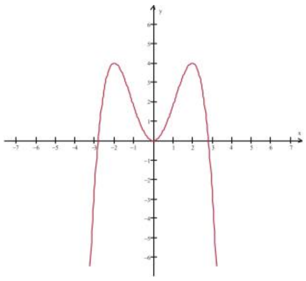
4. Relative maximum at $x = -\sqrt{3}$, located at $(-\sqrt{3}, -2\sqrt{3})$; relative minimum at $x = \sqrt{3}$, located at $(\sqrt{3}, 2\sqrt{3})$. There are no inflection points.



5. Relative maximum at x = -2; relative minimum at x = 2; the relative maximum is located at $(-2, 16)$; the relative minimum is located at $(2, 16)$. There is a point of inflection at $(0, 0)$.



6. Relative maxima at $x = \pm 2$, relative minimum at $x = 0$; the relative maxima are located at $(-2, 4)$ and $(2, 4)$; the relative minimum is located at $(0, 0)$. There are two inflection points, located at $\left(-\frac{2\sqrt{3}}{3}, \frac{20}{9}\right)$ and $\left(\frac{2\sqrt{3}}{3}, \frac{20}{9}\right)$.

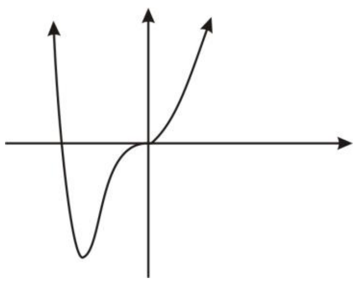


7.

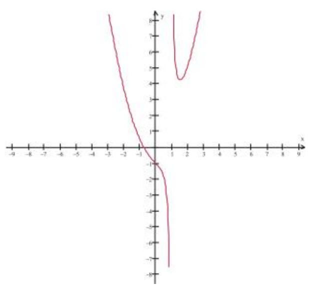
a. The graph is concave up in the interval.

b. There is a relative minimum at $(0.25, -0.10)$

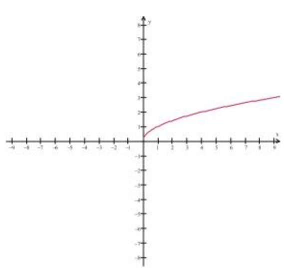
8. False: there are inflection points at $x = 0$ and $x = -2$. There is a relative minimum at $x = -3$.



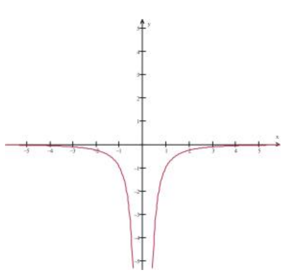
9. $f(x) = x^2 + \left(\frac{1}{x-1}\right)$



10. $f(x) = \sqrt{x}$ on $(0, +\infty)$



Also, $f(x) = -\frac{1}{x^2}$



Second Derivative Test Practice

State the intervals on which the function is concave up and concave down and state all points of inflection.

1.) $f(x) = x^2 - x - 2$

2.) $f(x) = \dfrac{24}{x^2 + 12}$

3.) $f(x) = x^3 - 9x^2 + 24x - 18$

4.) $f(x) = 2x^4 - 8x^3 + 12x^2 + 12x$

5.) $f(x) = (x-1)^3(x-5)$

6.) $f(x) = -x^4 + 18x^2 - 5$

7.) $f(x) = -4x^3 - 8x^2 + 32$

Answers:

	1	2	3	4	5	6	7
CU	$(-\infty,\infty)$	$(-\infty,-2)\cup(2,\infty)$	$(3,\infty)$	$(-\infty,1)\cup(1,\infty)$	$(-\infty,1)\cup(3,\infty)$	$(-\sqrt{3},\sqrt{3})$	$(-\infty,-\frac{2}{3})$
CD	nowhere	$(-2,2)$	$(-\infty,3)$	nowhere	$(1,3)$	$(-\infty,-\sqrt{3})\cup(\sqrt{3},\infty)$	$(-\frac{2}{3},\infty)$
POI	none	$(2,\frac{3}{2}),(-2,\frac{3}{2})$	$(3,\ 0)$	none	$(1,0),(3,-16)$	$(-\sqrt{3},40),(\sqrt{3},40)$	$(-\frac{2}{3},\frac{800}{27})$

Concavity HW

Determine intervals where each function is concave up and concave down, and find all inflection points for each function.

1.) $f(x) = x^2 - x - 2$

2.) $f(x) = \dfrac{24}{x^2 + 12}$

3.) $f(x) = x^3 - 9x^2 + 24x - 18$

4.) $f(x) = 2x^4 - 8x^3 + 12x^2 + 12x$

5.) $f(x) = (x-1)^3(x-5)$

6.) $f(x) = -x^4 + 18x^2 - 5$

7.) $f(x) = -4x^3 - 8x^2 + 32$

Answers!

	1	2	3	4	5	6	7
C.Up	$\mathbb{R}$	$(-\infty,-2)\cup(2,\infty)$	$(3,\infty)$	$\mathbb{R}$	$(-\infty,1)\cup(3,\infty)$	$(-\sqrt{3},\sqrt{3})$	$(-\infty,-\frac{2}{3})$
C.Down	nowhere	$(-2,2)$	$(-\infty,3)$	nowhere	$(1,3)$	$(-\infty,-\sqrt{3})\cup(\sqrt{3},\infty)$	$(-\frac{2}{3},\infty)$
IP	none	$(2,\frac{3}{2}),(-2,\frac{3}{2})$	$(3, 0)$	none	$(1,0),(3,-16)$	$(-\sqrt{3},40),(\sqrt{3},40)$	$(-\frac{2}{3},\frac{800}{27})$

Curve Sketching Practice

For each function, use the First and Second Derivative Tests to find the intervals where the function is increasing/decreasing/concave up/concave down, all extrema, and all points of inflection. Then, use that information to sketch the graph, labeling the important points. Feel free to use your calculator to **check** your sketches.

1.) $f(x) = \frac{1}{4}x^4 - 2x^2$

2.) $f(x) = (x-2)(x+1)^2$

3.) $f(x) = x\sqrt{x+3}$

4.) $f(x) = \dfrac{4}{1+x^2}$

“Big Problem” HW

For each function, find intervals of increasing / decreasing / concave up / concave down, all extrema, and all inflection points. (Do this without using your calculator.) Then, use that information to sketch the graph. Feel free to use your calculator to check your sketches.

1.) $f(x) = \frac{1}{4}x^4 - 2x^2$

2.) $f(x) = (x-2)(x+1)^2$

3.) $f(x) = x\sqrt{x+3}$

4.) $f(x) = \dfrac{4}{1+x^2}$

5.4 Limits at Infinity

Learning Objectives

A student will be able to:

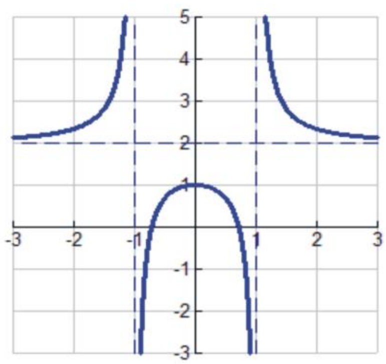
- Examine end behavior of functions on infinite intervals.
- Determine horizontal asymptotes.
- Examine indeterminate forms of limits of rational functions.
- Apply L'Hospital's Rule to find limits.
- Examine infinite limits at infinity.

Introduction

In this lesson we will return to the topics of infinite limits and end behavior of functions and introduce a new method that we can use to determine limits that have indeterminate forms.

Examine End Behavior of Functions on Infinite Intervals

Suppose we are trying to analyze the end behavior of rational functions. In Lesson on Infinite Limits we looked at some rational functions such as $f(x) = \frac{2x^2-1}{x^2-1}$ and showed that $\lim_{x\to+\infty} f(x) = 2$ and $\lim_{x\to-\infty} f(x) = 2$. We required an analysis of the end behavior of f since computing the limit by direct substitution yielded the indeterminate form $\frac{\infty}{\infty}$. Our approach to compute the infinite limit was to look at actual values of the function $f(x)$ as x approached $\pm\infty$. We interpreted the result graphically as the function having a horizontal asymptote at $f(x) = 2.$



We were then able to find infinite limits of more complicated rational functions such as $\lim_{x\to\infty} \frac{3x^4-2x^2+3x+1}{2x^4-2x^2+x-3} = \frac{3}{2}$ using the fact that $\lim_{x\to\infty} \frac{1}{x^p} = \theta, p > \theta$. Similarly, we used such an approach to compute limits whenever direct substitution resulted in the indeterminate form $\frac{\theta}{\theta}$, such as $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$.

Now let's consider other functions of the form $(f(x)/g(x))$ where we get the indeterminate forms $\frac{\theta}{\theta}$ and $\frac{\theta}{\theta}$ and determine an appropriate analytical method for computing the limits.

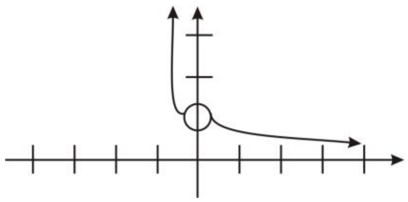
Example 1:

Consider the function $f(x) = \frac{\ln(x+1)}{x}$ and suppose we wish to find $\lim_{x\to 0} \frac{\ln(x+1)}{x}$ and $\lim_{x\to\infty} \frac{\ln(x+1)}{x}$ We note the following:

1. Direct substitution leads to the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.
2. The function in the numerator is not a polynomial function, so we cannot use our previous methods such as applying $\lim_{x\to\infty} \frac{1}{x^p} = \theta.$

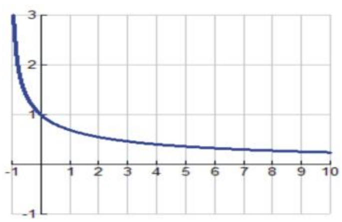
Let's examine both the graph and values of the function for appropriate x values, to see if they cluster around particular y values. Here is a sketch of the graph and a table of extreme values.

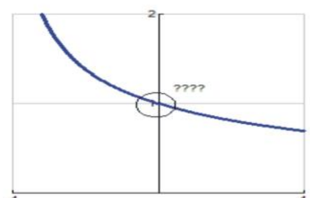
We first note that domain of the function is $(-1, 0) \cup (0, +\infty)$ and is indicated in the graph as follows:



So, $\lim_{x\to 0} \frac{\ln(x+1)}{x}$ appears to approach the value 1 as the following table suggests.

Note: Please see Differentiation and Integration of Logarithmic and Exponential Functions in Chapter 6 for more on derivatives of Logarithmic functions.





x	$\ln(x+1)/x$
-0.1	1.05361
-0.001	1.0005
0	undef
0.001	0.9995
0.1	0.953102

So we infer that $\lim_{x\to 0} \frac{\ln(x+1)}{x} = 1$.

For the infinite limit, $\lim_{x\to\infty} \frac{\ln(x+1)}{x} = 1$, the inference of the limit is not as obvious. The function appears to approach the value 0 but does so very slowly, as the following table suggests.

x	$\ln(x+1)/x$
10	0.23979
50	0.078637
100	0.046151
1000	0.006909
10000	0.000921

This unpredictable situation will apply to many other functions of the form. Hence we need another method that will provide a different tool for analyzing functions of the form $\frac{f(x)}{g(x)}$.

L'Hospital's Rule: Let functions f and g be differentiable at every number other than c in some interval, with $g'(x) \neq 0$ if $x \neq c$. If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$, or if $\lim_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) = \pm\infty$, then:

1. $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$ as long as this latter limit exists or is infinite.
2. If f and g are differentiable at every number x greater than some number a, with $g'(x) \neq 0$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ as long as this latter limit exists or is infinite.

Let's look at applying the rule to some examples.

Example 2:

We will start by reconsidering the previous example, $f(x) = \frac{\ln(x+1)}{x}$, and verify the following limits using L'Hospital's Rule:

$$\lim_{x\to 0} \frac{\ln(x+1)}{x} = 1.$$
$$\lim_{x\to \infty} \frac{\ln(x+1)}{x} = 0.$$

Solution:

Since $\lim_{x\to 0} \ln(x+1) = \lim_{x\to 0} x = 0$, L'Hospital's Rule applies and we have

$$\lim_{x\to 0} \frac{\ln(x+1)}{x} = \lim_{x\to 0} \frac{\frac{1}{x+1}}{1} = \frac{1}{1} = 1.$$

Likewise,

$$\lim_{x\to \infty} \frac{\ln(x+1)}{x} = \lim_{x\to \infty} \frac{\frac{1}{x+1}}{1} = \frac{0}{1} = 0.$$

Now let's look at some more examples.

Example 3:

Evaluate $\lim_{x\to 0} \frac{e^x - 1}{x}$.

Solution:

Since $\lim_{x\to 0}(e^x - 1) = \lim_{x\to 0} x = 0$, L'Hospital's Rule applies and we have

$$\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\frac{e^x}{1} = \frac{1}{1} = 1.$$

Let's look at an example with trigonometric functions.

Example 4:

Evaluate $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

Solution:

Since $\lim_{x\to 0}(1-\cos x) = \lim_{x\to 0} x^2 = 0$, L'Hospital's Rule applies and we have

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{\sin x}{2x} = \lim_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}.$$

Example 5: Evaluate $\lim_{x\to +\infty}\frac{x^2}{e^x}$

Solution:

Since $\lim_{x\to +\infty} x^2 = \lim_{x\to +\infty} e^x = +\infty$, L'Hospital's Rule applies and we have

$$\lim_{x\to +\infty}\frac{x^2}{e^x} = \lim_{x\to +\infty}\frac{2x}{e^x}.$$

Here we observe that we still have the indeterminate form $\frac{\infty}{\infty}$. So we apply L'Hospital's Rule again to find the limit as follows:

$$\lim_{x\to +\infty}\frac{x^2}{e^x} = \lim_{x\to +\infty}\frac{2x}{e^x} = \lim_{x\to +\infty}\frac{2}{e^x} = 0$$

L'Hospital's Rule can be used repeatedly on functions like this. It is often useful because polynomial functions can be reduced to a constant.

Lesson Summary

1. We learned to examine end behavior of functions on infinite intervals.
2. We determined horizontal asymptotes of rational functions.
3. We examined indeterminate forms of limits of rational functions.
4. We applied L'Hospital's Rule to find limits of rational functions.
5. We examined infinite limits at infinity.

Review Questions

1. Use your graphing calculator to estimate $\lim_{x\to +\infty}\left[x[\ln(x+3) - \ln(x)]\right]$.
2. Use your graphing calculator to estimate $\lim_{x\to +\infty}\frac{x}{\ln(1+2e^x)}$.

In problems #3–10, use L'Hospital's Rule to compute the limits, if they exist.

3. $\lim_{x\to 3} \frac{x^2-9}{x-3}$
4. $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$
5. $\lim_{x\to +\infty} \frac{\ln(x)}{\sqrt{x}}$
6. $\lim_{x\to +\infty} x^2 e^{-2x}$
7. $\lim_{x\to 0}(1-x)^{\frac{1}{x}}$
8. $\lim_{x\to 0} \frac{e^x-1-x}{x^2}$
9. $\lim_{x\to -\infty} \frac{e^x-1-x}{x^2}$
10. $\lim_{x\to \infty} x^{\frac{1}{4}}\ln(x)$

Review Answers

1. $\lim_{x\to +\infty} [x[\ln(x+3)-\ln(x)]] = 3$
2. $\lim_{x\to +\infty} \frac{x}{\ln(1+2e^x)} = 1$
3. $\lim_{x\to 3} \frac{x^2-9}{x-3} = 6$
4. $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = 1$
5. $\lim_{x\to +\infty} \frac{\ln(x)}{\sqrt{x}} = 0$
6. $\lim_{x\to +\infty} x^2 e^{-2x} = 0$
7. $\lim_{x\to 0}(1-x)^{\frac{1}{x}} = \frac{1}{e}$ Hint: Let $(1-x)^{\frac{1}{x}} = e^{\ln(1-x)^{\frac{1}{x}}}$, so $\lim_{x\to 0}(1-x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\ln(1-x)^{\frac{1}{x}}} = e^{\lim_{x\to 0}\frac{1}{x}\ln(1-x)}$
8. $\lim_{x\to 0} \frac{e^x-1-x}{x^2} = \frac{1}{2}$
9. $\lim_{x\to -\infty} \frac{e^x-1-x}{x^2} = 0$
10. $\lim_{x\to \infty} x^{\frac{1}{4}}\ln(x) = 0$

Asymptote & Limits Review HW

For each function, find all requested info, if possible.

1.) $f(x)=\dfrac{x^2+1}{x^2}$

Roots: $\lim_{x\to\pm\infty} f(x)=$

Holes: $\lim_{x\to VA^+} f(x)=$

VA: $\lim_{x\to VA^-} f(x)=$

y-int:

2.) $f(x)=\dfrac{x^2-2}{x^2-x-2}$

Roots: $\lim_{x\to\pm\infty} f(x)=$

Holes: $\lim_{x\to VA^+} f(x)=$

VA: $\lim_{x\to VA^-} f(x)=$

y-int:

3.) $f(x)=\dfrac{x^2-1}{2x^2-8}$

Roots: $\lim_{x\to\pm\infty} f(x)=$

Holes: $\lim_{x\to VA^+} f(x)=$

VA: $\lim_{x\to VA^-} f(x)=$

y-int:

4.) $f(x)=\dfrac{x^2-x-12}{x^2-9}$

Roots: $\lim_{x\to\pm\infty} f(x)=$

Holes: $\lim_{x\to VA^+} f(x)=$

VA: $\lim_{x\to VA^-} f(x)=$

y-int:

5.) $f(x)=\dfrac{3x}{4x^2-1}$

Roots: $\lim_{x\to\pm\infty} f(x)=$

Holes: $\lim_{x\to VA^+} f(x)=$

VA: $\lim_{x\to VA^-} f(x)=$

y-int:

6.) $f(x)=\dfrac{1}{(x+2)^2}$

Roots: $\lim_{x\to\pm\infty} f(x)=$

Holes: $\lim_{x\to VA^+} f(x)=$

VA: $\lim_{x\to VA^-} f(x)=$

y-int:

Answers:

	Roots	Holes	VA	y-int	$\lim_{x\to\pm\infty} f(x)=$	$\lim_{x\to VA^+} f(x)=$	$\lim_{x\to VA^-} f(x)=$
1.)	none	none	$x = 0$	none	1	∞	∞
2.)	$(\pm\sqrt{2},0)$	none	$x = 2$, $x = -1$	(0, 1)	1	2: ∞; -1: ∞	2: $-\infty$; -1: $-\infty$
3.)	$(\pm1,0)$	none	$x=\pm2$	$(0,\frac{1}{8})$	$\frac{1}{2}$	2: ∞; -2: $-\infty$	2: $-\infty$; -2: ∞
4.)	(4, 0)	$(-3,\frac{7}{6})$	$x = 3$	$(0,\frac{4}{3})$	1	$-\infty$	∞
5.)	(0, 0)	none	$x=\pm\frac{1}{2}$	(0, 0)	0	$\frac{1}{2}:\infty; -\frac{1}{2}:\infty$	$\frac{1}{2}:-\infty; -\frac{1}{2}:-\infty$
6.)	none	none	$x=-2$	$(0,\frac{1}{4})$	0	∞	∞

5.5 Analyzing the Graph of a Function

Learning Objectives

A student will be able to:

- Summarize the properties of function including intercepts, domain, range, continuity, asymptotes, relative extreme, concavity, points of inflection, limits at infinity.
- Apply the First and Second Derivative Tests to sketch graphs.

Introduction

In this lesson we summarize what we have learned about using derivatives to analyze the graphs of functions. We will demonstrate how these various methods can be applied to help us examine a function's behavior and sketch its graph. Since we have already discussed the various techniques, this lesson will provide examples of using the techniques to analyze the examples of representative functions we introduced in the Lesson on Relations and Functions, particularly rational, polynomial, radical, and trigonometric functions. Before we begin our work on these examples, it may be useful to summarize the kind of information about functions we now can generate based on our previous discussions. Let's summarize our results in a table like the one shown because it provides a useful template with which to organize our findings.

Table Summary	
$f(x)$	Analysis
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	
Differentiability	
Intervals where f is increasing	
Intervals where f is decreasing	
Relative extrema	
Concavity	
Inflection points	

Example 1: ***Analyzing Rational Functions***

Consider the function $f(x) = \frac{x^2-4}{x^2-2x-8}$.

General Properties: The function appears to have zeros at $x = \pm 2$.However, once we factor the expression we see

$$\frac{x^2-4}{x^2-2x-8} = \frac{(x+2)(x-2)}{(x-4)(x+2)} = \frac{x-2}{x-4}.$$

Hence, the function has a zero at $x = 2$; there is a hole in the graph at $x = -2$; the domain is $(-\infty, -2) \cup (-2, 4) \cup (4, +\infty)$; and the $y-$intercept is at $(0, \frac{1}{2})$.

Asymptotes and Limits at Infinity

Given the domain, we note that there is a vertical asymptote at $x = 4$. To determine other asymptotes, we examine the limit of f as $x \to \infty$ and $x \to -\infty$. We have

$$\lim_{x\to\infty} \frac{x^2-4}{x^2-2x-8} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2}-\frac{4}{x^2}}{\frac{x^2}{x^2}-\frac{2x}{x^2}-\frac{8}{x^2}} = \lim_{x\to\infty} \frac{1-\frac{4}{x^2}}{1-\frac{2}{x}-\frac{8}{x^2}} = 1.$$

Similarly, we see that $\lim_{x\to-\infty} \frac{x^2-4}{x^2-2x-8} = 1$. We also note that $y \neq 1$ since $x \neq -2$.

Hence we have a horizontal asymptote at $y = 1$.

Differentiability

$f'(x) = \frac{-2x^2-8x-8}{(x^2-2x-8)} = \frac{-2}{(x-4)^2} < 0$. Hence the function is differentiable at every point of its domain, and since $f'(x) < 0$ on its domain, then f is decreasing on its domain, $(-\infty, -2) \cup (-2, 4) \cup (4, +\infty)$.

$$f''(x) = \frac{4}{(x-4)^3}.$$

$f''(x) \neq 0$ in the domain of f. Hence there are no relative extrema and no inflection points.

So $f''(x) > 0$ when $x > 4$. Hence the graph is concave up for $x > 4$.

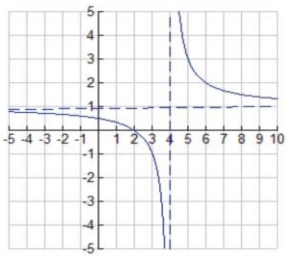
Similarly, $f''(x) < 0$ when $x < 4$. Hence the graph is concave down for $x < 4, x \neq -2$.

Let's summarize our results in the table before we sketch the graph.

Table Summary	
$f(x) = \frac{x^2-4}{x^2-2x-8}$	*Analysis*
Domain and Range	$D = (-\infty, -2) \cup (-2, 4) \cup (4, +\infty) R = \{\text{all reals} \neq 1\}$
Intercepts and Zeros	zero at $x = 2$, $y-$intercept at $(0, \frac{1}{2})$
Asymptotes and limits at infinity	VA at $x = 4$, HA at $y = 1$, hole in the graph at $x = -2$
Differentiability	differentiable at every point of its domain
Intervals where f is increasing	nowhere
Intervals where f is decreasing	$(-\infty, -2) \cup (-2, 4) \cup (4, +\infty)$

Table Summary	
$f(x) = \frac{x^2-4}{x^2-2x-8}$	*Analysis*
Relative extrema	none
Concavity	concave up in $(4, +\infty)$; concave down in $(-\infty, -2) \cup (-2, 4)$
Inflection points	none

Finally, we sketch the graph as follows:



Let's look at examples of the other representative functions we introduced in Lesson 1.2.

Example 2:

Analyzing Polynomial Functions

Consider the function $f(x) = x^3 + 2x^2 - x - 2.$

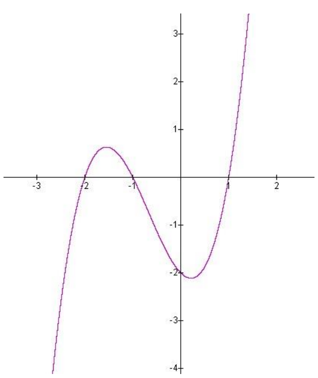
General Properties

The domain of f is $(-\infty, +\infty)$ and the $y-$intercept at $(0, -2).$

The function can be factored

$$f(x) = x^3 + 2x^2 - x - 2 = x^2(x+2) - 1(x+2) = (x^2-1)(x+2) = (x-1)(x+1)(x+2)$$

and thus has zeros at $x = \pm 1, -2.$



Asymptotes and limits at infinity

Given the domain, we note that there are no vertical asymptotes. We note that $\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty.$

Differentiability

$f'(x) = 3x^2 + 4x - 1 = 0$ if $x = \frac{-4\pm\sqrt{28}}{6} = \frac{-2\pm\sqrt{7}}{3}$. These are the critical values. We note that the function is differentiable at every point of its domain.

$f'(x) > 0$ on $\left(-\infty, \frac{-2-\sqrt{7}}{3}\right)$ and $\left(\frac{-2+\sqrt{7}}{3}, +\infty\right)$; hence the function is increasing in these intervals.

Similarly, $f'(x) < 0$ on $\left(\frac{-2-\sqrt{7}}{3}, \frac{-2+\sqrt{7}}{3}\right)$ and thus is f decreasing there.

$f''(x) = 6x + 4 = 0$ if $x = -\frac{2}{3}$, where there is an inflection point.

In addition, $f''\left(\frac{-2-\sqrt{7}}{3}\right) < 0$. Hence the graph has a relative maximum at $x = \frac{-2-\sqrt{7}}{3}$ and located at the point $(-1.55, 0.63)$.

We note that $f''(x) < 0$ for $x < -\frac{2}{3}$. The graph is concave down in $\left(-\infty, -\frac{2}{3}\right)$.

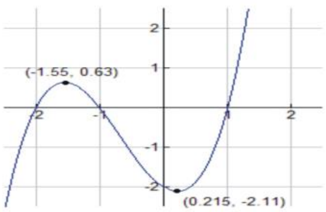
And we have $f''\left(\frac{-2+\sqrt{7}}{3}\right) > 0$; hence the graph has a relative minimum at $x = \frac{-2+\sqrt{7}}{3}$ and located at the point $(0.22, -2.11)$.

We note that $f''(x) > 0$ for $x > -\frac{2}{3}$. The graph is concave up in $\left(-\frac{2}{3}, +\infty\right)$.

Table Summary	
$f(x) = x^3 + 2x^2 - x - 2$	*Analysis*
Domain and Range	$D = (-\infty, +\infty), R = \{\text{ all reals }\}$
Intercepts and Zeros	zeros at $x = \pm1, -2$, y-intercept at $(0, -2)$
Asymptotes and limits at infinity	no asymptotes
Differentiability	differentiable at every point of it's domain
Intervals where f is increasing	$\left(-\infty, \frac{-2-\sqrt{7}}{3}\right)$ and $\left(\frac{-2+\sqrt{7}}{3}, +\infty\right)$
Intervals where f is decreasing	$\left(\frac{-2-\sqrt{7}}{3}, \frac{-2+\sqrt{7}}{3}\right)$
Relative extrema	relative maximum at $x = \frac{-2-\sqrt{7}}{3}$ and located at the point $(-1.55, 0.63)$;

Table Summary	
$f(x) = x^3 + 2x^2 - x - 2$	*Analysis*
	relative minimum at $x = \frac{-2+\sqrt{7}}{3}$ and located at the point $(0.22, -2.11)$.
Concavity	concave up in $\left(-\frac{2}{3}, +\infty\right)$. concave down in $\left(-\infty, -\frac{2}{3}\right)$.
Inflection points	$x = -\frac{2}{3}$, located at the point $\left(-\frac{2}{3}, -.74\right)$

Here is a sketch of the graph:



Example 3: ***Analyzing Radical Functions***

Consider the function $f(x) = \sqrt{2x-1}$.

General Properties

The domain of f is $\left(\frac{1}{2}, +\infty\right)$, and it has a zero at $x = \frac{1}{2}$.

Asymptotes and Limits at Infinity

Given the domain, we note that there are no vertical asymptotes. We note that $\lim_{x\to\infty} f(x) = +\infty$.

Differentiability

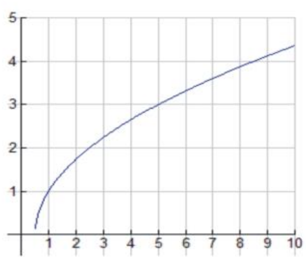
$f'(x) = \frac{1}{\sqrt{2x-1}} > 0$ for the entire domain of f. Hence f is increasing everywhere in its domain. $f'(x)$ is not defined at $x = \frac{1}{2}$, so $x = \frac{1}{2}$ is a critical value.

$f''(x) = \frac{-1}{\sqrt{(2x-1)^3}} < 0$ everywhere in $\left(\frac{1}{2}, +\infty\right)$. Hence f is concave down in $\left(\frac{1}{2}, +\infty\right)$. $f'(x)$ is not defined at $x = \frac{1}{2}$, so $x = \frac{1}{2}$ is an absolute minimum.

Table Summary	
$f(x) = \sqrt{2x-1}$	*Analysis*
Domain and Range	$D = \left(\frac{1}{2}, +\infty\right), R = \{y \geq 0\}$

Table Summary	
$f(x)=\sqrt{2x-1}$	*Analysis*
Intercepts and Zeros	zeros at $x=\frac{1}{2}$, no $y-$intercept
Asymptotes and limits at infinity	no asymptotes
Differentiability	differentiable in $\left(\frac{1}{2},+\infty\right)$
Intervals where f is increasing	everywhere in $D=\left(\frac{1}{2},+\infty\right)$
Intervals where f is decreasing	nowhere
Relative extrema	none absolute minimum at $x=\frac{1}{2}$, located at $\left(\frac{1}{2},0\right)$
Concavity	concave down in $\left(\frac{1}{2},+\infty\right)$
Inflection points	none

Here is a sketch of the graph:



Example 4: ***Analyzing Trigonometric Functions***

We will see that while trigonometric functions can be analyzed using what we know about derivatives, they will provide some interesting challenges that we will need to address. Consider the function $f(x)=x-2\sin x$ on the interval $[-\pi,\pi]$.

General Properties

We note that f is a continuous function and thus attains an absolute maximum and minimum in $[-\pi,\pi]$. Its domain is $[-\pi,\pi]$ and its range is $R=\{-\pi\leq y\leq\pi\}$.

Differentiability

$f'(x)=1-2\cos x=0$ at $x=-\frac{\pi}{3},\frac{\pi}{3}$.

Note that $f'(x)>0$ on $\left(\frac{\pi}{3},\pi\right)$ and $\left(-\pi,\frac{\pi}{3}\right)$; therefore the function is increasing in $\left(\frac{\pi}{3},\pi\right)$ and $\left(-\pi,-\frac{\pi}{3}\right)$.

Note that $f'(x) < 0$ on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$; therefore the function is decreasing in $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$f''(x) = 2\sin x = 0$ if $x = 0, \pi, -\pi$. Hence the critical values are at $x = -\pi, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \pi$.

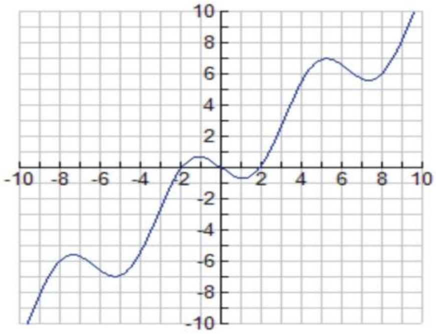
$f''\left(\frac{\pi}{3}\right) > 0$; hence there is a relative minimum at $x = \frac{\pi}{3}$.

$f''\left(-\frac{\pi}{3}\right) < 0$; hence there is a relative maximum at $x = -\frac{\pi}{3}$.

$f''(x) > 0$ on $(0, \pi)$ and $f''(x) < 0$ on $(-\pi, 0)$. Hence the graph is concave up and decreasing on $(0, \pi)$ and concave down on $(-\pi, 0)$. There is an inflection point at $x = 0$, located at the point $(0, 0)$.

Finally, there is absolute minimum at $x = -\pi$, located at $(-\pi, -\pi)$, and an absolute maximum at $x = \pi$, located at (π, π).

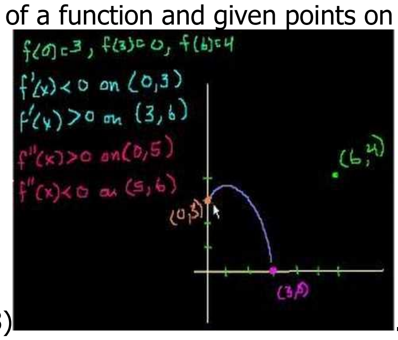
Table Summary	
$f(x) = x - 2\sin x$	*Analysis*
Domain and Range	$D = [-\pi, \pi]$, $R = \{-\pi \leq y \leq \pi\}$
Intercepts and Zeros	$x = -\frac{\pi}{3}, \frac{\pi}{3}$
Asymptotes and limits at infinity	no asymptotes
Differentiability	differentiable in $D = [-\pi, \pi]$
Intervals where f ***is increasing***	$\left(\frac{\pi}{3}, \pi\right)$ and $\left(-\pi, -\frac{\pi}{3}\right)$
Intervals where f ***is decreasing***	$\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$
Relative extrema	relative maximum at $x = -\pi/3$ relative minimum at $x = \pi/3$ absolute maximum at $x = \pi$ absolute minimum at $x = \frac{1}{2}$, located at $\left(\frac{1}{2}, 0\right)$
Concavity	concave up in $(0, \pi)$
Inflection points	$x = 0$, located at the point $(0, 0)$



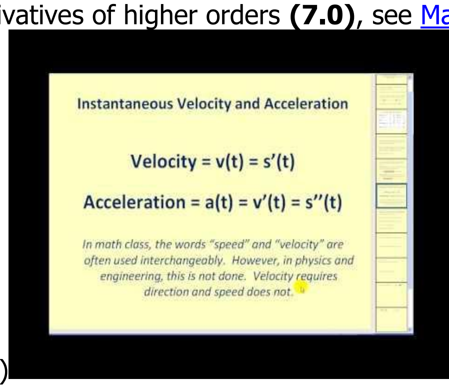
Lesson Summary

1. We summarized the properties of function, including intercepts, domain, range, continuity, asymptotes, relative extreme, concavity, points of inflection, and limits at infinity.
2. We applied the First and Second Derivative Tests to sketch graphs.

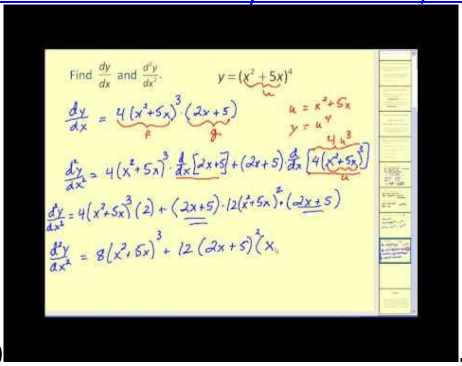
Multimedia Links

Each of the problems above started with a function and then we analyzed its zeros, derivative, and concavity. Even without the function definition it is possible to sketch the graph if you know some key pieces of information. In the following video the narrator illustrates how to use information about the derivative of a function and given points on the function graph to sketch the function. Khan Academy Graphing with Calculus (9:43).

Another approach to this analysis is to look at a function, its derivative, and its second derivative on the same set of axes. This interactive applet called Curve Analysis allows you to trace function points on a graph and its first and second derivative. You can also enter new functions (including the ones from the examples above) to analyze the functions and their derivatives.

For more information about computing derivatives of higher orders **(7.0)**, see Math Video Tutorials by James Sousa, Higher-Order Derivatives: Part 1 of 2 (7:34)

and Math Video Tutorials by James Sousa, Higher-Order Derivatives: Part 2 of 2

(5:21)

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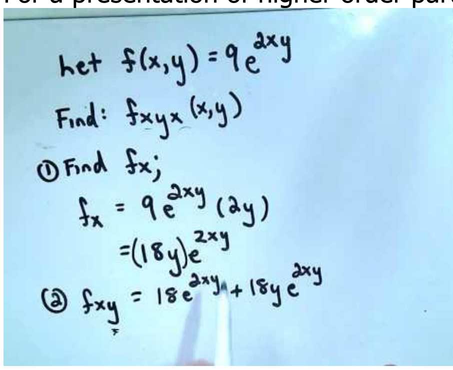
For a presentation of higher order partial derivatives **(7.0)**, see Calculus, Higher Order Derivatives (8:09)



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Review Questions

Summarize each of the following functions by filling out the table. Use the information to sketch a graph of the function.

	Analysis
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	
Differentiability	
Intervals where f ***is increasing***	
Intervals where f ***is decreasing***	
Relative extrema	
Concavity	
Inflection points	

1. $f(x) = x^3 + 3x^2 - x - 3$
2. $f(x) = -x^4 + 4x^3 - 4x^2$

3. $f(x) = \frac{2x-2}{x^2}$

4. $f(x) = x - x^{\frac{1}{3}}$

5. $f(x) = -\sqrt{2x-6} + 3$

6. $f(x) = x^2 - 2\sqrt{x}$

7. $f'(x) = 1 + \cos x = 0$ on [-π, π]

Review Answers

1.

$f(x) = x^3 + 3x^2 - x - 3$	*Analysis*
Domain and Range	$D = (-\infty, +\infty), R = \{\text{all reals}\}$
Intercepts and Zeros	zeros at $x = \pm 1, -3$, y-intercept at $(0, -3)$
Asymptotes and limits at infinity	no asymptotes
Differentiability	differentiable at every point of its domain
Intervals where f is increasing	$\left(-\infty, \frac{-3-2\sqrt{3}}{3}\right)$ and $\left(\frac{-3+2\sqrt{3}}{3}, +\infty\right)$
Intervals where f is decreasing	$\left(\frac{-3-2\sqrt{3}}{3}, \frac{-3+2\sqrt{3}}{3}\right)$
Relative extrema	relative maximum at $x = \frac{-3-2\sqrt{3}}{3}$, located at the point $(-2.15, 3.07)$; relative minimum at $x = \frac{-3+2\sqrt{3}}{3}$, located at the point $(0.15, -3.07)$
Concavity	concave up in $(-1, +\infty)$ concave down in $(-\infty, -1)$
Inflection points	$x = -1$, located at the point $(-1, 0)$

2.

$f(x) = -x^4 + 4x^3 - 4x^2$	*Analysis*
Domain and Range	$D = (-\infty, +\infty), R = \{y \leq 0\}$
Intercepts and Zeros	zeros at $x = 0, 2$, y-intercept at $(0, 0)$

$f(x) = -x^4 + 4x^3 - 4x^2$	*Analysis*
Asymptotes and limits at infinity	no asymptotes
Differentiability	differentiable at every point of its domain
Intervals where f is increasing	$(-\infty, 0)$ and $(1, 2)$
Intervals where f is decreasing	$(0, 1)$ and $(2, +\infty)$
Relative extrema	relative maximum at $x = 0$, located at the point $(0, 0)$; and at $x = 2$ located at the point $(2, 0)$ relative minimum at $x = 1$, located at the point $(1, -1)$
Concavity	concave up in $\left(\frac{2}{3}, \frac{5}{3}\right)$ concave down in $\left(-\infty, \frac{2}{3}\right)$ and $\left(\frac{5}{3}, +\infty\right)$
Inflection points	$x = \frac{2}{3}, \frac{5}{3}$, located at the points $\left(\frac{2}{3}, -0.79\right)$ and $\left(\frac{5}{3}, -0.30\right)$

3.

$f(x) = \frac{2x-2}{x^2}$	*Analysis*
Domain and Range	$D = (-\infty, 0) \cup (0, +\infty). R = \{y \neq 0\}$
Intercepts and Zeros	zeros at $x = 1$, no y-intercept
Asymptotes and limits at infinity	HA $y = 0$
Differentiability	differentiable at every point of its domain
Intervals where f is increasing	$(0, 2)$
Intervals where f is decreasing	$(-\infty, 0)$ and $(2, +\infty)$
Relative extrema	relative maximum at $x = 2$, located at the point $(2, 0.5)$
Concavity	concave up in $(3, +\infty)$ concave down in $(-\infty, 0)$ and $(0, 3)$
Inflection points	$x = 3$, located at the point $\left(3, \frac{4}{9}\right)$

4.

$f(x) = x - x^{\frac{1}{3}}$	*Analysis*
Domain and Range	$D = (-\infty, +\infty), R = \{\text{all reals}\}$
Intercepts and Zeros	zeros at $x = \pm 1, 0,$ y-intercept at $(0, 0)$
Asymptotes and limits at infinity	no asymptotes
Differentiability	differentiable in $(\infty, 0) \cup (0, +\infty)$
Intervals where f is increasing	$\left(-\infty, \frac{-\sqrt{3}}{9}\right)$ and $\left(\frac{\sqrt{3}}{9}, +\infty\right)$
Intervals where f is decreasing	$\left(\frac{-\sqrt{3}}{9}, \frac{\sqrt{3}}{9}\right)$
Relative extrema	relative maximum at $x = \frac{-\sqrt{3}}{9}$, located at the point $\left(\frac{-\sqrt{3}}{9}, 0.384\right)$ relative minimum at $x = \frac{\sqrt{3}}{9}$, located at the point $\left(\frac{\sqrt{3}}{9}, -0.384\right)$
Concavity	concave up in $(0, +\infty)$ concave down in $(-\infty, 0)$
Inflection points	$x = 0$, located at the point $(0, 0)$

5.

$f(x) = -\sqrt{2x - 6} + 3$	*Analysis*
Domain and Range	$D = (3, +\infty), R = \{y \leq 3\}$
Intercepts and Zeros	zero at $x = \frac{15}{2}$, no y-intercept
Asymptotes and limits at infinity	no asymptotes
Differentiability	differentiable in $(3, +\infty)$
Intervals where f is increasing	nowhere
Intervals where f is decreasing	everywhere in $D = (3, +\infty)$
Relative extrema	none absolute maximum at $x = 3$, located at $(3, 3)$
Concavity	concave up in $(3, +\infty)$

$f(x) = -\sqrt{2x-6}+3$	*Analysis*
Inflection points	none

6.

$f(x) = x^2 - 2\sqrt{x}$	*Analysis*
Domain and Range	$D = (0, +\infty), R = \{y \geq -1.19\}$
Intercepts and Zeros	zero at $x = \sqrt[3]{4}$; no y-intercept
Asymptotes and limits at infinity	no asymptotes
Differentiability	differentiable in $(0, +\infty)$
Intervals where f is increasing	$\left(\frac{\sqrt[3]{16}}{4}, +\infty\right)$
Intervals where f is decreasing	$\left(0, \frac{\sqrt[3]{16}}{4}\right)$
Relative extrema	relative minimum at $x = \frac{\sqrt[3]{16}}{4}$; located at the point $x = \left(\frac{\sqrt[3]{16}}{4}, -1.19\right)$
Concavity	concave up in $(0, +\infty)$
Inflection points	none

7.

$f'(x) = 1 + \cos x$	*Analysis*
Domain and Range	$D = [-\pi, \pi], R = \{0 \leq y \leq 2\}$
Intercepts and Zeros	zeros at $x = -\pi, \pi$;, y-intercept at $(0, 2)$
Asymptotes and limits at infinity	no asymptotes; $\lim_{x \leftrightarrow \infty} f(x)$ does not exist
Differentiability	differentiable at every point of its domain
Intervals where f is increasing	$(-\pi, 0)$
Intervals where f is decreasing	$(0, \pi)$

$f'(x) = 1 + \cos x$	*Analysis*
Relative extrema	absolute max at $x = 0$, located at the point $(0, 2)$ absolute minimums at $x = \pm\pi$, located at the points $(-\pi, 0)$ and $(\pi, 0)$
Concavity	concave down in $(-\pi, \pi)$
Inflection points	$x = \pm\frac{\pi}{2}$, located at the points $\left(-\frac{\pi}{2}, 1\right)$ and $\left(\frac{\pi}{2}, 1\right)$

Curve Sketching HW 1

For each function, find all requested info, and sketch the graph. Use your calculator to check your sketch.

1.) $f(x)=\dfrac{2+x}{1-x}$

INC:

DEC:

MAX:

MIN:

C. UP:

C. DOWN:

IP:

Roots:

Holes:

VA:

$\lim\limits_{x\to\pm\infty} f(x)=$

$\lim\limits_{x\to \text{VA}^+} f(x)=$

$\lim\limits_{x\to \text{VA}^-} f(x)=$

y-int:

Other point (if needed):

2.) $f(x)=\dfrac{x^2}{x^2-16}$

INC:

DEC:

MAX:

MIN:

C. UP:

C. DOWN:

IP:

Roots:

Holes:

VA:

$\lim\limits_{x\to\pm\infty} f(x)=$

$\lim\limits_{x\to \text{VA}^+} f(x)=$

$\lim\limits_{x\to \text{VA}^-} f(x)=$

y-int:

Other point (if needed):

3.) $f(x)=\dfrac{2x}{1-x}$

INC:

DEC:

MAX:

MIN:

C. UP:

C. DOWN:

IP:

Roots:

Holes:

VA:

$\lim\limits_{x\to\pm\infty} f(x)=$

$\lim\limits_{x\to \text{VA}^+} f(x)=$

$\lim\limits_{x\to \text{VA}^-} f(x)=$

y-int:

Other point (if needed):

Hints:

1.) $f'(x)=\dfrac{3}{(1-x)^2}$, $f''(x)=\dfrac{6}{(1-x)^3}$

2.) $f'(x)=-\dfrac{32x}{(x^2-16)^2}$, $f''(x)=\dfrac{32(3x^2+16)}{(x^2-16)^3}$

3.) $f'(x)=\dfrac{2}{(1-x)^2}$, $f''(x)=\dfrac{4}{(1-x)^3}$

Curve Sketching HW 2

For each function, find all requested info, and sketch the graph. Use your calculator to check your sketch.

1.) $f(x) = \dfrac{x^2}{x^2 + 9}$

INC:

DEC:

MAX:

MIN:

C. UP:

C. DOWN:

IP:

Roots:

Holes:

VA:

$\lim_{x \to \pm\infty} f(x) =$

$\lim_{x \to \text{VA}^+} f(x) =$

$\lim_{x \to \text{VA}^-} f(x) =$

y-int:

Other point (if needed):

2.) $f(x) = \dfrac{3x^2 - 3}{x^2}$

INC:

DEC:

MAX:

MIN:

C. UP:

C. DOWN:

IP:

Roots:

Holes:

VA:

$\lim_{x \to \pm\infty} f(x) =$

$\lim_{x \to \text{VA}^+} f(x) =$

$\lim_{x \to \text{VA}^-} f(x) =$

y-int:

Other point (if needed):

3.) $f(x)=\dfrac{2x^2-6}{(x-1)^2}$

INC:	Roots:
DEC:	Holes:
MAX:	VA:
MIN:	$\lim_{x\to\pm\infty} f(x)=$
C. UP:	$\lim_{x\to\text{VA}^+} f(x)=$
C. DOWN:	$\lim_{x\to\text{VA}^-} f(x)=$
IP:	y-int:
	Other point (if needed):

Hints:

1.) $f'(x)=\dfrac{18x}{(x^2+9)^2}$, $f''(x)=-\dfrac{54(x^2-3)}{(x^2+9)^3}$

2.) $f'(x)=\dfrac{6}{x^3}$, $f''(x)=-\dfrac{18}{x^4}$

3.) $f'(x)=-\dfrac{4(x-3)}{(x-1)^3}$, $f''(x)=\dfrac{8(x-4)}{(x-1)^4}$