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Extrema:

A Curriculum Module for AP® Calculus

2010

Curriculum Module

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AP® Calculus Extrema

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In 1995 when graphing calculators were first allowed on the AP® Calculus Exams, I remember thinking, "Well, there go all the good extrema problems. The kids will just use the max/min feature of their calculators and won't have to know any calculus at all." That thought seems ridiculous in retrospect, as extrema problems are alive and well and much more interesting than they have ever been before. The AP Calculus Exam uses multiple representations of functions to test students' understanding of extrema in so many different ways now, some calculator active and some inactive.

I am convinced that students comprehend the topic at greater depth than ever before, rather than simply applying a rote process (set derivative equal to zero and solve) to problems similar to ones they have solved previously. The creativity of the AP Calculus Development Committee continues to amaze me as they write problems I would never have considered and have certainly never seen in a textbook. As these problems have evolved, I have tried to change my teaching in various ways to keep pace. I used to think of the topic of extrema as a few sections of my textbook. Now I see it as one of the major themes of the course, permeating every topic that I teach. When I teach motion, I think, "How can extrema questions be asked in this context?" When I teach f, f', f'' relationships, I think, "How can extrema questions be asked in this context?" My thoughts are the same with functions defined by integrals, differential equations, etc.

Because this topic is so important, it is frequently discussed on the AP Calculus LISTSERV (electronic discussion group, or EDG), for which you can sign up through AP Central® (http://apcentral.collegeboard.com/apc/public/homepage/7173.html). Some questions come up repeatedly as new teachers struggle with the concepts surrounding extrema. One such question is: "Can an endpoint be considered a relative extreme point?" The answer depends on the definition of "relative extreme" being used, which varies from textbook to textbook. Therefore, the AP Calculus Development Committee carefully crafts the questions for the AP Exam to avoid this issue entirely. It's often a good idea to search the archives of the EDG for answers to these frequently discussed issues. There is also an excellent article on justification of extrema, "On the Role of Sign Charts in AP Calculus Exams," which can be accessed through AP Central (http://apcentral.collegeboard.com/apc/members/courses/teachers_corner/36693.html). Every teacher of AP Calculus should read this article.

AP Exam Questions

An inspection of recently released AP Calculus Exam free-response questions provides an excellent overview of the rich variety of ways that students' understanding of extrema can be assessed. All of these questions, along with solutions and student samples, are available on AP Central (AP Calculus AB items are found at http://apcentral.collegeboard.com/apc/members/exam/exam_questions/1997.html; AP Calculus BC items are found at http://apcentral.collegeboard.com/apc/members/exam/exam_questions/8031.html).

2008 Exam

- 3b and 3c (AB only): Accumulating rates of change, determine the location and value of the maximum volume of an oil slick on an interval of infinite length.
- 4a (Motion): Find the time that the particle is farthest left (minimum position), given velocity defined graphically.
- 5a (BC only): Given an analytic definition of f', determine the type of relative extreme.
- 6b (AB only): Given analytic definitions of f and f', determine the location and type of relative extreme.

2008 Form B Exam

- 4c (AB only): Determine the location and value of maximum value of composition of two functions, one defined by an integral.
- 5b: Given the graph of g', find the absolute maximum value of g on a closed interval.

2007 Exam

- 2c: Accumulating rates of change, determine the location and value of maximum on a closed interval.
- 4a (Motion): Find the time at which the particle is farthest left (minimum position).
- 6b (Analytic): For a given critical point, determine the type of relative extreme using second derivative test.

2007 Form B Exam:

- 2d (Motion): Find the time that the particle is farthest right (maximum position), given velocity defined analytically and graphically.

- 4a and 4d: Given the graph of f', determine x-value of the relative maximum and minimum values of $f(x)$.
- 5c (Differential equation): For a given point, determine the type of relative extreme using the second derivative test.

2006 Exam

- 3b: Function defined by integral, determine the type of relative extreme of a particular x-value.

2006 Form B Exam

- 2b: Given the graph of f', determine the x-value of absolute maximum.
- 4b: Determine the time at which f is increasing at its greatest rate.

2005 Exam

- 2d: Accumulating rates of change, determine the time and value when the amount of sand is at its minimum.
- 4a and 4c: Given the table of information on f, f', f'', determine the locations and types of relative extrema; given a function defined by integral, do the same.

2005 Form B Exam

- 2c: Accumulating rates of change, determine the time at which the amount of water is at an absolute minimum.

2004 Exam

- 3c: Given velocity defined analytically, determine the maximum position.
- 4c: Given an implicitly defined function, determine the type of relative extreme using the second derivative test.
- 5b and 5c: Given a function defined by integral, determine x-values of relative maximum and absolute minimum values of the function.

2004 Form B Exam

- 2d: Given a rate of change, determine the maximum number of mosquitoes.
- 4b: Given a graph of f', determine x-values of absolute minimum and maximum.
- 6c: Determine the value of n that will maximize the area of region S.

Student Worksheets and Teaching Examples

On the following pages, the first two student worksheets can be used any time after students have learned the closed interval test (or candidates test), the first derivative test and the second derivative test for extrema. I would not use them back-to-back, but would space them apart by a few weeks. The teaching examples and third student worksheet can be used either as each of the relevant topics is covered or after all relevant topics have been covered and the students are reviewing for the AP Exam.

Worksheet 1: Classifying Critical Points, Part I

Complete each statement by choosing one of the four phrases from the box below. Phrases may be used more than once. Unless otherwise specified, assume each function is defined and continuous for all real numbers.

the absolute (global) maximum	the absolute (global) minimum
a local (relative) maximum	a local (relative) minimum

1. A function f defined and continuous on the interval $-2 \le x \le 5$ has critical points (or critical numbers) only at $x = -1$ and $x = 2$. The function f has values as given in the table below.

x	$f(x)$
-2	1
-1	2
2	0
5	2

   The value $x = 2$ locates ____________________ value of the function. The value $f(x) = 2$ is ____________________ value of the function.

2. If $x = 2$ is the only critical point of a function f and $f''(2) < 0$, then $x = 2$ locates ____________________ value of the function.

3. If $f'(2) = 0$ and $f'(x)$ changes from negative to positive at $x = 2$, then $x = 2$ locates

   ____________________ value of the function f.

4. If $f'(2) = 0$ and $f''(2) > 0$, then $x = 2$ locates ____________________ value of the function f.

5. If $x = 2$ is a critical point of the function f, and $f'(x)$ decreases through $x = 2$, then $x = 2$ locates ____________________ value of the function.

6. If a continuous function f increases throughout a closed interval, then the left endpoint of the graph of f on the interval is ____________________ point of the function.

7. A student found the critical points of a function f to be $x = 2$ and $x = 4$, and

produced the chart below.

Interval	$x < 2$	$2 < x < 4$	$x > 4$
$f'(x)$	positive	negative	positive

The value $x = 2$ locates ______________________________ value of the function.

8. If $x = 2$ is the only critical point of a function f and $f''(2) = 3$, then $x = 2$ locates a ______________________ value of the function.

Worksheet 1: Solutions

Note that critical points also are referred to in some texts as *critical numbers* or *critical values*. They are values of x at which a function f satisfies $f'(x) = 0$ or at which $f'(x)$ is not defined (does not exist).

1. the absolute (global) minimum, the absolute (global) maximum
   One also can say, "The value $x = 2$ locates, or is the x-coordinate of, the absolute (global) minimum point of the graph of the function." Note that the graph of f has two absolute (global) maximum points, $(-1, 2)$ and $(5, 2)$, but just one absolute (global) maximum value, $f(-1) = 2 = f(5)$.
2. the absolute (global) maximum
3. a local (relative) minimum
4. a local (relative) minimum
5. a local (relative) maximum
6. the absolute (global) minimum
   One also can say, "The left endpoint of the interval locates the absolute (global) minimum value of the function," or "The left endpoint of the interval locates, or is the x-coordinate of, the absolute minimum point of the graph of the function."
7. a local (relative) maximum
8. the absolute (global) minimum

Worksheet 2: Classifying Critical Points, Part II

Determine if each of the following statements is true or false. If you decide a statement is false, provide a counterexample to show why it is false and then rewrite the statement in order to make it true. Unless otherwise specified, assume each function is defined and continuous for all real numbers.

1. A critical point (or critical number) of a function f of a variable x is the x-coordinate of a relative maximum or minimum value of the function.
2. A continuous function on a closed interval can have only one maximum value.
3. If $f''(x)$ is always positive, then the function f must have a relative minimum value.
4. If a function f has a local minimum value at $x = $ c, then $f'(c) = 0$.
5. If $f'(2) = 0$ and $f''(2) < 0$, then $x = 2$ locates a relative maximum value of f.
6. If $f''(2) < 0$, then $x = c$ is a point of inflection for the function f and cannot be the x-coordinate of a maximum or minimum point on the graph of f.
7. If a function f is defined on a closed interval and $f'(x) > 0$ for all x in the interval, then the absolute maximum value of the function will occur at the right endpoint of the interval.
8. The absolute minimum value of a continuous function on a closed interval can occur at only one point.
9. If $x = 2$ is the only critical point of a function f and $f''(2) > 0$, then $f(2)$ is the minimum value of the function.
10. To locate the absolute extrema of a continuous function on a closed interval, you need only compare the y-values of all critical points.
11. If $f'(c) = 0$ and $f'(x)$ decreases through $x = c$, then $x = c$ locates a local minimum value for the function.
12. Absolute extrema of a continuous function on a closed interval can occur only at endpoints or critical points.

Worksheet 2: Solutions and Teacher Notes

Note to instructors: This is a great worksheet to have students work on in pairs. The discussions can get quite heated! There is certainly more than one counterexample to each false statement and often more than one way to rewrite the false statements; the solutions below provide you with one idea for each. Students might have other ideas that make for good discussion items. To make the work go faster, I often require students only to provide a counterexample or to rewrite the false statement so that it is true.

1. **False**

   Counterexample: Note that $x = 0$ is a critical point for the function $f(x) = x^3$, but that $x = 0$ corresponds to neither a relative maximum nor a relative minimum value of f.

   Rewrite: A critical point is a POSSIBLE location for a relative maximum or minimum value of a function.

2. **True.** Note that the maximum value can occur at more than one x-value but that the maximum value itself is unique. See Problem 8 for an example.

3. **False**

   Counterexample: For $f(x) = e^x$, $f''(x)$ is always positive, but the function $f(x) = e^x$ has no relative extrema.

   Rewrite: If $f'(c) = 0$ and $f''(x)$ is always positive, then the function must have a relative minimum at $x = c$.

4. **False**

   Counterexample: The function $f(x) = |x|$ has a local minimum at $x = 0$, but $f'(0)$ is not defined.

   Rewrite: If a DIFFERENTIABLE function has a local minimum value at $x = c$, then $f'(c) = 0$.

5. **True**

6. **False**

   Counterexample: For the function $f(x) = x^4$, $f''(0) = 0$, but $x = 0$ is not a point of inflection. Note that $x = 0$ does correspond to a relative and absolute minimum value of f.

   Rewrite: If $f''(c) = 0$ for a function f, then $x = c$ may or may not be an inflection point for f and $x = c$ may or may not correspond to a relative minimum or maximum value of f.

7. **True**
8. **False**

   Counterexample: The function $f(x) = \sin(x)$ takes on its minimum value of -1 at the points $x = \frac{3\pi}{2}$, $x = \frac{7\pi}{2}$, and $x = \frac{11\pi}{2}$ in the interval $0 < x < 6\pi$.

   Rewrite: There is exactly one absolute minimum value of a continuous function on a closed interval, but this minimum value can occur at more than one point in the interval. See Problem 2.

9. **True**
10. **False**

    Counterexample: The function $f(x) = x^2$ on the interval $2 \le x \le 6$ has its absolute minimum value at $x = 2$ and its absolute maximum value at $x = 6$. Neither $x = 2$ nor $x = 6$ is a critical point of the function.

    Rewrite: To locate the absolute extrema of a continuous function on a closed interval, you must compare the y-values of all critical points AND ENDPOINTS.

11. **False**

    Counterexample: For $f(x) = 7 - x^2$, $f'(0) = 0$, $f'(x) = -2x$ decreases through $x = 0$, and f has a local (and global) maximum value at $x = 0$.

    Rewrite: If $f'(c) = 0$ and $f'(x)$ decreases through $x = c$, then $x = c$ locates a local MAXIMUM value for the function. Or, if $f'(c) = 0$ and $f'(x)$ INCREASES through $x = c$, then $x = c$ locates a local minimum value for the function.

12. **True**

Teaching Examples for Extrema

Example 1 (noncalculator): Given the function $f(x) = -x - e^{(1-x)}$ on the closed interval [0,3], convince me in three different ways that the maximum value of $f(x)$ occurs at $x = 1$.

Solution and explanation:

Method 1: Closed Interval (or Candidates) Test
Since $f'(x) = -1 + e^{1-x}$ only for $x = 1$, then $x = 1$ is the only critical point (or critical number) for f.

x	$f(x)$
0	$2e$
1	-2
3	$-3 - 1/e^2$

Since f is continuous on the interval [0,3] and $f(1) = -2$ is greater than the values of $f(x)$ at either endpoint, the absolute maximum occurs at $x = 1$. Note that the absolute maximum value is $f(1) = -2$ and the absolute maximum point is $(1, -2)$.

Method 2: First Derivative Test
Since $f'(x) = -1 + e^{1-x}$ only for $x = 1$, then $x = 1$ is the only critical point for f.

Interval	$0 \le x < 1$	$1 < x \le 3$
$f'(x)$	positive	negative
$f(x)$	increasing	decreasing

Since $x = 1$ is the only critical point in the interval, f is continuous at $x = 1$, and $f'(x)$ goes from positive to negative at $x = 1$, then the function f must go from increasing to decreasing there, thus establishing an absolute maximum at $x = 1$.

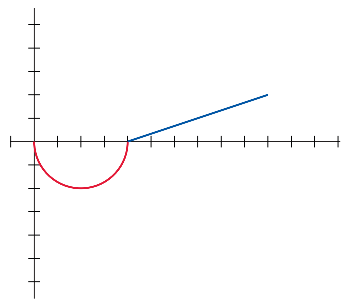
Note that the First Derivative Test (normally a test for relative, or local, extrema) works here because $x = 1$ is the only critical point in the interval, so the function f can change from increasing to decreasing or from decreasing to increasing at most once in the interval. We also could argue that since $f'(x)$ is positive for $0 \le x < 1$ and negative for $1 < x \le 3$, then $f(x)$ increases for $0 < x < 1$ and decreases for $1 < x \le 3$. Since f is continuous at $x = 1$, this behavior guarantees that f has an absolute maximum value at $x = 1$.

Method 3: Second Derivative Test
Since $f'(x) = -1 + e^{1-x} = 0$ only for $x = 1$, then $x = 1$ is the only critical point for f. Since $f''(x) = -e^{1-x}$, then $f''(1) = -1 < 0$. Since $x = 1$ is the only critical point, f is continuous near $x = 1$, $f'(1) = 0$ and $f''(1)$ is negative, we have established an absolute maximum at $x = 1$.

Note that the Second Derivative Test (normally a test for relative, or local, extrema) works here because $x = 1$ is the only critical point in the interval, so the function f can change from increasing to decreasing or from decreasing to increasing at most once in the interval. We also could argue that since $f''(x)$ is negative for $0 \le x \le 3$ and $f'(1) = 0$, then the graph of $f(x)$ is concave down throughout the interval with a horizontal tangent at the point $(1, -2)$. This behavior guarantees that f has an absolute maximum value of -2 at $x = 1$.

Note to instructors: By doing the same problem using three different methods, students can discuss the relative merits of each method and the importance of particular phrasing, such as specifying in the first and second derivative tests that $x = 1$ is the ONLY critical point.

Example 2 (graphical, noncalculator): The graph of $f'(x)$, shown below on the interval [0,10], consists of a semicircle of radius 2 and a line segment meeting at the point (4,0). The line segment has right endpoint (10,2).



Graph of f'

a. For what value of x, $0 < x < 10$, does f have a local minimum? Justify your answer.

b. For what value of x, $0 \le x \le 10$, does f have an absolute maximum? Justify your answer.

Solution and explanation:

a. f has a local minimum at $x = 4$ since $f'(x)$ goes from negative (graph of f' below the x-axis) to positive (graph of f' above the x-axis) at that point, indicating that f goes from decreasing to increasing at $x = 4$, thus establishing a local minimum there.

b. There are no values of x for which $f'(x)$ goes from positive to negative. This means there are no internal candidates for the absolute maximum, hence the maximum must occur at one of the endpoints. Since the amount of decrease (represented by the area below the x-axis or 2π) is greater than the amount of increase (represented by the area above the x-axis or 6), the absolute maximum value of f must be at the left endpoint $x = 0$. More specifically, suppose $f(0) = y_0$. Then $f(10) = y_0 - 2\pi + 6 < y_0$. Hence f has its absolute maximum value at $x = 0$. Note also that $f(4) = y_0 - 2\pi$. Since $f(4) = y_0 - 2\pi$ is smaller than $f(0) = y_0$ and smaller than $f(10) = y_0 - 2\pi + 6$, then f has its absolute minimum value at $x = 4$.

Example 3 (accumulating rates of change, calculator active): The rate, in gallons per hour, at which water flows into a tank over the time interval $0 \le t \le 3$, where t is measured in hours, is given by $R(t) = t^2 \sin(t)$ and the rate, in gallons per hour, at which water flows out of the tank over the same time interval is given by $S(t) = \dfrac{2}{(t+1)^2}$. The tank initially holds 10 gallons of water.

a. When is the rate at which water flows into the tank the greatest? Justify your answer.

b. Write an expression that represents the amount $A(t)$ of water in the tank at any given time t.

c. Determine when there is the least amount of water in the tank. How much water is in the tank at this time? Show the analysis that leads to your conclusion.

Solution and explanation:

a. To find the value of t for which the rate at which water flows into the tank is greatest, we seek the maximum value of $R(t)$. Start by setting $R'(t) = 0$: $R'(t) = 2t\sin t + t^2 \cos t = 0$. Solving on our calculator, we see that $R'(t) = 0$ when $t = 0$ or when $t = 2.289$. We can now choose among three methods of justification (closed interval test, first derivative test or second derivative test). You can have different groups of students use each method and then have them display their work for the rest of the class to discuss and critique. I choose to use the second derivative test since it gives me the opportunity to review the keystrokes for NDeriv. Since $t = 2.289$ is the only critical point in the interior of the interval, $R'(2.289) = 0$, and $R''(2.289) = -8.464$, which is negative, we have established an absolute maximum for R at $t = 2.289$.

b. $A(t) = 10 + \int_0^t \left(x^2 \sin x - \dfrac{2}{(x+1)^2} \right) dx,\ 0 \le t \le 3.$

c. The least amount of water means the absolute minimum value of $A(t)$. Start by using the Second Fundamental Theorem of Calculus to find $A'(t) = t^2 \sin t - \dfrac{2}{(t+1)^2}$. Set $A'(t) = 0$ and solve using the calculator to find $t = 0.867$ hours. Again, we have three possible methods of justification. Since $t = 0.867$ is our only critical point and $A'(t)$ goes from negative to positive there, we have established an absolute minimum at $t = 0.867$ hours. Finally, evaluate $A(0.867)$ using the FnInt feature of the calculator to find that there were 9.201 gallons of water in the tank at that time.

Example 4 (motion, noncalculator): The velocity of a particle moving on the x-axis is given by $v(t) = t^3 - 6t^2$ for the time interval $0 \le t \le 10$.

a. When is the particle farthest to the left?

b. When is the velocity of the particle increasing the fastest?

Solution and explanation:

a. The phrase "farthest to the left" tells us we are looking for the minimum position. To find minimum position, we start by seeing where the derivative of position (which is velocity) equals zero or fails to exist: $x'(t) = v(t) = t^3 - 6t^2 = 0$ for $t = 0$ or $t = 6$.

Time interval	$0 < t < 6$	$6 < t \le 10$
Sign of velocity	negative	positive
Direction of movement	left	right

Since the particle moves to the left for $0 < t < 6$ (indicated by negative velocity) and moves to the right for $6 < t \le 10$, it is farthest to the left at $t = 6$.

b. To find where velocity is increasing the fastest, we really are looking for the maximum acceleration. Acceleration is given by $a(t) = 3t^2 - 12t$. Since $a'(t) = 6t - 12 = 0$ for $t = 2$ only, the only critical point is at $t = 2$. Compare the values of $a(t)$ at the critical point and endpoints:

t	$a(t)$
0	0
2	-12
10	180

Therefore, velocity is increasing the fastest at $t = 10$.

Or, students may note that $a(t)$ is a quadratic function for which the graph is a parabola opening upward. Since the vertex of the parabola has t-coordinate $t = 2$ in the interval $[0,10]$, then the minimum acceleration occurs at $t = 2$ and the maximum acceleration occurs at one of the endpoints of the interval. Comparing values as above, we see that the maximum acceleration occurs at $t = 10$.

Example 5 (differential equation, noncalculator): Let $\frac{dy}{dx} = 2x + y$.

a. How do we know that $(1, -2)$ is a critical point for the function $y = f(x)$?

b. Is the point $(1, -2)$ a relative maximum point, relative minimum point, or neither? Justify your answer.

Solution and explanation:

This is a great problem for which to have students draw a slope field, so they can see that the particular solution passing through $(1, -2)$ looks as if it has a relative minimum at this point. In calculus, though, seeing is not enough to make us believe. We need analysis to verify our answer.

a. We know that $(1, -2)$ is a critical point since the value of the derivative $\frac{dy}{dx}$ at $x = 1, y = -2$ is zero $\left(\frac{dy}{dx} = 2x + y = 2 \cdot 1 + (-2) = 0\right)$.

b. Find $\frac{d^2y}{dx^2} = 2 + \frac{dy}{dx}$. Since we already know the first derivative at $(1, -2)$ is zero, we now know that the second derivative at this point is positive $\left(\frac{d^2y}{dx^2} = 2 + \frac{dy}{dx} = 2 + 0 = 2 > 0\right)$. Therefore, $(1, -2)$ is a relative minimum point.

You might discuss with students why a closed interval test or first derivative test would be impossible for this problem.

Worksheet 3: Extrema in a Variety of Settings

1. (noncalculator) The minimum value of $f(x) = x^4 - 4x^3 + k$ is 7. Determine the value of k.

2. (noncalculator) Suppose that $P(t)$ measures the proportion of the normal oxygen level in a pond, with $P(t) = 1$ corresponding to the normal (unpolluted) level and $0 \le P(t) \le 1$. The time t is measured in weeks with $t \ge 0$. At time $t = 0$, organic waste is dumped into the pond and, as the waste material oxidizes, the proportion of the normal oxygen level in the pond is given by $P(t) = \dfrac{t^2 - t + 1}{t^2 + 1}$, t > 0.

   a. At what time t is the proportion of the normal oxygen level in the pond the least?

   b. At what time t is the proportion of the normal oxygen level in the pond increasing most rapidly?

3. (noncalculator) The curve with derivative $\dfrac{dy}{dx} = \dfrac{3 - x}{y + 2}$ has $y = -3$ as a tangent line.

   At what point is the line tangent to the curve? Determine if the point that you found is a relative maximum point, relative minimum point or neither for the curve. Justify your answer.

4. (calculator active) The rate of change R, in kilometers per hour, of the altitude of a hot air balloon is given by $R(t) = t^3 - 4t^2 + 6$ for time $0 \le t \le 4$, where t is measured in hours. Assume the balloon is initially at ground level.

   a. What is the maximum altitude of the balloon during the interval $0 \le t \le 4$?

   b. At what time is the altitude of the balloon increasing most rapidly?

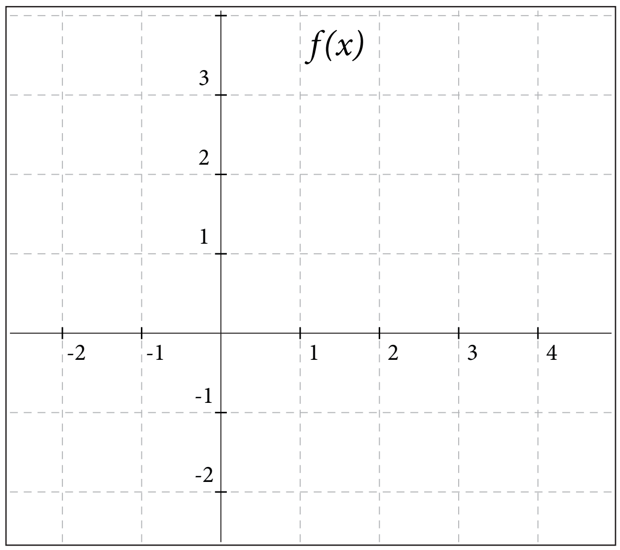
5. (noncalculator) Let f be a function that is continuous on the interval $[-1, 4]$. The function f is twice differentiable except at $x = 1$, and f and its derivatives have the properties indicated in the table below, where "DNE" indicates that the derivatives of f do not exist at $x = 1$.

x	-1	$-1<x<0$	0	$0<x<1$	1	$1<x<2$	2	$2<x<4$	4
$f(x)$	1	positive	0	negative	-1	negative	0	positive	3
$f'(x)$	-6	negative	-1	negative	DNE	positive	-1	positive	8
$f''(x)$	3	positive	0	negative	DNE	negative	0	positive	4

a. For $-1 < x < 4$, find all values of x at which f has a relative extreme. For each of these x-values, determine whether f has a relative maximum or minimum. Justify your answer.

b. For $-1 \le x \le 4$, find the maximum value of f. Justify your answer.

c. On the axes provided, sketch the graph of a function that has all the characteristics of f.



d. Let h be the function defined by $h(x) = \int_{-1}^{x} f(t)\,dt$ on the interval $[-1, 4]$.

For $-1 < x < 4$, find all values of x at which h has a relative extreme. For each of these x-values, determine whether h has a relative maximum or minimum. Justify your answers.

6. (calculator active) The velocity of a particle moving along the x-axis is given by $v(t) = t^{1/2}\cos t$ for time $0 \le t \le 6.5$. When $t = 0$, the particle is at $x = -1$.

   a. Write an expression for the position of the particle at any time t.

   b. Determine when the particle is farthest from the origin.

   c. When the particle is farthest from the origin, is the velocity increasing or decreasing?

Worksheet 3: Solutions and Teacher Notes

1. First, set the derivative equal to zero to find critical numbers. That is, solve the equation $f'(x) = 4x^3 - 12x^2 = 0$ to obtain the critical numbers $x = 0$ and $x = 3$. A first derivative test reveals that the minimum must occur at $x = 3$.

	$x < 0$	$0 < x < 3$	$x > 3$
$f'(x)$	negative	negative	positive
$f(x)$	decreasing	decreasing	increasing

Substituting $x = 3$ into the original function and setting the result equal to 7 yields $k = 34$. That is, the equation $f(3) = 3^4 - 4 \cdot 3^3 + k = 7$ has solution $k = 34$.

2. a. Since we are looking for the minimum of P, we compute $P'(t) = \dfrac{t^2 - 1}{(t^2+1)^2}$ and solve $P'(t) = 0$ to obtain $t = 1$ and $t = -1$. Since $t = -1$ is not in our domain, our only critical point is $t = 1$. Since $t = 1$ is the only critical point for $t \geq 0$ and is negative for $0 \leq t < 1$ and positive for $t > 1$, we have established that the minimum proportion of the normal oxygen level in the pond occurs at $t = 1$ week.

   b. Since we are looking for the maximum of $P'(t)$, we need to compute the derivative of $P'(t)$, $P''(t) = \dfrac{-2t(t^2-3)}{(t^2+1)^3} = 0$. We then solve $P''(t) = 0$ to obtain $t = 0$ and $t = \pm\sqrt{3}$. Since $t = \sqrt{3}$ is the only critical point in the interior of our domain and $P''(t)$ changes from positive to negative at $t = \sqrt{3}$ (which means $P'(t)$ goes from increasing to decreasing), then the proportion of the normal oxygen level in the pond is increasing most rapidly at $t = \sqrt{3}$.

3. The key to understanding this problem is in realizing that the tangent line to the curve is horizontal, which means that the slope, $\dfrac{dy}{dx}$, must equal zero. Note that $\dfrac{dy}{dx} = \dfrac{3-x}{y+2} = 0$ only at $x = 3$, $y \neq -2$. We conclude that the line $y = -3$ is tangent to the curve at the point $(3, -3)$. To determine the type of point we are working with, we must use the second derivative test. Using the quotient rule to find $\dfrac{d^2y}{dx^2} = \dfrac{-(y+2) - (3-x)\dfrac{dy}{dx}}{(y+2)^2}$ and evaluating at the point $(3, -3)$, we find the value of the second derivative to be positive $\left(\dfrac{d^2y}{dx^2} = 1\right)$, which means the point $(3, -3)$ must be a relative (local) minimum point.

4. a. By examining the graph of R on the graphing calculator, you can see that $R(t)$ is positive for $0 < t < 1.572$, negative for $1.572 < t < 3.514$, and positive for $3.514 < t < 4$. This means that the balloon rises, falls and then rises. The maximum altitude is either at $t = 1.572$ or $t = 4$. Since the area below the x-axis for $1.572 < t < 3.514$ is larger than the area above the x-axis for $3.514 < t < 4$, the decrease in altitude is greater than the increase in altitude during the last time interval $(3.514 < t < 4)$, so the maximum altitude occurs at $t = 1.572$.

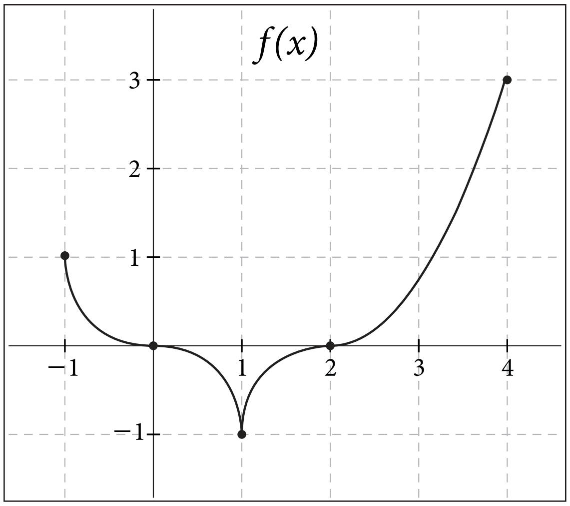
   Indeed, we can check that the altitude at $t = 1.572$ hours is $\int_0^{1.572} R(t)\,dt = 5.779$ kilometers, while the altitude at $t = 4$ hours is $\int_0^4 R(t)\,dt = 2.667$ km.

   b. We are looking for the time when $R(t)$ is a maximum. Therefore, we need to set the derivative of $R(t)$, $R'(t)$, equal to zero: $R'(t) = 3t^2 - 8t = 0$ when $t = 0$ or $t = 8/3$. Since $t = 8/3$ is the only critical point in the interior of our domain and $R''(t) = 6t$ is positive at $t = 8/3$, R has a local minimum there. (In fact, $R(8/3) = -3.481$ km/hr, showing that the altitude of the balloon is decreasing at $t = 8/3$.) Therefore, we should check the values of $R(t)$ at the two endpoints $t = 0$ and $t = 4$. Since $R(0) = R(4) = 6$ kilometers per hour, the altitude of the balloon is increasing most rapidly at $t = 0$ and $t = 4$.

   I like to include problems like this because students always assume the maximum or minimum will occur at the critical point they have found (and many textbooks reinforce this idea because that's how ALL of the problems turn out). It's good to occasionally have problems with endpoint answers.

5. a. f has a relative minimum at $x = 1$ since the sign of $f'(x)$ changes from negative to positive at that point, which indicates that the function changes from decreasing to increasing at $x = 1$, giving us a relative minimum.

   b. Since there are no points where $f'(x)$ changes from positive to negative there are no interior candidates for our maximum value. Therefore, the maximum must occur at an endpoint. Since $f(4) = 3 > 1 = f(-1)$, the maximum value of the function is $f(4) = 3$.

c. 

d. Using the Fundamental Theorem of Calculus, $h'(x) = f(x)$. Since $h'(x) = f(x)$ changes from positive to negative at $x = 0$, there is a relative maximum at $x = 0$. Since $h'(x) = f(x)$ changes from negative to positive at $x = 2$, there is relative minimum at $x = 2$.

6. a. $x(t) = x(0) + \int_0^t v(u)\,du = -1 + \int_0^t u^{1/2}\cos u\ du$. Integrating the velocity gives the net change in position and, if we add the initial position to the change in position, we get the new position.

b. We compare values of the position function x at the endpoints and at all critical points. Critical points for the position function occur when the derivative of the position, which is the velocity, equals zero. Since $x'(t) = v(t) = t^{1/2}\cos t = 0$ for $t = 0$, $t = \pi/2$, and $t = 3\pi/2$, the critical points are $t = 0$, $t = \pi/2$ and $t = 3\pi/2$. We use our solution from part a to find the position $x(t)$ at the critical points and endpoints.

t	0	$\pi/2$	$3\pi/2$	6.5
$x(t)$	-1	-0.296	-3.819	-0.887

The particle is farthest from the origin at $t = 3\pi/2$.

c. To determine if velocity is increasing or decreasing, we should check the sign of $v'(t)$. Using the NDeriv feature of the calculator, $v'\left(\frac{3\pi}{2}\right) = 2.171$, which indicates velocity is increasing because its derivative is positive.

This part of the problem has very little to do with extrema, but I try to find every opportunity I can to reinforce students' calculator skills. They seem to get lots of practice evaluating definite integrals on the calculator but much less practice evaluating derivatives.

About the Author

Dixie Ross is a classroom teacher at Pflugerville High School in Pflugerville, Texas. She started the Advanced Placement® program at Taylor High School in Taylor, Texas, in 1989, and began working as a College Board consultant and workshop leader in 1994. She assisted with the development of the AP Mathematics Vertical Teams Toolkit, the Setting the Cornerstones™ workshop, the Building Success in Mathematics workshop, and the *AP Vertical Teams® Guide for Mathematics and Statistics*. Ross has served as a Reader for the AP Calculus Exam.