7.8 Factoring Polynomials Completely

Essential Question

How can you factor a polynomial completely?

EXPLORATION 1 Writing a Product of Linear Factors

Work with a partner. Write the product represented by the algebra tiles. Then multiply to write the polynomial in standard form.

a. 

b. 

c. 

d. 

e. 

f. 

REASONING ABSTRACTLY

To be proficient in math, you need to know and flexibly use different properties of operations and objects.

EXPLORATION 2 Matching Standard and Factored Forms

Work with a partner. Match the standard form of the polynomial with the equivalent factored form. Explain your strategy.

a. $x^3 + x^2$	**A.** $x(x + 1)(x - 1)$
b. $x^3 - x$	**B.** $x(x - 1)^2$
c. $x^3 + x^2 - 2x$	**C.** $x(x + 1)^2$
d. $x^3 - 4x^2 + 4x$	**D.** $x(x + 2)(x - 1)$
e. $x^3 - 2x^2 - 3x$	**E.** $x(x - 1)(x - 2)$
f. $x^3 - 2x^2 + x$	**F.** $x(x + 2)(x - 2)$
g. $x^3 - 4x$	**G.** $x(x - 2)^2$
h. $x^3 + 2x^2$	**H.** $x(x + 2)^2$
i. $x^3 - x^2$	**I.** $x^2(x - 1)$
j. $x^3 - 3x^2 + 2x$	**J.** $x^2(x + 1)$
k. $x^3 + 2x^2 - 3x$	**K.** $x^2(x - 2)$
l. $x^3 - 4x^2 + 3x$	**L.** $x^2(x + 2)$
m. $x^3 - 2x^2$	**M.** $x(x + 3)(x - 1)$
n. $x^3 + 4x^2 + 4x$	**N.** $x(x + 1)(x - 3)$
o. $x^3 + 2x^2 + x$	**O.** $x(x - 1)(x - 3)$

Communicate Your Answer

3. How can you factor a polynomial completely?

4. Use your answer to Question 3 to factor each polynomial completely.

a. $x^3 + 4x^2 + 3x$ **b.** $x^3 - 6x^2 + 9x$ **c.** $x^3 + 6x^2 + 9x$

7.8 Lesson

Core Vocabulary

factoring by grouping, *p. 404*
factored completely, *p. 404*

Previous
polynomial
binomial

What You Will Learn

- Factor polynomials by grouping.
- Factor polynomials completely.
- Use factoring to solve real-life problems.

Factoring Polynomials by Grouping

You have used the Distributive Property to factor out a greatest common monomial from a polynomial. Sometimes, you can factor out a common binomial. You may be able to use the Distributive Property to factor polynomials with four terms, as described below.

Core Concept

Factoring by Grouping

To factor a polynomial with four terms, group the terms into pairs. Factor the GCF out of each pair of terms. Look for and factor out the common binomial factor. This process is called **factoring by grouping**.

EXAMPLE 1 Factoring by Grouping

Factor each polynomial by grouping.

a. $x^3 + 3x^2 + 2x + 6$ **b.** $x^2 + y + x + xy$

SOLUTION

a. $x^3 + 3x^2 + 2x + 6 = (x^3 + 3x^2) + (2x + 6)$ Group terms with common factors.

Common binomial factor is $x + 3$. → $= x^2(x + 3) + 2(x + 3)$ Factor out GCF of each pair of terms.

$= (x + 3)(x^2 + 2)$ Factor out $(x + 3)$.

▶ So, $x^3 + 3x^2 + 2x + 6 = (x + 3)(x^2 + 2)$.

b. $x^2 + y + x + xy = x^2 + x + xy + y$ Rewrite polynomial.

$= (x^2 + x) + (xy + y)$ Group terms with common factors.

Common binomial factor is $x + 1$. → $= x(x + 1) + y(x + 1)$ Factor out GCF of each pair of terms.

$= (x + 1)(x + y)$ Factor out $(x + 1)$.

▶ So, $x^2 + y + x + xy = (x + 1)(x + y)$.

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Factor the polynomial by grouping.

1. $a^3 + 3a^2 + a + 3$ **2.** $y^2 + 2x + yx + 2y$

Factoring Polynomials Completely

You have seen that the polynomial $x^2 - 1$ can be factored as $(x + 1)(x - 1)$. This polynomial is factorable. Notice that the polynomial $x^2 + 1$ cannot be written as the product of polynomials with integer coefficients. This polynomial is unfactorable. A factorable polynomial with integer coefficients is **factored completely** when it is written as a product of unfactorable polynomials with integer coefficients.

Concept Summary

Guidelines for Factoring Polynomials Completely

To factor a polynomial completely, you should try each of these steps.

1.	Factor out the greatest common monomial factor.	$3x^2 + 6x = 3x(x + 2)$
2.	Look for a difference of two squares or a perfect square trinomial.	$x^2 + 4x + 4 = (x + 2)^2$
3.	Factor a trinomial of the form $ax^2 + bx + c$ into a product of binomial factors.	$3x^2 - 5x - 2 = (3x + 1)(x - 2)$
4.	Factor a polynomial with four terms by grouping.	$x^3 + x - 4x^2 - 4 = (x^2 + 1)(x - 4)$

EXAMPLE 2 Factoring Completely

Factor (a) $3x^3 + 6x^2 - 18x$ and (b) $7x^4 - 28x^2$.

SOLUTION

a. $3x^3 + 6x^2 - 18x = 3x(x^2 + 2x - 6)$ Factor out $3x$.

$x^2 + 2x - 6$ is unfactorable, so the polynomial is factored completely.

▶ So, $3x^3 + 6x^2 - 18x = 3x(x^2 + 2x - 6)$.

b. $7x^4 - 28x^2 = 7x^2(x^2 - 4)$ Factor out $7x^2$.

$= 7x^2(x^2 - 2^2)$ Write as $a^2 - b^2$.

$= 7x^2(x + 2)(x - 2)$ Difference of two squares pattern

▶ So, $7x^4 - 28x^2 = 7x^2(x + 2)(x - 2)$.

EXAMPLE 3 Solving an Equation by Factoring Completely

Solve $2x^3 + 8x^2 = 10x$.

SOLUTION

$2x^3 + 8x^2 = 10x$ Original equation

$2x^3 + 8x^2 - 10x = 0$ Subtract $10x$ from each side.

$2x(x^2 + 4x - 5) = 0$ Factor out $2x$.

$2x(x + 5)(x - 1) = 0$ Factor $x^2 + 4x - 5$.

$2x = 0$ *or* $x + 5 = 0$ *or* $x - 1 = 0$ Zero-Product Property

$x = 0$ *or* $x = -5$ *or* $x = 1$ Solve for x.

▶ The roots are $x = -5$, $x = 0$, and $x = 1$.

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Factor the polynomial completely.

3. $3x^3 - 12x$ **4.** $2y^3 - 12y^2 + 18y$ **5.** $m^3 - 2m^2 - 8m$

Solve the equation.

6. $w^3 - 8w^2 + 16w = 0$ **7.** $x^3 - 25x = 0$ **8.** $c^3 - 7c^2 + 12c = 0$

Solving Real-Life Problems

EXAMPLE 4 **Modeling with Mathematics**

A terrarium in the shape of a rectangular prism has a volume of 4608 cubic inches. Its length is more than 10 inches. The dimensions of the terrarium in terms of its width are shown. Find the length, width, and height of the terrarium.

SOLUTION

1. **Understand the Problem** You are given the volume of a terrarium in the shape of a rectangular prism and a description of the length. The dimensions are written in terms of its width. You are asked to find the length, width, and height of the terrarium.

2. **Make a Plan** Use the formula for the volume of a rectangular prism to write and solve an equation for the width of the terrarium. Then substitute that value in the expressions for the length and height of the terrarium.

3. **Solve the Problem**

$\text{Volume} = \text{length} \cdot \text{width} \cdot \text{height}$ — Volume of a rectangular prism

$4608 = (36 - w)(w)(w + 4)$ — Write equation.

$4608 = 32w^2 + 144w - w^3$ — Multiply.

$0 = 32w^2 + 144w - w^3 - 4608$ — Subtract 4608 from each side.

$0 = (-w^3 + 32w^2) + (144w - 4608)$ — Group terms with common factors.

$0 = -w^2(w - 32) + 144(w - 32)$ — Factor out GCF of each pair of terms.

$0 = (w - 32)(-w^2 + 144)$ — Factor out $(w - 32)$.

$0 = -1(w - 32)(w^2 - 144)$ — Factor -1 from $-w^2 + 144$.

$0 = -1(w - 32)(w - 12)(w + 12)$ — Difference of two squares pattern

$w - 32 = 0$ *or* $w - 12 = 0$ *or* $w + 12 = 0$ — Zero-Product Property

$w = 32$ *or* $w = 12$ *or* $w = -12$ — Solve for w.

Disregard $w = -12$ because a negative width does not make sense. You know that the length is more than 10 inches. Test the solutions of the equation, 12 and 32, in the expression for the length.

$\text{length} = 36 - w = 36 - 12 = 24$ ✓ *or* $\text{length} = 36 - w = 36 - 32 = 4$ ✗

The solution 12 gives a length of 24 inches, so 12 is the correct value of w.

Use $w = 12$ to find the height, as shown.

$\text{height} = w + 4 = 12 + 4 = 16$

▶ The width is 12 inches, the length is 24 inches, and the height is 16 inches.

4. **Look Back** Check your solution. Substitute the values for the length, width, and height when the width is 12 inches into the formula for volume. The volume of the terrarium should be 4608 cubic inches.

Check

$V = \ell wh$

$4608 \stackrel{?}{=} 24(12)(16)$

$4608 = 4608$ ✓

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9. A box in the shape of a rectangular prism has a volume of 72 cubic feet. The box has a length of x feet, a width of $(x - 1)$ feet, and a height of $(x + 9)$ feet. Find the dimensions of the box.

7.8 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** What does it mean for a polynomial to be factored completely?

2. **WRITING** Explain how to choose which terms to group together when factoring by grouping.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–10, factor the polynomial by grouping. *(See Example 1.)*

3. $x^3 + x^2 + 2x + 2$
4. $y^3 - 9y^2 + y - 9$
5. $3z^3 + 2z - 12z^2 - 8$
6. $2s^3 - 27 - 18s + 3s^2$
7. $x^2 + xy + 8x + 8y$
8. $q^2 + q + 5pq + 5p$
9. $m^2 - 3m + mn - 3n$
10. $2a^2 + 8ab - 3a - 12b$

In Exercises 11–22, factor the polynomial completely. *(See Example 2.)*

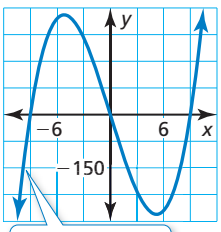
11. $2x^3 - 2x$
12. $36a^4 - 4a^2$
13. $2c^2 - 7c + 19$
14. $m^2 - 5m - 35$
15. $6g^3 - 24g^2 + 24g$
16. $-15d^3 + 21d^2 - 6d$
17. $3r^5 + 3r^4 - 90r^3$
18. $5w^4 - 40w^3 + 80w^2$
19. $-4c^4 + 8c^3 - 28c^2$
20. $8t^2 + 8t - 72$
21. $b^3 - 5b^2 - 4b + 20$
22. $h^3 + 4h^2 - 25h - 100$

In Exercises 23–28, solve the equation. *(See Example 3.)*

23. $5n^3 - 30n^2 + 40n = 0$
24. $k^4 - 100k^2 = 0$
25. $x^3 + x^2 = 4x + 4$
26. $2t^5 + 2t^4 - 144t^3 = 0$
27. $12s - 3s^3 = 0$
28. $4y^3 - 7y^2 + 28 = 16y$

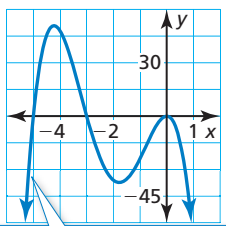
In Exercises 29–32, find the x-coordinates of the points where the graph crosses the x-axis.

29.



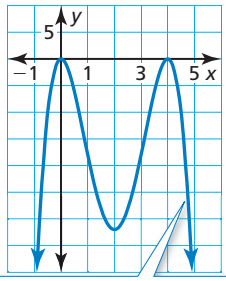
$y = x^3 - 81x$

30.



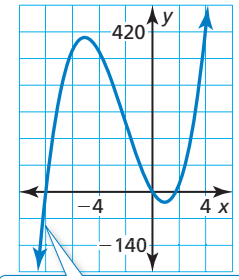
$y = -3x^4 - 24x^3 - 45x^2$

31.



$y = -2x^4 + 16x^3 - 32x^2$

32.



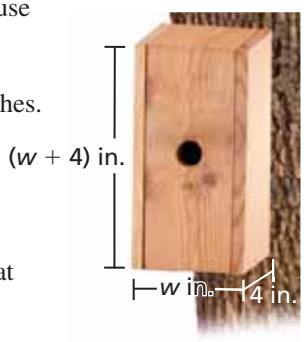
$y = 4x^3 + 25x^2 - 56x$

ERROR ANALYSIS **In Exercises 33 and 34, describe and correct the error in factoring the polynomial completely.**

33. ✗ $a^3 + 8a^2 - 6a - 48 = a^2(a + 8) + 6(a + 8)$
$= (a + 8)(a^2 + 6)$

34. ✗ $x^3 - 6x^2 - 9x + 54 = x^2(x - 6) - 9(x - 6)$
$= (x - 6)(x^2 - 9)$

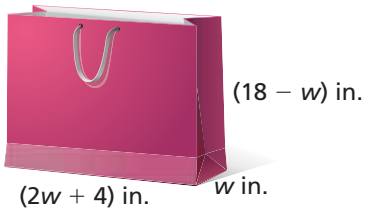
35. **MODELING WITH MATHEMATICS** You are building a birdhouse in the shape of a rectangular prism that has a volume of 128 cubic inches. The dimensions of the birdhouse in terms of its width are shown. *(See Example 4.)*



a. Write a polynomial that represents the volume of the birdhouse.

b. What are the dimensions of the birdhouse?

36. MODELING WITH MATHEMATICS A gift bag shaped like a rectangular prism has a volume of 1152 cubic inches. The dimensions of the gift bag in terms of its width are shown. The height is greater than the width. What are the dimensions of the gift bag?



In Exercises 37–40, factor the polynomial completely.

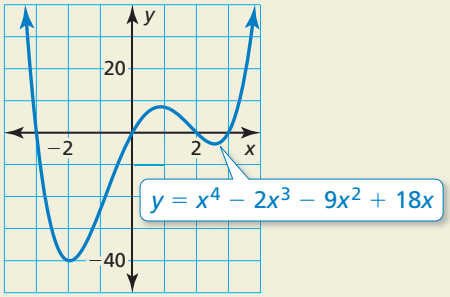
37. $x^3 + 2x^2y - x - 2y$ **38.** $8b^3 - 4b^2a - 18b + 9a$

39. $4s^2 - s + 12st - 3t$

40. $6m^3 - 12mn + m^2n - 2n^2$

41. WRITING Is it possible to find three real solutions of the equation $x^3 + 2x^2 + 3x + 6 = 0$? Explain your reasoning.

42. HOW DO YOU SEE IT? How can you use the factored form of the polynomial $x^4 - 2x^3 - 9x^2 + 18x = x(x - 3)(x + 3)(x - 2)$ to find the x-intercepts of the graph of the function?



43. OPEN-ENDED Write a polynomial of degree 3 that satisfies each of the given conditions.

a. is not factorable **b.** can be factored by grouping

44. MAKING AN ARGUMENT Your friend says that if a trinomial cannot be factored as the product of two binomials, then the trinomial is factored completely. Is your friend correct? Explain.

45. PROBLEM SOLVING The volume (in cubic feet) of a room in the shape of a rectangular prism is represented by $12z^3 - 27z$. Find expressions that could represent the dimensions of the room.

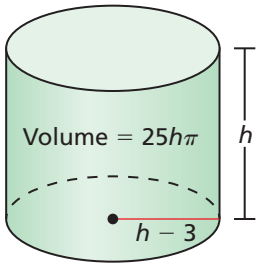
46. MATHEMATICAL CONNECTIONS The width of a box in the shape of a rectangular prism is 4 inches more than the height h. The length is the difference of 9 inches and the height.

a. Write a polynomial that represents the volume of the box in terms of its height (in inches).

b. The volume of the box is 180 cubic inches. What are the possible dimensions of the box?

c. Which dimensions result in a box with the least possible surface area? Explain your reasoning.

47. MATHEMATICAL CONNECTIONS The volume of a cylinder is given by $V = \pi r^2 h$, where r is the radius of the base of the cylinder and h is the height of the cylinder. Find the dimensions of the cylinder.



48. THOUGHT PROVOKING Factor the polynomial $x^5 - x^4 - 5x^3 + 5x^2 + 4x - 4$ completely.

49. REASONING Find a value for w so that the equation has (a) two solutions and (b) three solutions. Explain your reasoning.

$$5x^3 + wx^2 + 80x = 0$$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the system of linear equations by graphing. *(Section 5.1)*

50. $y = x - 4$, $y = -2x + 2$ **51.** $y = \frac{1}{2}x + 2$, $y = 3x - 3$ **52.** $5x - y = 12$, $\frac{1}{4}x + y = 9$ **53.** $x = 3y$, $y - 10 = 2x$

Graph the function. Describe the domain and range. *(Section 6.3)*

54. $f(x) = 5^x$ **55.** $y = 9\left(\frac{1}{3}\right)^x$ **56.** $y = -3(0.5)^x$ **57.** $f(x) = -3(4)^x$